TROPICAL CURVES WITH A SINGULARITY IN A FIXED POINT

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Abstract. In this paper, we study tropicalisations of families of curves with a singularity in a fixed point. The tropicalisation of such a family is a linear tropical variety. We describe its maximal dimensional cones using results about linear tropical varieties from [1] and [5]. We show that a singularity tropicalises either to a vertex of higher valence or of higher multiplicity, or to an edge of higher weight. We then classify maximal dimensional types of singular tropical curves. For those, the singularity is either a crossing of two edges, or a 3-valent vertex of multiplicity 3, or a point on an edge of weight 2 whose distances to the neighbouring vertices satisfy a certain metric condition. We also study algebraic preimages of our singular tropical curves.

1. Introduction

Fix a non-degenerate convex lattice polygon $\Delta \subset \mathbb{R}^2$ and denote by $\mathcal{A} = \Delta \cap \mathbb{Z}^2$ the lattice points of Δ. For any field $\mathbb{K}$ there is a toric surface $\Sigma = \mathrm{Tor}_{\mathbb{K}}(\Delta)$ associated to Δ and it comes with the tautological line bundle $\mathcal{L}_\Delta$ generated by the global sections $\{x^i y^j \;:\; (i,j) \in \mathcal{A}\}$. The torus $(\mathbb{K}^*)^2$ is embedded in Σ via

$$\Psi_{\mathcal{A}} : (\mathbb{K}^*)^2 \longrightarrow \mathbb{P}^{\mathcal{A}}_{\mathbb{K}} : (x,y) \mapsto \big(x^i y^j \mid (i,j) \in \mathcal{A}\big)$$

and inside the torus the elements in the linear system $|\mathcal{L}_\Delta|$ are defined by the equations

$$f_{\underline{a}} = \sum_{(i,j)\in\mathcal{A}} a_{i,j} \cdot x^i \cdot y^j = 0.$$

$|\mathcal{L}_\Delta|$ contains a nonempty linear subsystem $\mathrm{Sing}_{\mathbb{K}}(\Delta)$ of curves with a singularity at the point $\boldsymbol{p} = (1,1)$. The equations for this subsystem are the linear equations

$$f_{\underline{a}}(\boldsymbol{p}) = 0, \quad \frac{\partial f_{\underline{a}}}{\partial x}(\boldsymbol{p}) = 0, \quad \frac{\partial f_{\underline{a}}}{\partial y}(\boldsymbol{p}) = 0.$$

In this paper, we describe the geometry of $\mathrm{Trop}(\mathrm{Sing}_{\mathbb{K}}(\Delta))$, the tropicalisation of $\mathrm{Sing}_{\mathbb{K}}(\Delta)$, as a tropical variety (i.e. a balanced fan) in $\mathbb{R}^{s-1} = \mathbb{R}^{\mathcal{A}}/(1,\ldots,1)\cdot\mathbb{R}$ (where $s = \#\mathcal{A}$) and we analyse the underlying tropical curves (i.e. the tropicalisations of the singular algebraic curves $C \in \mathrm{Sing}_{\mathbb{K}}(\Delta)$).

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In order to be able to tropicalise we have to use an algebraically closed field $\mathbb{K}$ with a valuation

$$\mathrm{val} : \mathbb{K}^* \longrightarrow \mathbb{R}$$

whose value group is dense in $\mathbb{R}$, e.g. the algebraic closure $\overline{\mathbb{C}(t)}$ of the field of rational functions over $\mathbb{C}$, or $\mathbb{C}\{\{t\}\}$ the field of Puiseux series, or a field of generalised Puiseux series as in [7]. In each of these cases the elements of the field can be represented by generalised power series of the form

$$p = a_1 t^{q_1} + a_2 t^{q_2} + \ldots$$

with complex coefficients and real exponents, and the valuation maps p to the least exponent q_1 whose coefficient a_1 is non-zero.

For an ideal $I \subset \mathbb{K}[x_1, \ldots, x_n]$ determining an affine variety $V = V(I) \subset \mathbb{K}^n$ we define the *tropicalisation* of V to be

$$\mathrm{Trop}(V) := \overline{\{(-\mathrm{val}(x_1), \ldots, -\mathrm{val}(x_n)) \mid (x_1, \ldots, x_n) \in V(I) \cap (\mathbb{K}^*)^n\}},$$

i.e. we map V componentwise with the negative of the valuation map and take the topological closure in $\mathbb{R}^n$. If the ideal I is homogeneous and defines a projective variety, we may consider $\mathrm{Trop}(V)$ modulo the linear space $(1, \ldots, 1) \cdot \mathbb{R}$, i.e. we identify $\mathrm{Trop}(V)$ with its image in $\mathbb{R}^n/(1, \ldots, 1) \cdot \mathbb{R}$.

In our paper, we use this definition in two situations:

- *The constant coefficient case:* Assume I is generated by polynomials in $\mathbb{C}[x_1, \ldots, x_s]$, then $\mathrm{Trop}(V(I))$ is a subfan of the Gröbner fan of I (see [2]). We will in fact only consider the situation where I is generated by linear forms. In this case, the tropicalisation is called the *Bergman fan* of I and has been well-studied e.g. in [5] and [1]. We describe it further in Section 3.1. Note that since I is homogeneous, we will consider $\mathrm{Trop}(V(I))$ as a fan in $\mathbb{R}^s/(1, \ldots, 1) \cdot \mathbb{R}$.
- *The case of plane tropical curves:* Assume $I = \langle f \rangle \subset \mathbb{K}[x, y]$ and

  $$f = \sum a_{ij} x^i y^j,$$

  then $\mathrm{Trop}(V(f))$ equals the locus of non-differentiability of the *tropical polynomial*

  $$\mathrm{trop}\, f := \max\{-\mathrm{val}(a_{ij}) + ix + jy\}$$

  by Kapranov's Theorem (see [4, Theorem 2.1.1]). More details about plane tropical curves follow in the Section 2.

Let us now give an example for the tropicalisation of a singular curve.

Example 1.1
We consider the polynomial

$$f = xy^2 - tx^2 - (2 + t^3) \cdot xy + (1 + 2t + t^3) \cdot x + t^3 y - (t + t^3) \in \mathbb{K}[x, y].$$

One easily verifies that $\boldsymbol{p} = (1, 1) \in (\mathbb{K}^*)^2$ is a singular point of the curve $V(f)$. f defines a curve in the toric surface $\mathrm{Tor}_{\mathbb{K}}(\Delta)$, where Δ is as in the left hand side of Figure 1. The tropicalisation of $V(f)$ is shown in the right hand side of Figure 1. The singularity $\boldsymbol{p} = (1, 1)$ tropicalises to $\boldsymbol{x}_0 = (0, 0)$. It sits precisely in the middle of an edge of weight two, i.e. it has the same distance to both neighbouring

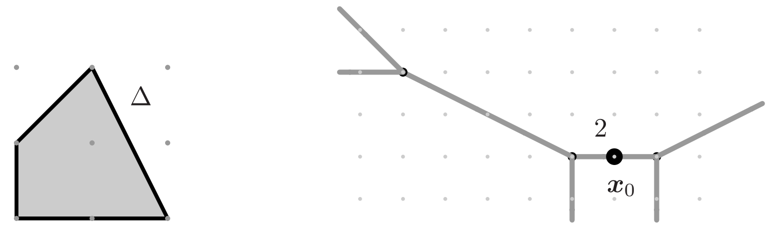

FIGURE 1. Tropicalisation of a singular curve

vertices. This metric condition and the fact that the edge has weight two are no coincidences, they are a general phenomenon, as we will see in Section 4.

Since $\mathrm{Sing}_{\mathbb{K}}(\Delta)$ is given by a linear ideal (see also Section 3), we can use results of [1] and [5] to study its tropicalisation $\mathrm{Trop}(\mathrm{Sing}_{\mathbb{K}}(\Delta))$. We classify the maximal cones of this tropical variety. Section 4 presents our main result, the classification of singular tropical curves of maximal dimensional type (for a definition of maximal dimensional type, see Subsection 2.2). We show that the singularity of such a tropical curve of maximal dimensional type is either a crossing of two edges, or a 3-valent vertex of multiplicity 3, or a point on an edge of weight 2 whose distances to the neighbouring vertices satisfy a certain metric condition. We also study algebraic preimages of our singular tropical curves, and in particular we thus give a conceptual explanation for the metric condition mentioned above. Moreover, we give suggestions how the singular tropical curves should be interpreted as parametrised tropical curves.

Note that our result does not depend on the choice of the singular point, as long as it is a point in the torus $(\mathbb{C}^*)^2$, like $\boldsymbol{p} = (1, 1)$ (see Remark 3.1).

The paper is organised as follows. In Section 2 we repeat well-known facts about the secondary fan and its connection to tropical curves. We study the dimension of cones of the secondary fan in Subsection 2.1. We define the dimension of types of tropical curves in Subsection 2.2. In Section 3, we introduce the family of curves with a singularity in a fixed point and its defining ideal. We repeat facts about tropicalisations of linear ideals in 3.1 and study the top-dimensional cones of $\mathrm{Trop}(\mathrm{Sing}_{\mathbb{K}}(\Delta))$ in 3.2. We relate these top-dimensional cones to cones of the secondary fan in Subsection 3.3. We study the connection of $\mathrm{Trop}(\mathrm{Sing}_{\mathbb{K}}(\Delta))$ to the tropical discriminant in 3.4. In Section 4, we present our main result: the classification of singular tropical curves of maximal dimensional type. As noted above, our result does not depend on the singular point, as long as it is in the complex torus. We study what happens if we move the point to a coordinate line in Section 5. In Section 6 finally, we study algebraic preimages of singular tropical curves and suggest how the latter can be interpreted as parametrised tropical curves.

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2. The secondary fan and its relation to plane tropical curves

Here, we repeat shortly some basic definitions. For more details, see [6, Chapter 7] or [8]. In the case of plane tropical curves, we can conclude from Kapranov's theorem that $\text{Trop}(V(f))$ is a piece-wise linear graph in $\mathbb{R}^2$. An important fact is that this graph is *dual* to a subdivision of the *Newton polygon* $\Delta = \text{conv}\{(i,j) \mid a_{ij} \neq 0\}$ of f.

A *marked polygon* is a 2-dimensional convex lattice polygon Q in $\mathbb{R}^2$ together with a subset $\mathcal{A}$ of the lattice points $Q \cap \mathbb{Z}^2$ containing the vertices of Q.

A *marked subdivision* of a polygon Δ is a collection of marked polygons, $T = \{(Q_1, \mathcal{A}_1), \ldots, (Q_k, \mathcal{A}_k)\}$, such that

- $\Delta = \bigcup_{i=1}^k Q_i$,
- $Q_i \cap Q_j$ is a face (possibly empty) of Q_i and of Q_j for all $i, j = 1, \ldots, k$,
- $\mathcal{A}_i \subset \Delta \cap \mathbb{Z}^2$ for $i = 1, \ldots, k$, and
- $\mathcal{A}_i \cap (Q_i \cap Q_j) = \mathcal{A}_j \cap (Q_i \cap Q_j)$ for all $i, j = 1, \ldots, k$.

We do not require that $\bigcup_{i=1}^k \mathcal{A}_i = \Delta \cap \mathbb{Z}^2$.

Definition 2.1
We define the *type* of a marked subdivision to be the subdivision, i.e. the collection of the Q_i, without the markings.

Figure 2 shows an example of a marked subdivision and its type. The subset of lattice points which are marked in each Q_i are drawn in black, the lattice points $\Delta \cap \mathbb{Z}^2$ which are not marked are white. We will stick to this convention throughout the paper.

Figure 2. A marked subdivision and its type

For a finite subset $\mathcal{A}$ of the lattice $\mathbb{Z}^2$ we denote by $\mathbb{R}^{\mathcal{A}}$ the set of vectors indexed by the lattice points in $\mathcal{A}$. A point $u \in \mathbb{R}^{\mathcal{A}}$ induces a *marked subdivision* of Δ by considering the convex hull of

$$\big\{(i, j, u_{ij}) \bigm| (i,j) \in \mathcal{A}\big\} \subset \mathbb{R}^3 \tag{1}$$

in $\mathbb{R}^3$, and projecting the upper faces onto the xy-plane. A lattice point (i,j) is marked if the point (i, j, u_{ij}) is contained in one of the upper faces. Marked subdivisions of Δ obtained in this way are called *regular* or *coherent*. We say two points u and u' in $\mathbb{R}^{\mathcal{A}}$ are equivalent if and only if they induce the same regular marked subdivision of Δ. This defines an equivalence relation on $\mathbb{R}^{\mathcal{A}}$ whose equivalence classes are the relative interiors of convex cones. The collection of these cones is the *secondary fan* of Δ.

Marked subdivisions of Δ are dual to plane tropical curves (see e.g. [8, Prop. 3.11]). Given a point $u \in \mathbb{R}^{\mathcal{A}}$ it defines a plane tropical curve C_F as the locus of non-differentiability of the tropical polynomial

$$F = \max\{u_{ij} + i \cdot x + j \cdot y \mid (i,j) \in \mathcal{A}\},$$

and it defines a regular subdivision of Δ. Each marked polygon of the subdivision is dual to a vertex of C_F, and each edge e of a marked polygon is dual to an edge E of C_F. Moreover, the edge E is orthogonal to its dual edge e. Finally, the edge E is unbounded if and only if its dual edge e is contained in the boundary of the polygon Δ. The *weight* of an edge E is equal to $\#(e \cap \mathbb{Z}^2) - 1$.

The duality implies that we can deduce the type of the marked subdivision from the plane tropical curve C_F, but not the markings. To deduce the markings, we need to know the coefficients u_{ij}.

Obviously, the vector $(1, \ldots, 1)$ is contained in the lineality space of the secondary fan. Therefore we can mod out this vector and consider the resulting fan in $\mathbb{R}^{s-1} = \mathbb{R}^{\mathcal{A}}/(1, \ldots, 1) \cdot \mathbb{R}$ with $s = \#\mathcal{A}$. We have seen above that every point in $\mathbb{R}^{\mathcal{A}}$ defines a tropical curve via the tropical polynomial $\max\{u_{ij} + i \cdot x + j \cdot y\}$. Of course, adding 1 to each coefficient u_{ij} does not change the tropical curve associated to this polynomial. Hence if we consider $\mathbb{R}^{\mathcal{A}}$ as a parametrising space for tropical curves, it makes sense to mod out $(1, \ldots, 1) \cdot \mathbb{R}$, and we will do so in what follows. By abuse of notation, we call the fan in $\mathbb{R}^{s-1}$ that we get from the secondary fan in this way also the *secondary fan.*

2.1. **The dimension of cones.** Let $T = \{(Q_l, \mathcal{A}_l) \mid l = 1, \ldots, k\}$, be a marked subdivision of Δ. Let

$$L := \left\{(\lambda_{ij}) \in \mathbb{R}^{\mathcal{A}} \;\middle|\; \sum\nolimits_{ij} \lambda_{ij} \cdot (i,j) = 0, \; \sum\nolimits_{ij} \lambda_{ij} = 0\right\}$$

be the space of affine relations among the lattice points (i,j) of Δ. For any l, let

$$L_{\mathcal{A}_l} = \{(\lambda_{ij}) \in L \mid \lambda_{ij} = 0 \text{ for } (i,j) \notin \mathcal{A}_l\}$$

be the space of affine relations among the elements of $\mathcal{A}_l$. Let L_T be the sum $\sum_l L_{\mathcal{A}_l}$.

Lemma 2.2
The codimension of the cone of the secondary fan corresponding to the marked subdivision T equals $\dim(L_T)$.

In particular, a cone in the secondary fan corresponding to a marked subdivision is top-dimensional if and only if the marked subdivision is a triangulation, i.e. all polygons Q_i are triangles and in each Q_i no other point besides the vertices is marked.

For a proof, see [6, Corollary 2.7].

Example 2.3
Let $T = \{(Q_1, \mathcal{A}_1), (Q_2, \mathcal{A}_2)\}$ be the subdivision shown in Figure 3, then L is the kernel of the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 2 \end{pmatrix},$$

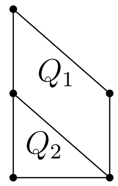


FIGURE 3. A marked subdivision

where the second and third entry of each column of A corresponds to the coordinates of a lattice point. Thus L is generated by

$$(1,-1,-1,1,0) \quad \text{and} \quad (0,1,0,-2,1).$$

Then $L_{\mathcal{A}_2}$ is set of vectors in L where the fourth and the fifth component vanish, and it is thus the zero space, while in $L_{\mathcal{A}_1}$ the first component has to vanish and it is thus generated by $(0,1,0,-2,1)$. We get

$$L_T = L_{\mathcal{A}_1} = (0,1,0,-2,1) \cdot \mathbb{R}$$

and the codimension of the cone in the secondary fan corresponding to the marked subdivision T is one.

Remark 2.4
A cone in the secondary fan is of codimension one if and only if exactly one of the $\mathcal{A}_i$ of the corresponding marked subdivision contains a circuit and it contains exactly one circuit. Here, a *circuit* is a set of lattice points that is affinely dependent but such that each proper subset is affinely independent. Figure 4 shows all types of circuits that can appear for point configurations in the plane together with some marked subdivisions of codimension one.

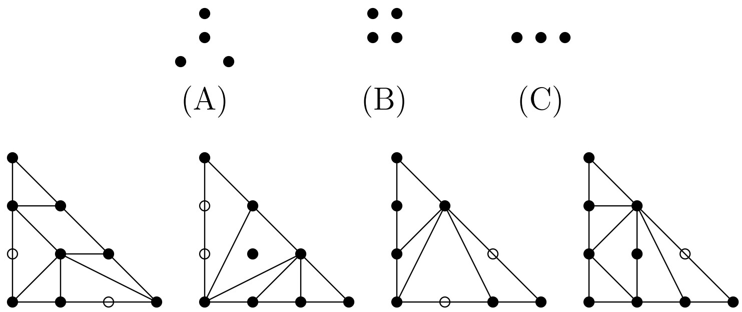


FIGURE 4. Planar circuits and marked subdivisions of codimension one

2.2. **The dimension of types of tropical curves.** Given a tropical curve C, we have seen above that it is dual to a type $\alpha = \{Q_1, \ldots, Q_k\}$ of a marked subdivision. We call α also the *type of the tropical curve*. We can parametrise all tropical curves of a given type by an unbounded polyhedron in $\mathbb{R}^{2+b}$ where b denotes the number of bounded edges of C. This is true because we can move the curve in the plane, and we can change the lengths of the bounded edges without changing the type. However, the lengths cannot be changed independently if the tropical curve is of genus $g \geq 1$. We get $2g$ (not necessarily independent) equations in $\mathbb{R}^{2+b}$ that tell us that the loops of C have to close up. We define the *dimension* $\dim(\alpha)$ of a type α to be the dimension of the parametrising polyhedron.

For the following lemma recall that we consider the secondary fan of Δ as a fan in $\mathbb{R}^{\mathcal{A}}/(1,\ldots,1)\cdot\mathbb{R}$.

Lemma 2.5
Given a marked subdivision $T = \{(Q_l, \mathcal{A}_l)\}$ of Δ of type α, we have

$$\dim(\alpha) \leq \dim(C_T),$$

where C_T denotes the cone of the secondary fan corresponding to T.

Equality holds if and only if in T all lattice points of Δ are marked, i.e. there are no white points.

Proof:
To a point $\overline{u} \in C_T$ with representative $u \in \mathbb{R}^{\mathcal{A}}$ we associate a tropical polynomial $\max\{u_{ij}+ix+jy\}$ and thus a tropical curve of type α. If we fix one of the polygons in T and assign to the tropical curve the coordinates of the vertex corresponding to this polygon and the lattice lengths of the b bounded edges, we get a map

$$\Phi_T : C_T \longrightarrow \mathbb{R}^{2+b}$$

from C_T to the parameter space of the type α. This map is given by rational functions, since the coordinates of the vertices of the tropical curve and thus the lattice lengths of the edges are solutions of systems of linear equations of the form $u_{ij} + ix + jy = u_{kl} + kx + ly$.

We have to show that Φ_S is a bijection onto the polyhedron parametrising the type α if S is the subdivision of type α where all lattice points are marked. Then $\dim(\alpha) = \dim(C_S)$ in this case, and if T is any other subdivision of type α then C_S lies in the boundary of C_T and has strictly smaller dimension.

Every tropical curve of type α comes from a point u (i.e. is the tropical curve associated to the tropical polynomial $\max\{u_{ij} + ix + jy\}$) which has to be inside a cone of the secondary fan corresponding to a marked subdivision of type α. Assume now there is a lattice point which is not marked in this subdivision. That means that the corresponding term in the tropical polynomial can never be the maximum. Therefore we can vary the coefficient without changing the tropical curve, until it reaches the upper faces of the convex hull of $\{(i, j, u_{ij})\} \subset \mathbb{R}^3$. Thus every tropical curve of type α comes in fact from a point u inside the cone C_S corresponding to the subdivision of type α where all lattice points are marked. This shows that Φ_S is surjective.

In order to see that Φ_S is injective it suffices to show that the tropical curve defined by $\max\{u_{ij} + ix + jy\}$ determines the class $\overline{u}$ of u in C_S uniquely, since the polyhedron associated to the type α parametrises the tropical curves of type α.

The vertices of the tropical curve are the solutions of a system of linear equations of the form

$$\begin{pmatrix} i-k & j-l \\ i-m & j-n \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} u_{kl} - u_{ij} \\ u_{mn} - u_{ij} \end{pmatrix}$$

with an invertible coefficient matrix defined by an arbitrary choice of three vertices of the polygon dual to the vertex of interest in the tropical curve. The solution thus determines the inhomogeneity of the system, i.e. the u_{ij} for (i,j) a vertex in a fixed polygon Q_k of S are determined up to a common summand. Since each polygon Q_i shares a vertex with some other Q_j, this shows that the u_{ij} corresponding to vertices of some Q_k are all determined up to a common summand. If (i,j) is a lattice point in Q_k which is not a vertex of Q_k, then u_{ij} is determined by the u_{mn} corresponding to vertices of Q_k, since (i,j,u_{ij}) is supposed to be visible in the upper face which projects to Q_k. But then u_{ij} is determined up to the same summand as the $u_{m,n}$. Altogether this shows that u is determined by the curve up to adding a multiple of $(1,\ldots,1)$, and the class $\overline{u}$ is determined uniquely. □

3. The tropicalisation of the family of curves with a singularity in a fixed point

Fix a non-degenerate convex lattice polygon Δ and set $\mathcal{A}=\Delta\cap\mathbb{Z}^2=\{m_1,\ldots,m_s\}$. The closure of the image of the map

$$\Psi_{\mathcal{A}}:(\mathbb{K}^*)^2\to\mathbb{P}^{s-1}_{\mathbb{K}}:(x,y)\mapsto(x^{m_{1,1}}y^{m_{1,2}},\ldots,x^{m_{s,1}}y^{m_{s,2}})$$

is a toric surface $\Sigma=\mathrm{Tor}_{\mathbb{K}}(\Delta)$ and the hyperplane sections are the closure of the images of the curves in $(\mathbb{K}^*)^2$ given by

$$f_{\underline{a}}=a_1x^{m_{1,1}}y^{m_{1,2}}+\ldots+a_sx^{m_{s,1}}y^{m_{s,2}}=0,$$

with $\underline{a}=(a_1,\ldots,a_s)$.

The linear equations in the a_i for the family $\mathrm{Sing}_{\mathbb{K}}(\Delta)$ of such curves with a singularity in the fixed point $\boldsymbol{p}=(1,1)$ are

$$f_{\underline{a}}(1,1)=0,\quad \frac{\partial f_{\underline{a}}}{\partial x}(1,1)=0,\quad \frac{\partial f_{\underline{a}}}{\partial y}(1,1)=0,$$

or equivalently we can say that the family $\mathrm{Sing}_{\mathbb{K}}(\Delta)$ is the kernel of the $3\times s$ matrix

$$A=\begin{pmatrix}1 & \ldots & 1\\ m_1 & \ldots & m_s\end{pmatrix}.$$

Notice that A is just the matrix of our point configuration, after raising the points on the $\{t=1\}$-plane in $\mathbb{R}^3$, if we choose the coordinates (t,x,y) on $\mathbb{R}^3$.

We want to study the tropicalisation of $\ker(A)$,

$$\mathrm{Trop}(\ker(A))=\mathrm{Trop}(\mathrm{Sing}_{\mathbb{K}}(\Delta)).$$

Remark 3.1
If we choose a different point $(p,q)\in(\mathbb{C}^*)^2$ and consider the family of curves with a singularity in (p,q), then the coefficient matrix A of the above linear equations changes. More precisely, it will be multiplied from the right by a diagonal matrix $D(p,q)=(d_{ij})_{i,j=1,\ldots,s}$ with diagonal entry $d_{ii}=p^{m_{i1}}\cdot q^{m_{i2}}$ if $m_i=(m_{i1},m_{i2})$. Denote by $A(p,q)=A\cdot D(p,q)$ the new matrix, then the minors of $A(p,q)$ differ from the minors of A only by certain factors, and each of these factors is a monomial in p and q since the columns in $A(p,q)$ differ from the corresponding columns of A only by factor which is a monomial in p and q.

However, the matroid of $A(p,q)$ is determined by the question which minors of $A(p,q)$ vanish and which do not. So the matroids of A and of $A(p,q)$ coincide. The tropical variety $\mathrm{Trop}(\ker(A))$ respectively $\mathrm{Trop}(\ker(A(p,q))$ depends only on the matroid of A respectively of $A(p,q)$ (see [13], § 9.3). Thus, the tropical variety of $\ker(A(p,q))$ is independent of the chosen point $(p,q) \in (\mathbb{C}^*)^2$.

3.1. **The tropicalisation of** $\ker(A)$**.** Let us now study the tropicalisation of $\ker(A)$. As remarked in the introduction, we consider this tropical variety as a fan in $\mathbb{R}^s/(1,\ldots,1)\cdot\mathbb{R}$. By Section 2.5 of [12], the fan is balanced. To study $\mathrm{Trop}(\ker(A))$, we use the following known results about the tropicalisation of linear spaces.

It was observed in [13], § 9.3, that the tropicalisation of a linear space $\ker(A)$ depends only on the matroid M associated to the matrix A. By [5], this matroid can be specified by its collection of circuits, which are the minimal sets arising as supports of linear forms vanishing on $\ker(A)$ (resp. minimal sets arising as supports of elements in the row space of A). Equivalently, these are minimal sets $\{i_1,\ldots,i_r\} \subset \{1,\ldots,s\}$ such that the columns $b_{i_1},\ldots,b_{i_r}$ of a Gale dual B of A are linearly dependent. A *Gale dual* is a matrix B whose rows span the kernel of A. Thus we can also describe the matroid associated to A as the matroid of the point configuration given by the columns of a Gale dual of A.

Proposition 2.5 of [5] states that the set of all $w \in \mathbb{R}^s$ such that M_w (the matroid of bases σ of M for which $\sum_{i\in\sigma} w_i$ is maximal) contains no loop equals the tropicalisation of the linear space $\ker(A)$. In [1] the set of all w such that M_w contains no loop is called the *Bergman fan* of the matroid M. Given $u \in \mathbb{R}^s$, let $\mathcal{F}(u)$ denote the unique *flag of subsets*

$$\emptyset =: F_0 \subsetneq F_1 \subsetneq \ldots \subsetneq F_k \subsetneq F_{k+1} := \{1,\ldots,s\}$$

such that

$$u_i < u_j \quad \Longleftrightarrow \quad \exists\, m \;:\; i \in F_{m-1} \text{ and } j \notin F_{m-1}.$$

In particular,

$$u_i = u_j \quad \Longleftrightarrow \quad \exists\, m \;:\; i,j \in F_m \setminus F_{m-1}.$$

The *weight class* of a flag $\mathcal{F}$ is the set of all u such that $\mathcal{F}(u) = \mathcal{F}$. We can describe weight classes by their defining equalities and inequalities.

For example, the set of all vectors u satisfying $u_3 < u_1 = u_4 < u_2$ defines a weight class in $\mathbb{R}^4$. It corresponds to the flag $\{3\} \subset \{1,3,4\} \subset \{1,2,3,4\}$.

A flag $\mathcal{F}$ is a *flag of flats* of the Gale dual B of A respectively of the associated matroid M if the linear span of the vectors $\{b_j \mid j \in F_i\}$ contains no b_k with $k \notin F_i$. As before, the vectors b_j denote the columns of a Gale dual of A.

Theorem 1 of [1] states that the Bergman fan of a matroid M is the union of all weight classes of flags of flats of M. This result also follows from Theorem 4.1 of [5].

As a consequence, we can study our tropical linear space by studying weight classes of flags of flats of a Gale dual of A.

Construction 3.2
Choose three points of $\mathcal{A}$ which are affinely independent Then we can perform Gaussian elimination with the matrix A making the columns corresponding to these

three points the columns with pivots. To the point configuration in threespace given by the columns of A, this Gaussian elimination has the effect of an affine transformation. Denote by $\widetilde{m}_i$ the i-th column of the transformed matrix A. To simplify notation, we will assume without restriction that the three points we chose are m_1, m_2 and m_3.

Remark 3.3
Before we made the transformation from Construction 3.2, all columns of A lived on the $\{t = 1\}$ plane. Since the three special points are transformed to $\widetilde{m}_1 = (1,0,0)$, $\widetilde{m}_2 = (0,1,0)$ and $\widetilde{m}_3 = (0,0,1)$, the point configuration now sits on the $\{t+x+y=1\}$ plane spanned affinely by these three points.

Performing this Gaussian elimination makes it easy to read off generators for the kernel of the matrix, and thus a possible Gale dual.

Example 3.4
Let $\mathcal{A}$ be the point configuration in Figure 5. Then A is the matrix

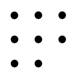


Figure 5. A point configuration

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 2 & 0 & 1 & 2 \\ 0 & 0 & 1 & 1 & 1 & 2 & 2 & 2 \end{pmatrix}.$$

We choose the first three points — $(0,0)$, $(1,0)$ and $(0,1)$ — to be the three special points in Construction 3.2. After performing Gaussian elimination, the matrix reads:

$$\widetilde{A} = \begin{pmatrix} 1 & 0 & 0 & -1 & -2 & -1 & -2 & -3 \\ 0 & 1 & 0 & 1 & 2 & 0 & 1 & 2 \\ 0 & 0 & 1 & 1 & 1 & 2 & 2 & 2 \end{pmatrix} = (\mathbb{1}_3 \mid A_1).$$

From this, we can easily read off a basis of the kernel.

$$B = \left(-A_1^t \mid \mathbb{1}_{s-3}\right) = \begin{pmatrix} 1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 2 & -2 & -1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & -2 & 0 & 0 & 1 & 0 & 0 \\ 2 & -1 & -2 & 0 & 0 & 0 & 1 & 0 \\ 3 & -2 & -2 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Note that by construction, the negative of the first three entries of the i-th row are just the coordinates of the transformed point $\widetilde{m}_{i+3}$.

Remark 3.5
Just as in Example 3.4 we have in general that the Gale dual B constructed in this way has the following form: It is an $(s-3) \times s$-matrix where the first 3 (column) vectors b_1, b_2 and b_3 are the (t,x,y)-coordinates of the $(s-3)$ points $-\widetilde{m}_i$, $i = 4, \dots, s$, and the remaining vectors are the unit vectors $b_4 = e_1, \dots, b_s = e_{s-3}$. The column b_i corresponds to the point $\widetilde{m}_i$ in the sense that the i-th entry of

the rows of B — which are in the kernel of $\widetilde{A}$ — gets multiplied with $\widetilde{m}_i$ when computing the product $\widetilde{A} \cdot B^t$.

In such a Gale dual we now want to find flags of flats, i.e. flags of $s-3$ subspaces $V_i \subset \mathbb{R}^{s-3}$:

$$\{0\} \subsetneqq V_1 \subsetneqq \ldots \subsetneqq V_{s-3},$$

where each V_i is generated by a subset of the column vectors b_j of the Gale dual indexed by the set F_i, and the vectors $\{b_j \mid j \in F_i\}$ are all the column vectors of the Gale dual that are contained in the subspace V_i. In particular, $F_{s-3} = \{1, \ldots, s\}$. We set $F_i' := F_i \setminus F_{i-1}$. Each F_i' must of course consist of at least one element j. Since we have s vectors in total, we have 3 "extra" vectors that can a priori belong to any of the F_i'. In the next lemma, we show that in fact we do not have that much choice.

Lemma 3.6
With the notation of Remark 3.5, for each flag of flats of a Gale dual B of A we have either

(a) $\#F_i' = 1$ *for all* $i = 1, \ldots, s-4$ *and* $\#F_{s-3}' = 4$, *or*
(b) $\#F_{s-3}' = 3$ *and there is a* $j \in \{1, \ldots, s-4\}$ *with* $\#F_j = 2$.

In the first case, if $F_{s-3}' = \{a, b, c, d\}$, then any proper subset of the points m_a, m_b, m_c and m_d is affinely independent (i.e. $\{m_a, m_b, m_c, m_d\}$ is a circuit of type (A) or (B) as in Remark 2.4).

In the second case, if $F_{s-3}' = \{a, b, d\}$, the points m_a, m_b, m_d are affinely dependent (i.e. $\{m_a, m_b, m_d\}$ is a circuit of type (C) as in Remark 2.4). Furthermore, all points m_r with $r \in F_l'$, $l > j$, are on the same line as m_a, m_b and m_d.

Proof:
First, we show that $\#F_{s-3}'$ cannot be 2 or 1. Assume it was. Then $\#F_{s-4} = s-2$ (resp. $s-1$) but the subspace V_{s-4} spanned by the vectors of F_{s-4} is only $(s-4)$-dimensional. Remember that $b_4 = e_1, \ldots, b_s = e_{s-3}$. To get an $(s-4)$-dimensional subspace with $s-2$ (resp. $s-1$) vectors, we in principal have 2 possibilities:

(a) $\{b_r \mid r \in F_{s-4}\}$ can contain $s-4$ of the unit vectors $b_4, \ldots, b_s$, and 2 of the special vectors b_1, b_2, b_3, or
(b) it can contain $s-5$ (resp. $s-4$) unit vectors and all special vectors b_1, b_2, b_3.

Let us consider case a) first. In $\{b_r \mid r \in F_{s-4}\}$, we are missing just one of the unit vectors, say $b_{j+3} = e_j$. Then the 2 special vectors which are also part of $\{b_r \mid r \in F_{s-4}\}$ must both have zeroes in the j-th component, or the dimension would be bigger than $s-4$. The j-th components of the two special vectors are two of the coordinates of the point $\widetilde{m}_{j+3}$. Thus without restriction $\widetilde{m}_{j+3} = (a, 0, 0)$ for some number a. But remember that the points $\widetilde{m}_i$ are all in the $\{t + x + y = 1\}$ plane by Remark 3.3, thus $a = 1$. But then $\widetilde{m}_{j+3} = \widetilde{m}_1 = (1, 0, 0)$ which is a contradiction.

Let us now consider case b). Now we are missing two of the unit vectors in $\{b_r \mid r \in F_{s-4}\}$, say $b_{j+3} = e_j$ and $b_{k+3} = e_k$ (resp. again just one). The three special vectors b_1, b_2, b_3 must have two linearly dependent rows in the j-th and k-th component, or the dimension would be bigger than $s-4$ (resp. they must all have a zero in some row which cannot be true since this row lives in the $\{t + x + y = 1\}$ plane). But both rows again are the coordinates of the points $\widetilde{m}_{j+3}$ and $\widetilde{m}_{k+3}$. Both points live in the $\{t+x+y = 1\}$ plane, so if they are linearly dependent, they are equal which is a contradiction.

We conclude that for each flag of flats, $\#F'_{s-3} = 4$ or 3. Pick a flag of flats. For the statement about the affine dependencies, we want to switch to another Gale dual of A which is more suitable for this particular flag of flats. First, choose two of the elements of F'_{s-3}, say a and b, and an arbitrary c such that m_a, m_b and m_c are affinely independent. We want to use these as the three pivot points in Construction 3.2. We thus produce a new Gale dual (that must contain the same flag of flats). To make the notation simple, as before we want to call the special vectors b_1, b_2 and b_3, i.e. we assume without restriction that $a = 1$, $b = 2$ and $c = 3$. Thus $1, 2 \in F'_{s-3}$.

Let d be a third index in F'_{s-3} and assume that m_1, m_2 and m_d are affinely dependent. Note that affine dependence is preserved under the transformation that we perform to produce $\widetilde{A}$. We want to show that there cannot be a fourth element in F'_{s-3}, i.e. if F'_{s-3} contains a, b and d with m_a, m_b and m_d affinely dependent, then $\#F'_{s-3} = 3$. To see this, remember that the coordinates of the transformed point $\widetilde{m}_d$ appear in the $(d-3)$-rd row of the special vectors b_1, b_2 and b_3. But since $\widetilde{m}_d$, $\widetilde{m}_1 = (1, 0, 0)$ and $\widetilde{m}_2 = (0, 1, 0)$ are affinely dependent, it follows that $\widetilde{m}_d$ has a 0 as third coordinate. Thus b_3 has a 0 in the $(d-3)$-rd row, and we have to show that $\{1, \ldots, s\} \setminus \{1, 2, d, i\} \subseteq F_{s-4}$ implies $i \in F_{s-4}$.

Assume first $i = 3$. But b_3 is in the subspace generated by $\{b_4 = e_1, \ldots, b_s = e_{s-3}\} \setminus \{b_d = e_{d-3}\}$, since it has a 0 in the $(d-3)$rd coordinate. Thus any set containing $\{1, \ldots, s\} \setminus \{1, 2, d, 3\}$ also contains 3.

Now assume $i \neq 3$. Suppose that i is not in F_{s-4}. Then the $s-4$ vectors $\{b_3, b_4 = e_1, \ldots, b_s = e_{s-3}\} \setminus \{b_i, b_d = e_{d-3}\}$ generate the $s-4$-dimensional space V_{s-4} and all vectors in V_{s-4} have a zero in the $d-3$-rd component, since b_3 has so. Then the $i-3$-rd component of b_3 cannot be zero as well, since otherwise also the $i-3$-rd component of all vectors in V_{s-4} would be zero in contradiction to the dimension being $s-4$. But then $b_i = e_{i-3}$ is a linear combination of $\{b_3, b_4 = e_1, \ldots, b_s = e_{s-3}\} \setminus \{b_i, b_d = e_{d-3}\}$ and it is in V_{s-4}, which implies that $i \in F_{s-4}$ in contradiction to our assumption. This shows that i is contained in F_{s-4}.

Hence we have shown that $F_{s-4} = \{1, \ldots, s\} \setminus \{1, 2, d\}$ and thus $\#F'_{s-3} = 3$, if we assume that $1, 2, d \in F'_{s-3}$ are affinely dependent.

If $\#F'_{s-3} = 4$, we thus have four points such that each proper subset is affinely independent, i.e. a circuit of type (A) or (B) from Remark 2.4.

Now assume that $F_{s-3} = \{a, b, d\}$. We have to show that m_a, m_b and m_d are affinely dependent, and that furthermore all points m_r with $r \in F'_l$, $l > j$, are on the same line as m_a, m_b and m_d, where j is such that $\#F'_j = 2$. Again, we want to pick a suitable Gale dual for this flag, i.e. we assume without restriction that $a = 1$ and $b = 2$. As the third pivot point, we pick a point m_c such that m_a, m_b and m_c are affinely independent, and such that c is in F'_k with k maximal. That means, if $i \in F'_l$, $l > k$, then m_a, m_b and m_i are affinely dependent, i.e. m_i lies on the line through m_a and m_b. Again, we assume $c = 3$. Now F_{s-4} must contain all elements except 1, 2 and d, thus $F_{s-4} = \{3, \dots, s\} \setminus \{d\}$. V_{s-4} is an $s-4$-dimensional subspace. The vectors $\{b_4 = e_1, \dots b_s = e_{s-3}\} \setminus \{b_d = e_{d-3}\}$ thus span this subspace, and hence b_3 is a linear combination of these vectors. This implies that b_3 has a 0 in the $(d-3)$-rd row, which in turn implies that $\widetilde{m}_d$, $\widetilde{m}_1 = (1,0,0)$ and $\widetilde{m}_2 = (0,1,0)$ are affinely dependent.

Actually, b_3 has a 0 in exactly those rows that correspond to points $\widetilde{m}_i$ which are affinely dependent of $\widetilde{m}_1 = (1,0,0)$ and $\widetilde{m}_2 = (0,1,0)$, i.e. that correspond to points m_i on the same line as m_1 and m_2. If we set

$$\begin{aligned} B =&\{b_i \mid i \geq 4, m_i \text{ is not on the line through } m_1 \text{ and } m_2\} \\ =&\{e_{i-3} \mid i \geq 4, m_i \text{ is not on the line through } m_1 \text{ and } m_2\}, \end{aligned}$$

then b_3 is a linear combination of the elements of B and non of the coefficients is zero. Thus any subset of $B \cup \{b_3\}$ of size $\#B$ is a basis of the span of $B \cup \{b_3\}$ which shows that some V_l contains $\#B$ of the vectors of $B \cup \{b_3\}$ if and only if it contains all of them. Above we defined k as the maximal index such that F'_k contains an i with m_i affinely independent of m_1 and m_2, then the previous considerations show that V_k contains all vectors in $B \cup \{b_3\}$ while in V_{k-1} two of them are missing. This shows that F'_k has size two, i.e. $F'_k = F'_j$, (and contains the index $c = 3$ by choice) and that for $l > k = j$ and $i \in F'_l$ the point m_i does not lie on the line through m_1 and m_2. □

Remark 3.7
The following reversed statement of 3.6 holds true as well:

(a) For any circuit $\{m_a, m_b, m_c, m_d\}$ there exist all flags of flats satisfying $\#F'_j = 1$ for all $j \neq s-4$ and $F'_{s-4} = \{a, b, c, d\}$.
(b) For any circuit $\{m_a, m_b, m_d\}$ and any choice of m_c and m_e which are not on the line of m_a, m_b, m_d, there exist all flags of flats satisfying $F'_{s-4} = \{a, b, d\}$, $F'_j = \{c, e\}$, and all $i \in F'_l$ with $l > j$ satisfy m_i is on the line.

This can be seen similar to the proof of Lemma 3.6 by picking a suitable Gale dual. In case (a), all vectors b_i with $i \notin F'_{s-4}$ are unit vectors and we can thus form any possible flag with them. In case (b), we can pick a, b and c as pivots, and then we can pick any flag such that c and e appear latest among all i such that m_i is not on the line of m_a and m_b.

3.2. **Steps towards the classification of tropical curves with a singularity in a fixed point.** As a consequence, we can try to classify all types of tropical curves with a singularity in a fixed point. To do this, let us first express the statement about the flags of flats from Lemma 3.6 in terms of weight classes and marked subdivisions. We keep the notation from Remark 3.5. The following list shows the important parts of the different weight classes we get and sums up what we can say about the marked subdivisions and their dual tropical curves.

(a) Assume we have a flag with $\#F'_{s-4} = 4$ and the corresponding circuit is of type (A) or (B) as in Remark 2.4. Then these points get the highest weight. Consequently, the triangle resp. quadrangle which is the convex hull of the circuit is part of the marked subdivision corresponding to any u in the weight class. Besides, in the tropical polynomial which has u as coefficients, the four terms corresponding to the four points have the same coefficients. The vertex dual to the triangle resp. quadrangle is at the point (x, y) where the maximum is attained by those four terms, in particular the four terms are equal at this vertex. That means, we can set the four terms equal and solve for x and y to get the position of the vertex. But since the coefficients are all equal, we get $x = y = 0$ when solving.

Thus the dual tropical curve has the point $\boldsymbol{x}_0 = (0, 0)$ as a vertex of multiplicity strictly larger than one (corresponding to a triangle with an interior point, and thus of area bigger 1/2), or it has a 4-valent vertex at $(0, 0)$ (see Figure 6).

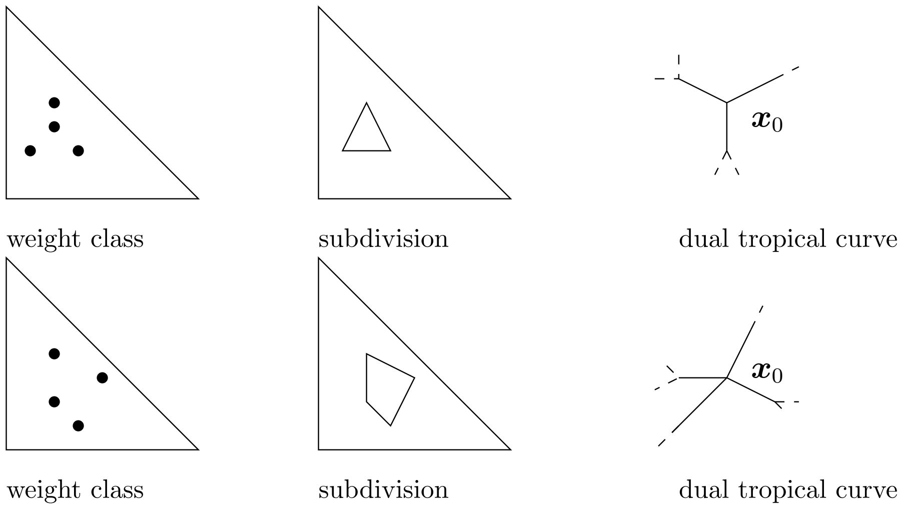


FIGURE 6. Weight classes of type (a) and their tropical curves

(b) Assume we have $F'_{s-4} = \{a, b, d\}$ and $F'_j = \{c, e\}$. In the picture, we draw the three points of highest weight black and the two points m_c and m_e in gray. Notice that the gray points are of the same height, namely the highest height of all points which are not on the line through m_a, m_b and m_d. The points on this line can have higher heights however.

Unfortunately, we cannot say much about the subdivision in this case. We can only be sure that the edge through m_a, m_b and m_d will be part of the subdivision. In the dual picture, this means we can see an edge of weight at least 2. Furthermore, this edge must pass through the point $\boldsymbol{x}_0 = (0, 0)$. The latter can be seen again by solving for the coordinates (x_1, y_1) and (x_2, y_2) of the two vertices adjacent to this edge. Since the heights of the three points m_a, m_b and m_d are equal, it follows that the line of which the dual edge is a segment passes through $(0, 0)$. Since the height of any point which belongs to an adjacent polygon of the edge through m_a, m_b and m_d is below the height of these, it follows that $x_1 < 0$ and $x_2 > 0$ (or vice versa) (see Figure 7).

Remark 3.8
The reason why we cannot say more than this is that we cannot predict how the

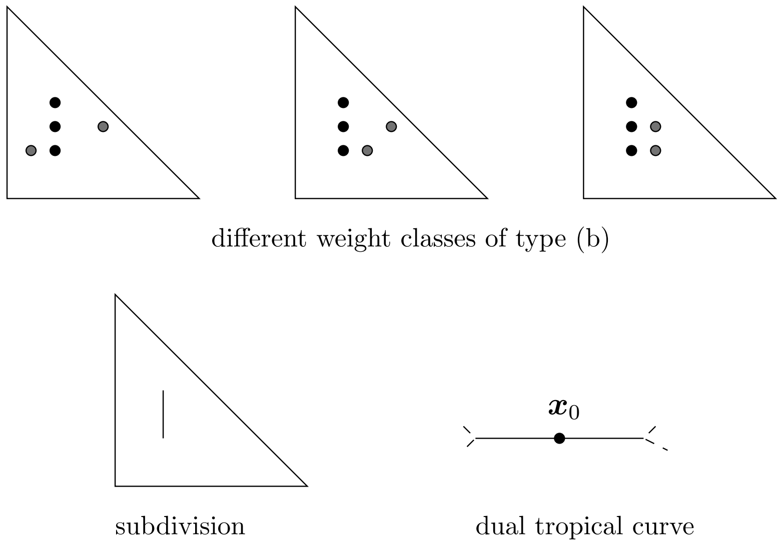
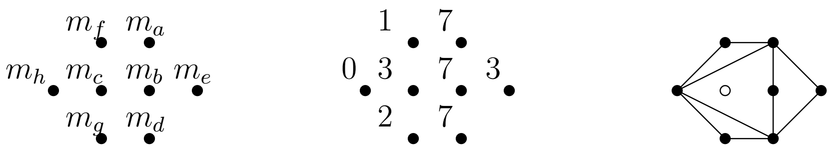


FIGURE 7. Different weight classes of type (b) and their tropical curve

polygons in the subdivision adjacent to the edge through m_a, m_b and m_d look like. It is possible that the gray points are not boundary points of an adjacent polygon. Even though they have the highest height of all points which are not on the line through m_a, m_b and m_d, (m_c, u_c) and (m_e, u_e) could still lie below the upper faces of the convex hull of the points (m_i, u_i). As an example, take the point configuration in the picture below (where $m_c = (1, 1)$) and take a weight vector u as depicted in the middle.



This weight vector is in the weight class

$$u_h < u_f < u_g < u_c = u_e < u_a = u_b = u_d$$

which comes from the flag indexed by

$$\{h\} \subsetneq \{h, f\} \subsetneq \{h, f, g\} \subsetneq \{h, f, g, c, e\} \subsetneq \{h, f, g, c, e, a, b, d\}.$$

In the picture, we can see the marked subdivision induced by u. Note that the point m_c is not part of a polygon adjacent to the edge through m_a, m_b and m_d.

A general point in a weightclass satisfies only the equalities given by the corresponding flag of flats, and strict inequalities otherwise. We can describe lower dimensional cones of the tropical variety $\mathrm{Trop}(\ker(A))$ by forcing some of the inequalities to become equalities. Here, we restrict ourselves to the classification of top-dimensional cones.

3.3. Trop(ker(A)) **and the secondary fan.** We have seen above that all the subdivisions we get in our family contain a circuit (either as a polygon Q_i, or as the face of a polygon Q_i). Hence the tropical variety Trop(ker(A)) lives inside the codimension-1-skeleton of the secondary fan. Furthermore, no weight class belonging to a flag of flats of type (a) of Subsection 3.2 contains the lineality space of the secondary fan. The following lemma then shows that in a sense it is just the lineality space which is missing to pass from the cone of a weight class to a codimension one cone of the secondary fan.

Remember that we have mod out the vector $(1,\dots,1)$ already. But the secondary fan still contains a 2-dimensional *lineality space* L spanned by the vector consisting of the x-coordinates of the points m_i, and the vector consisting of their y-coordinates. This is true because if we incline the heights u_i of the points (m_i,u_i) by a fixed multiple of the x-coordinates of the m_i respectively of the y-coordinates of the m_i, we do not change the projection of the upper faces of the convex hull.

Lemma 3.9
Let Δ be a convex lattice polygon in the plane with associated matrix A and Gale dual B of A, and let Z be a circuit in Δ of type (A) or (B) as in Remark 2.4, i.e. a circuit consisting of four elements $Z=\{m_a,m_b,m_c,m_d\}$.

Then the union of all weight classes $C_{\mathcal{F}}$ of flags of flats $\mathcal{F}$ of B that end with $F'_{s-4}=\{a,b,c,d\}$ (where again we use the notation from 3.5) plus the lineality space L of the secondary fan of Δ equals the union of all codimension one cones C_T of the secondary fan of Δ corresponding to subdivisions T that contain this circuit, i.e.

$$\left(\bigcup_{\mathcal{F}}\overline{C_{\mathcal{F}}}\right)+L=\bigcup_{T}\overline{C_T},$$

where the union on the left goes over all flags of flats $\mathcal{F}$ of B that end with $F'_{s-4}=\{a,b,c,d\}$ and the union on the right goes over all subdivisions T that contain the circuit Z.

Proof:
We have seen in our Classification 3.2 that the marked subdivision of a vector u in any weight class corresponding to such a flag of flats contains the circuit as a polygon. Thus "$\subset$" is obvious. Pick any u in C_T, then we can write it as a sum of a vector in the lineality space and a vector that satisfies that the heights of the four points m_a, m_b, m_c and m_d are equal and highest among all heights. This shows "$\supset$". □

Note also that the statement makes sense dimension-wise: The secondary fan is of dimension $s-1$ and the codimension 1 cone C_T of dimension $s-2$. Our tropical variety Trop(ker(A)) is of the same dimension as the "classical" variety ker(A) which is $s-4$-dimensional, since it lives in projective space of dimension $s-1$ and is given by 3 independent equations.

Remark 3.10
Next we want to understand the cones of the secondary fan of Δ which correspond to flags of flats respectively weight classes of type (b) in the Classification 3.2.

Let us thus assume that we have such a flag $\mathcal{F}$ of flats with $F'_{s-3} = \{a, b, d\}$ and $F'_j = \{c, e\}$ as in the proof of Lemma 3.6.

The case where the points m_c and m_e span a line parallel to the line through m_a, m_b and m_d, i.e. where we are in a situation as depicted in Figure 8, plays a special role.

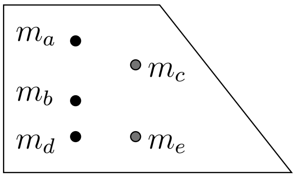


FIGURE 8. A weight class of type (b) in the boundary of others

Let $\mathcal{F} = \mathcal{F}(u)$, let T be the subdivision of Δ such that $u \in C_T$ and let Q be the polygon in T which contains the circuit $Z = \{m_a, m_b, m_d\}$ and lies on the same side of Z as the points m_c and m_e (see Figure 9). We then have to distinguish two subcases. Either Q contains a vertex whose distance to the line through m_a, m_b

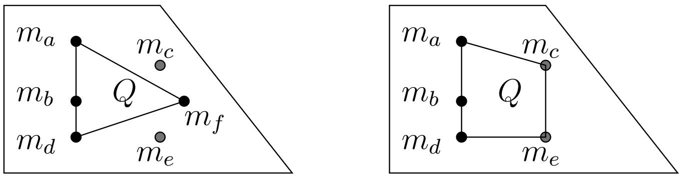


FIGURE 9. Two different types of the boundary type

and m_d is larger than the distance of m_c and m_e to this line, or Q is the polygon spanned by Z and the two vertices m_c and m_e (see Figure 9). To convince yourself that this is true recall that $u_a = u_b = u_d > u_c = u_e > u_i$ for all other i; thus, if Q has no vertex whose distance is larger than that of m_c and m_e the planar polygon spanned by (m_a, u_a), (m_b, u_b), (m_d, u_d), (m_c, u_c) and (m_e, u_e) in three space is an upper face of the extended Newton polytope corresponding to Δ and u.

If Q is spanned by $m_a, \ldots, m_e$ then the cone C_T is in the boundary of the cone C_S for a subdivision S as shown in Figure 10, where four of the lattice points form a quadrangle. Such quadrangles where already considered in Lemma 3.9, and thus together with the cone C_T the weight class $C_{\mathcal{F}} \subset C_T$ is contained in the boundary of cones of the secondary fan belonging to weight classes of type (a).

If Q instead contains a vertex which is further away from the line through m_a, m_b and m_d than m_c and m_e, then the cone C_T as well as the cone $C_{\mathcal{F}} \subset C_T$ of the weight class $\mathcal{F}$ are in the boundary of cones of the secondary fan belonging to weight classes of type (b) as considered in the following Lemma 3.11.

In any case it is not necessary to consider these weight classes in order to get a full picture of the codimension one cones of the secondary fan of Δ fixed by the weight classes of type (a) or (b).

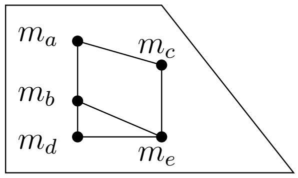


FIGURE 10. A subdivision such that C_S contains C_T in its boundary

Lemma 3.11
Let Δ be a convex lattice polygon in the plane with associated matrix A and Gale dual B of A, and let Z be a circuit of type (C) as in Remark 2.4, i.e. a circuit consisting of three elements $Z = \{m_a, m_b, m_d\}$.

Then

$$\left(\bigcup_{\mathcal{F}} \overline{C_{\mathcal{F}}}\right) + L = \bigcup_T \overline{C_T},$$

where

- *L is the lineality space of the secondary fan of Δ;*
- *the union on the left is the union of all weight classes $C_{\mathcal{F}}$ of flags of flats $\mathcal{F}$ of B as in (b) of the Classification 3.2, except for those considered in Remark 3.10; that is, the flags end with $F'_{s-4} = \{a, b, d\}$, have $F'_j = \{c, e\}$ where the line through m_c and m_e is not parallel to the line through m_a, m_b and m_d, while m_i is on the latter line for all $i \in F'_l$ for $l > j$;*
-
  - *if Z is not contained in the boundary of Δ, the union on the right is the union of all codimension one cones C_T of the secondary fan of Δ that correspond to subdivisions T containing Z;*
  - *if Z is contained in the boundary of Δ, then the union on the right is the union of all codimension one cones C_T of the secondary fan of Δ that correspond to subdivisions T containing Z, except those T for which the triangle containing Z has its third vertex at a point of minimal distance from Z.*

Figure 11 shows part of a triangulation corresponding to one of the codimension one cones we throw out of the union if Z is contained in the boundary of Δ.

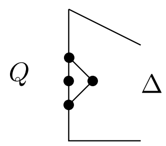


FIGURE 11. Triangulations we have to throw out

Proof:
The proof is analogous to 3.9. If u is a vector in one of the weight classes we chose, then u induces a subdivision containing Z. If Z is in the boundary of Δ, the two points m_e and m_c have to be on one side of the line through Z. We have to show that the polygon containing Z is not a triangle with its third vertex at minimal

distance. One of the points of m_e and m_c has to be at a non minimal distance, since we assume that they do not sit on a line parallel to the line through Z. Assume this point is m_c. One of the three lines connecting the points (m_a, u_a) and (m_c, u_c), resp. (m_b, u_b) and (m_c, u_c), resp. (m_d, u_d) and (m_c, u_c), certainly lives above any line connecting (m_a, u_a) with a point of minimal distance to Z. Hence Z cannot be the face of a triangle with its third vertex at minimal distance. This proves "$\subset$".

Conversely, we can write a vector u in C_T as a sum of a vector in the lineality space and a vector that satisfies that the heights of the three points m_a, m_b and m_d are equal and highest, and that there are two points which are not on the line through Z whose heights are equal and the highest among all points which are not on the line through Z. To do this, assume without restriction that Z is on the line $\{x = 0\}$. We can write u as a sum of a multiple of the vector of y-coordinates of the m_i and a vector u' such that $u'_a = u'_b = u'_d$.

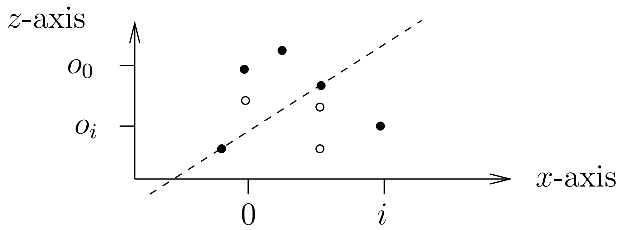


FIGURE 12. The projection of the (m_i, u'_i) to the xz-plane

Now let o_i be the maximum height on the line $\{x = i\}$, i.e.

$$o_i = \max\{u'_j \mid m_j \in \{x = i\}\},$$

see Figure 12. Note that since Z is part of the subdivision, o_0 must be the height of the points of Z. Even more, since Z is part of the subdivision, the point $(0, o_0)$ will be above each line through two points (k, u'_i) and (l, u'_j) with $m_i \in \{x = k\}$, $m_j \in \{x = l\}$ and $k < 0 < l$. This is true because otherwise there are two points in Z, say m_a and m_b, which are not on different sides of the line through m_i and m_j and where one of them, say m_a, has a strictly larger distance to this line, and then the point (m_b, u'_b) would be strictly below the plane spanned by (m_a, u'_a), (m_i, u'_i) and (m_j, u'_j) in contradiction to the assumption that Z is visible in the subdivision. Thus o_0 is contained in the boundary of the convex hull of the points (k, o_k). Now we can add a multiple of the vector of x-coordinates to u' to rotate the image in Figure 12 about $(0, o_0)$ such that o_0 becomes the largest among the o_k and such that the two next smaller o_k have the same height, i.e such that there are $j \neq l$ with $o_j = o_l \geq o_h$ for all $h \neq 0$. It is possible to make o_0 maximal, since $(0, o_0)$ is in the boundary of the convex hull of the points (k, o_k).

If Z is not contained in the boundary of Δ, we have points (k, o_k) with positive and negative k-coordinate. By rotating about $(0, o_0)$ we can ensure that the vertices in the convex hull of the (k, o_k) closest to the vertex $(0, o_0)$ on each side will have the same height, i.e. the two largest o_k on each side of Z will be equal. If Z is contained in the boundary, we have only points (k, o_k) with positive k-coordinate (without restriction). However, the point of minimal distance $(1, o_1)$ is not a vertex of the convex hull of the points (k, o_k). This is true since the triangle containing Z does not have its vertex on the line $\{x = 1\}$. This means again that we can make the two next largest heights o_k equal by rotating. The point u'' we get in this way lives in a weight class as in (b) of 3.2. This proves "$\supset$".

□

3.4. **The connection to the tropical discriminant.** The tropical discriminant has been studied by Dickenstein, Feichtner and Sturmfels ([3]). Their main result is that the tropicalisation of the discriminant of a point configuration $\mathcal{A} = \{m_1, \ldots, m_s\}$ — i.e. the locus of all parameters $\underline{a}$ for which the curves $V(f_{\underline{a}})$ given by a polynomial

$$f_{\underline{a}} = a_1 x^{m_{1,1}} y^{m_{1,2}} + \ldots + a_s x^{m_{s,1}} y^{m_{s,2}}$$

are singular — equals Trop(ker(A)) $\oplus$ rowspace(A). This follows by a tropical version of Horn uniformisation. A curve $V(f_{\underline{a}})$ is singular in a point (p, q) in the torus if and only if $V(f_{\Psi_{\mathcal{A}}(p,q)\cdot\underline{a}})$ (see beginning of Section 3) is singular in $(1, 1)$. This helps to express every point in the discriminant as the image under Horn uniformisation of a tuple consisting of a point in ker(A) and a point in the torus. Notice that the rowspace of A equals the lineality space L of the secondary fan of the point configuration. In the previous Section, we have described what cones of the secondary fan we get if we add this lineality space to our tropical variety Trop(ker(A)). Thus our result can also be seen as a description of the tropical discriminant. It follows that the tropical discriminant of a plane point configuration is a subfan of the secondary fan and consists of all closed codimension one cones of the secondary fan except the ones involving a circuit Z consisting of three points on the boundary of Δ such that the triangle containing Z has its third vertex at a point of minimal distance of Z. This description of the tropical discriminant was known before (see 11.3.9 of [6]). There, Δ-equivalent triangulations of the secondary fan are classified. Two triangulations are Δ-equivalent, if their corresponding cones lie in the same top-dimensional cone of the Gröbner fan of the discriminant. Since the tropicalisation of the discriminant equals the codimension 1-skeleton of the Gröbner fan, this means that two neighbouring triangulations are Δ-equivalent if and only if the codimension 1-cone that they meet in does not belong to the tropical discriminant. The only codimension 1-cones of the secondary fan which do not belong to the tropical discriminant are the ones containing a circuit Z of three points on the boundary such that the triangle containing Z has its third vertex at a point of minimal distance of Z. Hence two neighbouring triangulations are Δ-equivalent if and only if we can go from one to the other by a modification along such a circuit.

4. Classification of tropical curves of maximal dimensional type with a singularity in a fixed point

We can say more if we restrict ourselves to tropical curves of maximal dimensional type.

We have seen in Lemma 2.5 that the dimension of a cone C_T of the secondary fan equals the dimension of its type if and only if the marked subdivision T has no white points. Thus we can get tropical curves of maximal dimensional type only if we restrict ourselves to marked subdivisions corresponding to cones of smallest codimension and without any white points. Note that in the Case (b) of Classification 3.2, top dimensional cones of Trop(ker(A)) can also partly live in cones of codimension two of the secondary fan (see also Remark 3.10). This is true because the two gray points m_c and m_e can be on a line which is parallel to the circuit $Z = \{m_a, m_b, m_d\}$. If these two points can be seen in the subdivision, then it belongs to a cone of the secondary fan of codimension two (see Figure 13).

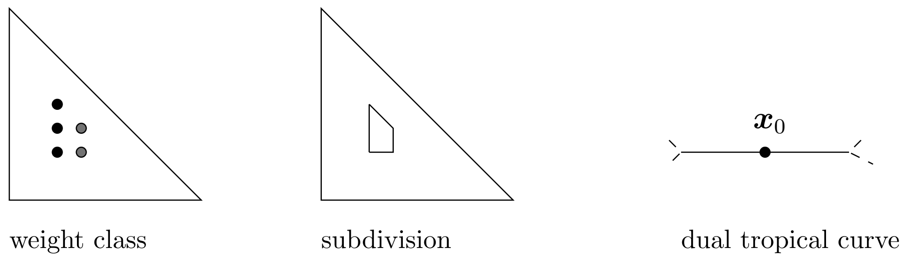


FIGURE 13. Subdivisions of codimension two

In fact, we can relate these weight classes (where we restrict to the parts where the two points can be seen) to the secondary fan in a way similar to Lemma 3.9 and 3.11. We have to add only part of the lineality space of the secondary fan however. We have to add the vector consisting of all y-coordinates of the m_i (if we assume without restriction that the circuit Z is on the line $\{x = 0\}$). On the right, we get the union over all codimension 2 cones C_T of the secondary fan whose corresponding marked subdivision T contains the polygon $\operatorname{conv}\{m_a, m_b, m_c, m_d, m_e\}$ and has all those points marked. This is true because for any vector $u \in C_T$, we can add a multiple of the vector of y-coordinates of the m_i to make the heights satisfy $u_a = u_b = u_d$ and $u_e = u_c$.

In order to get tropical curves of maximal dimensional type, we thus have to study codimension 1 cones of the secondary fan that are part of the tropical discriminant, and codimension 2 cones that correspond to a marked subdivision containing a polygon $\operatorname{conv}\{m_a, m_b, m_c, m_d, m_e\}$ with all those points marked and such that m_a, m_b and m_d are on a line and m_c and m_e are on a parallel line. We do not allow white points in the corresponding marked subdivisions.

We go through the classification in 3.2 and check what information on the dual tropical curve we can deduce in addition by assuming that there are no white points in the marked subdivision.

(a) Just as in 3.2 (a) we get tropical curves with a vertex of multiplicity 3 at $\boldsymbol{x}_0 = (0, 0)$ which is dual to a triangle with one interior lattice point, resp. with a 4-valent vertex at $(0, 0)$ whose dual polygon is a quadrangle not covering any other lattice points (see Figure 14).

(b.1) Let us consider a flag of flats as in 3.2 (b). Since we do not want any white points, the two gray points have to be of minimal distance to the circuit Z, and they have to be vertices of polygons of the subdivision. The first case is that they are on different sides of Z (see Figure 15). Again, we can solve for the coordinates (x_1, y_1) and (x_2, y_2) of the two vertices adjacent to the edge through $(0, 0)$. Now we know that dual to these vertices, we

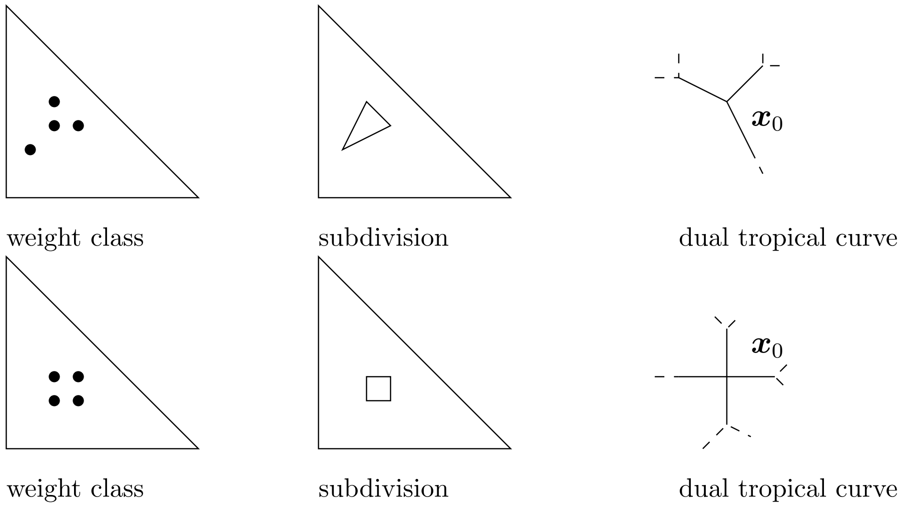


FIGURE 14. Types (a)

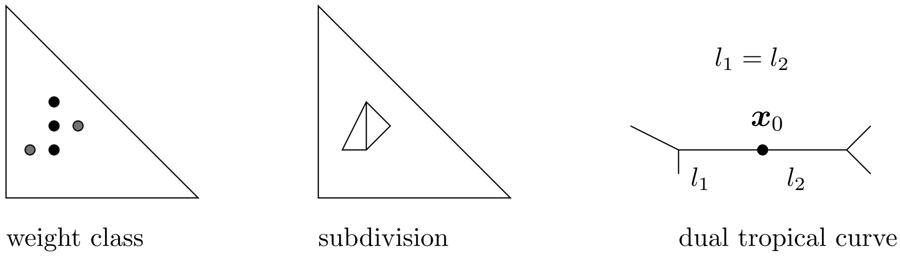


FIGURE 15. Type (b) with m_c and m_e on different sides

have two triangles whose third vertices are at the same height. If we assume without restriction that the circuit is on the line $\{x = 1\}$ and the vertex of the left triangle is at $m_c = (0, 0)$, then the equations to solve for (x_1, y_1) and (x_2, y_2) read:

$$\lambda = \mu + x_1 + (m_a)_2 \cdot y_1 = \mu + x_1 + (m_b)_2 \cdot y_1$$
$$\lambda + 2 \cdot x_2 + (m_e)_2 \cdot y_2 = \mu + x_2 + (m_a)_2 \cdot y_2 = \mu + x_2 + (m_b)_2 \cdot y_2$$

where λ is the height of the two gray points and μ is the height of the circuit points. Without restriction we can assume that $\lambda = 0$ and $\mu > 0$. Thus we conclude that the first vertex is at $(-\mu, 0)$ and the second at $(\mu, 0)$. In particular, the distances of both vertices to the singular point $(0, 0)$ on the edge are equal.

(b.2) Let us still consider flags as in 3.2(b), but now with the two gray points on the same side of the circuit Z. Again, the gray points have to be of minimal distance, and they have to be seen in the subdivision. Thus we can see a quadrangle with two parallel lines in the subdivision. If Z is not contained in the boundary of Δ, there must be a triangle whose vertex is of minimal distance in the subdivision on the other side of Z (see Figure 16). As above, we solve for the coordinates of the two vertices corresponding to

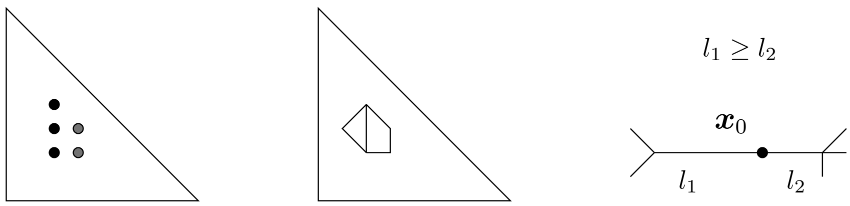


FIGURE 16. Type (b) with m_c and m_e on the same side

the quadrangle and the triangle. Again, without restriction let us assume that the circuit is on the line $\{x = 1\}$ and that the vertex of the left triangle is at $m_c = (0, 0)$. Then the equations read:

$$\nu = \mu + x_1 + (m_a)_2 \cdot y_1 = \mu + x_1 + (m_b)_2 \cdot y_1$$
$$\lambda + 2 \cdot x_2 + (m_e)_2 \cdot y_2 = \mu + x_2 + (m_a)_2 \cdot y_2 = \mu + x_2 + (m_b)_2 \cdot y_2$$

where λ is the height of the gray points, μ is the height of the circuit points and ν is the height of the vertex of the left triangle. Without restriction, we can assume $\nu = 0$, and $0 < \lambda < \mu$. Thus the 3-valent vertex is at $(-\mu, 0)$ and the 4-valent vertex is at $(\mu - \lambda, 0)$. In particular, the distance from the 4-valent vertex to the singular point $\boldsymbol{x}_0 = (0, 0)$ is smaller than the distance of the 3-valent vertex to $(0, 0)$. If Z is contained in the boundary, we see just the 4-valent vertex, and $(0, 0)$ on an infinite edge adjacent to this vertex.

Remark 4.1
The tropical variety $\text{Trop}(\ker(A))$ is of dimension $s - 4$. In the above, we describe (part of) the variety as subsets of cones of the secondary fan. The subsets are cut out in the case (a) by the two conditions that the 4-valent vertex (resp. the vertex of multiplicity 3) of the tropical curve has to be at the point $\boldsymbol{x}_0 = (0, 0)$. In the cases (b.1), we ask an edge to meet $(0, 0)$, and then in addition, we require the two adjacent lengths to be equal. In both cases we start with a codimension one cone of the secondary cone and then we cut out a codimension two subset. In the last case (b.2), we start with a codimension two cone of the secondary fan. But also, we cut out only a codimension one subset in here, since we only require an edge to meet $(0, 0)$. The lengths of adjacent edges have to satisfy an inequality, but this does not cut down the dimension. Thus also in the last case we describe tropical curves of maximal dimensional type in our family.

5. THE TROPICALISATION OF THE FAMILY OF CURVES WITH A SINGULARITY IN A FIXED POINT WHICH IS NOT A TORUS POINT

In Remark 3.1 we have seen that for any choice of singular point $(p, q) \in (\mathbb{C}^*)^2$, we get the same tropicalisation for the family of curves with a singularity in (p, q). What happens if we allow a point which is not in the torus, say $(p, q) = (1, 0)$?

The matrix A given by the three equations $f_{\underline{a}}(1,0) = 0$, $\frac{\partial f_{\underline{a}}(1,0)}{\partial x}(1,0) = 0$ and $\frac{\partial f_{\underline{a}}(1,0)}{\partial y}(1,0) = 0$ reads

$$A = \begin{pmatrix} 1 & \dots & 1 & 0 & \dots & 0 & 0 & \dots & 0 \\ m_{1,1} & \dots & m_{k,1} & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & \dots & 0 & 1 & \dots & 1 & 0 & \dots & 0 \end{pmatrix},$$

where we assume that the first block corresponds to the points $m_1, \dots, m_k \in \mathcal{A}$ that satisfy $m_{i,2} = 0$, the second block corresponds to the points with $m_{i,2} = 1$, and the last block corresponds to the points with $m_{i,2} > 1$. We assume that we have at least 3 points with $m_{i,2} = 0$ and at least 2 points with $m_{i,2} = 1$.

Let us compute a Gale dual for this matrix as in Section 3.1.

Example 5.1
Assume the point configuration is as depicted in Figure 17. Then the matrix A

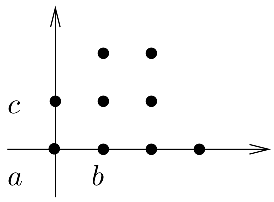


FIGURE 17. A point configuration

reads

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \end{pmatrix}.$$

Choose a, b and c as pivots and switch two columns such that the column of c becomes the third column. Then the reduced row echelon form reads

$$\begin{pmatrix} 1 & 0 & 0 & -1 & -2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \end{pmatrix}.$$

The Gale dual we can easily read off from this form is

$$B = \begin{pmatrix} 1 & -2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & -3 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

For the flags of flats, we can deduce the following:

- The vectors corresponding to points which are not on the line $\{y = 0\}$ or $\{y = 1\}$ (i.e. corresponding to the last block) are independent. They can be anywhere in a flag.
- For the vectors corresponding to the second block, i.e. to points on $\{y = 1\}$: if (and only if) in the flag we collected all but one of those vectors, then the last one belongs to the subspace, too.

- For the vectors corresponding to the first block, i.e. to points on $\{y = 0\}$: if (and only if) in the flag we collected all but two of those vectors, the last two belong to the subspace, too.

For the corresponding weight classes, we conclude:

- The maximum of heights appearing on the line $\{y = 1\}$ is attained twice, and
- the maximum of heights appearing on the line $\{y = 0\}$ is attained three times.

However, those two maxima can be in any relation to each other, and also in any relation to the heights of the points with y-coordinate larger than one.

What can we conclude for the possible subdivisions? The only thing we know for sure is that the three points on $\{y = 0\}$ must be seen in the subdivision. If they form a polygon with vertices in $\{y = 1\}$, then these vertices have to be the two maximal points on this line. But they do not have to form a polygon with vertices in $\{y = 1\}$, they could also form a polygon with a vertex in the line $\{y = 2\}$. In Figure 18, we show several possible subdivisions. The three maximal points on $\{y = 0\}$ are drawn in dark grey, the two maximal points on $\{y = 1\}$ in light gray - not depending on whether they can be seen in the subdivision or not.

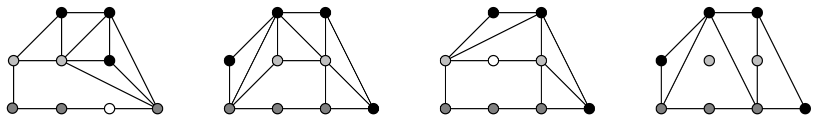


FIGURE 18. Some possible subdivisions

For the dual tropical curves, we can conclude that there is an end of weight at least 2 contained in the line $\{x = 0\}$. (The x-coordinate can be found by solving the system of linear equations given by the three points in the subdivision. The fact that they are of the same height implies that the x-coordinate is 0.) This fat end is either adjacent to an at least 4-valent vertex, or to a 3-valent vertex of multiplicity at least 4. If the tropical curve is of maximal dimension, it has to end at a 4-valent vertex. Since the negative of the valuation of the singular point $(1, 0)$ is $(0, -\infty)$, we expect the "singularity information" of the tropical curve to be contained in the ends. Local pictures of dual tropical curves are shown in Figure 19.

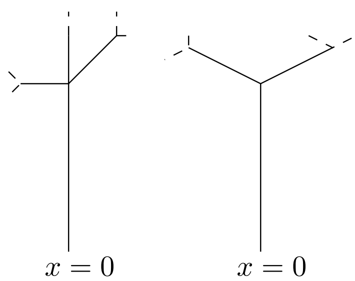


FIGURE 19. Dual tropical curves

Remark 5.2
It is easy to show that the essential features of example 5.1 hold in general. The matrix always consists of 3 blocks corresponding to points in $\{y = 0\}$, points in $\{y = 1\}$ and points with $y > 1$. The vectors in the Gale dual corresponding to the three blocks always behave like in the example, and we always get weight classes for the flags of flats where the maximal height in $\{y = 0\}$ is attained three times and the maximal height in $\{y = 1\}$ is attained twice. Thus, a tropical curve with a singularity in $(0, -\infty)$ always has a fat end at $\{x = 0\}$ with either an at least 4-valent vertex or a 3-valent vertex of multiplicity at least 4. If the curve is of maximal dimension, it has a 4-valent vertex adjacent to the fat end.

If we compare this to tropical curves with a singularity in $\boldsymbol{x}_0 = (0, 0)$ that we studied in Section 3.2, we can see that we only get subdivisions where the circuit is of type (C) as in Remark 2.4 and is contained in the boundary on the line $\{y = 0\}$.

6. Algebraic lifts of tropical curves of maximal dimensional type with a singularity in a fixed point

For the following considerations we assume that $\mathbb{K} = \overline{\mathbb{C}(t)}$ is the algebraic closure of the field of rational functions over the complex numbers.

We would like to describe algebraic curves $C \in \mathrm{Sing}_{\mathbb{K}}(\Delta)$ which correspond to tropical curves $T \in \mathrm{Trop}(\mathrm{Sing}_{\mathbb{K}}(\Delta))$ of maximal dimensional type. We furthermore assume the following generality condition:

(G) T is a generic member in the interior of a top-dimensional cone of the tropical variety $\mathrm{Trop}(\mathrm{Sing}_{\mathbb{K}}(\Delta))$ corresponding to a maximal dimensional type, and C is a generic element of $\mathrm{Sing}_{\mathbb{K}}(\Delta)$ with $\mathrm{Trop}(C) = T$.

Below we specify this generality assumption which breaks certain explicit relations.

As a particular consequence of our consideration, we give a conceptual explanation of the metric conditions for the type (b) curves (cf. Section 4).

Tropical limits of plane algebraic curves over $\mathbb{K}$. We shortly recall the definition of tropical limits used in the sequel following [10]. A tropical curve T uniquely determines a convex piece-wise linear function $\nu : \Delta \to \mathbb{R}$ with $\max \nu = 0$. Note that with the notation introduced in Section 2 on Page 5 ν determines a defining tropical polynomial $F = \max\{u_{ij} + ix + jy \mid (i, j) \in \Delta \cap \mathbb{Z}^2\}$ for T via $\nu(i, j) = u_{ij}$. Without loss of generality, assume that T is defined over $\mathbb{Q}$ and that $\nu(\Delta \cap \mathbb{Z}^2) \subset \mathbb{Z}$ (the latter can be achieved by a suitable stretching of T). An algebraic curve $C \in |\mathcal{L}_\Delta|$ with $\mathrm{Trop}(C) = T$ is then given by an equation

$$f(x, y) = \sum_{(i,j)\in\Delta\cap\mathbb{Z}^2} (a^0_{ij} + O(t))t^{-\nu(i,j)}x^i y^j = 0, \quad a_{ij}(t) \in \mathbb{K}\ , \tag{2}$$

where the $a^0_{ij} \in \mathbb{C}$ do not vanish since (i, j) is visible in the subdivision S_T of Δ induced by ν, and the $O(t)$ are analytic functions in the disc $D_\varepsilon = \{|t| < \varepsilon\}$. Evaluating (2) for $t \in D_\varepsilon\backslash\{0\}$, we obtain a family of curves $C^{(t)} \subset \mathrm{Tor}_{\mathbb{C}}(\Delta)$ which admits a flat extension to $t = 0$ in the form

$$\begin{array}{ccccccccc} & & & & \mathrm{Tor}(\widetilde{\Delta}) & & \\ & & & & \| & & \\ C^{(0)} & \hookrightarrow & C & \hookrightarrow & \widetilde{\Sigma} & \hookleftarrow & \Sigma^{(0)} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 0 & \in & D_\varepsilon & = & D_\varepsilon & \ni & 0 \end{array}$$

where $\widetilde{\Delta} = \{(\omega, z) \in \mathbb{R}^3 \mid \omega \in \Delta,\ z \le \nu(\omega)\}$ is the undergraph of ν, $\Sigma^{(0)} = \bigcup_\delta \mathrm{Tor}(\delta)$ with δ ranging over all polygons of the subdivision S_T, and $C^{(0)} \subset \Sigma^{(0)}$ splits into the *limit curves* $C_\delta \subset \mathrm{Tor}(\delta)$ given by the equations

$$f_\delta(x, y) \equiv \sum_{(i,j)\in\delta\cap\mathbb{Z}^2} a^0_{ij} x^i y^j = 0 \ .$$

The data $(T, \{C_\delta\}_{\delta\in S_T})$ is called the *tropical limit* of C.

The fact that C has a singularity at the point $\boldsymbol{p} = (1, 1)$ is equivalent to the fact that $C^{(t)}$ has a singularity at $(1, 1)$ for each $t \in D_\varepsilon\backslash\{0\}$ (cf. [10, Lemma 2.3]). The (constant) family of points $(1, 1) \in (\mathbb{C}^*)^2 \subset \mathrm{Tor}(\Delta) = \Sigma^{(t)}$, $t \in D_\varepsilon\backslash\{0\}$, has a limit point $p \in \Sigma^{(0)}$. Two cases are possible:

(i) $p \in (\mathbb{C}^*)^2 \subset \mathrm{Tor}(\delta)$ for some polygon δ in S_T; then, in particular, the equality $f(p) = f(1, 1) = 0$ implies that the initial form $\mathrm{in}_{(0,0)} f(1, 1)$ vanishes which forces the constancy of ν along δ, and hence ν vanishes there by our assumptions; furthermore, the limit curve C_δ has a singularity at $(1, 1)$;

(ii) $p \in \mathrm{Tor}(\sigma)$, where $\sigma = \delta' \cap \delta''$ is a common side of polygons δ', δ'' in S_T, $\mathrm{Tor}(\sigma) = \mathrm{Tor}(\delta') \cap \mathrm{Tor}(\delta'')$ is a common toric divisor; then ν vanishes along σ, and $p \in C_{\delta'} \cap C_{\delta''} \cap \mathrm{Tor}(\sigma)$, where the pairwise intersection multiplicities are ≥ 2 (a transverse intersection point with $\mathrm{Tor}(\sigma)$ smoothes out in the deformation $C^{(0)} \to C^{(t)}$, $t \ne 0$, cf. [10, Lemma 3.2]).

In the second case we shall *refine* the tropical limit of C as described in [10, Section 3.5 and 3.6].

Curves of type (a). Let T be of type (a) introduced in Section 4. Then

- either the dual subdivision S_T of Δ consists of a triangle δ_0 of lattice area 3, which up to $SL(2, \mathbb{Z})$-action and translations coincides with the triangle $\mathrm{conv}\{(0, 0), (2, 1), (1, 2)\}$ (cf. Figure 20(a)), and the remaining pieces are primitive lattice triangles (i.e. of unit lattice area);
- or S_T contains a quadrangle δ_0, which up to $SL(2, \mathbb{Z})$-action and translations coincides with the square $\mathrm{conv}\{(0, 0), (1, 0), (0, 1), (1, 1)\}$ (cf. Figure 20(b)), and the remaining pieces again are primitive lattice triangles.

Observe that in this case all the edges σ of the subdivision S_T have unit lattice length, and hence the limit curves C_δ can intersect the toric divisors $\mathrm{Tor}(\sigma) \subset \mathrm{Tor}(\delta)$ only transversally, which allows only the option (i) for the limit singular point $p \in \Sigma^{(0)}$ described above. More precisely, the curve C_{δ_0} has a node at $\boldsymbol{p} = (1, 1)$ being irreducible if δ_0 is a triangle (since the sides of Δ_0 have unit length), or reducible if δ_0 is a parallelogram, whereas the remaining limit curves (corresponding to primitive triangles) are nonsingular. In both the cases, C is a nodal curve of genus $g = \#(\mathrm{Int}(\Delta) \cap \mathbb{Z}^2) - 1$, and we can define a natural parametrisation of T of genus g (cf. with a canonical tropicalisation in [14]):

- if δ_0 is a triangle, then T is self-parameterising of genus g,
- if δ_0 is a parallelogram, then we resolve the four-valent vertex $\boldsymbol{x}_0 = (0,0)$ of T, obtaining the refined tropical curve $\widehat{T}$ of genus g as a parameterising graph of T (cf. Figure 20(c)).

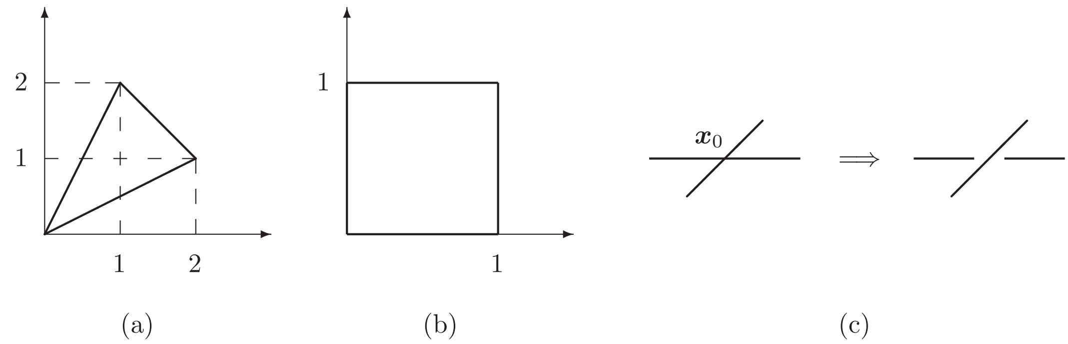


FIGURE 20. Curves of type (a)

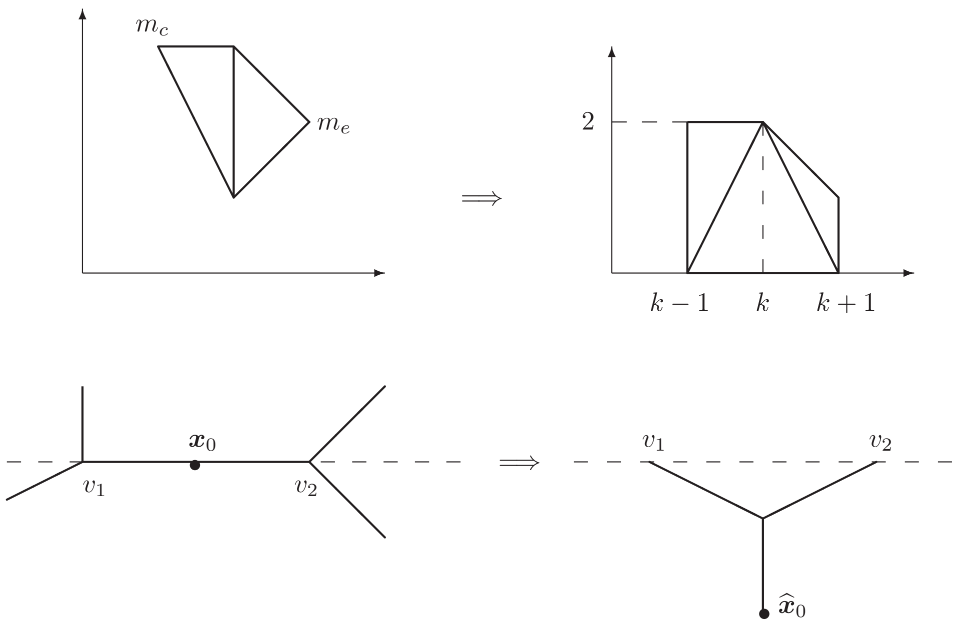
Curves of type (b.1). If T is of type (b.1) (see Section 4 and Figure 15), then S_T consists of triangles, all of them but two δ', δ'' shown in Figure 15 being primitive, and all the limit curves C_δ are nonsingular. Hence, we have the option (ii) for the limit singular point p, which then must belong to $\mathrm{Tor}(\sigma)$, where $\sigma = \delta' \cap \delta''$ is the common edge of length 2 spanned by the circuit Z (cf. Section 4). Furthermore, the limit curves $C_{\delta'}$, $C_{\delta''}$ must be quadratically tangent to $\mathrm{Tor}(\sigma)$ at p (formally, we have an option of two transverse intersection points, however, it is not possible in our situation, since such points are smoothed out in the deformation $C^{(t)}$, $t \in (\mathbb{C}, 0)$).

Now we are going to refine the tropical limit of C as described in [10, Section 3.5]. Geometrically it corresponds to a blow up resolving the considered singularity. Without loss of generality, assume that σ is vertical (cf. Figure 15). Consider the polynomial

$$\widehat{f}(x,y) = f(x, y+1) \; . \tag{3}$$

It defines a curve $\widehat{C} \subset \mathbb{K} \times \mathbb{K}^*$ with a singularity at $\widehat{p} = (1,0)$. Note that this refined tropical curve is a member of the family we described in Section 5, and indeed we will see that the refined tropical curve has a fat down end. Denote by $\widehat{T}$ its tropicalisation and by $\widehat{\nu} : \widehat{\Delta} \to \mathbb{R}$ the corresponding concave piece-wise linear function on its Newton polygon $\widehat{\Delta}$. The fragment $(\delta', \delta'', C_{\delta'}, C_{\delta''})$ of the tropical limit of C turns into the fragment $(\widehat{\delta'}, \widehat{\delta''}, \delta_\sigma, \widehat{C}_{\delta'}, \widehat{C}_{\delta''}, C_\sigma)$ of the tropical limit of $\widehat{C}$ (see Figure 21(a)), where C_σ is a curve in the toric surface $\mathrm{Tor}(\delta_\sigma)$, $\delta_\sigma = \mathrm{conv}\{(k-1,0), (k,2), (k+1,0)\}$, having a singularity at $\widehat{p} = (1,0)$. Observe that, since $\widehat{p}$ appears on the toric divisor $\mathrm{Tor}(\widehat{\sigma})$, where $\widehat{\sigma} = [(k-1,0),(k+1,0)]$, and since $\widehat{f}(1,0) = 0$, which implies $\mathrm{in}_{(0,-\infty)} \widehat{f}(1,0) = 0$ we can conclude that the values $\widehat{\nu}(k-1,0)$ and $\widehat{\nu}(k+1,0)$ must be equal. In view of the clear relations $\widehat{\nu}(k-1,0) = \nu(m_c)$ and $\widehat{\nu}(k+1,0) = \nu(m_e)$, this confirms the equality $\nu(m_c) =$

$\nu(m_e)$, equivalent to the metric relation $l_1 = l_2$ for the tropical curve T shown in Section 4.



(a)

(b)

FIGURE 21. Curves of type (b.1)

Furthermore, from the above refinement we derive a correct canonical parametrisation of T. It can be conveniently represented via the tropical blow-up[1], or *modification* in the terminology of [9, Section 1]. Notice that (under our assumptions) the edge E of T dual to σ lies on the x-axis of $\mathbb{R}^2$. Replace the (tropical) plane $\mathbb{R}^2$ by the tropical plane P in $\mathbb{R}^3$ which consists of three half-planes:

$$P_+ = \{y \ge 0,\ y = z\},\ P_- = \{y \le 0,\ z = 0\},\ P_0 = \{y = 0,\ z \le 0\}\ ,$$

and introduce the projection

$$\pi : P \to \mathbb{R}^2, \quad \pi(x, y, z) = (x, y)\ .$$

Then the tropical curve $T \subset \mathbb{R}^2$ lifts to a tropical curve $T^* \subset P$

- the part $T \backslash E$ lifts to $\pi^{-1}(T \backslash E) \cap (P_+ \cup P_-)$,
- the edge E is replaced by the fragment dual to the triangle δ_σ placed in P_0 (see Figure 21(b)), whereas the (tropical) singular point $\boldsymbol{x}_0 = (0, 0)$ lifts to the (infinite) univalent vertex $\widehat{\boldsymbol{x}}_0$ of the vertical ray of the above fragment.

The map $\pi : T^* \to T$ provides a canonical parametrisation of T of genus g.

Curves of type (b.2). In this case, the edge σ of the subdivision S_T spanned by the circuit Z has lattice length 2 and is common for a triangle δ' and a trapeze δ'', altogether containing 6 integral points (see Figure 22(a)). Without loss of generality

[1]The above refinement can be interpreted as the (weighted) blow-up of the toric variety $\mathrm{Tor}(\widetilde{\Delta})$ at the point p which replaces it by the exceptional divisor $\mathrm{Tor}(\delta_\sigma)$ (cf. [11, Section 2]).

assume that σ is vertical (see Figure 22(a)). Then the function $\nu : \Delta \to \mathbb{R}$ satisfies the following:

$$\begin{cases} \nu\big|_\sigma = 0, \quad \nu(k,j) < 0,\ (k,j) \notin \sigma, \\ \nu(m) = \alpha < 0, \quad \nu(k-1,j) < \alpha,\ (k,j) \ne m, \\ \nu(m_c) = \nu(m_e) = \beta < 0, \quad \nu(k+1,j) < \beta,\ (k,j) \ne m_c, m_e, \\ \nu(k+s,j) < s\beta, \quad s \ge 2\ . \end{cases} \tag{4}$$

Due to the generality condition (G), we can assume that $\alpha < \beta$ (the inequality $\alpha \le \beta$ is included in the definition of the corresponding cone of $\mathrm{Trop}(\mathrm{Sing}_{\mathbb{K}}(\Delta))$).

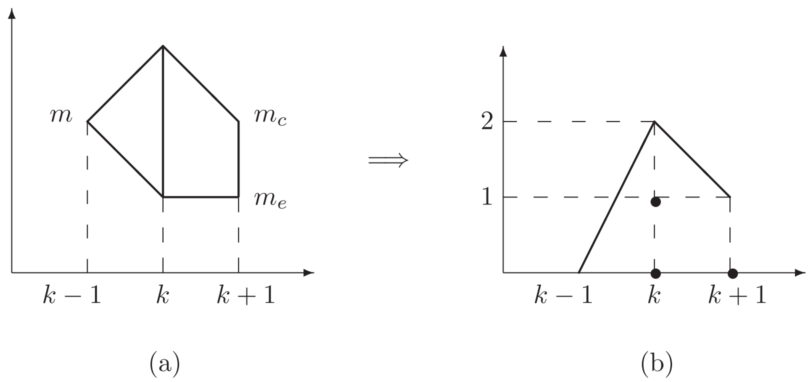


FIGURE 22. Curves of type (b.2), I

The limit point p belongs to $\mathrm{Tor}(\sigma)$, since the function $\nu : \Delta \to \mathbb{R}$ vanishes only along σ. Then the limit curve $C_{\delta'}$ is nonsingular and quadratically tangent to $\mathrm{Tor}(\sigma)$ at p. The limit curve $C_{\delta''}$ either is nonsingular, quadratically tangent to $\mathrm{Tor}(\sigma)$ at p, or splits into two components transversally intersecting at p. The former option is not possible, since, otherwise, the refinement used in the preceding stage would lead to a subdivision containing the triangle δ_σ as in Figure 21(a), and hence to the equality $\alpha = \beta$ against the assumption made. Thus, $C_{\delta''}$ is reducible as indicated above.

Let $\widehat{f}(x,y) = f(x,y+1)$. In view of (4), the fragment (δ', δ'') of the subdivision S_T of Δ turns into a fragment of the subdivision $S_{\widehat{T}}$ (in the notation of the preceding step) containing the two edges shown in Figure 22(b) with the following values of the function $\widehat{\nu} : \widehat{\Delta} \to \mathbb{R}$:

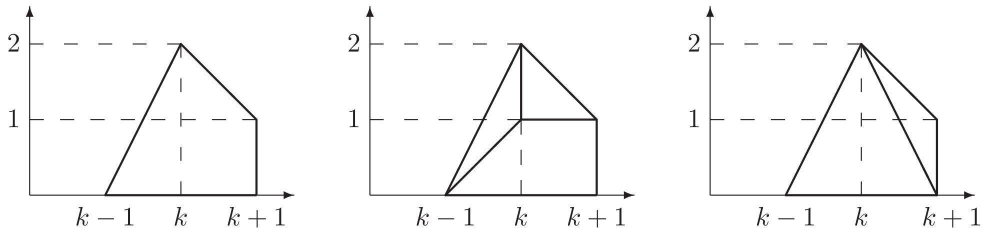
$$\widehat{\nu}(k-1,0) = \alpha,\ \widehat{\nu}(k,0), \widehat{\nu}(k,1) < \widehat{\nu}(k,2) = 0,\ \widehat{\nu}(k+1,0) < \widehat{\nu}(k+1,1) = \beta\ . \tag{5}$$

To derive these relations, notice, first, that the expansion of $f(x,y)$ into power series in t looks as $f(x,y) = x^k y^{k'}(y-1)^2 + O(t)$, where (k,k') is the bottom vertex of σ, and hence $\widehat{f}(x,y) = x^k y^2 + O(y^3) + O(t)$. Second, recall that the limit curve $C_{\delta''}$ is reducible and both its components hit the point $(0,1)$, which is the limit point of $\boldsymbol{p} = (1,1)$ in $\mathrm{Tor}(\sigma)$. The shape of δ'' dictates that one of the components is $\{y - 1 = 0\}$, in particular, the truncation of $f(x,y)$ to the edge $[m_c, m_e]$ is $x^{k+1}y^{k'}(y-1)t^\beta(1+O(t))$, and hence the substitution $y \to y+1$ produces the truncation $t^\beta x^{k+1}y(1+O(y)+O(t))$ of $\widehat{f}(x,y)$ to the segment $[(k+1,0),(k+1,1)]$.

(1) Assume that the segment $[m_c, m_e]$ *lies on* $\partial\Delta$. Then the subdivision $S_{\widehat{T}}$ contains a fragment bounded by the quadrangle

$$Q = \text{conv}\{(k-1,0),(k,2),(k+1,1),(k+1,0)\}$$

(see Figure 23(a)). Since the point $\widehat{\boldsymbol{p}} = (1,0)$ is singular for $\widehat{C}$, the limit point $\widehat{p}$ is singular for the corresponding limit curve of $\widehat{C}$. Thus,



(a) (b) (c)

FIGURE 23. Curves of type (b.2), II

- the entire segment $[(k-1,0),(k+1,0)]$ must be an edge of the induced subdivision of Q, and
- the limit curve C' corresponding to the polygon which has the segment $[(k-1,0),(k+1,0)]$ as a face must be singular at $\widehat{p}$.

This allows one only the following subdivisions of Q and relations on $\widehat{\nu}$:

(i) the subdivision shown in Figure 23(b), where the limit curve C' splits into two components transversally intersecting at $\widehat{p}$, and

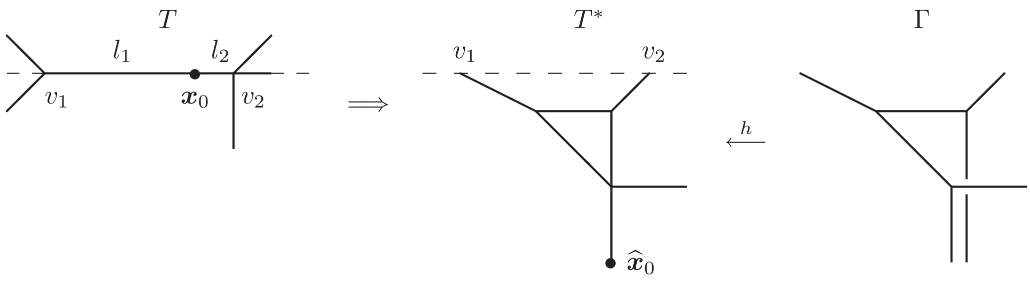
$$\widehat{\nu}(k-1,0) = \widehat{\nu}(k,0) = \widehat{\nu}(k+1,0) = \alpha,\ \widehat{\nu}(k,1) = \widehat{\nu}(k+1,1) = \beta \in \left(\frac{\alpha}{2}, 0\right)\ ;$$

(ii) the subdivision shown in Figure 23(c), where the limit curve C' is irreducible with a node at $\widehat{p}$, and

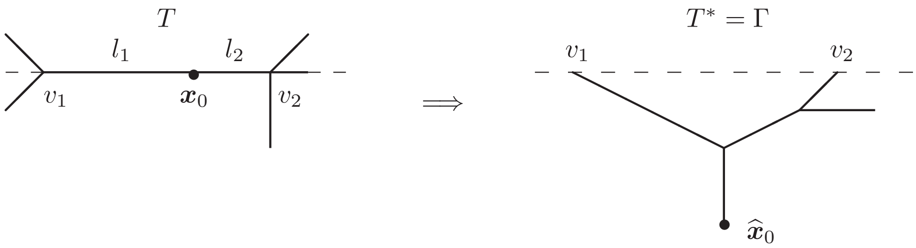
$$\widehat{\nu}(k-1,0) = \widehat{\nu}(k,0) = \widehat{\nu}(k+1,0) = \alpha,\ \widehat{\nu}(k,1) = \frac{\alpha}{2},\ \widehat{\nu}(k+1,1) = \beta \in \left(\alpha, \frac{\alpha}{2}\right)\ .$$

Notice that the relation $\alpha > \beta$ is not possible due to the last inequality in (5) which confirms the same conclusion of Section 4. We complete the study of this case with a canonical parametrisation of the tropical curve T: we perform the modification of the plane as for the type (b.1) curves and replace the edge E of T passing through the origin with the fragment dual to the subdivisions shown in Figure 23(b,c) - see Figure 24(a,b): in the first case, we have the parametrisation $\Gamma \xrightarrow{h} T^* \xrightarrow{\pi} T$, and, in the second case, the parametrisation $\Gamma = \widehat{T} \xrightarrow{\pi} T$. The geometry of those fragments of $\widehat{T}$ imply the metric conditions on the position of the tropical singularity $\boldsymbol{x}_0$ as indicated in Figure 24.

(2) Assume that the segment $[m_c, m_e]$ *does not lie on* $\partial\Delta$. Then the fragment of the subdivision $S_{\widehat{T}}$ we are interested in may include the point $(k+2,0)$ too (see Figure 25(a)), where due to (4), $\widehat{\nu}(k+2,0) < 2\beta = 2\widehat{\nu}(k+1,1)$. If $\widehat{\nu}(k+2,0) < \widehat{\nu}(k-1,0) = \alpha$, then the preceding argument leaves us with the only possible subdivisions



(a): $l_2 < l_1/2, \quad \alpha/2 < \beta < 0$



(b): $l_1/2 < l_2 \le l_1, \quad \alpha < \beta < \alpha/2$

Figure 24. Curves of type (b.2), III

shown in Figure 25(b,c) with the same conclusions as for the subdivisions in Figure 23(b,c) analysed before. If $\widehat{\nu}(k+2,0) > \alpha$ (what, in particular, yields $\beta > \alpha/2$), then the only suitable subdivision is shown in Figure 25(d), where

$$\widehat{\nu}(k,0) = \widehat{\nu}(k+1,0) = \widehat{\nu}(k+2,0),\ \widehat{\nu}(k,1) = \widehat{\nu}(k+1,1) = \beta\ ,$$

and the limit curve with the Newton trapeze splits into two components transversally intersecting the toric divisor $\mathrm{Tor}([(k,0),(k+2,0)])$ at the same point $\widehat{\boldsymbol{p}}$.

The canonical parametrisation of T again is built in the form $\Gamma \xrightarrow{h} T^* \xrightarrow{\pi} T$, where $\widehat{T}$ appears in the modification of the plane along the x-axis: the edge E of T (with the endpoints v_1, v_3 in Figure 26(a)) is replaced by the fragment dual to the subdivision in Figure 25(d) and lying in the half-plane P_0 (shown in Figure 26(b)). At last, the parameterising graph Γ is obtained by the resolving the double ray with the endpoint $\widehat{\boldsymbol{x}}$ (see Figure 26(c)). Notice that this case corresponds to the metric relation $l_2 < l_1/2$.

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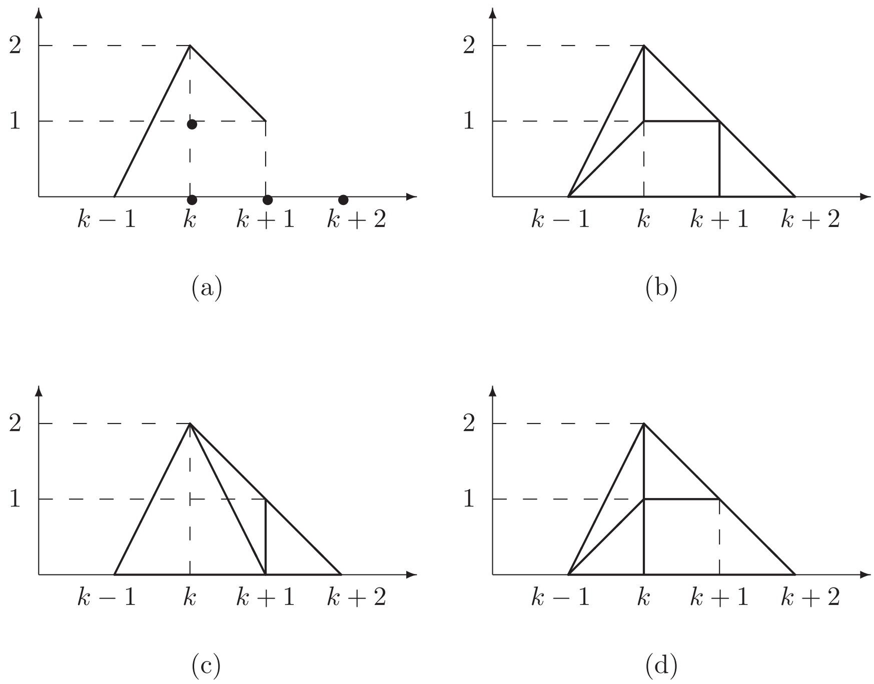

FIGURE 25. Curves of type (b.2), IV

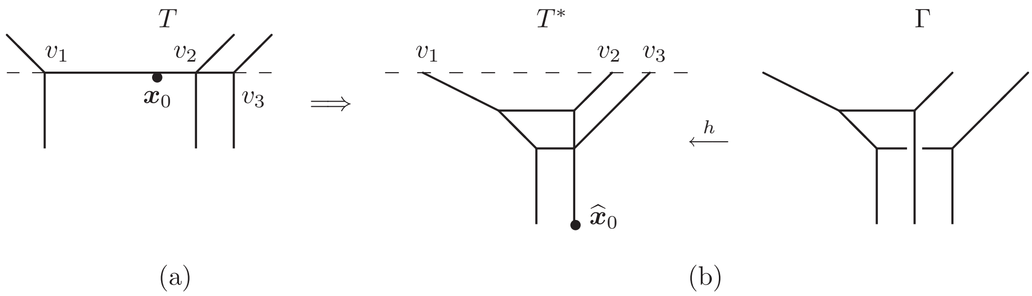

FIGURE 26. Curves of type (b.2), V

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