Fachbereich Mathematik
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Winter Semester 2021/22, Set 5

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Commutative Algebra

Submit by: Monday, 22/11/2021, 10 am

Exercise 15: (No rigorous proof is needed.)

a. Consider the $\mathbb{Z}$-modules $M = \mathbb{Z}/2\mathbb{Z}$ and $N = \mathbb{Z}/4\mathbb{Z}$. How many elements does $M \otimes_{\mathbb{Z}} N$ have? Is it isomorphic to a $\mathbb{Z}$-module that you know?

b. Consider the $\mathbb{Z}$-module $M = \mathbb{Z}^3 \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/5\mathbb{Z}$ and the $\mathbb{Q}$-vector space $M \otimes_{\mathbb{Z}} \mathbb{Q}$. What is its dimension?

Exercise 16: Let R be a ring, M a finitely generated R-module and $\varphi \in \mathrm{Hom}_R(M, R^n)$ surjective. Show that $\ker(\varphi)$ is finitely generated as an R-module.

Hint, consider the exact sequence $0 \to \ker(\varphi) \to M \to R^n \to 0$.

Exercise 17: Let R be a ring, M and N be R-modules, and suppose $N = \langle n_\lambda \mid \lambda \in \Lambda \rangle$. Show:

a. $M \otimes_R N = \left\{ \sum_{\lambda \in \Lambda} m_\lambda \otimes n_\lambda \;\middle|\; m_\lambda \in M \text{ and only finitely many } m_\lambda \neq 0 \right\}$.

b. Let $x = \sum_{\lambda \in \Lambda} m_\lambda \otimes n_\lambda \in M \otimes_R N$ with $m_\lambda \in M$ and only finitely many $m_\lambda \neq 0$.

Then $x = 0$ if and only if there exist $m'_\theta \in M$ and $a_{\lambda,\theta} \in R$, $\theta \in \Theta$ some index set, such that

$$m_\lambda = \sum_{\theta \in \Theta} a_{\lambda,\theta} \cdot m'_\theta \quad \text{for all} \quad \lambda \in \Lambda$$

and

$$\sum_{\lambda \in \Lambda} a_{\lambda,\theta} \cdot n_\lambda = 0 \quad \text{for all} \quad \theta \in \Theta.$$

Hint, for part b. consider first the case that N is free in the $(n_\lambda \mid \lambda \in \Lambda)$ and show that in that case actually all m_λ are zero. Then consider a free presentation $\bigoplus_{\theta \in \Theta} R \to \bigoplus_{\lambda \in \Lambda} R \to N \to 0$ of N and tensorize this with M.

In class exercise 12: Let K be a field. Is the K-vector space $K[x] \otimes_K K[y]$ isomorphic to a K-vector space that you know very well? Can you define a multiplication on the tensor product, such that it becomes a K-algebra that you know?

In class exercise 13: Suppose that $(R, \mathfrak{m})$ is local ring and that $M \oplus R^m \cong R^n$ for some $n \geq m$. Show that then $M \cong R^{n-m}$.