Fachbereich Mathematik
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Winter Semester 2021/22, Set 13

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Commutative Algebra

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Exercise 40: Let R be a ring, $Q, Q' \in \text{Spec}(R)$ such that $\dim(R_{Q'}/Q_{Q'}) = 1$ and suppose that R is integral over $K[y_1, \ldots, y_n]$ and $K[y_1, \ldots, y_n]/Q^c$ is integral over $K[z_1, \ldots, z_m]$. Show, there exists no prime ideal $P \in \text{Spec}(K[y_1, \ldots, y_n])$ such that $Q^c \subsetneqq P \subsetneqq (Q')^c$.

Exercise 41: Find all maximal ideals in $\mathbb{C}[x, y]/\langle x^3 - x^2, x^2y - 2x^2\rangle$.

Exercise 42:

a. Show that every maximal ideal in $\mathbb{Z}[x]$ can be generated by a prime number $p \in \mathbb{Z}$ and a polynomial $f \in \mathbb{Z}[x]$ such that its residue class in $\mathbb{Z}_p[x]$ is irreducible.

b. Show that $\dim(\mathbb{Z}[x]) = 2$.

Hint, show in a. first that a maximal ideal cannot be principal, then that it contains a prime number and finally that modulo that prime number it will be generated by a single polynomial. Recall that $\mathbb{Z}[x]/\langle p\rangle \cong \mathbb{Z}_p[x]$ is a PID and that in $\mathbb{Q}[x]$ the Bézout identity holds.

In class exercise 28: Find a Noether normalisation of $R = K[x, y]/\langle x^3 - y^2\rangle$ and compute the normalisation of R.