5.4 Algorithm (Radical Membership)

Input: $g \in K[\underline{x}]$, $G = (f_1, \ldots, f_k)$, $f_i \in K[\underline{x}] \setminus \{0\}$, $>$ any monomial orderin

Output: 1 if $g \in \sqrt{\langle G \rangle_{K[\underline{x}]_>}}$ and 0 else.

Instructions:

- Choose any elimination ordering $>_1$ on $\mathrm{Mon}(t, \underline{x})$ w.r.t. t such that $>_1$ co
- Compute a minimal SB $S = (g_1, \ldots, g_l)$ of $\langle f_1, \ldots, f_k, 1 - tg \rangle_{K[\underline{x},t]_{>_1}}$ w.r.
- IF $\mathrm{lm}(g_1) = 1$
  - THEN RETURN(1)
  - ELSE RETURN(0)

5.6 Algorithm (Extended Ideal/Submodule Membership)

Input: $G = (f_1, \ldots, f_k)$, $f_i \in K[\underline{x}]^m \setminus \{0\}$, $g \in \langle G \rangle$, $>$ a monomial ordering

Output: $(u, q_1, \ldots, q_k) \in K[\underline{x}]^{k+1}$ with $u \in K[\underline{x}]^*_>$ s.t. $g = \sum_{i=1}^{k} \frac{g_i}{u} \cdot f_i$.

Instructions:

- Compute an SB S of $\mathrm{syz}(g, f_1, \ldots, f_k)$ with respect to $(c, >)$ using 5.10.
- Return a syzygy $(u, -q_1, \ldots, -q_k)^t \in S$ where u is a unit.

5.8 Algorithm (Elimination of Components)

Input: $G = (f_1, \ldots, f_k)$, $f_i \in K[\underline{x}]^m \setminus \{0\}$, $s \in \{1, \ldots, m\}$, $>$ a monomial o

Output: S, an SB of $\langle G \rangle_{K[\underline{x}]_>} \cap \bigoplus_{j>s} e_j K[\underline{x}]_>$

Instructions:

- Choose an elimination ordering $>_m$ on Mon_n^m w.r.t. $e_1, \ldots, e_s$ and $>$, e.g.
- Compute an SB S of $\langle G \rangle_{K[\underline{x}]_>}$ w.r.t. $>_m$.
- Return $S' = (f_i \in S \mid \mathrm{lm}(f_i)$ does not involve $e_1, \ldots, e_s)$.

5.10 Algorithm (Syzygies)

Input: $G = (f_1, \dots, f_k)$, $f_i \in K[\underline{x}]^m \setminus \{0\}$, $>$ a monomial ordering on Mon_n

Output: S, an SB of $\mathrm{syz}(f_1, \dots, f_k) \le K[\underline{x}]^k_>$ w.r.t. $(c, >)$

Instructions (using the notation of 5.9):

- Eliminate $e_1, \dots, e_m$ from $M = \langle i(f_j) + e_{m+j} \mid j = 1, \dots, k \rangle_{K[\underline{x}]_>}$ by 5.8 we get.
- Return $S = \big(\pi(g) \mid g \in G'\big)$.

5.13 Algorithm (Elimination of Variables)

Input: $G = (f_1, \ldots, f_k)$, $f_i \in K[\underline{x}, \underline{y}] \setminus \{0\}$, $>$ a monomial ordering on Mon^{m}

Output: S, an SB of $I \cap K[\underline{y}]^m_>$, where $I = \langle f_1, \ldots, f_k \rangle_{K[\underline{y}]_>[\underline{x}]}$

Instructions:

- Choose any elimination ordering $>_m$ on $\mathrm{Mon}^m(\underline{x}, \underline{y})$ w.r.t. $\underline{x}$ which coinci
  e.g. $>_m = \big((>_{dp}, c), >\big)$.
- Compute an SB S' of I w.r.t. $>_m$.
- Return $S = \big(g \in S \;\big|\; \mathrm{lm}(g) \in K[\underline{y}]^m\big)$.

5.18 Algorithm (Intersection)

Input: $F = (f_1, \ldots, f_k), G = (g_1, \ldots, g_l), f_i, g_j \in K[\underline{x}]^m$, $>$ a monomial ord

Output: S, a generating system of $\langle f_1, \ldots, f_k \rangle_{K[\underline{x}]_>} \cap \langle g_1, \ldots, g_l \rangle_{K[\underline{x}]_>}$.

Instructions: Algorithm I:

- Choose an elimination ordering $>_m$ on $\mathrm{Mon}^m(t, \underline{x})$ w.r.t. t which coincide e.g. $\big((>_{lp}, c), >\big)$.
- Eliminate t from $\langle t \cdot f_1, \ldots, t \cdot f_k, (1-t) \cdot g_1, \ldots, (1-t) \cdot g_l \rangle_{K[t,\underline{x}]_{>_1}}$ by 5 where $>_1$ is induced on $\mathrm{Mon}(t, \underline{x})$ by $>_m$.

Instructions: Algorithm II:

- Compute with 5.10 an SB $\tilde{S} = (s_1, \ldots, s_t)$ of

$$\mathrm{syz}\left(\begin{pmatrix} e_1 \\ e_1 \end{pmatrix}, \ldots, \begin{pmatrix} e_m \\ e_m \end{pmatrix}, \begin{pmatrix} f_1 \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} f_k \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ g_1 \end{pmatrix}, \ldots\right.$$

- Return $S = \big(\pi(s_1), \ldots, \pi(s_t)\big)$, where π projects onto the first m compon

5.21 Algorithm (Quotients)

Input: $F = (f_1, \ldots, f_k), G = (g_0, \ldots, g_l), f_i, g_j \in K[\underline{x}]^m \setminus \{0\}$, $>$ a monomi

Output: S, a generating system of $I : J \trianglelefteq K[\underline{x}]_>$, where $I = \langle F \rangle_{K[\underline{x}]_>}$ and J

Instructions: Algorithm I:

1. FOR $i = 0, \ldots, l$ compute a generating system G_i of $I \cap \langle g_i \rangle_{K[\underline{x}]_>}$ by 5.18
2. FOR $i = 0, \ldots, l$ set $G'_i = \left(\frac{g}{g_i} \mid g \in G_i \right)$.
3. - $S_0 = \langle G'_0 \rangle$
   - FOR $i = 1, \ldots, l$ compute a generating system of $\langle S_{i-1} \rangle_{K[\underline{x}]_>} \cap \langle G'_i \rangle_{K}$
   - RETURN(S_l)

Instructions: Algorithm II:

1. As in Algorithm I, with $G_i = (h_{i1}, \ldots, h_{in_i})$.
2. FOR $i = 0, \ldots, l$ DO
   - Compute a generating system $(s_{i1}, \ldots, s_{it_i})$ of $\text{syz}(h_{i1}, \ldots, h_{in_i}, g_i)$ by
   - $G'_i = \big(\pi_i(s_{i1}), \ldots, \pi_i(s_{it_i})\big)$, where π_i projects onto the last component
3. As in Algorithm I.

Instructions: Algorithm III:

1. Choose an elimination ordering $>_1$ on $\mathrm{Mon}^m(t, \underline{x})$ w.r.t. t s.t. $>_1$ coincide
2. Compute a generating system S' of $\langle I \rangle_{K[t,\underline{x}]_{>_1}} \cap \left\langle \sum_{i=0}^{l} t^i g_i \right\rangle_{K[t,\underline{x}]_{>_1}}$ by 5.1
3. Divide the elements of S' by $\sum_{i=0}^{l} t^i g_i$.
4. Eliminate t from $\langle S' \rangle_{K[t,\underline{x}]_{>_1}}$ by 5.10.

Instructions: Algorithm IV:

1. Choose an elimination ordering $>_1$ on $\mathrm{Mon}^m(t_1, \ldots, t_l, \underline{x})$ w.r.t. $(t_1, \ldots,$ on $\mathrm{Mon}^m(\underline{x})$.
2. Compute a generating system S' of $\langle I \rangle_{K[\underline{t},\underline{x}]_{>_1}} \cap \left\langle g_0 + \sum_{i=1}^{l} t_i g_i \right\rangle_{K[\underline{t},\underline{x}]_{>_1}}$
3. Divide the elements of S' by $g_0 + \sum_{i=1}^{l} t_i g_i$.
4. Eliminate $t_1, \ldots, t_l$ from $\langle S' \rangle_{K[\underline{t},\underline{x}]_{>_1}}$ by 5.10.

5.25 Algorithm (Saturation)

Input: $F = (f_1, \ldots, f_k), G = (g_1, \ldots, g_l), f_i, g_j \in K[\underline{x}]^m \setminus \{0\}$, $>$ a monomi

Output: (S, N), with S an SB of $I : J^\infty$ and N the saturation index of I w.
where $I = \langle f_1, \ldots, f_k \rangle_{K[\underline{x}]_>}$ and $J = \langle g_1, \ldots, g_l \rangle_{K[\underline{x}]_>}$

Instructions:

- $I_0 = I$, $N = 0$.
- Compute an SB S_0 of $I_1 = I : J$ by 5.21 + 4.12.
- WHILE ($S_N \not\subseteq I_N$, checked by 5.1) DO
  - Compute an SB S_N of $I_{N+1} = I_N : J$.
  - $N = N + 1$.
- RETURN((S_N, N))

5.27 Algorithm (Kernel of a Ringhomomorphism)

Input: $F = (f_1, \dots, f_n)$, $f_i \in K[\underline{y}]$, $G = (g_1, \dots, g_k)$, $g_i \in K[\underline{x}]$, $H = (h_1, .$
$g_i(f_1, \dots, f_n) \in \langle h_1, \dots, h_l \rangle_{K[\underline{y}]}$.

Output: S, a generating system of $\ker(\varphi)$, where

$$\varphi : K[\underline{x}]/I \longrightarrow K[\underline{y}]/J : \overline{x_i} \mapsto \overline{f_i},$$

$I = \langle g_1, \dots, g_k \rangle \trianglelefteq K[\underline{x}]$ and $J = \langle h_1, \dots, h_l \rangle \trianglelefteq K[\underline{y}]$.

Instructions:

- Choose an elimination ordering $>$ on $\mathrm{Mon}(\underline{x}, \underline{y})$ w.r.t. $\underline{y}$ inducing a *global*
- Compute by 5.13 an SB S of

$$\langle h_1, \dots, h_l, x_1 - f_1, \dots, x_n - f_n \rangle_{K[\underline{x}, \underline{y}]} \cap K[\underline{x}].$$

- RETURN(S).

5.30 Algorithm (Algebraic Dependence)

Input: $F = (f_1, \ldots, f_n), f_i \in K[\underline{y}]$

Output: 0 if $f_1, \ldots, f_n$ are algebraically independent, 1 otherwise

Instructions:

- Compute a generating system S of $\ker(\varphi)$ with $\varphi : K[\underline{x}] \longrightarrow K[\underline{y}] : x_i \mapsto$
- IF $(S = 0)$
  - THEN RETURN(0)
  - ELSE RETURN(1)

5.32 Algorithm (Kernel of a Linear Map)

Input: $>$ any monomial ordering on $\mathrm{Mon}(\underline{x})$,

- $f_1, \ldots, f_k \in K[\underline{x}]^m$,
- $g_1, \ldots, g_t \in K[\underline{x}]^m$,
- $h_1, \ldots, h_s \in K[\underline{x}]$,
- $k_1, \ldots, k_r \in K[\underline{x}]^k$ such that $\sum_{j=1}^k k_{ij} \cdot f_j \in M$ for $i = 1, \ldots, r$,

where $I = \langle h_1, \ldots, h_s \rangle_{K[\underline{x}]_>}$, $R = K[\underline{x}]_>/I$ and $M = \langle \overline{g_1}, \ldots, \overline{g_t} \rangle_R$.

Output: $S = (s_1, \ldots, s_\tau)$ with $s_i \in K[\underline{x}]^k$ such that $(\overline{s_1}, \ldots, \overline{s_\tau})$ is a generat

$$\alpha : R^k/N \longrightarrow R^m/M : (\overline{a_1}, \ldots, \overline{a_k}) \mapsto \sum_{j=1}^{k} \overline{a_i} \cdot$$

where $N = \langle \overline{k_1}, \ldots, \overline{k_r} \rangle_R$.

Instructions:

- Compute a generating system $(b_1, \ldots, b_\tau)$ of

$$\mathrm{syz}(f_1, \ldots, f_k, g_1, \ldots, g_t, h_1 \cdot e_1, \ldots, h_1 \cdot e_m, \ldots, h_s \cdot e_1, \ldots$$

- FOR $i = 1, \ldots, \tau$ set $s_i = \pi(b_i)$, where π projects to the first k componen
- Return $S = (s_1, \ldots, s_\tau)$.

5.33 Algorithm (Solving Linear Systems of Equations)

Input: $h_1, \ldots, h_s \in K[\underline{x}]$, $A \in \mathrm{Mat}(m \times k, K[\underline{x}])$, $b \in K[\underline{x}]^m$, $>$ a monomia

Output: (y_0, S) with $S = (s_1, \ldots, s_\tau)$, $y_0, s_1, \ldots, s_\tau \in K[\underline{x}]^k$ and $R = K[\underline{x}]$

$$\overline{y_0} + \langle \overline{s_1}, \ldots, \overline{s_\tau} \rangle_R = \left\{ \overline{y} \in R^k \mid \overline{A} \cdot \overline{y} = \overline{b} \right\},$$

if $\overline{A} \cdot \overline{y} = \overline{b}$ **is solvable**.

Instructions:

- Denote by $\underline{a}^i$ the i-th column of A for $i = 1, \ldots, k$.
- IF $(b \in \langle \underline{a}^1, \ldots, \underline{a}^k, h_1 \cdot e_1, \ldots, h_1 \cdot e_m, \ldots, h_s \cdot e_1, \ldots, h_s \cdot e_m \rangle_{K[\underline{x}]}$ – chec
  - THEN compute $(y, z)^t \in K[\underline{x}]^{k+sm}$ s.t. $b = \sum_{i=1}^k y_i \cdot \underline{a}^i + \sum_{\kappa=1}^s \sum_{\lambda=1}^m$
  - ELSE return that the system is not solvable.
- Compute a generating system S of $\ker(\alpha)$ where $\alpha : R^k \longrightarrow R^m : \overline{y} \mapsto \overline{A}$
- RETURN$((y, S))$.