Fachbereich Mathematik
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Winter Semester 2021/22, Set 6

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Commutative Algebra

Submit by: Monday, 29/11/2021, 10 am

Exercise 18:

a. Let K be a field, $R = K[x, y, z]/\langle xz, yz\rangle$ and $P = \langle x, y, z-1\rangle \trianglelefteq R$. Show $R_P \cong K[z]_{\langle z-1\rangle}$.

b. Let R be a ring, $f \in R$ a non-zero-divisor. Show $R_f \cong R[x]/\langle fx - 1\rangle$.

Exercise 19: Let R be a ring and $\mathcal{N}(R)$ its nilradical. Show:

a. If $S \subseteq R$ multiplicatively closed, then $\mathcal{N}(S^{-1}R) = S^{-1}\mathcal{N}(R)$.

b. A ring is called *reduced* if it has no nilpotent elements except 0. Show that "being reduced" is a local property, i.e. the following are equivalent:

   (1) R is reduced.

   (2) R_P is reduced for each $P \in \operatorname{Spec}(R)$.

   (3) $R_\mathfrak{m}$ is reduced for each $\mathfrak{m} \lhd \cdot R$.

Exercise 20: Show that "being flat" is a local property, i.e. if M is an R-module, then the following are equivalent:

a. M is a flat R-module.

b. M_P is a flat R_P module for each $P \in \operatorname{Spec}(R)$.

c. $M_\mathfrak{m}$ is a flat $R_\mathfrak{m}$ module for each $\mathfrak{m} \lhd \cdot R$.

Hint, use in class exercise 14 and note that any R_P-module N is also an R-module and that $N_P = N$.

In class exercise 14: Let R' be an R-algebra and M and N be R-modules. Show that there is an isomorphism of R'-modules

$$\Phi : \big(M \otimes_R N\big) \otimes_R R' \longrightarrow \big(M \otimes_R R'\big) \otimes_{R'} \big(N \otimes_R R'\big) : m \otimes n \otimes r' \mapsto (m \otimes r') \otimes (n \otimes 1).$$

Recall that $M \otimes_R R'$ is an R'-module via $r' \cdot (m \otimes s') := m \otimes (r' \cdot s')$.

In class exercise 15: Let $R = K[x, y]$ and $P = K[x, y, z]/\langle xz - x, yz - y - z + 1\rangle$. Is P a flat R-module?

In class exercise 16: Let $\mathfrak{m} = \langle x, y\rangle \lhd K[x, y]$. Localize the ring R and the module P in In-Class Exercise 13 at $\mathfrak{m}$. Is the resulting module $P_\mathfrak{m}$ a flat $R_\mathfrak{m}$-module?