Fachbereich Mathematik | Winter Semester 2007/08, Set 2
Thomas Markwig | Henning Meyer

Computer Algebra

Due date: Tuesday, 13/11/2007, 10h00

Exercise 5: We define the *degree lexicographical ordering* $>_{Dp}$ on Mon_n by

$$\underline{x}^\alpha >_{Dp} \underline{x}^\beta \iff |\alpha| > |\beta| \text{ or } \big(|\alpha| = |\beta| \text{ and } \exists\, k \,:\, \alpha_1 = \beta_1, \ldots, \alpha_{k-1} = \beta_{k-1}, \alpha_k > \beta_k\big).$$

Show that the orderings $>_{lp}$, $>_{Dp}$ and $>_{dp}$ are described by the following characterising properties. Let $>$ be a monomial ordering on Mon_n, then:

a. $> = >_{lp}$ if and only if $>$ is an elimination ordering for $\{x_1, \ldots, x_k\}$ for all $k = 1, \ldots, n-1$, i. e. if $\text{lm}(f) \in R[x_{k+1}, \ldots, x_n]$ implies $f \in R[x_{k+1}, \ldots, x_n]$.

b. $> = >_{Dp}$ if and only if $>$ is a degree ordering and for any *homogeneous* $f \in R[\underline{x}]$ with $\text{lm}(f) \in R[x_k, \ldots, x_n]$ we have $f \in R[x_k, \ldots, x_n]$, $k = 1, \ldots, n$.

c. $> = >_{dp}$ if and only if $>$ is a degree ordering and for any *homogeneous* $f \in R[\underline{x}]$ with $\text{lm}(f) \in \langle x_k, \ldots, x_n \rangle$ we have $f \in \langle x_k, \ldots, x_n \rangle$, $k = 1, \ldots, n$.

Exercise 6:

a. Let R be a unique factorisation domain and $S \subset R$ a multiplicatively closed subset. Show that $S^{-1}R$ is a unique factorisation domain.

b. Let $>$ be a local ordering on $\text{Mon}(x_1, \ldots, x_n)$. Show that

$$K(y_1, \ldots, y_m)[x_1, \ldots, x_n]_> = K[x_1, \ldots, x_n, y_1, \ldots, y_m]_{\langle x_1, \ldots, x_n \rangle}.$$

Hint, for part a. use the one-to-one correspondance of prime ideals under localisation.

Exercise 7: Give one possible realization of the following rings within SINGULAR:

a. $\mathbb{Q}[x, y, z]$,

b. $\mathbb{F}_5[x, y, z]$,

c. $\mathbb{Q}[x, y, z]/\langle x^5 + y^3 + z^2 \rangle$,

d. $\mathbb{Q}(i)[x, y]$, where i is the imaginary unit,

e. $\mathbb{F}_{27}[x_1, \ldots, x_{10}]_{\langle x_1, \ldots, x_{10} \rangle}$,

f. $\mathbb{F}_{32003}[x, y, z]_{\langle x,y,z \rangle}/\langle x^5 + y^3 + z^2, xy \rangle$,

g. $\mathbb{Q}(t)[x, y, z]$,

h. $\big(\mathbb{Q}[t]/\langle t^3 + t^2 + 1 \rangle\big)[x, y, z]_{\langle x,y,z \rangle}$,

i. $\mathbb{Q}[x, y, z]_{\langle x,y \rangle}$.

Exercise 8: Write a SINGULAR procedure `spolynomial` which takes as input two polynomials and returns their s-polynomial.