Gröbner Fans of x-homogeneous Ideals in $R[\![t]\!][\mathbf{x}]$

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Abstract

We generalise the notion of Gröbner fan to ideals in $R[\![t]\!][x_1,\ldots,x_n]$ for certain classes of coefficient rings R and give a constructive proof that the Gröbner fan is a rational polyhedral fan. For this we introduce the notion of initially reduced standard bases and show how these can be computed in finite time. We deduce algorithms for computing the Gröbner fan, implemented in the computer algebra system Singular. The problem is motivated by the wish to compute tropical varieties over the p-adic numbers, which are the intersection of a subfan of a Gröbner fan as studied in this paper by some affine hyperplane, as shown in a forthcoming paper.

Keywords: Gröbner fan, standard basis over base rings, local standard fan, tropical variety
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1. Introduction

Gröbner fans of ideals I in the polynomial ring over a field were first introduced and studied by Mora and Robbiano in [1] as an invariant associated to the ideal. The Gröbner fan of I is a convex rational polyhedral fan classifying all possible leading ideals of I w.r.t. arbitrary global monomial orderings and encoding the impact of all these orderings on the ideal. It provides an interesting link between commutative algebra and convex geometry, opening the rich tool box of the latter for the first. Moreover, tropical varieties, which have gained lots of interest recently, can be described often as subcomplexes of certain Gröbner fans and can be computed that way. The latter is the main motivation for our paper, as we explain further down.

Mora and Robbiano describe in their paper an algorithm to compute the Gröbner fan. The underlying structure was then used efficiently by Collart, Kalkbrenner and Mall in [2] to transform a standard basis w.r.t. one global monomial ordering into a standard basis w.r.t. another one by passing through several cones of the Gröbner fan. At a common facet of two cones a local change of the standard basis was necessary making use of the fact that the monomial orderings of the neighbouring cones can be seen as a refinement of a common partial ordering on the monomials. Their methods were later refined by many others (see e.g. [3, 4, 5, 6, 7]).

For homogeneous ideals the Gröbner fan is complete and Sturmfels showed in [8] that it is the normal fan of a polytope, the state polytope of I. If the ideal is not homogeneous the Gröbner fan is in general neither complete, nor is the part in the positive orthant the normal fan of a polyhedron, as was shown by Jensen in [9].

Since the notion of the Gröbner fan turned out to be so powerful in the polynomial ring it was in the sequel generalised to further classes of rings. Assi, Castro-Jiménez and Granger (see [10]) and Saito, Sturmfels and Takayama (see [11]) studied an analogue of the Gröbner fan for ideals in the ring of algebraic differential operators. In a subsequent paper the first three authors generalised the notion to the ring of analytic differential operators (see [12]), proving that the equivalence classes of weight vectors yet again are convex rational polyhedral cones. Bahloul and Takayama (see [13, 14]) then show that these cones glue to give a fan and they give an algorithm to compute this fan. They show that their techniques apply to ideals in the subrings of convergent or formal power series over a field and treat this case explicitly. This leads to the notion of the local standard fan which covers the negative orthant and whose cones characterise the impact of the local monomial orderings on the ideal in the power series ring.

Even though the approach is algorithmical, it cannot be applied in practice right away, since the computation of the standard cones heavily relies on the computation of a reduced standard basis, which even for polynomial input data in general contains power series and is not feasible in practice. If the input data is polynomial Bahloul and Takayama, therefore, propose to homogenise the ideal, compute the Gröbner fan with the usual techniques and then to cut down the additional variable again. This will lead to a refinement of the actual local standard fan, but for each pair of neighbouring cones one can check with a standard basis computation, if the cones should be glued in the local standard fan. Since the number of fulldimensional cones in the refined fan may be larger by an order of magnitude, this approach is very expensive.

In our paper we address a situation which in some respects is more general and in some is much more specialised than the above. It is motivated by a very particular application that we have in mind, the computation of tropical varieties over the p-adic numbers. These appear as the intersection of a subfan of the Gröbner fans studied in this paper with an affine hyperplane (see [15]). Here we lay the theoretical and the algorithmical foundation for this approach to compute tropical varieties over the p-adic numbers, leading to the only currently available software for computing these varieties.

In this paper we allow as coefficient domain a ring R satisfying some additional technical properties which ensure that standard bases over R can be computed (see Page 2 and [16]). We then consider $\mathbf{x}$-homogeneous ideals I in the mixed power series polynomial ring $R[\![t]\!][\mathbf{x}] = R[\![t]\!][x_1, \ldots, x_n]$, that is, we consider one local variable and any finite number of global variables. We then define the Gröbner fan of I as usual with some necessary adjustments. The main theoretical result of this paper shows that the Gröbner fan is indeed a rational polyhedral fan covering all of the half space $\mathbb{R}_{\leq 0} \times \mathbb{R}^n$ (see Theorem 3.19). For the theory the generators of I may be arbitrary power series in the local variable, for the practice we restrict to input data which is polynomial in t as well as in $\mathbf{x}$, but homogeneity is only required w.r.t. $\mathbf{x}$. A major point when it comes to actually computing the Gröbner fans is that restricting to one local variable allows us to replace reduced standard bases by the weaker notion of initially reduced standard bases. We show that these are sufficiently strong to let us read off the Gröbner cones (see Section 3), yet weak enough to be computable for polynomial input data with a finite number of steps at the same time in important cases.

Note that for polynomial input data we could have followed the approach of Bahloul and Takayama (see [13, 14]) by homogenising first, cutting down and gluing cones. However, not only is the gluing very costly, the Gröbner fan of the homogenised ideal has way more cones and these have plenty more facets that have to be traversed. For a simple tropical linear space in an example we have 20 full-dimensional cones without homogenisation and 1393 for the homogenised ideal, and the number of facets that have to be traversed has increase by a factor way larger than 100. Thus, already computing the Gröbner fan of the homogenised ideal is much more expensive than computing the Gröbner fan of I directly via our approach.

In Section 2 we introduce the basic notions used throughout the paper and we show that also in our situation there are only finitely many possible leading ideals. Section 3 is devoted to proving that the Gröbner fan is a rational polyhedral fan. We provide a constructive approach for the Gröbner cones using initially reduced standard bases. In Section 4 we present algorithms to reduce standard bases initially in finite time under some additional hypotheses on R and the ideal (see Page 23), and in Section 5 we finally provide algorithms to compute Gröbner fans of $\mathbf{x}$-homogeneous ideals, where for the latter we follow the lines of [17]. The algorithms are implemented in and distributed with Singular and they complement the software package `gfan` (see [18]) by Jensen which is specialised in computing Gröbner fans for ideals in polynomial rings and their tropical varieties.

2. Basic notions

Throughout this paper we assume that R is a noetherian ring and that linear equations in R are solvable, that is, for any choice of $c_1, \ldots, c_k \in R$ we can decide the ideal membership problem $b \in \langle c_1, \ldots, c_k \rangle$, if applicable represent b as $b = a_1 \cdot c_1 + \cdots + a_k \cdot c_k$, and compute a finite generating set of the syzygy module $\operatorname{syz}_R(c_1, \ldots, c_k)$. The most important example that we have in mind is the ring of integers. For further classes of interesting examples see [16, Ex. 1.2]. Due to [16] this assumption ensures that in the mixed power series polynomial ring

$$R[\![t]\!][\mathbf{x}] := R[\![t]\!][x_1, \ldots, x_n],$$

with a single variable t, standard bases exist and are computable in finite time and with polynomial output, if the ideal is generated by polynomials.

We represent an element f of $R[\![t]\!][\mathbf{x}]$ in the usual multiindex notation as

$$f = \sum_{\beta,\alpha} c_{\alpha,\beta} \cdot t^\beta \mathbf{x}^\alpha$$

with $\beta \in \mathbb{N}$ and $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ where $\mathbf{x}^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, and we sometimes represent it as

$$f = \sum_{\alpha} g_\alpha \cdot \mathbf{x}^\alpha$$

with

$$g_\alpha = \sum_\beta c_{\alpha,\beta} \cdot t^\beta \in R[\![t]\!]$$

as an element in the polynomial ring in $\mathbf{x}$ over the ring $R[\![t]\!]$. We then call f *x-homogeneous* if all monomials $\mathbf{x}^\alpha$ have the same degree, and we call an ideal $I \trianglelefteq R[\![t]\!][\mathbf{x}]$ *x-homogeneous* if it is generated by $\mathbf{x}$-homogeneous elements. In what follows we will construct Gröbner fans of $\mathbf{x}$-homogeneous ideals $I \trianglelefteq R[\![t]\!][\mathbf{x}]$ as fans on the closed half space $\mathbb{R}_{\leq 0} \times \mathbb{R}^n$.

Let us now fix some standard notation used in the context of standard bases and Gröbner fans. We denote by

$$\mathrm{Mon}(t, \mathbf{x}) = \left\{ t^\beta \cdot \mathbf{x}^\alpha \;\middle|\; \beta \in \mathbb{N}, \alpha \in \mathbb{N}^n \right\}$$

the multiplicative semigroup of monomials in the variables t and $\mathbf{x}$. A *monomial ordering* on $\mathrm{Mon}(t, \mathbf{x})$ is a total ordering $>$ which is compatible with the semigroup structure on $\mathrm{Mon}(t, \mathbf{x})$, and we call it *$t$-local* if $1 > t$. The least monomial $t^\beta \mathbf{x}^\alpha$ w.r.t. a t-local monomial ordering $>$ occuring in $0 \neq f \in R[\![t]\!][\mathbf{x}]$ is called the *leading monomial* $\mathrm{LM}_>(f) = t^\beta \mathbf{x}^\alpha$ of f, the corresponding coefficient is its *leading coefficient* $\mathrm{LC}_>(f) = c_{\beta,\alpha}$, the term $\mathrm{LT}_>(f) = \mathrm{LC}_>(f) \cdot \mathrm{LM}_>(f)$ is its *leading term* and $\mathrm{tail}_>(f) = f - \mathrm{LT}_>(f)$ its *tail*, and we set $\mathrm{LT}_>(0) = 0$. We call the ideal

$$\mathrm{LT}_>(I) := \langle \mathrm{LT}_>(f) \mid f \in I \rangle \trianglelefteq R[t, \mathbf{x}]$$

the *leading ideal* of I w.r.t. $>$. Note, that it is an ideal generated by terms, but in general not by monomials, since R is only a ring. However, as in the case of base fields the number of possible leading ideals w.r.t. t-local monomial orderings is finite, which will essentially imply that the Gröbner fan of I has only finitely many cones. The proof of is an adaptation of the proof of [19, Thm. 4.1].

Proposition 2.1
Any $\mathbf{x}$-homogeneous ideal $I \trianglelefteq R[\![t]\!][\mathbf{x}]$ has only finitely many leading ideals.

Proof. Observe that an element $g \in R[\![t]\!][\mathbf{x}]$ has only finitely many possible leading terms, since there are only finitely many distinct monomials in $\mathbf{x}$ and a leading term w.r.t. a t-local monomial ordering has to have minimal power in t.

Now assume there are infinitely many leading ideals. For each leading ideal J, let $>_J$ be a t-local monomial ordering such that $\mathrm{LT}_{>_J}(I) = J$. Set

$\Delta_0 := \{>_J \mid\ J \text{ leading ideal of } I\}$, so that different orderings in Δ_0 yield different leading ideals. By our assumption, Δ_0 is infinite.

Let $G_1 \subseteq I$ be a finite $\mathbf{x}$-homogeneous generating set of I and set Σ_1 to be the union of all potential leading terms of elements of G_1. Then Σ_1 is finite and hence, by the pigeonhole principle, there must be infinitely many monomial orderings $\Delta_1 \subseteq \Delta_0$ which agree on Σ_1. [16, Cor. 2.8] now implies that if $G_1 \subseteq I$ was a standard basis for one of them, it would be a standard basis for all of them. As this cannot be the case, given an ordering $>_1 \in \Delta_1$ there must be an element $g_2 \in I$ such that $\mathrm{LT}_{>_1}(g_2) \notin J_1 := \langle \mathrm{LT}_{>_1}(g) \mid g \in G_1 \rangle$ with J_1 being independent from the ordering chosen.

Since I is $\mathbf{x}$-homogeneous, we may choose g_2 to be $\mathbf{x}$-homogeneous. Moreover, by computing a determinate division with remainder w.r.t. G_1 and $>_1$, we may assume that no term of g_2 lies in J_1 (see e.g. condition (DD2) in [16, Alg. 1.13]). In particular,

$$\mathrm{LT}_>(g_2) \notin J_1 := \langle \mathrm{LT}_>(g) \mid g \in G_1 \rangle \text{ for any ordering } > \in \Delta_1.$$

Setting $G_2 := G_1 \cup \{g_2\}$, we can repeat the entire process, and find an infinite subset of monomial orderings $\Delta_2 \subseteq \Delta_1$ such that G_2 is either a standard basis for all of them or for none of them. Consequently, there is a $g_3 \in I$ such that $\mathrm{LT}_>(g_3) \notin J_2 := \langle \mathrm{LT}_>(g) \mid g \in G_2 \rangle$ for all monomial orderings $> \in \Delta_2$. We thus obtain an infinite chain of strictly ascending ideals $J_1 \subsetneq J_2 \subsetneq \ldots$, which contradicts the ascending chain condition of our noetherian ring $R[t, \mathbf{x}]$. $\square$

A weight vector $w = (w_0, \ldots, w_n) \in \mathbb{R}_{<0} \times \mathbb{R}^n$ induces a partial ordering on $\mathrm{Mon}(t, \mathbf{x})$ via

$$t^\beta \mathbf{x}^\alpha \geq t^\delta \mathbf{x}^\gamma \quad :\Longleftrightarrow \quad w \cdot (\beta, \alpha) \geq w \cdot (\delta, \gamma),$$

where "$\cdot$" denotes the canonical scalar product. Any monomial ordering $>$ on $\mathrm{Mon}(t, \mathbf{x})$ can be used as a tie breaker to refine this partial ordering to a t-local monomial ordering $>_w$. Given $w \in \mathbb{R}_{<0} \times \mathbb{R}^n$ we denote by

$$\mathrm{in}_w(f) = \sum_{w \cdot (\beta,\alpha) \text{ maximal}} c_{\beta,\alpha} \cdot t^\beta \mathbf{x}^\alpha \in R[t, \mathbf{x}]$$

the *initial form* of f w.r.t. w and by

$$\mathrm{in}_w(I) = \langle \mathrm{in}_w(f) \mid f \in I \rangle \trianglelefteq R[t, \mathbf{x}]$$

the *initial ideal* of I.

Initial ideals of I can be used to define an equivalence relation on the space of weight vectors $\mathbb{R}_{<0} \times \mathbb{R}^n$, by setting

$$w \sim v \quad :\Longleftrightarrow \quad \mathrm{in}_w(I) = \mathrm{in}_v(I).$$

We denote the closure the equivalence class of a weight vector $w \in \mathbb{R}_{<0} \times \mathbb{R}^n$ in the Euclidean topology by

$$C_w(I) := \overline{\{v \in \mathbb{R}_{<0} \times \mathbb{R}^n \mid \mathrm{in}_v(I) = \mathrm{in}_w(I)\}} \subseteq \mathbb{R}_{\leq 0} \times \mathbb{R}^n,$$

and call it an *interior Gröbner cone* of I. We then call the intersection of $C_w(I)$ with the boundary,

$$C_w^0(I) := C_w(I) \cap (\{0\} \times \mathbb{R}^n),$$

a *boundary Gröbner cone* of I, and given any t-local monomial ordering $>$, we set

$$C_>(I) := \overline{\{v \in \mathbb{R}_{<0} \times \mathbb{R}^n \mid \mathrm{in}_v(I) = \mathrm{LT}_>(I)\}} \subseteq \mathbb{R}_{\leq 0} \times \mathbb{R}^n.$$

Finally, we refer to the collection

$$\Sigma(I) := \{C_w(I) \mid w \in \mathbb{R}_{<0} \times \mathbb{R}^n\} \cup \{C_w^0(I) \mid w \in \mathbb{R}_{<0} \times \mathbb{R}^n\},$$

of all cones as the *Gröbner fan* of I. It is this object whose properties we want to study and that we want to compute.

Example 2.2
Consider the principal ideal $I = \langle g \rangle \trianglelefteq \mathbb{Z}[\![t]\!][x,y]$ with $g = tx^2 + xy + ty^2$. Because $\mathrm{in}_w(I) = \langle \mathrm{in}_w(g) \rangle$ for any $w \in \mathbb{R}_{<0} \times \mathbb{R}^2$ and g is (x,y)-homogeneous, it is easy to see that every Gröbner cone of I is invariant under translation by $(0,1,1)$. Its Gröbner fan divides the weight space $\mathbb{R}_{\leq 0} \times \mathbb{R}^2$ into three distinct maximal Gröbner cones, see Figure 1. Note that the two red maximal cones intersect each other solely in the boundary $\{0\} \times \mathbb{R}^2$, while the third maximal cone intersects the boundary in codimension 2.

3. The Gröbner fan

This section is devoted to the study of the Gröbner fan of an $\mathbf{x}$-homogeneous ideal I in $R[\![t]\!][\mathbf{x}]$. We will show that it is a rational polyhedral fan

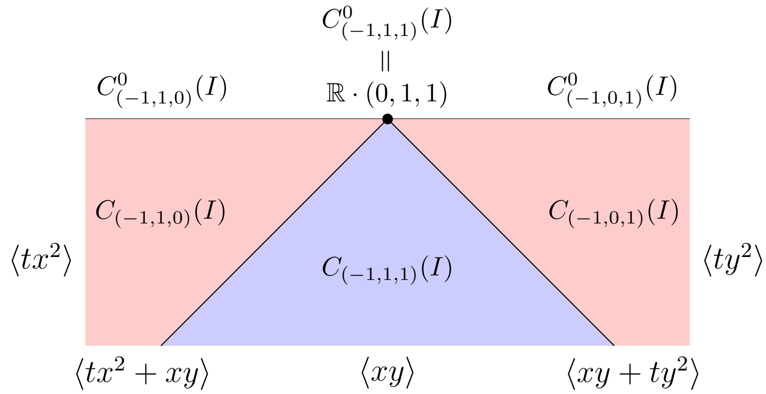

Figure 1: $\Sigma(\langle tx^2+xy+ty^2\rangle)$ projected along $\mathbb{R}\cdot(0,1,1)$

(see Theorem 3.19), i.e. it is a finite collection of rational polyhedral cones containing all faces of each cone in the collection and such that the intersection of each two cones in the collection is a face of both. For this we introduce the notion of an initially reduced standard basis of I w.r.t. a t-local monomial ordering, and show how such a standard basis can be used to read off the Gröbner cone $C_w(I)$ (see Proposition 3.13). All proofs in this section, except that of Proposition 3.3, are constructive, so that we end up with algorithms to compute Gröbner cones, provided that we can compute initially reduced standard bases.

Let us first recall that a *standard basis* of an ideal $I \trianglelefteq R[\![t]\!][\mathbf{x}]$ w.r.t. a t-local monomial ordering $>$ is a finite subset G of I, such the leading terms of its elements w.r.t. $>$ generate the leading ideal of I. A standard basis of I is automatically a generating set of I. A standard basis G of I is called *reduced* if no term of the tail of any element of G is in $\mathrm{LT}_>(I)$ and if it is minimal, i.e. $\mathrm{LT}_>(I)$ cannot be generated by any proper subset of the set of leading terms of G. Observe that we forego any kind of normalisation of the leading coefficients that is normally done in polynomial rings over fields. By [16, Alg. 4.2] reduced standard bases of $\mathbf{x}$-homogeneous ideals in $R[\![t]\!][\mathbf{x}]$ exist. However, even if the ideal I is generated by polynomials in $R[t,\mathbf{x}]$ the elements in a reduced standard of I will in general be power series in t. We, therefore, now introduce a weaker notion.

Definition 3.1 (Initially reduced standard bases)
Let $>$ be a t-local monomial ordering on $\mathrm{Mon}(t,\mathbf{x})$, and let $G, H \subseteq R[\![t]\!][\mathbf{x}]$ be

finite subsets where $G = \{g_1, \dots, g_k\}$ with $g_i = \sum_{\alpha\in\mathbb{N}^n} g_{i,\alpha} \cdot \mathbf{x}^\alpha$, $g_{i,\alpha} \in R[\![t]\!]$.

1. G is *reduced* w.r.t. H, if no term of $\operatorname{tail}_>(g_i)$ lies in $\mathrm{LT}_>(H)$ for any i.
2. We call G *initially reduced* w.r.t. H, if the set
$$G' := \Big\{ g_i' := \sum_{\alpha\in\mathbb{N}} \mathrm{LT}_>(g_{i,\alpha}) \cdot \mathbf{x}^\alpha \;\Big|\; i = 1, \dots, k \Big\},$$
is reduced w.r.t. H, i.e. no term of $\operatorname{tail}_>(g_i')$ is in $\mathrm{LT}_>(H)$ for any i.
3. We call a standard basis G *initially reduced*, if it is minimal and initially reduced w.r.t. itself.

Example 3.2
Obviously, any reduced standard basis is initially reduced. The converse is false, since $G = \{1 - t\}$ is initially reduced w.r.t. any t-local monomial ordering, but it is not reduced.

Proposition 3.3 (Existence of initially reduced standard bases)
Any $\mathbf{x}$-homogeneous ideal in $R[\![t]\!][\mathbf{x}]$ has an initially reduced standard basis w.r.t. any t-local monomial ordering.

Proof. By [16, Alg. 4.2] reduced standard bases exist. □

Algorithm 4.2 in [16] does not produce the basis G in finite time, even if the input data is polynomial. The question, how to achieve this, is postponed to Section 4, where we treat a case of particular interest for the computation of tropical varieties over the p-adic numbers (see [15]). Instead we will now use initially reduced standard bases to give a constructive proof that the Gröbner fan of an $\mathbf{x}$-homogeneous ideal indeed yields a polyhedral fan.

Lemma 3.4
Let G be an initially reduced standard basis of the $\mathbf{x}$-homogeneous ideal $I \trianglelefteq R[\![t]\!][\mathbf{x}]$ w.r.t. a t-local monomial ordering $>$. Then for all $w \in \mathbb{R}_{<0} \times \mathbb{R}^n$ we have
$$\mathrm{in}_w(I) = \mathrm{LT}_>(I) \quad \Longleftrightarrow \quad \forall g \in G : \mathrm{in}_w(g) = \mathrm{LT}_>(g).$$

Proof. $\Rightarrow$ Let $g \in G$. Then $\mathrm{in}_w(g) \in \mathrm{in}_w(I) = \mathrm{LT}_>(I)$. Writing $g = \sum_{\alpha\in\mathbb{N}^n} g_\alpha \cdot \mathbf{x}^\alpha$ with $g_\alpha \in R[\![t]\!]$, note that the only terms of g which can occur in $\mathrm{in}_w(g)$ are of the form $\mathrm{LT}_>(g_\alpha) \cdot \mathbf{x}^\alpha$ for some $\alpha \in \mathbb{N}^n$. And since our leading ideal is naturally generated by terms, these terms of $\mathrm{in}_w(g)$ also lie in $\mathrm{LT}_>(I)$. Because G is initially reduced, we see that the only term of g which can occur in $\mathrm{in}_w(g)$ is $\mathrm{LT}_>(g)$, i.e. $\mathrm{in}_w(g) = \mathrm{LT}_>(g)$.

$\Leftarrow$ It is clear that $\mathrm{in}_w(I) \supseteq \mathrm{LT}_>(I)$. For the converse, it suffices to show $\mathrm{in}_w(f) \in \mathrm{LT}_>(I)$ for all $f \in I$. For that, consider the weighted ordering $>_w$ with weight vector w and tiebreaker $>$, and note that G is also a standard basis w.r.t. that ordering. Hence any $f \in I$ will have a weak division with remainder 0 w.r.t. G and $>_w$:

$$u \cdot f = q_1 \cdot g_1 + \ldots + q_k \cdot g_k.$$

The weighted monomial ordering ensures, that there is no cancellation of highest weighted degree terms on the right hand side, and that 1 is amongst the highest weighted degree terms in u. Taking the initial form w.r.t. w on both sides then yields:

$$\begin{aligned} \mathrm{in}_w(u) \cdot \mathrm{in}_w(f) &= \mathrm{in}_w(q_{i_1}) \cdot \mathrm{in}_w(g_{i_1}) + \ldots + \mathrm{in}_w(q_{i_l}) \cdot \mathrm{in}_w(g_{i_l}) \\ &= \mathrm{in}_w(q_{i_1}) \cdot \mathrm{LT}_>(g_{i_1}) + \ldots + \mathrm{in}_w(q_{i_l}) \cdot \mathrm{LT}_>(g_{i_l}) \in \mathrm{LT}_>(I) \end{aligned}$$

for the $1 \leq i_1 < \ldots < i_l \leq k$ whose terms contribute to the highest weighted degree. Now since $\mathrm{LT}_>(I)$ is generated by terms, any term of $\mathrm{in}_w(u) \cdot \mathrm{in}_w(f)$ is contained in it. In particular, that means $\mathrm{in}_w(f) \in \mathrm{LT}_>(I)$. $\square$

Example 3.5
Consider the ideal

$$\langle g_1 = x - t^3x + t^3z - t^4z, g_2 = y - t^3y + t^2z - t^4z \rangle \trianglelefteq \mathbb{Z}[\![t]\!][x,y,z]$$

and the weighted ordering $> = >_v$ on $\mathrm{Mon}(t,x,y,z)$ with weight vector $v = (-1,3,3,3) \in \mathbb{R}_{<0} \times \mathbb{R}^3$ and t-local lexicographical ordering $x > y > z > 1 > t$ as tiebreaker.

Since g_1 and g_2 already form an initially reduced standard basis, the set whose Euclidean closure yields $C_>(I)$ is, due to Lemma 3.4 given by

$$\{w \in \mathbb{R}_{<0} \times \mathbb{R}^3 \mid \mathrm{in}_w(g_1) = x, \mathrm{in}_w(g_2) = y\}.$$

Hence it is cut out by the following two systems of inequalities:

$$\mathrm{in}_w(g_1) = x \iff \begin{cases} \deg_w(x) > \deg_w(t^3x) \\ \deg_w(x) > \deg_w(t^3z) \\ \deg_w(x) > \deg_w(t^4z) \end{cases} \iff \begin{cases} 0 > w_0 & (*) \\ w_1 > 3w_0 + w_3 \\ w_1 > 4w_0 + w_3 & (*) \end{cases}$$

and

$$\mathrm{in}_w(g_2) = y \iff \begin{cases} \deg_w(y) > \deg_w(t^3 y) \\ \deg_w(y) > \deg_w(t^2 z) \\ \deg_w(y) > \deg_w(t^4 z) \end{cases} \iff \begin{cases} 0 > w_0 & (*) \\ w_2 > 2w_0 + w_3 \\ w_2 > 4w_0 + w_3. & (*) \end{cases}$$

The inequalities marked with $(*)$ are redundant, which is why the terms from which they arise are ignored in the definition of initial reducedness. Figure 2 shows an image in which we restrict ourselves to the affine subspace $\{w_0 = -1, w_3 = 1\}$. Because the set is invariant under translation by $(0, 1, 1, 1)$, no information is lost by doing so.

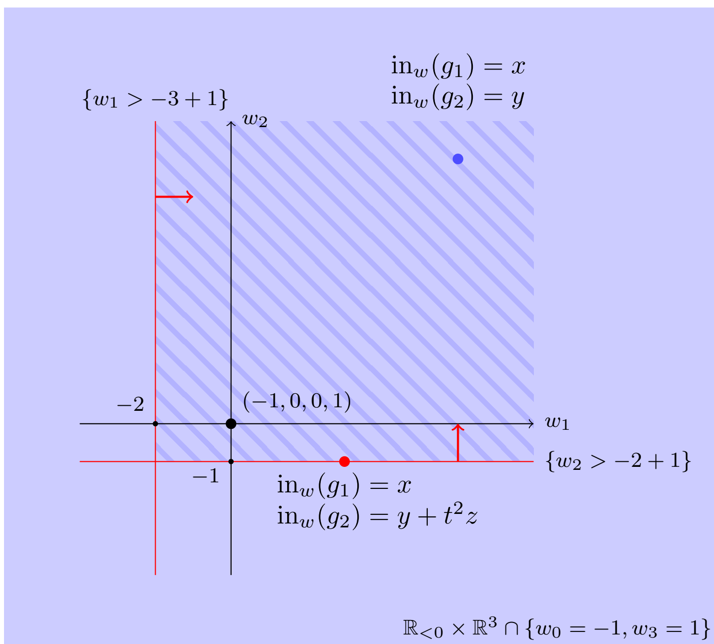

Figure 2: $C_>(I)$ having the structure of a polyhedral cone

Also note that while the weight vectors on the Euclidean boundary may not induce initial forms of g_1 and g_2 coinciding to the leading terms, the initial forms still contain the leading terms.

Lemma 3.6
Let G be an initially reduced standard basis of the $\mathbf{x}$-homogeneous ideal $I \trianglelefteq R[\![t]\!][\mathbf{x}]$ w.r.t. a t-local monomial ordering $>$. Then for all $w \in \mathbb{R}_{<0} \times \mathbb{R}^n$ we

have

$$w \in C_>(I) \quad \iff \quad \forall g \in G : \mathrm{LT}_>(\mathrm{in}_w(g)) = \mathrm{LT}_>(g).$$

Proof. Suppose $G = \{g_1, \ldots, g_k\}$. Similar to Example 3.5, Lemma 3.4 implies that the set $\{w \in \mathbb{R}_{<0} \times \mathbb{R}^n \mid \mathrm{in}_w(I) = \mathrm{LT}_>(I)\}$ is cut out by a system of strict inequalities of the form:

$$\deg_w(\mathrm{LT}_>(g_i)) > \deg_w(\mathrm{tail}_>(g_i)), \quad i = 1, \ldots, k.$$

Note that each line, despite $g_i \in R[\![t]\!][\mathbf{x}]$, only yields a finite amount of minimal inequalities, since higher degrees of t yield redundant inequalities. Therefore, its Euclidean closure $C_>(I)$ is given by a system of inequalities of the form

$$\deg_w(\mathrm{LT}_>(g_i)) \geq \deg_w(\mathrm{tail}_>(g_i)), \quad i = 1, \ldots, k,$$

which is equivalent to $\mathrm{LT}_>(g_i)$ occuring in $\mathrm{in}_w(g)$ and translates to the condition in the claim. $\square$

We can use this result to generalise the statement of Lemma 3.4 to weight vectors in the boundary of $C_>(I)$.

Lemma 3.7
Let $>$ be a t-local monomial ordering and let $I \trianglelefteq R[\![t]\!][\mathbf{x}]$ be an $\mathbf{x}$-homogeneous ideal. Then for all $w \in C_>(I)$, $w \in \mathbb{R}_{<0} \times \mathbb{R}^n$, we have

$$\mathrm{LT}_>(\mathrm{in}_w(I)) = \mathrm{LT}_>(I).$$

Proof. Let G be an initially reduced standard basis of I w.r.t. $>$. Since $\mathrm{LT}_>(\mathrm{in}_w(g)) = \mathrm{LT}_>(g)$ for all $g \in G$ by Lemma 3.6, we have

$$\mathrm{LT}_>(I) = \langle \mathrm{LT}_>(g) \mid g \in G \rangle \overset{\text{Lem.}}{\underset{3.6}{=}} \langle \mathrm{LT}_>(\mathrm{in}_w(g)) \mid g \in G \rangle \subseteq \mathrm{LT}_>(\mathrm{in}_w(I)).$$

For the opposite inclusion, we can again consider the weighted ordering $>_w$. Given any $h \in \mathrm{in}_w(I)$ with $h = \mathrm{in}_w(f)$ for some $f \in I$, this f has a weak division with remainder 0 w.r.t. $G = \{g_1, \ldots, g_k\}$ under $>_w$:

$$u \cdot f = q_1 \cdot g_1 + \ldots + q_k \cdot g_k.$$

Because no cancellation of highest weighted degree terms occurs on the right, taking the initial forms on both sides yields:

$$\mathrm{in}_w(u) \cdot \mathrm{in}_w(f) = \mathrm{in}_w(q_{i_1}) \cdot \mathrm{in}_w(g_{i_1}) + \ldots + \mathrm{in}_w(q_{i_l}) \cdot \mathrm{in}_w(g_{i_l})$$

for the $1 \leq i_1 < \ldots < i_l \leq k$ whose terms contribute to the highest weighted degree. Moreover, $\mathrm{LT}_>(\mathrm{in}_w(u)) = \mathrm{LT}_{>_w}(u) = 1$. Therefore taking the leading terms on both sides produces:

$$\begin{aligned}
\mathrm{LT}_>(\mathrm{in}_w(f)) &= q'_{i_1} \cdot \mathrm{LT}_>(\mathrm{in}_w(g_{i_1})) + \ldots + q'_{i_l} \cdot \mathrm{LT}_>(\mathrm{in}_w(g_{i_l})) \\
&\overset{\text{Lem.}}{\underset{3.6}{=}} q'_{i_1} \cdot \mathrm{LT}_>(g_{i_1}) + \ldots + q'_{i_l} \cdot \mathrm{LT}_>(g_{i_l}) \in \mathrm{LT}_>(I),
\end{aligned}$$

where we abbreviated $q'_{i_j} := \mathrm{LT}_>(\mathrm{in}_w(q_{i_j}))$ for $j = 1, \ldots, l$. □

Combining the previous lemmata we deduce how initially reduced standard bases of restrict to initially reduced standard bases of initial ideals.

Proposition 3.8
Let G be an initially reduced standard basis of the $\mathbf{x}$-homogeneous ideal $I \trianglelefteq R[\![t]\!][\mathbf{x}]$ w.r.t. a t-local monomial ordering $>$. Then for all $w \in C_>(I)$ with $w_0 < 0$ the set

$$H := \{\mathrm{in}_w(g) \mid g \in G\}$$

is an initially reduced standard basis of $\mathrm{in}_w(I)$ w.r.t. the same ordering.

Proof. By the previous Lemmata, we have

$$\mathrm{LT}_>(\mathrm{in}_w(I)) \overset{\text{Lem.}}{\underset{3.7}{=}} \mathrm{LT}_>(I) = \langle \mathrm{LT}_>(g) \mid g \in G\rangle \overset{\text{Lem.}}{\underset{3.6}{=}} \langle \mathrm{LT}_>(\mathrm{in}_w(g)) \mid g \in G\rangle,$$

and therefore H is a standard basis of $\mathrm{in}_w(I)$. Moreover, because G was initially reduced, so is H. □

Example 3.9
Given the same ideal and ordering as in Example 3.5, g_1 and g_2 form an initially reduced standard basis. Because $w := (-1, 2, -1, 1) \in C_>(I)$, Proposition 3.8 implies that the initial ideal $\mathrm{in}_w(I)$ has the initially reduced standard bases $\{\mathrm{in}_w(g_1), \mathrm{in}_w(g_2)\} = \{x, y + t^2 z\}$. As we go over all weight vectors in $C_>(I)$ in the affine subspace, we obtain four distinct initial ideals as illustrated in Figure 3.

Corollary 3.10
Any $\mathbf{x}$-homogeneous ideal $I \trianglelefteq R[\![t]\!][\mathbf{x}]$ has only finitely many distinct initial ideals. In particular, I has only finitely many Gröbner cones.

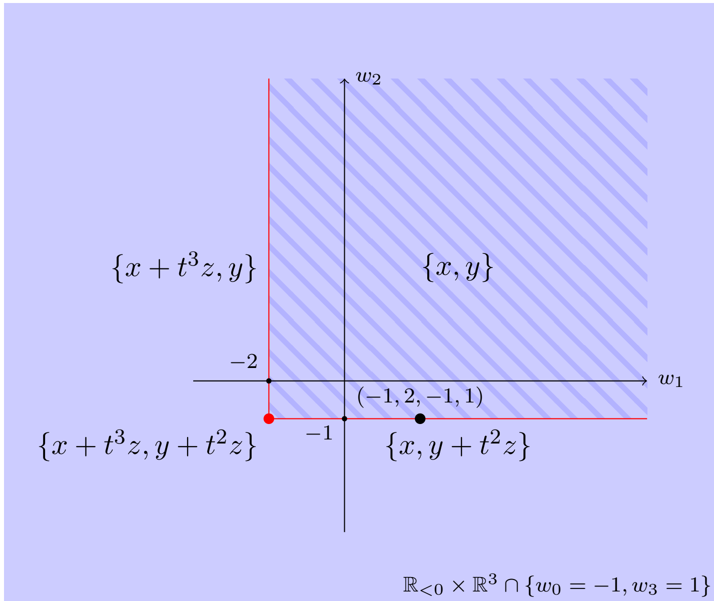

Figure 3: standard bases of initial ideals with various weights

Proof. Note first that due to Lemma 3.6 every weight vector w is contained in $C_>(I)$ for some t-local monomial ordering $>$, just choose any refinement $>_w$ of the partial ordering induced by w. By Proposition 3.8 the initial ideal $\mathrm{in}_w(I)$ is determined by any initially reduced standard basis of I w.r.t. $>$. Since by Proposition 2.1 there are only finitely many distinct $C_>(I)$, it suffices to argue why a fixed $C_>(I)$ can only lead to finitely many distinct initial ideals $\mathrm{in}_w(I)$, since this implies that there are only finitely many $C_w(I)$ and hence only finitely many $C_w^0(I)$.

To this end note that an arbitrary element $g = \sum_{\alpha\in\mathbb{N}^n} g_\alpha \mathbf{x}^\alpha \in R[\![t]\!][\mathbf{x}]$ with $g_\alpha \in R[\![t]\!]$ has only finitely many distinct initial forms. Consider a weight vector $w \in \mathbb{R}_{<0} \times \mathbb{R}^n$, and let $>$ be a t-local monomial ordering. The initial forms of g w.r.t. $>$ are of the form

$$\mathrm{in}_w(g) = \sum_{\alpha\in\Lambda} \mathrm{LT}_>(g_\alpha) \cdot \mathbf{x}^\alpha$$

for a finite set $\Lambda \subseteq \{\alpha \in \mathbb{N}^n \mid g_\alpha \neq 0\}$. Thus a fixed initially reduced standard basis of I w.r.t. $>$ admits by Proposition 3.8 only finitely many choices for generating sets of initial ideals $\mathrm{in}_w(I)$ and hence only finitely many initial ideals. $\square$

The next proposition allows us to read off the inequalities and equations of the Gröbner cones, from which we can derive the remaining properties needed to show that they form a polyhedral fan.

Proposition 3.11
Let G be an initially reduced standard basis of the $\mathbf{x}$*-homogeneous ideal* $I \trianglelefteq R[\![t]\!][\mathbf{x}]$ *w.r.t. a* t*-local monomial ordering* $>$ *and let* $w \in C_>(I)$ *with* $w_0 < 0$*. Then for all* $v \in \mathbb{R}_{<0} \times \mathbb{R}^n$ *we have*

$$\mathrm{in}_v(I) = \mathrm{in}_w(I) \quad \Longleftrightarrow \quad \forall g \in G : \mathrm{in}_v(g) = \mathrm{in}_w(g).$$

Proof. $\Leftarrow$ For $g \in G$ note that

$$\mathrm{LT}_>(\mathrm{in}_v(g)) = \mathrm{LT}_>(\mathrm{in}_w(g)) \overset{\text{Lem.}}{\underset{3.6}{=}} \mathrm{LT}_>(g),$$

thus $v \in C_>(I)$, again by Lemma 3.6. This allows us to use Proposition 3.8, which says that $\mathrm{in}_w(I)$ and $\mathrm{in}_v(I)$ share a common standard basis, therefore they must coincide.

$\Rightarrow$ Let $g \in G$. On the one hand, Lemma 3.6 implies that $\mathrm{LT}_>(g)$ is a term of $\mathrm{in}_w(g)$. On the other hand,

$$\mathrm{LT}_>(\mathrm{in}_v(g)) \in \mathrm{LT}_>(\mathrm{in}_v(I)) = \mathrm{LT}_>(\mathrm{in}_w(I)) \overset{\text{Lem.}}{\underset{3.7}{=}} \mathrm{LT}_>(I).$$

But because G is initially reduced, the only term of g occurring in $\mathrm{in}_v(g)$ and $\mathrm{LT}_>(I)$ is $\mathrm{LT}_>(g)$. Thus $\mathrm{LT}_>(g)$ is also a term of $\mathrm{in}_v(g)$.

Now consider $\mathrm{in}_w(g) - \mathrm{in}_v(g) \in \mathrm{in}_w(I) = \mathrm{in}_v(I)$. Our previous arguments show that $\mathrm{LT}_>(\mathrm{in}_w(g) - \mathrm{in}_v(g)) \neq \mathrm{LT}_>(g)$. However, because

$$\mathrm{LT}_>(\mathrm{in}_w(g) - \mathrm{in}_v(g)) \in \mathrm{LT}_>(\mathrm{in}_w(I)) \overset{\text{Lem.}}{\underset{3.7}{=}} \mathrm{LT}_>(I),$$

it is another term of $\mathrm{in}_w(g)$ or $\mathrm{in}_v(g)$ in $\mathrm{LT}_>(I)$, which must be 0. □

Example 3.12
Consider the same ideal and ordering as in Example 3.5 and Example 3.9, where $g_1 = x - t^3x + t^3z - t^4z$ and $g_2 = y - t^3y + t^2z - t^4z$ form an initially reduced standard basis.

For $w = (-1, 2, -1, 1) \in C_>(I)$ we have by Proposition 3.11:

$$\mathrm{in}_{w'}(I) = \mathrm{in}_w(I) = \langle x, y + t^2z \rangle \quad \Longleftrightarrow \quad \begin{cases} \mathrm{in}_{w'}(g_1) = x, \\ \mathrm{in}_{w'}(g_2) = y + t^2z. \end{cases}$$

Therefore, its equivalence class of weight vectors $w' \in \mathbb{R}_{<0} \times \mathbb{R}^3$ such that $\mathrm{in}_{w'}(I) = \mathrm{in}_w(I)$ is determined by the following system of inequalities and equations:

$$\mathrm{in}_{w'}(g_1) = x \quad \Longleftrightarrow \quad \begin{cases} \deg_{w'}(x) > \deg_{w'}(t^3 x) \\ \deg_{w'}(x) > \deg_{w'}(t^3 z) \\ \deg_{w'}(x) > \deg_{w'}(t^4 z) \end{cases} \quad \Longleftrightarrow \quad \begin{cases} 0 > w'_0 \\ w'_1 > 3w'_0 + w'_3 \\ w'_1 > 4w'_0 + w'_3 \end{cases}$$

$$\mathrm{in}_{w}(g_2) = y{+}t^2 z \quad \Longleftrightarrow \quad \begin{cases} \deg_{w'}(y) > \deg_{w'}(t^3 y) \\ \deg_{w'}(y) = \deg_{w'}(t^2 z) \\ \deg_{w'}(y) > \deg_{w'}(t^4 z) \end{cases} \quad \Longleftrightarrow \quad \begin{cases} 0 > w'_0 \\ w'_2 = 2w'_0 + w'_3 \\ w'_2 > 4w'_0 + w'_3 \end{cases}$$

In particular, its Euclidean closure, the Gröbner cone $C_w(I)$, is the face of $C_>(I)$ cut out by the hyperplane $\{w'_2 = 2w'_0 + w'_3\}$.

In fact, Proposition 3.11 implies that $C_>(I)$ is stratified by equivalence classes of weight vectors as Figure 3 already suggested. Each class is an open polyhedral cone whose Euclidean closure yields a face of $C_>(I)$.

Proposition 3.13
For any $\mathbf{x}$-homogeneous ideal $I \trianglelefteq R[\![t]\!][\mathbf{x}]$ and for any $w \in \mathbb{R}_{<0} \times \mathbb{R}^n$, the Gröbner cones $C_w(I)$ and $C^0_w(I)$ are closed rational polyhedral cones.

Proof. Let $>$ be a t-local weighted monomial ordering w.r.t. a weight vector w, and let G be an initially reduced standard basis of I w.r.t. $>$.

Suppose $G = \{g_1, \ldots, g_k\}$ with $g_i = \sum_{\beta,\alpha} c_{\alpha,\beta,i} \cdot t^\beta \mathbf{x}^\alpha$. Let Λ_i be the finite set of exponent vectors with minimal entry in t,

$$\Lambda_i := \{(\beta, \alpha) \in \mathbb{N} \times \mathbb{N}^n \mid \alpha \in \mathbb{N}^n, \beta = \min\{\beta' \in \mathbb{N} \mid c_{\alpha,\beta',i} \neq 0\}\}.$$

Similar to Example 3.12, Proposition 3.11 implies that the equivalence class of w, $\{v \in \mathbb{R}_{<0} \times \mathbb{R}^n \mid \mathrm{in}_v(I) = \mathrm{in}_w(I)\}$, is cut out by a system of inequalities and equations

$$\begin{aligned} v \cdot (\beta, \alpha) > v \cdot (\delta, \gamma), &\quad \text{for all } (\beta, \alpha), (\delta, \gamma) \in \Lambda_i \text{ with } w \cdot (\beta, \alpha) > w \cdot (\delta, \gamma), \\ v \cdot (\beta, \alpha) = v \cdot (\delta, \gamma), &\quad \text{for all } (\beta, \alpha), (\delta, \gamma) \in \Lambda_i \text{ with } w \cdot (\beta, \alpha) = w \cdot (\delta, \gamma). \end{aligned}$$

Therefore, the equivalence class forms a relative open polyhedral cone contained in the open lower half space $\mathbb{R}_{<0} \times \mathbb{R}^n$ and its closure $C_w(I)$ yields a closed polyhedral cone in the closed lower half space $\mathbb{R}_{\leq 0} \times \mathbb{R}^n$. In particular, $C^0_w(I) = C_w(I) \cap (\{0\} \times \mathbb{R}^n)$ is also a closed polyhedral cone. □

Corollary 3.14
Let $I \trianglelefteq R[\![t]\!][\mathbf{x}]$ be an $\mathbf{x}$-homogeneous ideal and let $w \in \mathbb{R}_{<0} \times \mathbb{R}^n$. Then any face $\tau \leq C_w(I)$ with $\tau \nsubseteq \{0\} \times \mathbb{R}^n$ coincides with the closure of the equivalence class of any weight vector in its relative interior.

In particular, each face $\tau \leq C_w(I)$ is a Gröbner cone of the form $\tau = C_v(I)$ or $\tau = C_v^0(I)$ for some $v \in C_w(I)$ and each face $\tau \leq C_w^0(I)$ is a Gröbner cone of the form $\tau = C_v^0(I)$ for some $v \in C_w(I)$.

Proof. Consider again the system of inequalities and equations that cut out $C_w(I)$ in the proof of the previous Proposition 3.13, which we obtained from the sets of exponent vectors $\Lambda_1, \ldots, \Lambda_k$ of an initially reduced standard basis w.r.t. a weighted ordering $>_w$.

A face $\tau \leq C_w(I)$ is cut out by supporting hyperplanes, on which some of the inequalities above become equations. Assuming that $\tau \nsubseteq \{0\} \times \mathbb{R}^n$, all weight vectors in the relative interior yield the same initial forms on $g_1, \ldots, g_k \in G$, since they satisfy the same equations and inequalities on the exponent vectors $\Lambda_1, \ldots, \Lambda_k$. This implies that they belong to the same equivalence class whose closure is then τ. In particular, $\tau = C_v(I)$.

And any face $\tau \leq C_w^0(I) \leq C_w(I)$ can be cut out by a supporting hyperplane which also cuts out a face $C_v(I) \leq C_w(I)$. It is then clear that $\tau = C_v^0(I)$. □

Proposition 3.15
Let $I \trianglelefteq R[\![t]\!][\mathbf{x}]$ be an $\mathbf{x}$-homogeneous ideal and let $C_u(I)$ and $C_v(I)$ be two interior Gröbner cones of I such that $C_u(I) \cap C_v(I) \nsubseteq \{0\} \times \mathbb{R}^n$. Then $C_u(I) \cap C_v(I)$ is an interior Gröbner cone and it is a face of both.

Proof. By Proposition 3.13, both $C_u(I) \cap (\mathbb{R}_{<0} \times \mathbb{R}^n)$ and $C_v(I) \cap (\mathbb{R}_{<0} \times \mathbb{R}^n)$ can be decomposed into a union of equivalence classes, and hence so can $(C_u(I) \cap C_v(I)) \cap (\mathbb{R}_{<0} \times \mathbb{R}^n) \neq \emptyset$.

Let $k := \dim(C_u(I) \cap C_v(I))$. Then the intersection contains exactly one equivalence class of dimension k: If there were none, then the intersection would be covered by a collection of lower dimensional open cones of which there are, however, only finitely many by Corollary 3.10. If there were more than one, then that would contradict Proposition 3.13, which states that the closure of each equivalence class yields a distinct face of both $C_u(I)$ and $C_v(I)$, and no two k-dimensional faces of a polyhedral cone may be cut out by the same k-dimensional supporting hyperplane.

So let w be in the maximal equivalence class in $C_u(I) \cap C_v(I)$. Taking the Euclidean closure, we necessarily have $C_w(I) = C_u(I) \cap C_v(I)$, and, by Corollary 3.13, it is a face of both $C_u(I)$ and $C_v(I)$. $\square$

Note that the proposition above falls a bit short in proving that the intersection of two Gröbner cones yields a face of both, as it only covers Gröbner cones with an intersection in the open part of the lower halfspace. To cover the remaining intersection, we need some results on recession fans.

Definition 3.16
Let $I \trianglelefteq R[\![t]\!][\mathbf{x}]$ be an $\mathbf{x}$-homogeneous ideal and $w \in \mathbb{R}_{<0} \times \mathbb{R}^n$. For an interior Gröbner cone $C_w(I)$ let $C_w^{-1}(I)$ denote the intersection

$$C_w^{-1}(I) := C_w(I) \cap (\{-1\} \times \mathbb{R}^n).$$

It is a polytope whose *recession cone* is defined to be the set of all weight vectors in $\mathbb{R}_{\leq 0} \times \mathbb{R}^n$ under whose translation it is closed,

$$\mathrm{rec}(C_w^{-1}(I)) := \{v \in \mathbb{R}_{\leq 0} \times \mathbb{R}^n \mid v + C_w^{-1}(I) \subseteq C_w^{-1}(I)\}.$$

Note that $C_w^{-1}(I) \subseteq \{-1\} \times \mathbb{R}^n$ necessarily implies $\mathrm{rec}(C_w^{-1}(I)) \subseteq \{0\} \times \mathbb{R}^n$.

Proposition 3.17
Let $I \trianglelefteq R[\![t]\!][\mathbf{x}]$ be an $\mathbf{x}$-homogeneous ideal.

1. *The collection $\{C_w^{-1}(I) \mid w \in \mathbb{R}_{<0} \times \mathbb{R}^n\}$ is a polyhedral complex whose support is the affine hyperplane $\{-1\} \times \mathbb{R}^n$.*
2. *For any weight vector $w \in \mathbb{R}_{<0} \times \mathbb{R}^n$, $C_w^0(I) = \mathrm{rec}(C_w^{-1}(I))$.*
3. *The collection $\{C_w^0(I) \mid w \in \mathbb{R}_{<0} \times \mathbb{R}^n\}$ is a polyhedral fan whose support is the boundary hyperplane $\{0\} \times \mathbb{R}^n$.*

Proof. 1. follows from Proposition 3.13, Corollary 3.14 and Proposition 3.15. 2. is clear, and 3. follows from [20, Cor. 3.10]. $\square$

We can now supplement the missing intersections in Proposition 3.15.

Corollary 3.18
Let $I \trianglelefteq R[\![t]\!][\mathbf{x}]$ be an $\mathbf{x}$-homogeneous ideal and let $u, v \in \mathbb{R}_{<0} \times \mathbb{R}^n$. Then the intersections $C_u^0(I) \cap C_v^0(I)$, $C_u^0(I) \cap C_v(I)$ are boundary Gröbner cones of I and they are faces of the intersected cones.

Proof. Since the boundary Gröbner cones form a polyhedral fan by Proposition 3.17, the intersection $C_u^0(I) \cap C_v^0(I)$ is a face of both. In particular, by Corollary 3.14, there is a weight vector $w \in \mathbb{R}_{<0} \times \mathbb{R}^n$ with

$$C_u^0(I) \cap C_v^0(I) = C_w^0(I).$$

And for the intersection of a boundary Gröbner cone and an interior Gröbner cone, note that

$$C_u^0(I) \cap C_v(I) = C_u^0(I) \cap C_v^0(I) = C_w^0(I). \qquad \square$$

We are now able to prove the main theoretical result of the paper.

Theorem 3.19
Let $I \trianglelefteq R[\![t]\!][\mathbf{x}]$ be an $\mathbf{x}$-homogeneous ideal, then the Gröbner fan

$$\Sigma(I) = \{C_w(I) \mid w \in \mathbb{R}_{<0} \times \mathbb{R}^n\} \cup \{C_w^0(I) \mid w \in \mathbb{R}_{<0} \times \mathbb{R}^n\}$$

is a rational polyhedral fan with support $\mathbb{R}_{\leq 0} \times \mathbb{R}^n$.

Proof. Proposition 3.13 shows that each Gröbner cone is a polyhedral cone, while Corollary 3.14 proves that each face of a Gröbner cone is again a Gröbner cone. Proposition 3.15 and Corollary 3.18 infer that the intersection of two Gröbner cones is a face of each, and Corollary 3.10 shows that there are only finitely many of them. $\square$

Example 3.20
Consider the following ideal generated by polynomials

$$\langle 2x + 2y, t + 2 \rangle \trianglelefteq \mathbb{Z}[\![t]\!][x, y].$$

Now because the ideal is generated by elements in $\mathbb{Z}[t, x, y]$, one might be tempted to believe that restricting ourselves to the polynomial ideal

$$\langle 2x + 2y, t + 2 \rangle \trianglelefteq \mathbb{Z}[t, x, y],$$

might allow us to work with weight vectors $\mathbb{R}_{\geq 0} \times \mathbb{R}^2$ with positive weight in t, obtain similar results about the existence of a Gröbner fan there and patch the two Gröbner fans in $\mathbb{R}_{\leq 0} \times \mathbb{R}^2$ and in $\mathbb{R}_{\geq 0} \times \mathbb{R}^2$ together.

While the existence of a Gröbner fan in the positive halfspace is true for our specific example, note that the two Gröbner fans cannot be glued together to a polyhedral fan on $\mathbb{R} \times \mathbb{R}^2$, as illustrated in Figure 4.

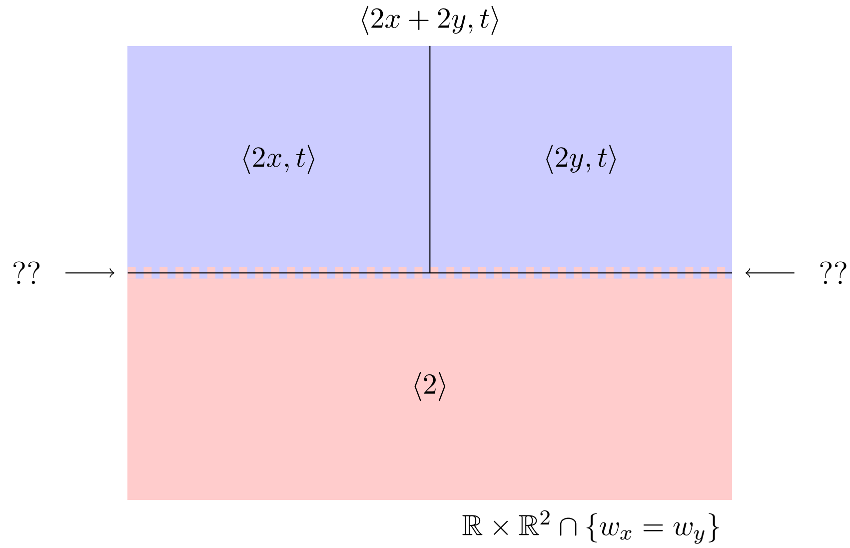

Figure 4: $\Sigma(\langle 2x+2y, t+2\rangle)$ on $\mathbb{R}\times\mathbb{R}^2$...?

As demonstrated in Example 3.12 and used in the proof of Proposition 3.13, Proposition 3.11 allows us to read of the inequalities and equations of a Gröbner cone from an initially reduced standard basis. This can be done as described in the following algorithm.

Algorithm 3.21 (Inequalities and equations of a Gröbner cone)
Input: $(H, G, >)$, where for an $\mathbf{x}$-homogeneous ideal I and an undetermined weight vector $w \in \mathbb{R}_{<0}\times\mathbb{R}^n$

1. $>$ a t-local monomial ordering such that $w \in C_>(I)$,
2. $G = \{g_1, \dots, g_k\}$ an initially reduced standard basis of I w.r.t. $>$,
3. $H = \{h_1, \dots, h_k\}$ with $h_i = \mathrm{in}_w(g_i)$.

Output: (A, B), a pair of matrices

$$A \in \mathrm{Mat}(l_A \times (n+1), \mathbb{R}), \quad B \in \mathrm{Mat}(l_B \times (n+1), \mathbb{R})$$

such that

$$C_w(I) = \{v \in \mathbb{R}_{\leq 0}\times\mathbb{R}^n \mid A\cdot v \in (\mathbb{R}_{\geq 0})^{l_A} \text{ and } B\cdot v = 0 \in \mathbb{R}^{l_B}\}.$$

1: **for** $i = 1, \dots, k$ **do**

Suppose $g_i = \sum_{\beta,\alpha} c_{\alpha,\beta,i} \cdot t^\beta \mathbf{x}^\alpha$ and $\mathrm{LM}_>(g) = t^\delta \mathbf{x}^\gamma$.
Construct the set of exponent vectors with minimal entry in t,

$$\Lambda_i := \{(\beta,\alpha) \in \mathbb{N} \times \mathbb{N}^n \mid \alpha \in \mathbb{N}^n, \beta = \min\{\beta' \in \mathbb{N} \mid c_{\alpha,\beta',i} \neq 0\}\}.$$

Construct a set of vectors that will yield the inequalities,

$$\Omega_i := \{(\delta,\gamma) - (\alpha,\beta) \in \mathbb{R} \times \mathbb{R}^n \mid (\alpha,\beta) \in \Lambda_i,\ \ (\alpha,\beta) \neq (\delta,\gamma)\}.$$

Let A be a matrix whose row vectors consist of $\bigcup_{i=1}^k \Omega_i$.
for $i = 1, \ldots, k$ **do**
Suppose $h_i = \sum_{\beta,\alpha} d_{\alpha,\beta,i} \cdot t^\beta \mathbf{x}^\alpha$.
Construct the set of exponent vectors with minimal entry in t,

$$\Lambda'_i := \{(\beta,\alpha) \in \mathbb{N} \times \mathbb{N}^n \mid \alpha \in \mathbb{N}^n, \beta = \min\{\beta' \in \mathbb{N} \mid d_{\alpha,\beta',i} \neq 0\}\}.$$

Construct a set of vectors that will yield the equations,

$$\Theta_i := \{a - b \in \mathbb{R} \times \mathbb{R}^n \mid a, b \in \Lambda'_i\}.$$

Let B be a matrix whose row vectors consist of $\bigcup_{i=1}^k \Theta_i$.
return (A, B).

We close the section with an example which shows why it is important that the standard basis is initially reduced in order to determine the corresponding Gröbner cone. It is an example abiding to the special assumptions on R and I considered in Section 4 (see Page 23).

Example 3.22
Let $R[\![t]\!][\mathbf{x}] = \mathbb{Z}[\![t]\!][x,y,z]$ and let $> = >_v$ be a weighted ordering with weight vector $v = (-1,1,1,1) \in \mathbb{R}_{<0} \times \mathbb{R}^3$ and the t-local lexicographical ordering $x > y > 1 > t$ as tiebreaker. We consider the ideal

$$I = \langle g_0 = 2 - t, g_1 = x + t^2 y + t^3 z, g_2 = y + tx + t^2 z \rangle \trianglelefteq \mathbb{Z}[\![t]\!][x,y,z],$$

to illustrate that the initial reduction of the standard basis ist important for determining the inequalities and equations of the corresponding Gröbner cone (see Algorithm 3.21).

Note that the generating set is a standard basis w.r.t. $>$, but it is not yet initially reduced as the terms $t^2 y$ in g_1 and tx in g_2 still lie in $\mathrm{LT}_>(I) = \langle 2, x, y \rangle$. Consequently, these two terms yield meddling inequalities, so that

$$C := \overline{\{w \in \mathbb{R}_{<0} \times \mathbb{R}^3 \mid \mathrm{in}_w(g_i) = \mathrm{in}_v(g_i) \text{ for } i = 0,1,2\}} \subsetneq C_v(I).$$

Ignoring g_0, as it yields no non-trivial inequalities in $\mathbb{R}_{\leq 0} \times \mathbb{R}^3$, C is the polyhedral cone given by the inequalities (see Figure 5)

$$\operatorname{in}_w(g_1) = x \quad\Longleftrightarrow\quad \begin{cases} \deg_w(x) \geq \deg_w(t^2 y) \\ \deg_w(x) \geq \deg_w(t^3 z) \end{cases} \quad\Longleftrightarrow\quad \begin{cases} w_1 \geq 2w_0 + w_2 \\ w_1 \geq 3w_0 + w_3 \end{cases}$$

$$\operatorname{in}_w(g_2) = y \quad\Longleftrightarrow\quad \begin{cases} \deg_w(y) \geq \deg_w(tx) \\ \deg_w(y) \geq \deg_w(t^2 z) \end{cases} \quad\Longleftrightarrow\quad \begin{cases} w_2 \geq w_0 + w_1 \\ w_2 \geq 2w_0 + w_3 \end{cases}$$

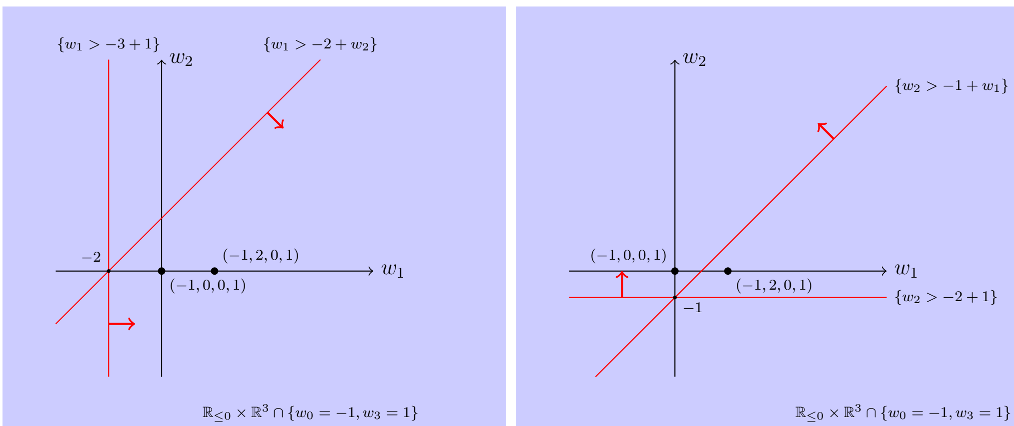

Figure 5: inequalities given by $\operatorname{in}_w(g_1) = x$ resp. $\operatorname{in}_w(g_2) = y$

Clearly, $w := (-1, 2, 0, 1) \notin C$, even though $\operatorname{in}_w(I) = \operatorname{in}_v(I)$, since

$$\operatorname{in}_w(g_1 - t^2 \cdot g_2) = \operatorname{in}_w(x - t^3 x + t^3 z - t^4 z) = x,$$
$$\operatorname{in}_w(g_2 - t \cdot g_1) = \operatorname{in}_w(y - t^3 y + t^2 z - t^4 z) = y,$$

implying that $w \in C_v(I)$. Replacing $\{g_1, g_2\}$ with the initially reduced standard basis $\{g_1 - t^2 \cdot g_2, g_2 - t \cdot g_1\}$, we see that we are replacing the unnecessary inequalities above, induced by $t^2 y$ and tx, with the redundant inequalities of Example 3.5, induced by $t^3 x$, $t^3 y$ and $t^4 z$.

4. Initially reduced standard bases

In this section, we present an algorithm for the initial reduction of a polynomial $\mathbf{x}$-homogeneous standard basis in finite time. For the sake of simplicity, we will restrict ourselves to a special case which is of particular interest for the computation of tropical varieties over the p-adic numbers (see [15]), though the basic ideas behind the algorithm can be generalised.

Throughout this section we assume that K is some field with non-trivial discrete valuation, $\mathfrak{K}$ its residue field, R_ν its discrete valuation ring, $p \in R_\nu$ a uniformising parameter and $R \subset R_\nu$ a dense noetherian subring with $p \in R$. Both K and R_ν are assumed to be complete, so that we have exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \langle p-t\rangle \cdot R[\![t]\!]_{\langle p-t\rangle}[\mathbf{x}] & \longrightarrow & R[\![t]\!]_{\langle p-t\rangle}[\mathbf{x}] & \longrightarrow & K[\mathbf{x}] & \longrightarrow & 0, \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \langle p-t\rangle \cdot R[\![t]\!][\mathbf{x}] & \longrightarrow & R[\![t]\!][\mathbf{x}] & \xrightarrow[t \longmapsto p]{} & R_\nu[\mathbf{x}] & \longrightarrow & 0.
\end{array}$$

and $R/\langle p\rangle = \mathfrak{K}$. Moreover, we still require that linear equations in R are solvable, so that standard bases in $R[\![t]\!][\mathbf{x}]$ exist and are computable. If $R = \mathbb{Z}$ is the ring of integers, $p \in \mathbb{Z}$ a prime number and $K = \mathbb{Q}_p$ the field of p-adic numbers, all properties are fulfilled (see [15] for further interesting examples). We then fix the preimage $I \trianglelefteq R[\![t]\!][\mathbf{x}]$ of some homogeneous ideal in $K[\mathbf{x}]$, which in particular implies that I is $\mathbf{x}$-homogeneous and $p - t \in I$. It is our aim to provide an algorithm which computes an initially reduced standard basis of I w.r.t. some t-local monomial ordering $>$ on $\mathrm{Mon}(t, \mathbf{x})$, provided that the ideal I is generated by polynomials. See Example 3.22 for an example.

This section has a simple monolithic structure. Because our ideals are all $\mathbf{x}$-homogeneous, the problems that commonly arise when lacking a well-ordering actually root in the inhomogeneity in t alone. It turns out that these problems can be circumvented by reducing w.r.t. $p - t$ diligently. Hence we begin with an algorithm dedicated to that. Next, we continue with an algorithm for reducing a set of elements of the same $\mathbf{x}$-degree w.r.t. themselves and $p-t$. Having all elements sharing the same $\mathbf{x}$-degree makes the inhomogeneity in t easy to handle. Using it, we construct an algorithm for reducing a set of elements of the same $\mathbf{x}$-degree w.r.t. themselves, $p - t$ and another set of elements of strictly lower $\mathbf{x}$-degree. This is the part in which the difficulty of our lack of well-ordering becomes apparent. We then conclude the section with Algorithm 4.6 for computing an initially reduced standard basis by reducing a standard basis w.r.t. itself.

Algorithm 4.1 ($(p-t)$-Reduce)

Input: $(g, >)$, where $>$ is a t-local monomial ordering and $g \in R[t, \mathbf{x}]$ $\mathbf{x}$-homogeneous.

Output: $g' \in R[t, \mathbf{x}]$ $\mathbf{x}$-homogeneous with $\langle p-t, g' \rangle = \langle p-t, g \rangle \trianglelefteq R[t, \mathbf{x}]$, $\mathrm{LT}_>(g') = \mathrm{LT}_>(g)$ and initially reduced w.r.t. $p-t$ under $>$.

1: Suppose $g = \sum_\alpha g_\alpha \cdot \mathbf{x}^\alpha$ with $g_\alpha \in R[t]$ and $\mathrm{LT}_>(g) = \mathrm{LT}_>(g_\gamma) \cdot \mathbf{x}^\gamma$.
2: Set $g' := g_\gamma \cdot \mathbf{x}^\gamma$ and $g'' := g - g_\gamma \cdot \mathbf{x}^\gamma$, so that $g = g' + g''$.
3: **while** $g'' \neq 0$ **do**
4:     Suppose $g'' = \sum_\alpha g''_\alpha \cdot \mathbf{x}^\alpha$ with $g''_\alpha \in R[t]$ and $\mathrm{LT}_>(g'') = \mathrm{LT}_>(g''_\gamma) \cdot \mathbf{x}^\gamma$.
5:     **if** $p \mid \mathrm{LT}_>(g''_\gamma)$ **then**
6:         Let $l := \max\{m \in \mathbb{N} \mid p^m \text{ divides } \mathrm{LT}_>(g''_\gamma)\} > 0$.
7:         Set $g'' := g'' - \frac{\mathrm{LT}_>(g''_\gamma)}{p^l} \cdot (p^l - t^l)$.
8:     **else**
9:         Set $g' := g' + g''_\gamma \cdot \mathbf{x}^\gamma$ and $g'' := g'' - g''_\gamma \cdot \mathbf{x}^\gamma$.
10: **return** g'

Proof. Termination: We need to show that $g'' = 0$ eventually. Since all changes to g'' during a single iteration of the `while` loop happen at a distinct monomial in $\mathbf{x}$, namely that of $\mathrm{LM}_>(g'')$, we may assume for our argument that all terms of g'' have the same monomial in $\mathbf{x}$. Suppose, in the beginning of an iteration,

$$g'' = (c_{i_1} t^{i_1} + \ldots + c_{i_j} \cdot t^{i_j}) \cdot \mathbf{x}^\gamma \text{ with } i_1 < \ldots < i_j.$$

Now if $p \nmid \mathrm{LT}_>(c_{i_1})$, then g'' will be set to 0 in Step 9 and the algorithm terminates. If $p \mid \mathrm{LT}_>(c_{i_1})$, we substitute the term $c_{i_1} \cdot t^{i_1} \mathbf{x}^\gamma$ by the term $c_{i_1}/p^l \cdot t^{i_1+l} \mathbf{x}^\gamma$ in Step 7, increasing the minimal t-degree strictly.

Let $\nu_p(c) := \max\{m \in \mathbb{N} \mid p^m \text{ divides } c\}$ denote the p-adic valuation on R, so that $l = \nu_p(c_{i_1})$, and consider the valued degree of g'' defined by

$$\max\{\nu_p(c_{i_1}) + \deg(t^{i_1}), \ldots, \nu_p(c_{i_j}) + \deg(t^{i_j})\}.$$

This is a natural upper bound on the t-degree of our substitute, and hence also for the t-degree of all terms in our new g''.

If the monomial of the substitute, $t^{i_1+l}\mathbf{x}^\gamma$, does not occur in the original g'', then this upper bound remains the same for our new g''. If it does occur in the original g'', then this valued degree might increase depending on the sum of the coefficients, however the number of terms in g'' strictly decreases.

Because g'' has only finitely many terms to begin with, this upper bound may therefore only increase a finite number of times. And since the minimal t-degree is strictly increasing, if g'' is not set to 0, our algorithm terminates eventually.

Correctness: It is clear that g' remains polynomial and $\mathbf{x}$-homogeneous. And the only term of g' that might be divisible by $\mathrm{LT}_>(p-t) = p$ is $\mathrm{LT}_>(g') = \mathrm{LT}_>(g)$, since all other terms passed the check in Step 5 negatively. Hence g' is initially reduced w.r.t. $p-t$ under $>$. □

With this, we can begin formulating an algorithm for initially reducing a set of elements which are $\mathbf{x}$-homogeneous of same degree in $\mathbf{x}$.

Algorithm 4.2 (initial reduction, same degree in $\mathbf{x}$)

Input: $(G, >)$, where $>$ is a t-local monomial ordering and $G = \{g_1, \ldots, g_k\} \subseteq R[t, \mathbf{x}]$ a finite subset such that

1. $g_1, \ldots, g_k$ $\mathbf{x}$-homogeneous of the same $\mathbf{x}$-degree,
2. $\mathrm{LC}_>(g_i) = 1$ for $i = 1, \ldots, k$,
3. $\mathrm{LM}_>(g_i) \neq \mathrm{LM}_>(g_j)$ for $i \neq j$.

Output: $G' = \{g'_1, \ldots, g'_k\} \subseteq R[t, \mathbf{x}]$ such that

1. $g'_1, \ldots, g'_k$ $\mathbf{x}$-homogeneous of the same $\mathbf{x}$-degree,
2. $\mathrm{LT}_>(g'_i) = \mathrm{LT}_>(g_i)$ for $i = 1, \ldots, k$,
3. G' initially reduced w.r.t. itself and $p - t$,
4. $\langle p - t, g_1, \ldots, g_k \rangle = \langle p - t, g'_1, \ldots, g'_k \rangle \trianglelefteq R[\![t]\!][\mathbf{x}]$.

for $i = 1, \ldots, k$ **do**
    Run $g_i := (p - t)$-Reduce$(g_i, >)$.
Reorder $G = \{g_1, \ldots, g_k\}$ such that $\mathrm{LM}_>(g_1) > \ldots > \mathrm{LM}_>(g_k)$, and suppose

$$g_i := \sum_{\alpha \in \mathbb{N}} g_{i,\alpha} \cdot \mathbf{x}^\alpha \text{ with } g_{i,\alpha} \in R[\![t]\!] \text{ and } \mathrm{LT}_>(g_i) = t^{\beta_i} \mathbf{x}^{\alpha_i}.$$

for $i = 1, \ldots, k - 1$ **do**
    for $j = i + 1, \ldots, k$ **do**
        if $g_{j,\alpha_i} \neq 0$ **then**

7: Set

$$g_j := \frac{g_{i,\alpha_i}}{t^{\beta_i}} \cdot g_j - \frac{g_{j,\alpha_i}}{t^{\beta_i}} \cdot g_i.$$

8: Run $g_j := (p-t)$-Reduce$(g_j, >)$.
9: **for** $i = 1, \ldots, k-1$ **do**
10: **for** $j = i+1, \ldots, k$ **do**
11: **if** $t^{\beta_j} \mid g_{i,\alpha_j}$ **then**
12: Set

$$g_i := \frac{g_{j,\alpha_j}}{t^{\beta_j}} \cdot g_i - \frac{g_{i,\alpha_j}}{t^{\beta_j}} \cdot g_j.$$

13: Run $g_i := (p-t)$-Reduce$(g_i, >)$.
14: **return** $G' = \{g_1, \ldots, g_k\}$.

Proof. For the correctness of the instructions note that, by definition and because $>$ is t-local, g_{j,α_j} is divisible by t^{β_j} and g_{i,α_i} is divisible by t^{β_i} in Step 7. From the assumption in Step 11 it follows that g_{i,α_j} in Step 12 will be divisible by t^{β_j}. Observe that due to the reordering in Step 3 and $\mathrm{LM}_>(g_{j,\alpha_i}) \cdot \mathbf{x}^{\alpha_i}$ being a monomial in g_j we have for $i < j$:

$$t^{\beta_i} \cdot \mathbf{x}^{\alpha_i} = \mathrm{LM}_>(g_i) > \mathrm{LM}_>(g_j) > \mathrm{LM}_>(g_{j,\alpha_i}) \cdot \mathbf{x}^{\alpha_i}.$$

Now since $>$ is t-local, t^{β_i} divides $\mathrm{LM}_>(g_{j,\alpha_i})$, hence also g_{j,α_i}.

It is clear that the algorithm terminates since it only consists of a finite number of steps, and, for the correctness, that the output is $\mathbf{x}$-homogeneous, polynomial and generates the same ideal as the input.

Next, we show that the leading terms of the g_i are preserved. Observe that in Step 7 we have $\mathrm{LM}_>(\frac{g_{i,\alpha_i}}{t^{\beta_i}}) = 1$ by definition and $\mathrm{LM}_>(\frac{g_{j,\alpha_j}}{t^{\beta_i}}) < 1$ by the previous argument. Due to the assumption that $\mathrm{LC}_>(g_i) = \mathrm{LC}_>(g_{i,\alpha_i}) = 1$ we therefore have

$$\mathrm{LT}_>(g_j) = \mathrm{LT}_> \left(\frac{g_{i,\alpha_i}}{t^{\beta_i}} \cdot g_j \right)$$

and

$$\mathrm{LM}_>(g_j) > \mathrm{LM}_>(g_{j,\alpha_i}) \cdot \mathbf{x}^{\alpha_i} = \mathrm{LM}_> \left(\frac{g_{j,\alpha_i}}{t^{\beta_i}} \cdot g_i \right).$$

In Step 12 we similarly have $\mathrm{LM}_>(\frac{g_{j,\alpha_j}}{t^{\beta_j}}) = 1$ and $\mathrm{LM}_>(\frac{g_{i,\alpha_j}}{t^{\beta_j}}) \leq 1$, thus

$$\mathrm{LT}_>(g_i) = \mathrm{LT}_> \left(\frac{g_{j,\alpha_j}}{t^{\beta_j}} \cdot g_i \right)$$

and

$$\mathrm{LM}_>(g_i) > \mathrm{LM}_>(g_{i,\alpha_j}) \cdot \mathbf{x}^{\alpha_j} = \mathrm{LM}_> \left(\frac{g_{i,\alpha_j}}{t^{\beta_j}} \cdot g_j \right).$$

On the whole, the leading terms of the $g_1, \dots, g_k$ remain unchanged.

The output is initially reduced w.r.t. $p-t$. For that note that p does neither divide the leading terms as they are monic nor the latter terms because every element of the output was sent through the Algorithm 4.1.

To see that the output G' is initially reduced w.r.t. itself, observe that the first pair of nested `for` loops eliminates all terms in g_j with $\mathbf{x}^{\alpha_i}$ for $i < j$. In particular, each g_j is initially reduced w.r.t. $g_1, \dots, g_{j-1}$.

Additionally, it will stay reduced w.r.t. $g_1, \dots, g_{j-1}$ in the second pair of nested `for` loops, because $g_{j+1}, \dots, g_k$ contain no monomial $\mathbf{x}^{\alpha_i}$, $i < j$, either.

Moreover, once g_i is initially reduced w.r.t. g_j for $i < j$ in Step 12, reducing it initially w.r.t. say g_{j+1} will not change that out of two reasons. First, g_{j+1} contains no term with $\mathbf{x}^{\alpha_j}$, hence adding a multiple of it to g_i is unproblematic. Secondly, $\mathrm{LT}_>(g_{j,\alpha_j}/t^{\beta_j}) = 1$, which means multiplying g_i by it will not change $\mathrm{LT}_>(g_{i,\alpha_j})$. So if t^{β_j} does not divide g_{i,α_j} before, because g_i is initially reduced w.r.t. g_j, it does not divide g_{i,α_j} after as well.

This shows that the constant changes to g_i in the second pair of nested `for` loops are unproblematic. Once g_i has been initially reduced w.r.t. g_j, it will stay that way while being reduced initially w.r.t. $g_{j+1}, \dots, g_k$. $\square$

Example 4.3

Let $p = 2$ and consider the set $G = \{g_1, g_2, g_3\} \subseteq \mathbb{Z}[\![t]\!][x_1, x_2, x_3]$ with

$$\begin{aligned}
g_1 &:= x_1^2 + tx_2^2 - t^2x_3^2,\\
g_2 &:= x_2^2 + tx_1^2 + tx_3^2 + t^2x_3^2 = x_2^2 + tx_1^2 + (t + t^2)x_3^2,\\
g_3 &:= t^3x_3^2 + t^4x_1^2 + t^4x_2^2 + t^5x_2^2 = t^3x_3^2 + t^4x_1^2 + (t^4 + t^5)x_2^2,
\end{aligned}$$

and the weighted ordering $> = >_w$ on $\mathrm{Mon}(t, \mathbf{x})$ with weight vector $(-1, 1, 1, 1) \in \mathbb{R}_{<0} \times \mathbb{R}^3$ and the t-local lexicographical ordering with $x_1 > x_2 > x_3 > 1 > t$ as tiebreaker.

We can illustrate the process with the aid of the following 3×3-matrix:

$$\begin{pmatrix} 1 & t & -t^2 \\ t & 1 & t + t^2 \\ t^4 & t^4 + t^5 & t^3 \end{pmatrix}.$$

The entry in position (i, j) contains the $R[\![t]\!]$-coefficient of g_i w.r.t. the $\mathbf{x}$-monomial in the leading term of g_j.

In the first pass, we begin by taking g_1 and reducing g_2 and g_3 w.r.t. it. To eliminate the term tx_1^2 in g_2 and $t^4x_1^2$ in g_3 we set

$$\begin{aligned}
g_2 &:= g_2 - t \cdot g_1 = (x_2^2 + tx_1^2 + tx_3^2 + t^2x_3^2) - t \cdot (x_1^2 + tx_2^2 - t^2x_3^2) \\
&= (1 - t^2) \cdot x_2^2 + (t + t^2 + t^3) \cdot x_3^2, \\
g_3 &:= g_3 - t^4 \cdot g_1 = (t^3x_3^2 + t^4x_1^2 + (t^4 + t^5)x_2^2) - t^4 \cdot (x_1^2 + tx_2^2 - t^2x_3^2) \\
&= (t^3 + t^6) \cdot x_3^2 + t^4 \cdot x_2^2.
\end{aligned}$$

Note that both g_2 and g_3 remain initially reduced w.r.t. $2 - t$.

$$\begin{tikzcd}
g_1 \arrow[r, bend right] \arrow[rr, bend left] & g_2 & g_3
\end{tikzcd}$$

$$\begin{pmatrix} 1 & t & -t^2 \\ 0 & 1-t^2 & t+t^2+t^3 \\ 0 & t^4 & t^3+t^6 \end{pmatrix}$$

Next, we take g_2 and reduce g_3 w.r.t. it, i.e.

$$\begin{aligned}
g_3 &:= (1 - t^2) \cdot g_3 - t^4 \cdot g_2 \\
&= (1 - t^2) \cdot ((t^3 + t^6)x_3^2 + t^4x_2^2) - t^4 \cdot ((1 - t^2)x_2^2 + (t + t^2 + t^3)x_3^2) \\
&= (t^3 - 2t^5 - t^7 - t^8) \cdot x_3^2.
\end{aligned}$$

And even though g_3 contains a term divisible by 2, it still remains initially reduced w.r.t. $2 - t$.

$$\begin{tikzcd}
g_1 & g_2 \arrow[r, bend left] & g_3
\end{tikzcd}$$

$$\begin{pmatrix} 1 & t & -t^2 \\ 0 & 1-t^2 & t+t^2+t^3 \\ 0 & 0 & t^3-2t^5-t^7-t^8 \end{pmatrix}$$

This concludes our first pass. For the second pass, we begin by taking g_1 and reducing it w.r.t. first g_2 and then g_3. Reducing g_1 w.r.t. g_2 yields

$$\begin{aligned}
g_1 &:= (1 - t^2) \cdot g_1 - t \cdot g_2 \\
&= (1 - t^2) \cdot (x_1^2 + tx_2^2 - t^2x_3^2) - t \cdot ((1 - t^2)x_2^2 + (t + t^2 + t^3)x_3^2) \\
&= (1 - t^2) \cdot x_1^2 + (-2t^2 - t^3) \cdot x_3^2
\end{aligned}$$

and reducing that w.r.t. $2-t$ we obtain

$$g_1 := g_1 - (-t^2 - t^3)x_3^2 \cdot (2-t) = (1-t^2) \cdot x_1^2 - t^4 x_3^2.$$

Reducing g_1 w.r.t. g_3 yields,

$$\begin{aligned} g_1 :&= (1 - 2t^2 - t^4 - t^5) \cdot g_1 - t \cdot g_3 = (1 - 2t^2 - t^4 - t^5)(1-t^2)x_1^2 \\ &= (1 - 3t^2 + t^4 - t^5 + t^6 + t^7) \cdot x_1^2, \end{aligned}$$

which is initially reduced w.r.t. $2-t$.

$$\begin{tikzcd} {\color{red} g_1} & g_2 \arrow[l, bend left] & g_3 \arrow[ll, bend right] \end{tikzcd}$$

$$\begin{pmatrix} 1 - 3t^2 + t^4 - t^5 + t^6 + t^7 & {\color{red}0} & {\color{red}0} \\ 0 & 1 - t^2 & t + t^2 + t^3 \\ 0 & 0 & t^3 - t^6 - t^7 - t^8 \end{pmatrix}$$

Finally, note that while g_2 has a term $t^3 x_3^2$ divisible by the leading term $t^3 x_3$ of g_3, it is still initially reduced w.r.t. g_3. This concludes our second pass and we obtain the initially reduced set

$$\begin{aligned} g_1 &= (1 - 5t^2 + 3t^4 - t^5 + t^6 + t^7) \cdot x_1^2, \\ g_2 &= (1 - t^2) \cdot x_2^2 + (t + t^2 + t^3) \cdot x_3^2, \\ g_3 &= (t^3 - 2t^5 - t^7 - t^8) \cdot x_3^2. \end{aligned}$$

Observe that it is possible to reduce the number of terms at the cost of the coefficient size, by substituting p for some of the t. One alternative initially reduced set with the same leading monomials as above would therefore be

$$g_1 := 165 \cdot x_1^2, \quad g_2 := -3 \cdot x_2^2 + 7t \cdot x_3^2 \quad \text{and} \quad g_3 := -55t^3 \cdot x_3^2.$$

Next, we need to discuss how to reduce a set H of $\mathbf{x}$-homogeneous elements of the same degree in $\mathbf{x}$ w.r.t. themselves and a set G of $\mathbf{x}$-homogeneous elements of lower degree. The simplest way is multiplying the elements of G up to the same degree in $\mathbf{x}$ as the elements of H in all possible combinations and using Algorithm 4.2 on the resulting set. This resembles a brute force method in which we directly summon the worst case scenario to be resolved. A more sophisticated method multiplies the elements of G up to the same degree in $\mathbf{x}$ as the elements of H when they are needed. In the optimal case, we can reduce the complexity drastically with this strategy, in the worst case we are only delaying the inevitable.

Algorithm 4.4 (initial reduction, step by step)

Input: $(G, H, >)$, where $>$ a t-local monomial ordering, $H = \{h_1, \dots, h_k\}$ and G finite subsets of $R[t, \mathbf{x}]$ such that

1. $h_1, \dots, h_k$ are $\mathbf{x}$-homogeneous of the same $\mathbf{x}$-degree d,
2. all $g \in G$ are $\mathbf{x}$-homogeneous of $\mathbf{x}$-degree less than d,
3. $\mathrm{LC}_>(h_i) = 1$, for $i = 1, \dots, k$, and $\mathrm{LC}_>(g) = 1$ for all $g \in G$,
4. $\mathrm{LM}_>(h_i) \neq \mathrm{LM}_>(h_j)$ for $i \neq j$,
5. $\mathrm{LM}_>(h_i) \notin \langle \mathrm{LM}_>(g) \mid g \in G \rangle$ for $i = 1, \dots, k$.

Output: $H' = \{h'_1, \dots, h'_k\} \subseteq R[t, \mathbf{x}]$ such that

1. $h'_1, \dots, h'_k$ are $\mathbf{x}$-homogeneous of the same $\mathbf{x}$-degree d,
2. $\mathrm{LT}_>(h'_i) = \mathrm{LT}_>(h_i)$ for $i = 1, \dots, k$,
3. H' initially reduced w.r.t. G and itself,
4. $\langle p - t, G, H \rangle = \langle p - t, G, H' \rangle \trianglelefteq R[\![t]\!][\mathbf{x}]$.

Reduce H initially using Algorithm 4.2 and set $E = \emptyset$.
Suppose $h_i = \sum_{\alpha \in \mathbb{N}^n} h_{i,\alpha} \cdot \mathbf{x}^\alpha$ with $h_{i,\alpha} \in R[\![t]\!]$, create the disjoint union

$$T := \{(\mathrm{LT}_>(h_{i,\alpha}) \cdot \mathbf{x}^\alpha, i) \mid \alpha \in \mathbb{N}^n \text{ and } \mathrm{LT}_>(h_{i,\alpha}) \cdot \mathbf{x}^\alpha < \mathrm{LT}_>(h_i)\},$$

a working list of terms to be checked for potential reduction w.r.t. G.
while $T \neq \emptyset$ **do**
  Pick $(s, i) \in T$ with $\mathrm{LM}_>(s)$ maximal.
  if $\mathrm{LT}_>(g) \mid s$ for some $g \in G$ **then**
    Pick $g \in G$, $\mathrm{LT}_>(g) \mid s$, and set $E := E \cup \left\{ \frac{\mathrm{LM}_>(s)}{\mathrm{LM}_>(g)} \cdot g \right\}$.
    Reduce $H \cup E$ initially using Algorithm 4.2.
    Update the working list:

$$T := \{(\mathrm{LT}_>(h_{i,\alpha}) \cdot \mathbf{x}^\alpha, i) \mid \alpha \in \mathbb{N}^n \text{ and } \mathrm{LM}_>(h_{i,\alpha}) \cdot \mathbf{x}^\alpha < \mathrm{LM}_>(s)\}.$$

  else
    Set $T := T \setminus \{(h_i, s)\}$.
return H

Proof. For the termination note that in each iteration of the while loop either the set of extra polynomials E increases or the working list T decreases. Also because each s is chosen to be maximal, each other term in the working list T with the same $\mathbf{x}$-monomial must have a higher t-degree and is therefore eliminated alongside s in the initial reduction of $H \cup E$. Because the updated T only includes relevant terms smaller than s, the $\mathbf{x}$-monomial of s is effectively eliminated in all working lists to follow. Hence each elements of E will always have a distinct $\mathbf{x}$-monomial which is of degree d. Thus E has a maximal size after which the algorithm will terminate in a finite number of steps.

For the correctness of the instructions, observe that $H \cup E$ satisfies the conditions for Algorithm 4.2 by assumption. For the correctness of the output, it is obvious that the leading terms of H are preserved, that H is initially reduced w.r.t. itself and that its elements are $\mathbf{x}$-homogeneous as well as polynomial. To show that H is initially reduced w.r.t. G, observe that, apart from the terms eliminated, any term altered in the initial reduction of $H \cup E$ is strictly smaller than s. Because s was chosen to be maximal, the updated working list therefore contains all relevant terms that have been altered or that have yet to be checked for reduction. Thus in the output any relevant term has been negatively checked for divisibility by an element of G. $\square$

Remark 4.5

Note that in Step 6 of Algorithm 4.4, we multiply g by a power of t even though it is not necessary for correctness. The reason is as follows:

Recall Algorithm 4.2, which consists of two big nested `for` loops. In the first pass from Step 4 to 8 we take each g_i, $i = 1, \ldots, k-1$, and reduce all g_j, $i < j$, w.r.t. it. In the second pass from Step 9 to 13 we take each g_i, $i = 1, \ldots, k-1$, and reduce it w.r.t. all g_j, $i < j$.

Now suppose we enter the Algorithm with $H \cup \{g\}$, where $H = \{h_1, \ldots, h_k\}$ is already initially reduced w.r.t. itself and $p - t$. Suppose furthermore

$$\mathrm{LM}_>(h_1) > \ldots > \mathrm{LM}_>(h_l) > \mathrm{LM}_>(g) > \mathrm{LM}_>(h_{l+1}) > \ldots > \mathrm{LM}_>(h_k).$$

By assumption, taking each h_i, $i = 1, \ldots, l$, and reducing all h_j, $i < j$, w.r.t. it is obsolete. The first necessary action is reducing g w.r.t. $h_1, \ldots, h_l$.

$$\begin{tikzcd}
h_1 \arrow[rrrr, bend left] & h_2 \arrow[rrr, bend left] & \ldots & h_l \arrow[r] & g & h_{l+1} & \ldots & h_{k-1} & h_k
\end{tikzcd}$$

Next, we consider the h_i, $i = l+1, \dots, k$. Each h_i is already reduced w.r.t. $h_1, \dots, h_l$ and remains so after reducing it w.r.t. g, as the $\mathbf{x}$-monomials of their leading monomials were already completely eliminated in g previously. Hence we may reduce each h_i, $i = l+1, \dots, k$, w.r.t. g without inducing the need of reducing them w.r.t. $h_1, \dots, h_l$ again.

$$\begin{tikzcd}
h_1 & h_2 & \dots & h_l & g \arrow[r] \arrow[rrr, bend right] \arrow[rrrr, bend right] & h_{l+1} & \dots & h_{k-1} & h_k
\end{tikzcd}$$

However, g might contain a term with monomial t^2x, which might not be reducible w.r.t. $\mathrm{LT}_>(h_j) = t^3x$, but if g is multiplied by t while reducing another element w.r.t. it, we do create a term that is reducible. Thus, we need to reduce each h_i, $i = l+1, \dots, k$ w.r.t. h_j, $j = l+1, \dots, i$ again and this concludes our first pass.

$$\begin{tikzcd}
h_1 & h_2 & \dots & h_l & g & h_{l+1} \arrow[r] \arrow[rr, bend left] \arrow[rrr, bend left] & \dots \arrow[r] & h_{k-1} \arrow[r] & h_k
\end{tikzcd}$$

For the second pass, taking each h_i, $i = 1, \dots, l$, and reducing it w.r.t. all h_j, $j = i+1, \dots, l$, is unnecessary. The first necessary step is to take each h_i, $i = 1, \dots, l$, and reduce it w.r.t. the newly added g. Similar to a previous step, each h_i remains reduced w.r.t. all h_j, $j = i+1, \dots, l$.

$$\begin{tikzcd}
h_1 & h_2 & \dots & h_l & g \arrow[l] \arrow[lll, bend left] \arrow[llll, bend left] & h_{l+1} & \dots & h_{k-1} & h_k
\end{tikzcd}$$

Afterwards, while each h_i remains reduced w.r.t. all h_j, $j = i+1, \dots, l$, it nonetheless needs to be reduced w.r.t. $h_{l+1}, \dots, h_k$ again.

$$\begin{tikzcd}
h_1 & h_2 & \dots & h_l & g & h_{l+1} \arrow[lllll, bend right] \arrow[llll, bend left] & \dots & h_{k-1} \arrow[lllllll, bend right] \arrow[llllll, bend left] & h_k \arrow[llllllll, bend right] \arrow[lllllll, bend left]
\end{tikzcd}$$

Next in the second pass, we take g and reduce it w.r.t. $h_{l+1}, \dots, h_k$.

$$
\begin{tikzcd}[column sep=small]
h_1 & h_2 & \dots & h_l & g & h_{l+1} \arrow[l, bend right] & \dots & h_{k-1} \arrow[lll, bend right] & h_k \arrow[llll, bend right]
\end{tikzcd}
$$

And finally, we take each h_i, $i = l+1, \dots, k-1$ and reduce it w.r.t. all h_j, $i < j$, as reducing them w.r.t. g earlier might have broken their reducedness property.

$$
\begin{tikzcd}[column sep=small]
h_1 & h_2 & \dots & h_l & g & h_{l+1} & \dots \arrow[l, bend right] & h_{k-1} \arrow[l, bend left] \arrow[ll, bend right] & h_k \arrow[l, bend left] \arrow[lll, bend right]
\end{tikzcd}
$$

It can be seen that a position of g more to the right minimises the number of reductions needed. This implies that $\mathrm{LM}_>(g)$ should be as small as possible, and since its monomial in x is fixed, this means that it should have as high a degree in t as possible.

Note that increasing the degree in t to increase performance is not risk-free a priori. For example, suppose we had a $g \in G$ with $\mathrm{LT}_>(g) = x$ and we were to add $t^5 y \cdot g$ to E in order to reduce a term with monomial $t^5 xy$. Then any subsequent term with monomial $t^4 xy$ would require adding an additional multiple of g to E. However, since our working list T is worked off in an order induced by a t-local monomial ordering $>$, any later s' picked in Step 4 with the same monomial in x necessarily has to have a higher degree in t. Thus this cannot happen in our algorithm.

With Algorithm 4.4, writing an algorithm for computing an initially reduced standard basis becomes a straightforward task. All we need to adhere is to proceed $\mathbf{x}$-degree by $\mathbf{x}$-degree while repeatedly applying the previous algorithm.

Algorithm 4.6 (initially reduced standard basis)

Input: $(F, >)$, where $F \subset I$ an $\mathbf{x}$-homogeneous, polynomial generating set of I containing $p - t$.

Output: $G \subseteq I$ an $\mathbf{x}$-homogeneous, polynomial and initially reduced standard basis of I.

Compute an $\mathbf{x}$-homogeneous standard basis G'' of $I = \langle F \rangle$ with [16, Alg. 2.16 or Alg. 3.8].

Set $G' := \emptyset$.

for $g \in G''$ with $p \nmid \mathrm{LT}_>(g)$ **do**

if $\mathrm{LC}_>(g) \neq 1$ **then**

Since $1 \in \langle \mathrm{LC}_>(g), p\rangle$, find $a, b \in R$ such that

$$1 = a \cdot \mathrm{LC}_>(g) + b \cdot p.$$

Set

$$g := a \cdot g + b \cdot \mathrm{LM}_>(g) \cdot (p - t),$$

so that $\mathrm{LC}_>(g) = 1$.

Set $G' := G' \cup \{g\}$.

Minimise the standard basis G' by gradually removing elements $g \in G$ with $\mathrm{LM}_>(g') \mid \mathrm{LM}_>(g)$ for some $g' \in G$, $g' \neq g$.

Set $G := \emptyset$

while $G' \neq \emptyset$ **do**

Set

$$\begin{aligned} d &:= \min\{\deg_{\mathbf{x}}(g) \mid g \in G'\}, \\ H' &:= \{g \in G' \mid \deg_{\mathbf{x}}(g) = d\}, \\ G' &:= \{g \in G' \mid \deg_{\mathbf{x}}(g) > d\}. \end{aligned}$$

Reduce H' initially w.r.t. G, $p - t$ and itself using Algorithm 4.4 and let H be the output of that initial reduction.

Set $G := G \cup H$.

return $G \cup \{p - t\}$.

Proof. It is clear that G is a standard basis of I, as we are merely normalising the leading coefficients of the standard bases G''. It is also obvious that G is polynomial and $\mathbf{x}$-homogeneous. The initial reducedness of G follows from the correctness of Algorithm 4.4. $\square$

5. How to compute the Gröbner fan

In this section, we describe algorithms for computing the Gröbner fan of an ideal $I \trianglelefteq R[\![t]\!][\mathbf{x}]$ as in our convention on Page 4, provided that we are able to compute initially reduced standard bases where needed. While computing a Gröbner fan can be as seemingly simple as computing maximal Gröbner cones $C_>(I)$ w.r.t. random monomial orderings $>$ until the whole weight space $\mathbb{R}_{\leq 0} \times \mathbb{R}^n$ is filled, sensible algorithms avoid computing initially reduced standard bases of I from scratch. The algorithms in this section are adjusted versions of the algorithms found in Chapter 4 of Jensen's dissertation [21] (see also [17]), though some of the ideas involved originate in Collart, Kalkbrenner and Mall's work on the Gröbner walk [2].

We start with an algorithm for computing witnesses of weighted homogeneous elements in initial ideals, which can then be used to lift standard bases of initial ideals to initially reduced standard bases of the original ideal. Adding in some statements about the perturbation of initial ideals, we obtain an algorithm which allows us to flip initially reduced standard bases of one ordering to initially reduced standard bases of an adjacent ordering. This algorithm can then be used to construct the Gröbner fan, requiring us to compute the standard basis of I from scratch only once.

Note that all polynomial computations in our algorithms, if given polynomial input, terminate and return polynomial output themselves, provided that we are able to initially reduce a standard basis as e.g. in Algorithm 4.6.

Algorithm 5.1 (Witness)
Input: $(h, H, G, >)$, where

- $>$ a weighted t-local monomial ordering on $\mathrm{Mon}(t, \mathbf{x})$,
- $G = \{g_1, \dots, g_k\} \subseteq I$ an initially reduced standard basis of I w.r.t. $>$,
- $H = \{h_1, \dots, h_k\}$ with $h_i = \mathrm{in}_w(g_i)$ for some $w \in C_>(I)$ with $w_0 < 0$,
- $h \in \mathrm{in}_w(I)$ weighted homogeneous w.r.t. w.

Output: $f \in I$ such that $\mathrm{in}_w(f) = h$

1: Use [16, Alg. 1.13] to compute a homogeneous determinate division with remainder w.r.t. $>$,

$$(\{q_1, \dots, q_k\}, r) = \mathrm{HDDwR}(h, \{h_1, \dots, h_k\}, >),$$

so that $h = q_1 \cdot h_1 + \ldots + q_k \cdot h_k$ and $r = 0$.

2: **return** $f := q_1 \cdot g_1 + \ldots + q_k \cdot g_k$

Proof. By Proposition 3.8, H is a standard basis of $\mathrm{in}_w(I)$, therefore the division of h will always yield remainder 0.

Since $h, h_1, \dots, h_k$ are weighted homogeneous w.r.t. w, so are $q_1, \dots, q_k$. Hence

$$\mathrm{in}_w(f) = \underbrace{\mathrm{in}_w(q_1) \cdot \mathrm{in}_w(g_1)}_{=q_1 \cdot h_1} + \ldots + \underbrace{\mathrm{in}_w(q_k) \cdot \mathrm{in}_w(g_k)}_{=q_k \cdot h_k} = h.$$

Also note that the division with remainder will always terminate, as the weighted degree cannot become arbitrarily small since the ideal $\mathrm{in}_w(I)$ is homogeneous in $\mathbf{x}$ and weighted homogeneous overall. $\square$

As announced, we immediately obtain an algorithm which allows us to lift a standard basis of an initial ideal to an initially reduced standard basis of I, assuming we have a standard basis of I w.r.t. an adjacent ordering at our disposal.

Algorithm 5.2 (Lift)

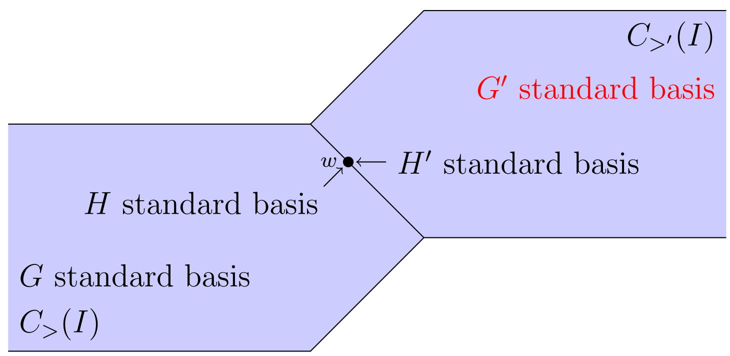

Figure 6: lift of standard bases

Input: $(H', >', H, G, >)$, where

- $>$ a weighted t-local monomial ordering on $\mathrm{Mon}(t, \mathbf{x})$ with weight vector in $\mathbb{R}_{<0} \times \mathbb{R}^n$,
- $G = \{g_1, \ldots, g_k\} \subseteq I$ an initially reduced standard basis of I w.r.t. $>$,
- $H = \{h_1, \ldots, h_k\}$ with $h_i = \mathrm{in}_w(g_i)$ for some $w \in C_>(I)$ with $w_0 < 0$,
- $>'$ a t-local monomial ordering such that $w \in C_>(I) \cap C_{>'}(I)$,
- $H' \subseteq \mathrm{in}_w(I)$ a weighted homogeneous standard basis w.r.t. $>'$.

Output: $G' \subseteq I$, an initially reduced standard basis of I w.r.t. $>'$.

Set $G'' := \{\mathrm{Witness}(h, H, G, >) \mid h \in H'\}$.
Reduce G'' initially w.r.t. $>'$ and obtain G'.
return G'.

Proof. Consider a witness $g := \text{Witness}(h, w, G, >)$ for some $h \in H'$. Then, by Lemma 3.6, we have $\text{LT}_{>'}(g) = \text{LT}_{>'}(\text{in}_w(g)) = \text{LT}_{>'}(h)$, and thus

$$\langle \text{LT}_{>'}(g) \mid g \in G'' \rangle = \langle \text{LT}_{>'}(h) \mid h \in H' \rangle = \text{LT}_{>'}(\text{in}_w(I)) \overset{\text{Lem.}}{\underset{3.7}{=}} \text{LT}_{>'}(I).$$

Thus G'' is a standard basis of I w.r.t. $>'$ and G' is even initially reduced. $\square$

Example 5.3
Consider again the ideal from Example 3.5

$$I = \langle g_1 = x - t^3x + t^3z - t^4z, g_2 = y - t^3y + t^2z - t^4z \rangle \trianglelefteq \mathbb{Z}[\![t]\!][x, y, z]$$

and the weighted monomial ordering $> = >_v$ on $\text{Mon}(t, x, y, z)$ with weight vector $v = (-1, 3, 3, 3) \in \mathbb{R}_{<0} \times \mathbb{R}^3$ and the t-local lexicographical ordering such that $x > y > z > 1 > t$ as tiebreaker. We have already seen that

$$C_>(I) = \overline{\{w \in \mathbb{R}_{<0} \times \mathbb{R}^n \mid w_1 \geq 3w_0 + w_3 \text{ and } w_2 \geq 2w_0 + w_3\}}.$$

Picking $w = (-1, 2, -1, 1)$ in a facet of $C_>(I)$, Proposition 3.8 implies

$$\text{in}_w(I) = \langle \text{in}_w(g_1), \text{in}_w(g_2) \rangle = \langle x, y + t^2z \rangle.$$

It is easy to see that $\{x, y + t^2z\}$ is a standard basis of $\text{in}_w(I)$ regardless which monomial ordering is chosen. Since using Algorithm 5.1 on $\text{in}_w(g_1)$ and $\text{in}_w(g_2)$ yields g_1 and g_2 respectively, Algorithm 5.2 therefore implies that $\{g_1, g_2\}$ is also a standard basis for the adjacent monomial ordering $>'$ on the other side of the facet containing w.

Moreover, since $>'$ has to induce a different leading ideal by definition, and the leading terms of g_1 and g_2 w.r.t. $>'$ have to occur in their initial forms by Lemma 3.6, we see that the adjacent leading ideal is $\langle x, t^2z \rangle$.

An easy way to construct orderings adjacent to $>$ is by connecting two weight vectors in series, the first a weight vector lying on a facet and the second an outer normal vector of the facet.

Proposition 5.4
Let $>$ be a t-local monomial ordering, $w \in C_>(I)$ with $w_0 < 0$ and $v \in \mathbb{R}^{n+1}$.

Let $>_{(w,v)}$ *denote the* t*-local monomial ordering given by*

$$\begin{aligned} t^\beta \cdot \mathbf{x}^\alpha >_{(w,v)} t^{\beta'} \cdot \mathbf{x}^{\alpha'} \quad &:\Longleftrightarrow \\ &(\beta,\alpha)\cdot w > (\beta',\alpha')\cdot w, \\ &\text{or } (\beta,\alpha)\cdot w = (\beta',\alpha')\cdot w \text{ and } (\beta,\alpha)\cdot v > (\beta',\alpha')\cdot v, \\ &\text{or } (\beta,\alpha)\cdot w = (\beta',\alpha')\cdot w \text{ and } (\beta,\alpha)\cdot v = (\beta',\alpha')\cdot v \\ &\qquad\qquad \text{and } t^\beta \cdot \mathbf{x}^\alpha > t^{\beta'} \cdot \mathbf{x}^{\alpha'}. \end{aligned}$$

Then $w = C_>(I) \cap C_{>_{(w,v)}}(I)$ *and for* $\varepsilon > 0$ *sufficiently small*

$$w + \varepsilon \cdot v \in C_{>_{(w,v)}}(I).$$

In particular for these ε *we have* $\mathrm{in}_{w+\varepsilon v}(I) = \mathrm{in}_v(\mathrm{in}_w(I))$.

Proof. By definition we have $\mathrm{LT}_{>_{(w,v)}}(g) = \mathrm{LT}_{>_{(w,v)}}(\mathrm{in}_w(g))$ for any $g \in R[\![t]\!][\mathbf{x}]$, which implies $w \in C_{>_{(w,v)}}(I)$ by Lemma 3.6.

Next, let G be an initially reduced standard basis of I w.r.t. that ordering. Observe that every $g \in G$,

$$g = \underbrace{\ldots\ldots\ldots\ldots\ldots}_{\mathrm{in}_w(g)} + \underbrace{\ldots\ldots\ldots\ldots\ldots}_{\text{rest}},$$

has a distinct degree gap between the terms of highest weighted degree and the rest. As the weighted degree varies continuously under the weight vector, choosing $\varepsilon > 0$ sufficiently small ensures that the $(w+\varepsilon\cdot v)$-weighted degrees of the terms in $\mathrm{in}_w(g)$ remain higher than those of the rest. Thus $\mathrm{in}_{w+\varepsilon\cdot v}(g)$ is the sum of those terms of $\mathrm{in}_w(g)$ that have maximal v-weighted degree, i.e. $\mathrm{in}_{w+\varepsilon\cdot v}(g) = \mathrm{in}_v(\mathrm{in}_w(g))$. In particular, we have

$$\mathrm{LT}_{>_{(w,v)}}(\mathrm{in}_{w+\varepsilon\cdot v}(g)) = \mathrm{LT}_{>_{(w,v)}}(g),$$

and hence $w + \varepsilon \cdot v \in C_{>_{(w,v)}}(I)$ by Lemma 3.6 again.

The final claim now follows from Proposition 3.8:

$$\mathrm{in}_{w+\varepsilon\cdot v}(I) \overset{\text{Prop.}}{\underset{3.8}{=}} \langle \mathrm{in}_{w+\varepsilon\cdot v}(g) \mid g \in G\rangle = \langle \mathrm{in}_v(\mathrm{in}_w(g)) \mid g \in G\rangle \overset{\text{Prop.}}{\underset{3.8}{=}} \mathrm{in}_v(\mathrm{in}_w(I)).$$

□

With this easy method of constructing adjacent orderings, we are now able to write an algorithm for flipping initially reduced standard bases.

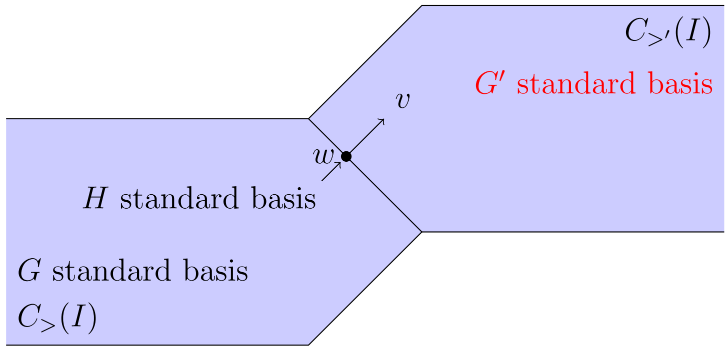

Figure 7: flip of standard bases

Algorithm 5.5 (Flip)
Input: $(G, H, v, >)$, where

- $>$ a weighted t-local monomial ordering on $\mathrm{Mon}(t, \mathbf{x})$ with weight vector in $\mathbb{R}_{<0} \times \mathbb{R}^n$,
- $G = \{g_1, \ldots, g_k\} \subseteq I$ an initially reduced standard basis of I w.r.t. $>$,
- $H = \{h_1, \ldots, h_k\}$ with $h_i = \mathrm{in}_w(g_i)$ for some relative interior point $w \in C_>(I)$ on a lower facet $\tau \leq C_>(I)$, $\tau \nsubseteq \{0\} \times \mathbb{R}^n$ and $w_0 < 0$.
- $v \in \mathbb{R} \times \mathbb{R}^n$ an outer normal vector of the facet τ.

Output: $(G', >')$, where $>'$ is an adjacent t-local monomial ordering with

$$\tau = C_>(I) \cap C_{>'}(I) \quad \text{and} \quad C_>(I) \neq C_{>'}(I),$$

and $G' \subseteq I$ is an initially reduced standard basis w.r.t. $>'$.
1: Compute a standard basis H' of $\langle H \rangle = \mathrm{in}_w(I)$ w.r.t. $>_{(w,v)}$.
2: Set $G' := \mathrm{Lift}(H', >_{(w,v)}, H, G, >)$.
3: **return** $(G', >_{(w,v)})$

Proof. By our Lifting Algorithm 5.2, G' is an initially reduced standard basis of I w.r.t. $>_{(w,v)}$. The remaining conditions follow from Proposition 5.4. $\square$

Example 5.6
Consider the ideal

$$I := \langle 2 - t, xy^2 - t^2y^3, x^2 - t^3y^2 \rangle \trianglelefteq \mathbb{Z}[\![t]\!][x, y]$$

and the weighted monomial ordering $>=>_u$ on $\mathrm{Mon}(t,x,y)$ with weight vector $u := (-1,1,1) \in \mathbb{R}_{<0} \times \mathbb{R}^2$ and t-local lexicographical ordering such that $x > y > 1 > t$ as tiebreaker. An initially reduced standard basis of I is then given by

$$G := \{2-t, xy^2 - t^2y^3, x^2 - t^3y^2, t^3y^4\}.$$

The maximal Gröbner cone $C_>(I) \subseteq \mathbb{R}_{\leq 0} \times \mathbb{R}^2$ is determined by the inequalities

$$\begin{aligned}(w_t, w_x, w_y) \in C_>(I) \quad &\Longleftrightarrow \quad \begin{cases} w_x + 2w_y \geq 2w_t + 3w_y \\ 2w_x \geq 3w_t + 2w_y \end{cases} \\ &\Longleftrightarrow \quad \begin{cases} w_x \geq 2w_t + w_y \\ 2w_x \geq 3w_t + 2w_y \end{cases}\end{aligned}$$

It is easy to see how $w := (-4,1,7)$ is contained in $C_>(I)$. In fact, it lies on its boundary since $2w_x = 3w_t + 2w_y = 2$. Then $v := (3,5,1) \in \mathbb{R}^3$ is an outer normal vector, as even for small $\varepsilon > 0$

$$\underbrace{2(w_x + \varepsilon \cdot v_x)}_{2+10\varepsilon} \ngeq \underbrace{3(w_t + \varepsilon \cdot v_t)}_{-12+9\varepsilon} + \underbrace{2(w_y + \varepsilon \cdot v_y)}_{14+2\varepsilon}.$$

An initially reduced standard basis of $\mathrm{in}_w(I)$ is then given by

$$H := \{\mathrm{in}_w(g) \mid g \in G\} = \{2, xy^2, x^2 - t^3y^2, t^3y^4\},$$

and computing a standard basis of $\mathrm{in}_w(I)$ w.r.t. the ordering $>_{(w,v)}$ yields

$$H' := \{2, xy^2, t^3y^2 - x^2, x^3\},$$

which can then be lifted to a standard basis of I w.r.t. the same ordering $>_{(w,v)}$ that is adjacent to $>$

$$G' = \{2-t, xy^2 - t^2y^3, t^3y^2 - x^2, x^3 - t^5y^3\}.$$

The Gröbner fan algorithm is a so-called fan traversal algorithm. We start with computing a starting cone and repeatedly use Algorithm 5.5 to compute adjacent cones until we obtain the whole fan. The whole process is commonly illustrated on a bipartite graph as shown in Figure 8. This bipartite graph also satisfies the so-called reverse search property, which can

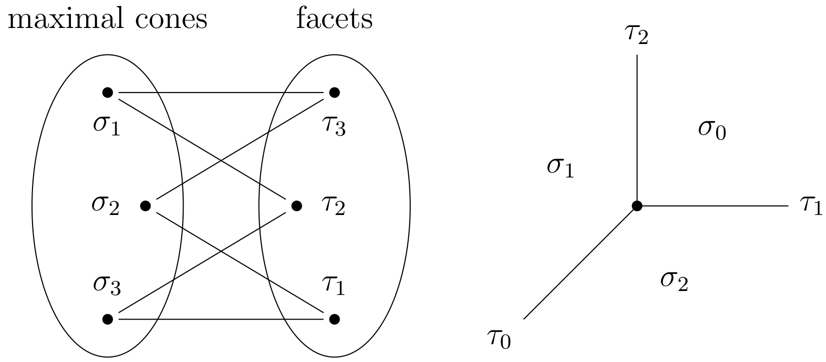

Figure 8: The bipartite graph of a Gröbner fan $\Sigma(\langle x+y+z\rangle)$

be used for further optimisation. See Chapter 3.2 in [21] for more information about the reverse search property of Gröbner fans.

Note that since the Gröbner fan spans the whole weight space $\mathbb{R}_{\leq 0} \times \mathbb{R}^n$, each lower facet is contained in exactly two maximal cones. That means, traversing a facet $\tau \leq C_>(I)$ of a Gröbner cone $C_>(I)$ can be omitted if τ is contained in a maximal Gröbner cone that was already computed.

Algorithm 5.7 (Gröbner fan)
Input: $F \subseteq I \trianglelefteq R[\![t]\!][\mathbf{x}]$ an $\mathbf{x}$-homogeneous generating set.
Output: Δ, the set of maximal cones in the Gröbner fan $\Sigma(I)$ of I.

Pick a random weight $u \in \mathbb{R}_{<0} \times \mathbb{R}^n$ and a t-local monomial ordering $>$.
Compute an initially reduced standard basis G of I w.r.t. $>_u$.
Construct the maximal Gröbner cone $C_{>_u}(I) = C(\mathrm{LT}_{>_u}(G), G, >_u)$ using Algorithm 3.21.
Initialise the Gröbner fan $\Sigma := \{C_{>_u}(I)\}$.
Initialise a working list $L := \{(G, >_u, C_{>_u}(I))\}$.
while $L \neq \emptyset$ **do**
  Pick $(G, >_u, C_{>_u}(I)) \in L$.
  for all facets $\tau \leq C_{>_u}(I)$, $\tau \nsubseteq \{0\} \times \mathbb{R}^n$ **do**
    Compute a relative interior point $w \in \tau$.
    if $w \notin C_{>'}(I)$ for all $C_{>'}(I) \in \Sigma \setminus \{C_{>_u}(I)\}$ **then**
      Compute an outer normal vector v of τ.
      Set $H := \{\mathrm{in}_w(g) \mid g \in G\}$.
      Compute $(G', >') := \mathrm{Flip}(G, H, v, >_u)$ using Algorithm 5.5.
      Construct the adjacent cone $C_{>'}(I) = C(\mathrm{LT}_{>'}(G'), G', >')$.

Compute a relative interior point $u' \in C_{>'}(I)$, so that G' is a standard basis w.r.t. $>_{u'}$ and $C_{>'}(I) = C_{>_{u'}}(I)$.
Set $\Sigma := \Sigma \cup \{C_{>'_u}(I)\}$.
Set $L := L \cup \{(G', >_{u'}, C_{>_{u'}}(I)\}$.
Set $L := L \setminus \{(G, >_u, C_{>_u}(I))\}$.
return Δ

Example 5.8
For an easy but clear example, consider the ideal

$$I := \langle x + z, y + z \rangle \trianglelefteq \mathbb{Z}[\![t]\!][x, y, z].$$

Because it is weighted homogeneous w.r.t. $(-1, 0, 0, 0) \in \mathbb{R}_{<0} \times \mathbb{R}^3$ and $(0, 1, 1, 1) \in \{0\} \times \mathbb{R}^3$, its Gröbner fan is closed under translation by $(-1, 0, 0, 0)$ and invariant under translation by $(0, 1, 1, 1)$. We therefore, concentrate on weight vectors on the hyperplane $\{0\} \times \mathbb{R}^2 \times \{0\}$, since any other weight vector in the closed lower halfspace can be generated out of them via the translations.

Looking only at potential leading terms of the generators, one might be led to believe that the Gröbner fan $\Sigma(I)$ restricted to $\{0\} \times \mathbb{R}^2 \times \{0\}$ is of the form as shown in Figure 9

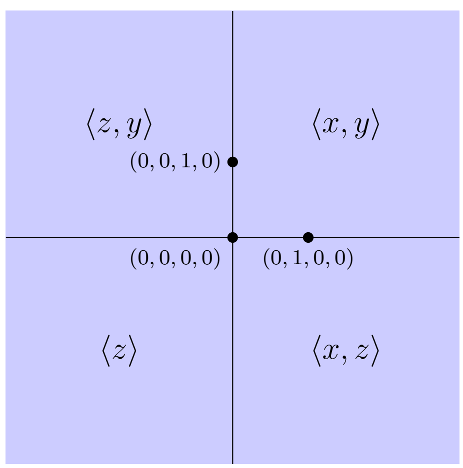

Figure 9: The Gröbner fan $\Sigma(I)$ restricted to $\{0\} \times \mathbb{R}^2 \times \{0\}$

Let us use our algorithm to see why this is not the case. We start with a random weight vector u, say $u = (0, 1, 1, 0)$, and a random t-local monomial ordering $>$ to be used as tiebreaker. Then $\{\underline{x} + z, \underline{y} + z\}$ already is an initially reduced standard basis w.r.t. $>_u$, leading terms underlined, so that by Lemma 3.6

$$w' \in C_u(I) \quad \Longleftrightarrow \quad \begin{cases} \deg_{w'}(x) \geq \deg_{w'}(z) = 0, \\ \deg_{w'}(y) \geq \deg_{w'}(z) = 0. \end{cases}$$

Hence, $C_u(I)$ is the upper left quadrant of the image above, with two facets available for the traversal. Picking τ to be the upper ray of $C_u(I)$, $w =$

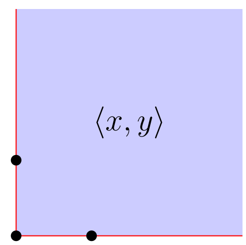

Figure 10: The first cone in the restricted Gröbner fan

$(0, 0, 1, 0)$ a relative interior point inside it and $v = (0, -1, 0, 0)$ an outer normal vector on it, we see that $\mathrm{in}_w(x+z) = \underline{z}+x$ and $\mathrm{in}_w(y+z) = \underline{y}$ already form an initially reduced standard basis of $\mathrm{in}_w(I)$ w.r.t. $>_{(w,v)}$. Therefore, this standard basis of $\mathrm{in}_w(I)$ lifts again to the very same standard basis $\{\underline{z} + x, \underline{y} + z\}$ of I for the adjacent ordering.

However that standard basis is not initially reduced anymore, and a quick calculation yields the initially reduced standard basis $\{\underline{z}+x, \underline{y}-x\}$ and hence

$$w' \in C_{>_{(w,v)}}(I) \quad \Longleftrightarrow \quad \begin{cases} 0 = \deg_{w'}(z) \geq \deg_{w'}(x), \\ \deg_{w'}(y) \geq \deg_{w'}(x). \end{cases}$$

Let τ be the lower ray of our new Gröbner cone (see Figure 11), $w = (0, -1, -1, 0)$ a relative interior point and $v = (0, 1, -1, 0)$ an outer normal vector. We see that $\mathrm{in}(z + x) = \underline{z}$ and $\mathrm{in}_w(y - x) = -\underline{x} + y$ already form an initially reduced standard basis of $\mathrm{in}_w(I)$ w.r.t. $>_{(w,v)}$, which is why it will lift again to the same standard basis $\{\underline{z} + x, -\underline{x} + y\}$ of I for the adjacent ordering.

As before, this standard basis is not initially reduced anymore, and a quick calculation yields the initially reduced standard basis $\{\underline{z} + y, -\underline{x} + y\}$,

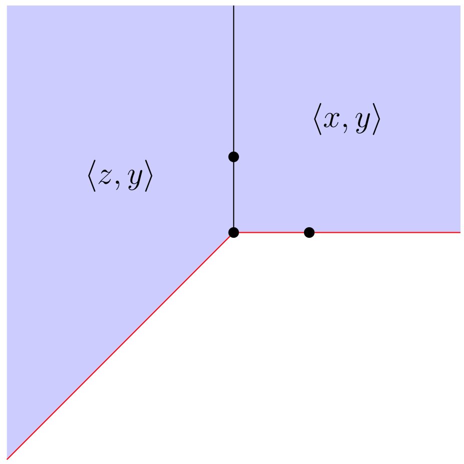

Figure 11: The first two cones in the restricted Gröbner fan

which means

$$w' \in C_{>_{(w,v)}}(I) \quad \Longleftrightarrow \quad \begin{cases} 0 = \deg_{w'}(z) \geq \deg_{w'}(y), \\ \deg_{w'}(x) \geq \deg_{w'}(y). \end{cases}$$

Figure 12 then shows how the Gröbner fan $\Sigma(I)$ actually looks like. The

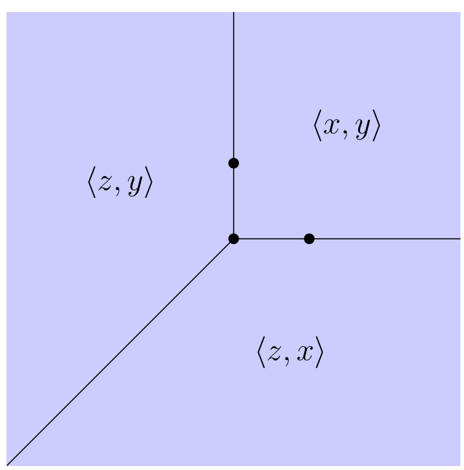

Figure 12: The Gröbner fan $\Sigma(I)$ restricted to $\{0\} \times \mathbb{R}^2 \times \{0\}$

misconception at the beginning of the example was due to the oversight that $\mathrm{in}_w(x+z) = \mathrm{in}_w(y+z) = z$ do not generate $\mathrm{in}_w(I)$, because $\{x + \underline{z}, y + \underline{z}\}$ is no initially reduced standard basis for $>_w$.

Remark 5.9
As we have already remarked, our main interest lies in the computation of tropical varieties over the p-adic numbers (see e.g. Section 4). For this we assume that $R = \mathbb{Z}$ and I contains the polynomial $p - t$ for some prime number p. At the beginning of this section we have mentioned that the traversal algorithm has the advantage that a standard basis of the ideal has to computed from scratch only once. All intermediate steps comprise of computing standard bases of initial ideals (see Algorithm 5.5), which are much simpler since they are weighted homogeneous, and lifting those via computing standard representations (see Algorithm 5.1). A priori, all these computations are computations over the integers as base ring, which is more expensive than computing over base fields. However, all the initial ideals involved contain the prime number $p = \mathrm{in}_w(p - t)$, and hence most computations can actually be done over the finite field $\mathbb{Z}/\langle p \rangle$. This reduces the overall cost drastically.

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