Fachbereich Mathematik Winter Semester 2007/08, Set 1
Thomas Markwig Henning Meyer

# Computer Algebra

Due date: Tuesday, 06/11/2007, 10h00

**Exercise 1:** Let $I = \langle \underline{x}^\alpha \cdot e_i \mid (\alpha, i) \in \Lambda \rangle$ and $J = \langle \underline{x}^\beta \cdot e_j \mid (\beta, j) \in \Lambda' \rangle$ be two monomial submodules of $R[\underline{x}]^m$ and let $\underline{x}^\gamma \cdot e_k \in R[\underline{x}]^m$ be a monomial. Show that

a. $I \cap J = \left\langle \operatorname{lcm}\left(\underline{x}^\alpha \cdot e_i, \underline{x}^\beta \cdot e_j\right) \mid (\alpha, i) \in \Lambda, (\beta, j) \in \Lambda' \right\rangle$, and

b. $I : \langle \underline{x}^\gamma \cdot e_k \rangle = \left\langle \frac{\operatorname{lcm}\left(\underline{x}^\alpha, \underline{x}^\gamma\right)}{\underline{x}^\gamma} \mid (\alpha, k) \in \Lambda \right\rangle$.

**Exercise 2:** Consider the polynomial ring $R = \mathbb{Q}[x_{ij} \mid 1 \leq i, j \leq 8]$, where the variable $x_{ij}$ represents the square with coordinates $(i, j)$ on a chess board. As in the introduction of the lecture we set

$$M = \{x_{ij} \cdot x_{kl} \mid \text{ queen on } (i,j) \text{ can move to } (k,l) \text{ in } \leq 1 \text{ moves}\}$$

and $I = \langle M \rangle$. We interprete a product $m_1 \cdots m_8$ with $m_i \in \{x_{ij} \mid 1 \leq i, j \leq 8\}$ as the positioning of 8 queens on a chess board, and we then set

$$B = \{m_1 \cdots m_8 \mid \text{ no 2 queens attack each other}\}.$$

Show that $B$ is a $\mathbb{Q}$-vector space basis of $R_8/(I \cap R_8)$.

**Exercise 3:** Let $>$ be any monomial ordering on $\operatorname{Mon}_n$.

a. Show that for a fixed $w = (w_1, \ldots, w_n) \in \mathbb{R}^n$ the following defines a monomial ordering on $\operatorname{Mon}_n$:

$$\underline{x}^\alpha >_{(w,>)} \underline{x}^\beta \;:\Longleftrightarrow\; \langle w, \alpha \rangle > \langle w, \beta \rangle, \;\text{ or }\; \left(\langle w, \alpha \rangle = \langle w, \beta \rangle \;\text{ and }\; \underline{x}^\alpha > \underline{x}^\beta\right).$$

Under which assumptions is the above definition independent of the chosen ordering $>$?

b. Let $A \in \operatorname{Gl}_n(\mathbb{R})$ be an invertible $n \times n$-matrix over the rational numbers. Show that the following defines a monomial ordering on $\operatorname{Mon}_n$:

$$\underline{x}^\alpha >_{(A,>)} \underline{x}^\beta \;:\Longleftrightarrow\; \underline{x}^{A\alpha} > \underline{x}^{A\beta}.$$

**Exercise 4:** Determine matrices $A \in \operatorname{Gl}_n(\mathbb{R})$ which define the following orderings:

a. $(>_{lp}, >_{ds})$, here $n = n_1 + n_2$ with $n_1$ variables for $lp$ and $n_2$ for $ds$;

b. $(>_{dp}, >_{ls})$, here $n = n_1 + n_2$ with $n_1$ variables for $dp$ and $n_2$ for $ls$;

c. $>_{wp(5,3,4)}$, here $n = 3$;

d. $>_{ws(5,3,4)}$, here $n = 3$.