Fachbereich Mathematik
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Winter Semester 2021/22, Set 10

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Commutative Algebra

Submit by: Monday, 10/01/2022, 10 am

Exercise 31: Let $R = K[x,y,z]/\langle xy - z^2\rangle$ and $P = \langle \overline{x}, \overline{z}\rangle$. Show

$$P^2 \subsetneqq P^{(2)} = \langle \overline{x}, \overline{z}^2\rangle \subsetneqq P.$$

Exercise 32: Let R be a noetherian integral domain, which is not a field, such that each ideal is a finite product of prime ideals.
Show that R is a PID if and only if R is a UFD and $\dim(R) = 1$.

Note, a noetherian ID of dimension one where each ideal is a product of primes is called a Dedekind domain and has by Exercise 28 “unique prime factorisation” for ideals!

Exercise 33: Let R be a noetherian integral domain, π be a set of prime elements in R and S be the set of all finite products of elements in π (including 1 as the empty product).

a. Show, if P is a prime ideal such that $P \cap S = \emptyset$ and such that $S^{-1}P$ is a principal ideal, and assume $p \in R$ is such that $\langle p\rangle \trianglelefteq R$ is maximal among the prinicipal ideals in R with $S^{-1}P = \langle \frac{p}{1}\rangle$, then $P = \langle p\rangle$.

b. Show, if $S^{-1}R$ is a unique factorisation domain, then so is R.

In class exercise 23: Give an example of a ring R with two “maximal” chains of prime ideals of different length.