Fachbereich Mathematik Winter Semester 2021/22, Set 12
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Commutative Algebra

Submit by: Monday, 24/01/2022, 10 am

Part a. and b. in Exercise 39 need not be handed in for correction.

Exercise 37: Let $R \subset R'$ be an integral ring extension and let $\mathfrak{m}' \lhd\cdot R'$ be a maximal ideal such that $\mathfrak{m} = \mathfrak{m}' \cap R \lhd\cdot R$ is maximal as well. Is then $R'_{\mathfrak{m}'}$ integral over $R_{\mathfrak{m}}$?

Hint, consider $R = K[x^2-1]$, $R' = K[x]$, $\mathfrak{m}' = \langle x-1\rangle$, and $f = \frac{1}{1+x} \in R'_{\mathfrak{m}'}$.

Exercise 38: Let $R \subseteq R'$ be an integral ring extension and suppose that R' is a finitely generated R-algebra. Show, for a prime $P \in \operatorname{Spec}(R)$ there is only a finite number of prime ideals $Q \in \operatorname{Spec}(R')$ lying over P, i.e. $Q \cap R = P$.

Hint, show that the intersection of all primes lying over P localised at $R \setminus P$ has a primary decomposition.

Exercise 39: [Rings of Integers of Quadratic Number Fields]

Let $d \in \mathbb{Z} \setminus \{0,1\}$ be a squarefree number (i.e. no square a^2 divides d), then $\mathbb{Q}\big[\sqrt{d}\big] = \{a + b\sqrt{d} \mid a, b \in \mathbb{Q}\}$ is a field extension of $\mathbb{Q}$ with $\dim_{\mathbb{Q}} \mathbb{Q}\big[\sqrt{d}\big] = 2$. Consider the *conjugation*

$$C : \mathbb{Q}\big[\sqrt{d}\big] \longrightarrow \mathbb{Q}\big[\sqrt{d}\big] : a + b\sqrt{d} \mapsto a - b\sqrt{d},$$

the *norm*

$$N : \mathbb{Q}\big[\sqrt{d}\big] \longrightarrow \mathbb{Q} : a + b\sqrt{d} \mapsto (a + b\sqrt{d}) \cdot C(a + b\sqrt{d}) = a^2 - b^2 d,$$

and the *trace*

$$T : \mathbb{Q}\big[\sqrt{d}\big] \longrightarrow \mathbb{Q} : a + b\sqrt{d} \mapsto (a + b\sqrt{d}) + C(a + b\sqrt{d}) = 2a.$$

Show:

a. $C(x \cdot y) = C(x) \cdot C(y)$ and $N(x \cdot y) = N(x) \cdot N(y)$ for $x, y \in \mathbb{Q}\big[\sqrt{d}\big]$.

b. C and T are $\mathbb{Q}$-linear.

c. If $x \in \mathbb{Q}\big[\sqrt{d}\big] \setminus \mathbb{Q}$, then $\mu_x = (t - x) \cdot \big(t - C(x)\big) = t^2 - T(x) \cdot t + N(x) \in \mathbb{Q}[t]$ is the minimal polynomial of x over $\mathbb{Q}$.

d. $x \in \mathbb{Q}\big[\sqrt{d}\big]$ is integral over $\mathbb{Z}$ if and only if $T(x)$ and $N(x)$ are integers.

e. $\operatorname{Int}_{\mathbb{Q}[\sqrt{d}]}(\mathbb{Z}) = \mathbb{Z}[\omega_d]$, where $\omega_d = \begin{cases} \sqrt{d}, & \text{if } d \equiv 2, 3 \bmod 4 \\ \frac{1+\sqrt{d}}{2}, & \text{if } d \equiv 1 \bmod 4. \end{cases}$

In class exercise 26: Es sei $R = \mathbb{R}[x]$ und $R' = \mathbb{R}[x, y]/\langle y^3 - x^2 y\rangle$. Wir betrachten ferner R' als Ringerweiterung von R mittels $R \hookrightarrow R' : f \mapsto \overline{f}$. Zeige, dass die Ringerweiterung die Voraussetzungen von Aufgabe 38 erfüllt und finde für die Primideale $P = \langle x - 1\rangle$ und $P' = \langle x\rangle$ in R alle Primideale in R', die darüber liegen.

In class exercise 27: Let $K \subseteq K'$ be a *field* extension, and let $T \subset K'$ be *algebraically independent* over K, i.e. for every finite subset $T' = \{t_1, \ldots, t_n\} \subseteq T$ and $0 \neq f \in K[x_1, \ldots, x_n]$ also $f(t_1, \ldots, t_n) \neq 0$. Moreover, denote by $\mathcal{T}$ the set of all finite subsets of T.

Show that the following are equivalent:

a. K' is integral over $K(T) = \bigcup_{n\in\mathbb{N}} \bigcup_{\{t_0,\ldots,t_n\}\in\mathcal{T}} \operatorname{Quot}\left(K[t_0, \ldots, t_n]\right)$.

b. For every $t' \in K' \setminus T$ the set $T \cup \{t'\}$ is not algebraically independent.