Fachbereich Mathematik
Thomas Markwig Winter Semester 2021/22, Set 11
Alheydis Geiger

Commutative Algebra

Submit by: Monday, 17/01/2022, 10 am

Exercise 34: Which of the following ring extensions is integral?

a. $K[x] \hookrightarrow K[x,y,z]/\langle z^2 - xy, y^3 - x^2\rangle : x \mapsto \overline{x}$.

b. $K[x] \hookrightarrow K[x,y,z]/\langle z^2 - xy, x^3 - yz\rangle : x \mapsto \overline{x}$.

Exercise 35: Let K be a field and $\overline{K}$ its algebraic closure and let $f \in K[x_1, \ldots, x_n]$.

a. Show that $\overline{K}[x_1, \ldots, x_n]$ is integral over $K[x_1, \ldots, x_n]$.

b. Show that $\overline{K}[x_1, \ldots, x_n]/\langle f\rangle$ is integral over $K[x_1, \ldots, x_n]/\langle f\rangle$.

Exercise 36: [Rings of Invariants]

Let G be a *finite* group, $R = K\big[\underline{x}\big]/I$ a finitely generated K-algebra, $G \to \mathrm{Aut}_{K-alg}(R)$ a group homomorphism (we say that G *acts* on R via K-algebra automorphisms), and write $g \cdot f := \alpha(g)(f)$ for $g \in G$ and $f \in R$. Moreover, let $R^G = \{f \in R \mid g \cdot f = f \ \forall\ g \in G\}$, the *ring of invariants of* G *in* R.

a. Show that R is integral over R^G.

b. Show that R^G is a finitely generated K-algebra, hence noetherian.

Hint, use Exercise 24 to solve part b.

In class exercise 24: [Remark: the results are needed for Exercise 34 and In Class Exercie 25.]

a. Let $\mathrm{Mon}(\underline{x}) = \{\underline{x}^\alpha \mid \alpha \in \mathbb{N}^n\}$ and $\mathrm{Mon}(f) = \{\underline{x}^\alpha \mid a_\alpha \neq 0\}$ for $0 \neq f = \sum_\alpha a_\alpha \underline{x}^\alpha \in K[\underline{x}]$. We define a *well-ordering* on $\mathrm{Mon}(\underline{x})$ by

$$\begin{aligned}\underline{x}^\alpha > \underline{x}^\beta \quad \Longleftrightarrow \quad & \deg(\underline{x}^\alpha) > \deg(\underline{x}^\beta) \quad \text{or} \\ & (\deg(\underline{x}^\alpha) = \deg(\underline{x}^\beta) \quad \text{and} \quad \exists\ i\ : \alpha_1 = \beta_1, \ldots, \alpha_{i-1} = \beta_{i-1}, \alpha_i > \beta_i),\end{aligned}$$

and we call $\mathrm{lm}(f) = \max\big(\mathrm{Mon}(f)\big)$ the *leading monomial of* f.

Show, $(\underline{x}^\alpha > \underline{x}^\beta \implies \underline{x}^\alpha \cdot \underline{x}^\gamma > \underline{x}^\beta \cdot \underline{x}^\gamma)$, and thus $\mathrm{lm}(f \cdot g) = \mathrm{lm}(f) \cdot \mathrm{lm}(g)$.

b. Let $K \subseteq L$ be a field extension and let $f \in K[x_1, \ldots, x_n]$. Show that

$$f \cdot L[x_1, \ldots, x_n] \cap K[x_1, \ldots, x_n] = f \cdot K[x_1, \ldots, x_n].$$

In class exercise 25: Consider the group homomorphism

$$\mathrm{Sym}(n) \longrightarrow \mathrm{Aut}_{K-alg}\left(K[x_1,\ldots,x_n]\right) : \sigma \mapsto (f \mapsto f(x_{\sigma(1)},\ldots,x_{\sigma(n)})),$$

and the polynomial $(X+x_1)\cdots(X+x_n) = X^n + s_1X^{n-1} + \ldots + s_n \in K[x_1,\ldots,x_n][X]$.

a. Show, that $\underline{x}^\alpha = \mathrm{lm}(f)$ for $f \in K[x_1,\ldots,x_n]^{\mathrm{Sym}(n)}$ implies $\alpha_1 \geq \ldots \geq \alpha_n$

b. Show, that for $f \in K[x_1,\ldots,x_n]^{\mathrm{Sym}(n)}$ there is a $g \in K[s_1,\ldots,s_n]$ such that $\mathrm{lm}(f) = \mathrm{lm}(g)$.

c. Show, $K[x_1,\ldots,x_n]^{\mathrm{Sym}(n)} = K[s_1,\ldots,s_n]$.

Hint, for part c. do induction on $\mathrm{lm}(f)$ for $f \in K[x_1,\ldots,x_n]^{\mathrm{Sym}(n)}$ in order to show that actually $f \in K[s_1,\ldots,s_n]$. Note that $s_i = \sum_{1\leq j_1<\ldots<j_i\leq n} x_{j_1}\cdots x_{j_i}$, so what is $\mathrm{lm}(s_i)$?