Fachbereich Mathematik — Winter Semester 2021/22, Set 8

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Commutative Algebra

Submit by: Monday, 13/12/2021, 10 am

Exercise 24: Let $R \subseteq R' \subseteq R''$ be rings, $R'' = R[a_1, \ldots, a_n]$ a finitely generated R-algebra and R'' finitely generated as an R'-module. Show, if R noetherian, then R' is finitely generated as an R-algebra and noetherian.

Recall, $R[a_1, \ldots, a_n] = \{f(a_1, \ldots, a_n) \mid f \in R[x_1, \ldots, x_n]\}$ is the set of all polynomial expressions in $a_1, \ldots, a_n$ with coefficients in R. Hint, if $R'' = \langle b_1, \ldots, b_m \rangle_{R'}$, then write a_i and $b_i \cdot b_j$ as linear combinations of the b_ν and consider the R-algebra generated by the coefficients.

Exercise 25:

a. Let $\varphi : R \longrightarrow R'$ be a ring homomorphism and $Q \lhd R'$ a P-primary ideal. Show that $Q^c = \varphi^{-1}(Q)$ is $P^c = \varphi^{-1}(P)$-primary.

b. Let R be a ring, $P \in \mathrm{Spec}(R)$, and $n \geq 1$. Show that the *symbolic power* $P^{(n)} := \{a \in R \mid \exists\, s \in R \setminus P \,:\, s \cdot a \in P^n\}$ is a P-primary ideal.

Note, if $\iota : R \longrightarrow R_P : a \mapsto \frac{a}{1}$, then $P^{(n)} = \big((P^n)^e\big)^c = \iota^{-1}\big(\langle P^n \rangle_{R_P}\big)$.

Exercise 26: Find a minimal primary decomposition of $I = \langle 6 \rangle \lhd \mathbb{Z}\big[\sqrt{-5}\big]$.

Hint, consider the ideals $P = \langle 2, 1 + \sqrt{-5} \rangle$, $Q = \langle 3, 1 + \sqrt{-5} \rangle$, and $Q' = \langle 3, 1 - \sqrt{-5} \rangle$. You may use Proposition 5.4.

Exercise 27: Let $R = K[x, y, z]$ for some field K. Calculate a primary decomposition of $J = \langle xz - y^2, y - x^2 \rangle$.

Hint, consider $\varphi : R \longrightarrow K[x]$ with $x \mapsto x, y \mapsto x^2, z \mapsto x^3$, $P = \ker(\varphi)$, and $Q = \langle x, y \rangle$. Show that $\ker(\varphi) = \langle y - x^2, z - x^3 \rangle$ using division with remainder.

In class exercise 19: Find the primary decomposition of $\langle x^3y^2 - xy^4 \rangle$ in $K[x, y]$.

In class exercise 20: Find the primary decomposition of $\langle x^2 - x, xy - x \rangle$ in $K[x, y]$.

In class exercise 21: Find the primary decomposition of the ideal $\langle x^3 - x^2 - x + 1, x^2y - x^2 - 2xy + 2x + y - 1, xy + y, y^2 - y \rangle$ in $K[x, y]$ and in $K[x, y]_{x+1}$.