Fachbereich Mathematik
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Commutative Algebra

Submit by: Monday, 20/12/2021, 10 am

Exercise 28: Let R be a integral domain of dimension $\dim(R) = 1$, and let $0 \neq I \trianglelefteq R$.

a. Show if $I = Q_1 \cap \ldots \cap Q_n$ is a minimal primary decomposition, then $I = Q_1 \cdots Q_n$.

b. If R is noetherian, then every non-zero ideal I is a finite product of primary ideals Q_i with $\sqrt{Q_i} \neq \sqrt{Q_j}$ for $i \neq j$, and the factors are unique up to ordering.

Hint, Chinese Remainder Theorem.

Exercise 29: Let R be a noetherian ring, $P \in \operatorname{Spec}(R)$, $I \lhd R$ be an ideal with minimal primary decomposition $I = Q_1 \cap \ldots \cap Q_r$ and set $I_i = \cap_{j \neq i} Q_j$. Show the following:

a. There exists an $m \geq 1$ such that

$$I_i \cdot \sqrt{Q_i}^m \subseteq I \not\supseteq I_i \cdot \sqrt{Q_i}^{m-1}.$$

b. For $a \in I_i \cdot \sqrt{Q_i}^{m-1} \setminus I$ we have

$$\sqrt{Q_i} = I : a.$$

c. $P \in \operatorname{Ass}(I)$ if and only if there exists an $a \in R$ such that $P = I : a$.

Exercise 30: Let $R = K[x, y, z]$ for some field K. Let $P = \langle x, y \rangle$ and $Q = \langle y, z \rangle$. Calculate a minimal primary decomposition of $I = P \cdot Q$. Which of the components are isolated, which are embedded?

In class exercise 22: Let $R = K[x, y, z]_{\langle x,y,z \rangle}$, $I = \langle x^2 - y^2, xz - y \rangle$, $J = \langle x^2 - y^2, xz - yz \rangle$. Compute $\dim(R/I)$ and $\dim(R/J)$.