COMPUTING TROPICAL VARIETIES OVER FIELDS WITH VALUATION

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Abstract. We show how tropical varieties of ideals $I \trianglelefteq K[x]$ over a field K with non-trivial valuation can always be traced back to tropical varieties of ideals $\pi^{-1}I \trianglelefteq R[\![t]\!][x]$ over some dense subring R in its ring of integers. Moreover, for homogeneous ideals, we present algorithms on how the latter can be computed in finite time, provided that $\pi^{-1}I$ is generated by elements in $R[t,x]$. While doing so, we also comment on the computation of the Gröbner polytope structure and p-adic Gröbner bases using our framework.

1. Introduction

Tropical varieties are commonly described as combinatorial shadows of their algebraic counterparts. Explicit computation of tropical varieties is not only of interest for practical applications, but also of theoretical importance on many occassions [24, 12, 3, 8]. However, computing tropical varieties is an algorithmically highly challenging task, requiring sophisticated techniques from computer algebra and convex geometry.

The first techniques were developed by Bogart, Jensen, Speyer, Surmfels and Thomas [4], who focused on homogeneous ideals over $\mathbb{C}$ with the trivial valuation, which allowed them to rely on classical Gröbner basis methods. Furthermore, the authors showed that, under sensible conditions, their techniques can be used over the field of Puiseux series $\mathbb{C}\{\{t\}\}$ with its natural valuation, by regarding t as a variable in the polynomial ring instead of a uniformizing parameter in the coefficient ring. The inhomogeneity of the resulting ideal in $\mathbb{C}[t,x]$ can be worked around through homogenization and dehomogenization. In order to adapt these techniques to the field of p-adic numbers and the p-adic valuation, Chan and Maclagan adapted the classical theory of Gröbner bases [7] to take the valuation on the ground field into account, instead of solely relying on monomial orderings.

In this article, in Section 2, we discuss another approach to compute tropical varieties over an arbitrary field with valuation, which can be regarded as a generalisation

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of the trick used for $\mathbb{C}\{\{t\}\}$. For that, we combine the existing notions of tropical varieties over power series [27, 1, 22] with the concept of tropical varieties over coefficient rings [19, Section 1.6]. Compared to [7], the approach relies on existing standard basis theory, which not only allows us to exploit the highly optimized implementations that exist in established computer algebra systems such as SINGULAR [9] or MACAULAY2 [10], it also connects us to a highly active field of research.
Moreover, in Section 3, we improve on the techniques in [4] by avoiding homogenization and dehomogenization. We also touch upon the topic on how to compute p-adic Gröbner bases in our framework. In Section 4 we present the algorithms for computing tropical varieties and in Section 5 we touch upon possible optimizations that are exclusive to non-trivial valuations.
We are not addressing issues on computational complexity as in [26, 17]. All algorithms in this article are implemented in the SINGULAR library `tropical.lib` [15], relying on the GFANLIB interface `gfan.lib` [14, 16] for computations in convex geometry. They are publicly available as part of the official SINGULAR distribution.

2. Tracing tropical varieties to a trivial valuation

The aim of this section is to show how tropical varieties over valued fields can be traced back to tropical varieties over integral power series. The linchpin of the section is to show that initial ideals over valued fields can be described through initial ideals over integral power series, the remaining results then follow naturally from this. To fix the notation, we will begin by recalling some very basic notions in tropical geometry that are of immediate relevance to us.

Convention 2.1
For the remainder of the article, fix a complete field K with non-trivial discrete valuation $\nu : K \to \mathbb{R} \cup \{\infty\}$ and a uniformizing parameter $p \in K$. Let $\mathcal{O}_K$ be its ring of integers and let $\mathfrak{K}$ denote its residue field. Let $R \leq \mathcal{O}_K$ be a dense, noetherian subring. By Cohen's Structure Theorem, we have two exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \langle p-t\rangle \cdot R[\![t]\!]_{\langle p-t\rangle} \arrow[r] & R[\![t]\!]_{\langle p-t\rangle} \arrow[r] & K \arrow[r] & 0, \\
0 \arrow[r] & \langle p-t\rangle \cdot R[\![t]\!] \arrow[r] \arrow[u] & R[\![t]\!] \arrow[r, "\pi", "t \longmapsto p"'] \arrow[u] & \mathcal{O}_K \arrow[r] \arrow[u] & 0.
\end{tikzcd}$$

Moreover, fix a multivariate polynomial ring $K[x] = K[x_1, \ldots, x_n]$. By abuse of notation, we will also use π to refer to both the map $R[\![t]\!][x] \to \mathcal{O}_K[x]$ as well as the composition $R[\![t]\!][x] \to \mathcal{O}_K[x] \hookrightarrow K[x]$.

Example 2.2 (p-adic numbers)
The most important example is the field $K := \mathbb{Q}_p$ of p-adic numbers with $\mathcal{O}_K := \mathbb{Z}_p$ the ring of p-adic integers. Then $R := \mathbb{Z} \leq \mathbb{Z}_p$ is a natural dense subring, which is

computationally easy to work over. The exact sequences in Convention 2.1 merely reflect the presentation of p-adic integers as power series in p:

$$\begin{tikzcd}
0 \arrow[r] & \langle p-t\rangle \cdot \mathbb{Z}[\![t]\!]_{\langle p-t\rangle}[x] \arrow[r] & \mathbb{Z}[\![t]\!]_{\langle p-t\rangle}[x] \arrow[r] & \mathbb{Q}_p[x] \arrow[r] & 0, \\
0 \arrow[r] & \langle p-t\rangle \cdot \mathbb{Z}[\![t]\!][x] \arrow[r] \arrow[u] & \mathbb{Z}[\![t]\!][x] \arrow[r, "\pi", "t \longmapsto p"'] \arrow[u] & \mathbb{Z}_p[x] \arrow[r] \arrow[u] & 0.
\end{tikzcd}$$

Example 2.3
Given the choice of $R \leq \mathcal{O}_K$ in Convention 2.1, choosing $R := \mathcal{O}_K$ is always possible. However, in many examples there are natural choices for R, which are computationally much easier to handle than $\mathcal{O}_K$ itself:

(1) $K = k(\!(t)\!)$ the field of Laurent series over a field k with $\mathcal{O}_K = k[\![t]\!]$ the ring of power series over k, $R = k[t]$ and $p = t$; e.g. $k = \mathbb{F}_q$ with q a prime power, as used in [24, Section 7] or [18], or $k = \mathbb{Q}$ as considered in [4], see Example 2.15.
(2) Finite extensions K of $\mathbb{Q}_p$ and $\mathbb{F}_q(\!(t)\!)$, i.e. all local fields with non-trivial valuation, and also all higher dimensional local fields.
(3) $\mathcal{O}_K$ any completion of a localization of a Dedekind domain R at a prime ideal $P \trianglelefteq R$, $p \in P$ a suitable element. Note that p does not need to generate P and hence $\mathcal{O}_K$ need not be the completion with respect to the ideal generated by p, e.g. $R = \mathbb{Z}[\sqrt{-5}]$, $P = \langle 2, 1+\sqrt{-5}\rangle$ and $p = 2$.
(4) For an odd choice of R, consider $K := \mathbb{Q}(s)(\!(t)\!)$ so that $\mathcal{O}_K = \mathbb{Q}(s)[\![t]\!]$. Set $R := S^{-1}\mathbb{Q}[s,t]$, where $S := \mathbb{Q}[s,t] \setminus (\langle t-1, s\rangle \cup \langle x\rangle)$ is multiplicatively closed as the complement of two prime ideals. Then R is a non-catenarian, dense subring of $\mathcal{O}_K$.

Definition 2.4 (initial forms, initial ideals, tropical varieties over valued fields)
For a polynomial $f = \sum_{\alpha\in\mathbb{N}^n} c_\alpha \cdot x^\alpha \in K[x]$ and a weight vector $w \in \mathbb{R}^n$, we define the *initial form* of f with respect to w to be:

$$\mathrm{in}_{\nu,w}(f) := \sum_{\substack{w\cdot\alpha-\nu(c_\alpha)\\ \text{maximal}}} \overline{c_\alpha \cdot p^{-\nu(c_\alpha)}} \cdot x^\alpha \in \mathfrak{K}[x].$$

For any subset $I \subseteq K[x]$ and a weight vector $w \in \mathbb{R}^n$, we define the *initial ideal* of I with respect to w to be:

$$\mathrm{in}_{\nu,w}(I) := \langle \mathrm{in}_{\nu,w}(f) \mid f \in I\rangle \trianglelefteq \mathfrak{K}[x].$$

We refer to the set of weight vectors for which the initial ideal contains no monomial as the *tropical variety* of I,

$$\mathcal{T}_\nu(I) := \{w \in \mathbb{R}^n \mid \mathrm{in}_{\nu,w}(I) \text{ monomial free}\}.$$

Theorem 2.5 (Structure Theorem for Tropical Varieties, [19, Theorem 3.3.5])
Let $I \trianglelefteq K[x]$ define an irreducible subvariety in $(K^*)^n$ of dimension d. Then $\mathcal{T}_\nu(I)$

is the support of a pure polyhedral complex of same dimension that is connected in codimension 1*.*

Remark 2.6
Note that [19] only focuses on the connectivity in codimension 1 in the characteristic 0 case. A general proof can be found in a work by Cartwright and Payne [5].

Next, we will introduce tropical varieties in $R[\![t]\!][x]$, and show how a certain class of them relates to tropical varieties in $K[x]$. In particular, we will note that those tropical varieties in $R[\![t]\!][x]$ are pure and connected in codimension 1. We begin by introducing initial forms and initial ideals in $R[\![t]\!][x]$ and show how they can be used to describe their pendants in $K[x]$.

Definition 2.7 (initial forms, initial ideals)
Given an element $f = \sum_{\beta,\alpha} c_{\alpha,\beta} \cdot t^\beta x^\alpha \trianglelefteq R[\![t]\!][x]$ and a weight vector $w \in \mathbb{R}_{<0} \times \mathbb{R}^n$, we define the *initial form* of f with respect to w to be

$$\operatorname{in}_w(f) := \sum_{w\cdot(\beta,\alpha) \text{ maximal}} c_\alpha t^\beta x^\alpha \in R[t,x].$$

Given an ideal $I \trianglelefteq R[\![t]\!][x]$ and a weight vector $w \in \mathbb{R}_{<0} \times \mathbb{R}^n$, we define the *initial ideal* of I with respect to w to be:

$$\operatorname{in}_w(J) := \langle \operatorname{in}_w(f) \mid f \in J \rangle \trianglelefteq R[t,x].$$

This can be thought of as a natural extension of Definition 2.4 with trivial valuation on the coefficients. Note that we only allow weight vectors with negative weight in t, so that our result lies in a polynomial ring.

Example 2.8 (p-adic numbers)
Let us consider the example in [6, Chapter 3.6], the ideal

$$I = \langle 2x_1^2 + 3x_1x_2 + 24x_3x_4, 8x_1^3 + x_2x_3x_4 + 18x_3^2x_4 \rangle \trianglelefteq \mathbb{Q}_3[x_1,\ldots,x_4]$$

over the 3-adic number Q_3, so that

$$\pi^{-1}I = \langle 3-t, 2x_1^2 + 3x_1x_2 + 24x_3x_4, 8x_1^3 + x_2x_3x_4 + 18x_3^2x_4 \rangle \trianglelefteq \mathbb{Z}[\![t]\!][x],$$

and the weight vector $(-1,w) \in \mathbb{R}_{<0} \times \mathbb{R}^4$, $w := (1,11,3,19)$. A short computation yields

$$\operatorname{in}_{(-1,w)}(\pi^{-1}I) = \langle 3, x_1^2, tx_1x_3x_4, t^3x_1x_2^2x_3, t^4x_1x_2^4, t^3x_3^4x_4^2 \rangle,$$

and the similarity to the initial ideal of I under the 3-adic valuation is no coincidence:

$$\operatorname{in}_{\nu_3,w}(I) = \langle x_1^2, x_1x_3x_4, x_1x_2^2x_3, x_1x_2^4, x_3^4x_4^2 \rangle \trianglelefteq \mathbb{F}_3[x].$$

Proposition 2.9
For any ideal $I \trianglelefteq \mathcal{O}_K[x]$ and any weight vector $w \in \mathbb{R}^n$, we have:

$$\overline{\operatorname{in}_{(-1,w)}(\pi^{-1}I)|_{t=1}} = \operatorname{in}_{\nu,w}(I),$$

where $\overline{(\cdot)}$ *denotes the canonical projection* $\overline{(\cdot)} : R[x] \to \mathfrak{K}[x]$.

Proof. $\supseteq$: Any term $s \in \mathcal{O}_K[x]$ is of the form $s = (\sum_\beta c_\beta p^\beta) \cdot x^\alpha$ with $p \nmid c_\beta$ for all $\beta \in \mathbb{N}$. Then the element $s' := (\sum_\beta c_\beta t^\beta) \cdot x^\alpha \in R[\![t]\!][x]$ is a natural preimage of it under π for which we have

$$\mathrm{in}_{\nu,w}(s) = \overline{c}_{\beta_0} \cdot x^\alpha = \overline{\mathrm{in}_{(-1,w)}(s')|_{t=1}}, \text{ where } \beta_0 = \min\{\beta \in \mathbb{N} \mid c_\beta \neq 0\}.$$

And because the valued weighted degree in $\mathcal{O}_K[x]$ and the weighted degree in $R[\![t]\!][x]$ coincide,

$$\deg_w(x^\alpha) - \mathrm{val}(\textstyle\sum_\beta c_\beta p^\beta) = \deg_{(-1,w)}(\textstyle\sum_\beta c_\beta \cdot t^\beta x^\alpha),$$

this implies any $f \in \mathcal{O}_K[x]$ has a preimage $f' \in R[\![t]\!][x]$ under π such that

$$\mathrm{in}_{\nu,w}(f) = \overline{\mathrm{in}_{(-1,w)}(f')|_{t=1}},$$

simply by applying the above argument to each of its terms.

$\subseteq$: Once again consider a term $s = \sum_\beta c_\beta p^\beta \cdot x^\alpha \in \mathcal{O}_K[x]$ with $p \nmid c_\beta$ for all $\beta \in \mathbb{N}$. Then any preimage of it under π is of the form $s' = \sum_\beta c_\beta t^\beta x^\alpha + r$ for some $r \in \langle t - p \rangle$.

If $\deg_{(-1,w)}(r) > \deg_{(-1,w)}(\sum_\beta c_\beta t^\beta x^\alpha)$, we would have

$$\overline{\mathrm{in}_{(-1,w)}(s')|_{t=1}} = \overline{\mathrm{in}_{(-1,w)}(r)|_{t=1}} = 0,$$

since $\mathrm{in}_{(-1,w)}(r) \in \mathrm{in}_{(-1,w)}\langle p - t \rangle = \langle p \rangle$.

And if $\deg_{(-1,w)}(r) < \deg_{(-1,w)}(\sum_\beta c_\beta t^\beta x^\alpha)$, we would have

$$\begin{aligned}\overline{\mathrm{in}_{(-1,w)}(s')|_{t=1}} &= \overline{\mathrm{in}_{(-1,w)}(\textstyle\sum_\beta c_\beta t^\beta x^\alpha)|_{t=1}} = \overline{c}_{\beta_0} \cdot x^\alpha \\ &= \mathrm{in}_{\nu,w}(\textstyle\sum_\beta c_\beta p^\beta \cdot x^\alpha) = \mathrm{in}_{\nu,w}(s),\end{aligned}$$

where $\beta_0 := \min\{\beta \in \mathbb{N} \mid c_\beta \neq 0\}$.

Now suppose $\deg_{(-1,w)}(r) = \deg_{(-1,w)}(\sum_\beta c_\beta t^\beta x^\alpha)$. First observe that because t is weighted negatively, there can be no cancellation amongst the highest weighted terms of r and the terms of $\sum_\beta c_\beta t^\beta x^\alpha$, as the terms of $\sum_\beta c_\beta t^\beta x^\alpha$ are not divisible by p, unlike the terms of the highest weighted terms of r. Therefore, we have

$$\overline{\mathrm{in}_{(-1,w)}(s')|_{t=1}} = \underbrace{\overline{\mathrm{in}_{(-1,w)}(\textstyle\sum_\beta c_\beta t^\beta x^\alpha)|_{t=1}}}_{=\mathrm{in}_{\nu,w}(\sum_\beta c_\beta p^\beta \cdot x^\alpha)} + \underbrace{\overline{\mathrm{in}_{(-1,w)}(r)|_{t=1}}}_{=\overline{0}} = \mathrm{in}_{\nu,w}(s).$$

Either way, we always have $\overline{\mathrm{in}_{(-1,w)}(s')|_{t=1}} \in \langle \mathrm{in}_{\nu,w}(s) \rangle$ for any arbitrary preimage $s' \in \pi^{-1}(s)$, and, as before, the same hence holds true for any arbitrary element $f \in \mathcal{O}_K[x]$. □

Corollary 2.10

For any ideal $I \trianglelefteq K[x]$ *and any weight vector* $w \in \mathbb{R}^n$, *we have:*

$$\overline{\mathrm{in}_{(-1,w)}(\pi^{-1}I)|_{t=1}} = \mathrm{in}_{\nu,w}(I).$$

Proof. Follows from $\mathrm{in}_{\nu,w}(I) = \mathrm{in}_{\nu,w}(I \cap \mathcal{O}_K[x])$. $\square$

With our previous considerations, we can define tropical varieties of ideals in $R[\![t]\!][x]$ and show how some of them relate to tropical varieties of ideals in $K[x]$.

Definition 2.11 (tropical variety)
For an ideal $I \trianglelefteq R[\![t]\!][x]$ we define its *tropical variety* to be

$$\mathcal{T}(I) = \overline{\{w \in \mathbb{R}_{<0} \times \mathbb{R}^n \mid \mathrm{in}_w(I) \text{ monomial free}\}} \subseteq \mathbb{R}_{\leq 0} \times \mathbb{R}^n,$$

where $\overline{(\cdot)}$ denotes the closure in the euclidean topology.

Example 2.12
Unlike over coefficient fields, initial ideals over coefficient rings may be devoid of monomials $t^\beta x^\alpha$, $\beta \in \mathbb{N}$ and $\alpha \in \mathbb{N}^n$ while containing terms $c \cdot t^\beta x^\alpha$, $c \notin R^*$. Consequently, tropical varieties over rings need not be pure.
Consider the principal ideal generated by $g := x + y + 2z \in \mathbb{Z}[\![t]\!][x, y, z]$. Figure 1 shows the intersection of its tropical variety with an affine subspace of codimension 2. Because g is homogeneous in x, y, z, its tropical variety is invariant under translation by $(0, 1, 1, 1)$, and since t does not occur in g, it is also closed under translation by $(-1, 0, 0, 0)$. Hence, the remaining points are then uniquely determined up to symmetry.

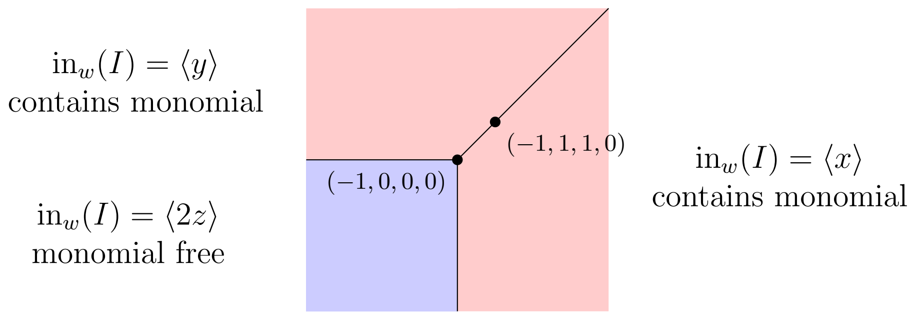


FIGURE 1. $\mathcal{T}(\langle x + y + z2 \rangle) \cap \{w_t = -1, w_z = 0\}$

Since $\mathrm{in}_{(-1,-1,-1,0)}(g) = 2z$ is no monomial, the entire lower left quadrant is included in our tropical variety, while the two other maximal cones are not. However, because $\mathrm{in}_{(-1,1,1,0)}(g) = x + y$ is no monomial either, the edge containing it is also part of our tropical variety. Therefore, the tropical variety cannot be the support of a pure polyhedral complex. Note, however, that I is not the type of ideal we are interested in, i.e. the type of ideal occurring in the following theorem.

Theorem 2.13
Let $I \trianglelefteq K[x]$ be an ideal. The projection $\mathbb{R}_{\leq 0} \times \mathbb{R}^n \to \mathbb{R}^n$ induces a natural bijection

$$\begin{aligned} \mathcal{T}(\pi^{-1} I) \cap (\{-1\} \times \mathbb{R}^n) &\xrightarrow{\sim} \mathcal{T}_\nu(I) \\ (-1, w_1, \ldots, w_n) &\longmapsto (w_1, \ldots, w_n). \end{aligned}$$

Proof. For the bijection, we show that

$$\mathrm{in}_{\nu,w}(I) \text{ monomial free} \quad \Longleftrightarrow \quad \mathrm{in}_{(-1,w)}(\pi^{-1}I) \text{ monomial free.}$$

$\Rightarrow$: Assume that $\mathrm{in}_{(-1,w)}(\pi^{-1}I) \trianglelefteq R[\![t]\!][x]$ contains a monomial $t^\beta x^\alpha$. Then, by Corollary 2.10, we have $\mathrm{in}_{\nu,w}(I) = \overline{\mathrm{in}_{(-1,w)}(\pi^{-1}I)|_{t=1}}$, which means $\mathrm{in}_{\nu,w}(I)$ must contain the monomial $x^\alpha \in \mathfrak{K}[x]$.

$\Leftarrow$: Assume that $\mathrm{in}_{\nu,w}(I) \trianglelefteq \mathfrak{K}[x]$ contains a monomial x^α. Then, by Corollary 2.10, $\mathrm{in}_{(-1,w)}(\pi^{-1}I)$ must contain an element of of the form $x^\alpha + (t-1)\cdot r + p \cdot s$, for some $r, s \in R[t,x]$. Recall that p lies in $\mathrm{in}_{(-1,w)}(\pi^{-1}I)$, therefore so does $p\cdot s$, and hence we have $x^\alpha + (t-1)\cdot r \in \mathrm{in}_{(-1,w)}(\pi^{-1}I)$.

Let $r = r_l + \ldots + r_1$ be a decomposition of r into its $(-1,w)$-homogeneous layers with $\deg_{(-1,w)}(r_1) < \ldots < \deg_{(-1,w)}(r_l)$. For sake of simplicity, we now distinguish between three cases:

1. $\deg_{(-1,w)}(x^\alpha) \geq \deg_{(-1,w)}(r_l)$: Set $g_1 := r - r_1 = r_l + \ldots + r_2$. Then

$$x^\alpha + (t-1)\cdot r = x^\alpha + (t-1)\cdot(g_1 + r_1) = \underbrace{x^\alpha + (t-1)\cdot g_1 - r_1}_{\text{higher weighted degree}} + t\cdot r_1.$$

Hence $x^\alpha + (t-1)\cdot g_1 - r_1, t\cdot r_1 \in \mathrm{in}_{(-1,w)}(\pi^{-1}I)$ and, more importantly,

$$t\cdot(x^\alpha + (t-1)\cdot g_1 - r_1) + t\cdot r_1 = tx^\alpha + (t-1)t\cdot g_1 \in \mathrm{in}_{(-1,w)}(\pi^{-1}I),$$

effectively shaving off the r_1 layer. We can continue this process by setting $g_2 := g_1 - r_2 = r_l + \ldots + r_3$. Then

$$\begin{aligned} tx^\alpha + (t-1)t\cdot g_1 &= tx^\alpha + (t-1)t\cdot(g_2 + r_2) \\ &= \underbrace{tx^\alpha + (t-1)t\cdot g_2 - t\cdot r_2}_{\text{higher weighted degree}} + t^2\cdot r_2. \end{aligned}$$

Hence $tx^\alpha + (t-1)t\cdot g_2 - t\cdot r_2, t^2\cdot r_2 \in \mathrm{in}_{(-1,w)}(\pi^{-1}I)$ and, as above,

$$\begin{aligned} &t\cdot(tx^\alpha + (t-1)t\cdot g_2 - t\cdot r_2) + t^2\cdot r_2 \\ &\quad = t^2x^\alpha + (t-1)t^2\cdot g_2 \in \mathrm{in}_{(-1,w)}(\pi^{-1}I) \end{aligned}$$

removing the r_2 layer. Eventually, we obtain $t^l x^\alpha \in \mathrm{in}_{(-1,w)}(\pi^{-1}I)$.

2. $\deg_{(-1,w)}(x^\alpha) \leq \deg_{(-1,w)}(r_1)$: Set $g_1 := r - r_l = r_{l-1} + \ldots + r_1$. Then

$$x^\alpha + (t-1)\cdot r = x^\alpha + (t-1)\cdot(g_1 + r_l) = \underbrace{x^\alpha + (t-1)\cdot g_1 + t\cdot r_l}_{\text{lower weighted degree}} - r_l.$$

Thus $r_l, x^\alpha + (t-1)\cdot g_1 + t\cdot r_l \in \mathrm{in}_{(-1,w)}(\pi^{-1}I)$ and, more importantly,

$$x^\alpha + (t-1)\cdot r_1 + t\cdot g_1 - t\cdot g_1 = x^\alpha + (t-1)\cdot r_1 \in \mathrm{in}_{(-1,w)}(\pi^{-1}I),$$

shaving off the the r_l layer this time. Continuing this pattern eventually yields $x^\alpha \in \mathrm{in}_{(-1,w)}(\pi^{-1}I)$.

3. $\deg_{(-1,w)}(r_1) < \deg_{(-1,w)}(x^\alpha) < \deg_{(-1,w)}(r_l)$: In this case we can use a combination of the steps in the previous cases to see $t^i \cdot x^\alpha \in \text{in}_{(-1,w)}(\pi^{-1}I)$ for the $1 \leq i \leq k$ such that $\deg_{(-1,w)}(r_{i-1}) < \deg_{(-1,w)}(x^\alpha) \leq \deg_{(-1,w)}(r_i)$.

In either case, we see that $\text{in}_{(-1,w)}(\pi^{-1}I)$ contains a monomial. □

Corollary 2.14
If $I \trianglelefteq K[x]$ defines an irreducible subvariety of $(K^*)^n$ of dimension d, then $\mathcal{T}(\pi^{-1}I)$ is the support of a pure polyhedral fan of dimension $d+1$ connected in codimension one.

Example 2.15
Let $K := \mathbb{Q}(\!(u)\!)$ be the field of Laurent series, equipped with is natural valuation ν_u, and let $I \trianglelefteq K[x,y]$ be the principal ideal generated by $(x+y+1)\cdot(u^2x+y+u)$. Then $\mathcal{T}_{\nu_u}(I)$ is the union of two tropical lines, one with vertex at $(0,0)$ and one with vertex at $(1,-1)$. Setting $R := \mathbb{Q}[t] \subseteq \mathbb{Q}[\![t]\!] = \mathcal{O}_K$, Proposition 2.9 implies that for any weight vector $w = (w_t, w_x, w_y) \in \mathbb{R}_{<0} \times \mathbb{R}^2$ in the lower open halfspace we have

$$w \in \mathcal{T}(\pi^{-1}I) \quad \Longleftrightarrow \quad \left(\frac{w_x}{|w_t|}, \frac{w_y}{|w_t|}\right) \in \mathcal{T}_{\nu_u}(I).$$

Hence $\mathcal{T}(\pi^{-1}I)$ is as shown in Figure 2, the cone over $\mathcal{T}_{\nu_u}(I)$. The polyhedral complex consists of 6 rays and 8 two-dimensional cones in a way that the intersection with the affine hyperplane yields a highlighted polyhedral complex, $\mathcal{T}_{\nu_u}(I)$.

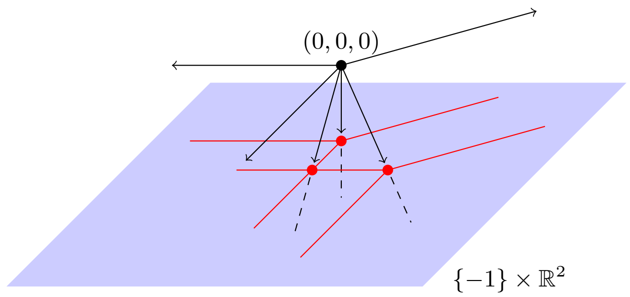


FIGURE 2. $\mathcal{T}(\pi^{-1}I)$ as cone over $\mathcal{T}_{\nu_u}(I)$

Example 2.16
Consider $I = \langle x_1 - 2x_2 + 3x_3, 3x_2 - 4x_3 + 5x_4 \rangle \trianglelefteq \mathbb{Q}_2[x_1,\ldots,x_4]$, whose preimage is given by

$$\pi^{-1}I = \langle x_1 - 2x_2 + 3x_3, 3x_2 - 4x_3 + 5x_4, 2-t \rangle \trianglelefteq \mathbb{Z}[\![t]\!][x_1,\ldots,x_4].$$

The tropical variety of the preimage is combinatorially of the form shown in Figure 3 and is invariant under the one-dimensional subspace generated by $(0,1,1,1,1)$. Hence each of the six vertices represents a two-dimensional cone and each of the five edges represents a three-dimensional cone.

Intersected with the affine hyperplane $\{-1\} \times \mathbb{R}^4$, we obtain a polyhedral complex as shown in Figure 4, any vertex of Figure 3 in $\{0\} \times \mathbb{R}^4$ becoming a point at infinity.

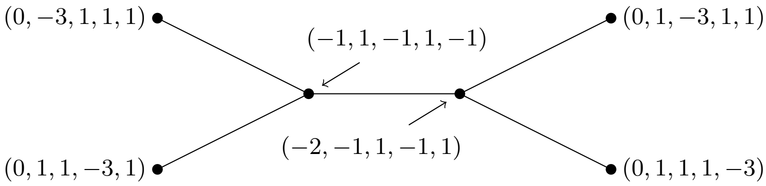


FIGURE 3. $\mathcal{T}(\langle x_1 - 2x_2 + 3x_3, 3x_2 - 4x_3 + 5x_4, 2 - t\rangle)$

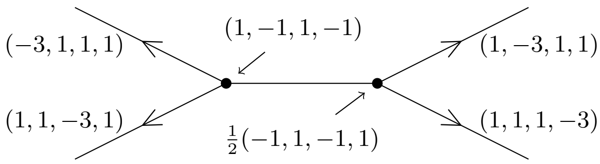


FIGURE 4. $\mathcal{T}_{\nu_2}(\langle x_1 - 2x_2 + 3x_3, 3x_2 - 4x_3 + 5x_4\rangle)$

3. Tracing Gröbner complexes to a trivial valuation

In this section, we show how the Gröbner complexes of ideals in $K[x]$ can be traced back to the Gröbner fans of ideals in $R[\![t]\!][x]$. We will show how the Gröbner fan induces a refinement of the Gröbner complex and how to determine whether two integral Gröbner cones map to the same valued Gröbner polytope. For the latter, we will need to delve into some basics in Gröbner bases. We close this section with a remark on p-adic Gröbner bases as introduced by Chan and Maclagan [7].

Definition 3.1 (Gröbner polyhedra, Gröbner complexes over valued fields)
For a homogeneous ideal $I \trianglelefteq K[x]$ and a weight vector $w \in \mathbb{R}^n$ we define its *Gröbner polytope* to be

$$C_{\nu,w}(I) := \overline{\{v \in \mathbb{R}^n \mid \mathrm{in}_{\nu,v}(I) = \mathrm{in}_{\nu,w}(I)\}} \subseteq \mathbb{R}^n,$$

where $\overline{(\cdot)}$ denotes the closure in the euclidean topology. We will refer to the collection $\Sigma_\nu(I) := \{C_{\nu,w}(I) \mid w \in \mathbb{R}^n\}$ as the *Gröbner complex* of I.

Theorem 3.2 (Gröbner complex, [19, Theorem 2.5.3])
Let $I \trianglelefteq K[x]$ be a homogeneous ideal. Then all $C_{\nu,w}(I)$ are convex polytopes and $\Sigma_\nu(I)$ is a finite polyhedral complex.

Definition 3.3
For an x-homogenous ideal $I \trianglelefteq R[\![t]\!][x]$, i.e. an ideal generated by elements which are homogeneous if considered as polynomials in x with coefficients in $R[\![t]\!]$, and a weight vector $w \in \mathbb{R}_{<0} \times \mathbb{R}^n$ we define its Gröbner cone to be

$$C_w(I) := \overline{\{v \in \mathbb{R}_{<0} \times \mathbb{R}^n \mid \mathrm{in}_v(I) = \mathrm{in}_w(I)\}},$$

where $\overline{(\cdot)}$ denotes the closure in the euclidean topology. We will refer to the collection $\Sigma(I) := \{C_w(I), C_w(I) \cap \{0\} \times \mathbb{R}^n \mid w \in \mathbb{R}_{<0} \times \mathbb{R}^n\}$ as the *Gröbner fan* of I.

Proposition 3.4 ([20], Theorem 3.19)
Let $I \trianglelefteq R[\![t]\!][x]$ be an x-homogeneous ideal. Then all $C_w(I)$ are polyhedral cones and $\Sigma(I)$ is a finite polyehdral fan.

Corollary 3.5
The map $\{-1\} \times \mathbb{R}^n \xrightarrow{\sim} \mathbb{R}^n, (-1, w) \longmapsto w$ is compatible with the Gröbner fan $\Sigma(\pi^{-1}I)$ and the Gröbner complex $\Sigma_\nu(I)$, i.e. it maps the restriction of a Gröbner cone $C_{(-1,w)}(\pi^{-1}I) \cap \big(\{-1\} \times \mathbb{R}^n\big)$ into a Gröbner polytope $C_{\nu,w}(I)$.

Proof. Follows directly from Proposition 2.9. □

Note that it may very well happen that several cones are mapped into the same Gröbner polytope, i.e. that the image of the restricted Gröbner fan is a refinement of the Gröbner complex (see Example 3.10).
We will now recall the notion of initially reduced standard bases of ideals in $R[\![t]\!][x]$ from [21] and how they determine the inequalities and equations of Gröbner cones as shown in [20]. We will then use them to decide whether two Gröbner cones are mapped to the same Gröbner polytope and, by doing so, show that no separate standard basis computation is required for it.

Definition 3.6 (initially reduced standard bases)
Fix the t-local lexicographical ordering $>$ such that $x_1 > \ldots > x_n > 1 > t$.
Given a weight vector $w \in \mathbb{R}_{<0} \times \mathbb{R}^n$ we define the *weighted ordering* $>_w$ to be

$$t^\beta x^\alpha >_w t^\delta x^\gamma \quad :\Longleftrightarrow \quad \begin{aligned} & w \cdot (\beta, \alpha) > w \cdot (\delta, \gamma) \text{ or} \\ & w \cdot (\beta, \alpha) = w \cdot (\delta, \gamma) \text{ and } t^\beta x^\alpha > t^\delta x^\gamma. \end{aligned}$$

For $g \in R[\![t]\!][x]$, the *leading term* $\mathrm{LT}_{>_w}(g)$ is the unique term of g with maximal monomial under $>_w$ and for $I \trianglelefteq R[\![t]\!][x]$, the *leading ideal* $\mathrm{LT}_{>_w}(I)$ is the ideal generated by the leading terms of all its elements. A finite subset $G \subseteq I$ is called a *standard basis* of I with respect to $>_w$, if the leading terms of its elements generate $\mathrm{LT}_{>_w}(I)$.
Suppose $G = \{g_1, \ldots, g_k\}$ with $g_i = \sum_{\alpha \in \mathbb{N}^n} g_{i,\alpha} \cdot x^\alpha$, $g_{i,\alpha} \in R[\![t]\!]$. We call G *initially reduced*, if the set

$$G' := \Big\{ \sum_{\alpha \in \mathbb{N}} \mathrm{LT}_>(g_{i,\alpha}) \cdot x^\alpha \;\Big|\; i = 1, \ldots, k \Big\} \subseteq R[t, x],$$

is reduced in the classical sense.

Proposition 3.7 ([20, Algorithm 4.6])
Let $I \trianglelefteq R[\![t]\!][x]$ be an x-homogeneous ideal and $w \in \mathbb{R}_{<0} \times \mathbb{R}^n$ a weight vector. Then an initially reduced standard basis G of I with respect to $>_w$ exists.
Moreover, if I can be generated by elements in $R[t, x]$, then G can be computed in finite time.

Proposition 3.8 ([20, Proposition 3.8, 3.11])
Let $I \trianglelefteq R[\![t]\!][x]$ be an x-homogeneous ideal, let $w \in \mathbb{R}_{<0} \times \mathbb{R}^n$ be a weight vector and let G an initially reduced standard basis of I with respect to $>_w$. Then the set of its initial forms $\{\mathrm{in}_w(g) \mid g \in G\}$ is an initially reduced standard basis of $\mathrm{in}_w(I)$ with respect to $>_w$, and the Gröbner cone of I around w is given by

$$C_w(I) = \overline{\{v \in \mathbb{R}_{<0} \times \mathbb{R}^n \mid \mathrm{in}_v(g) = \mathrm{in}_w(g) \text{ for all } g \in G\}}.$$

We now show that our standard bases of $\pi^{-1}I \trianglelefteq R[\![t]\!][x]$ yield Gröbner bases of initial ideals of $I \trianglelefteq K[x]$, allowing us to immediately decide whether two Gröbner cones of the former are mapped to the same Gröbner polytope of the latter.

Corollary 3.9
Let $I \trianglelefteq K[x]$ be a homogeneous ideal, let $w \in \mathbb{R}^n$ be a weight vector and let G be an initially reduced standard basis of $\pi^{-1}I$ with respect to the weighted ordering $>_{(-1,w)}$. Then

$$\left\{ \overline{\mathrm{in}_{(-1,w)}(g)}|_{t=1} \;\middle|\; g \in G \right\}$$

is a standard basis of $\mathrm{in}_{\nu,w}(I)$ with respect to the fixed lexicographical ordering $>$ restricted to monomials in x.

Proof. By Proposition 3.8, the set $\mathrm{in}_{(-1,w)}(G) := \{\mathrm{in}_{(-1,w)}(g) \mid g \in G\}$ is an initially reduced standard basis of $\mathrm{in}_{(-1,w)}(\pi^{-1}I)$ with respect to $>_{(-1,w)}$. And because it is homogeneous with respect to weight vector $(-1,w)$, it is also an initially reduced standard basis with respect to $>$. By choice of $>$, the set $\mathrm{in}_{(-1,w)}(G)|_{t=1}$ remains a standard basis of $\mathrm{in}_{(-1,w)}(\pi^{-1}I)|_{t=1}$ with respect to the restriction of $>$ to monomials in x. And since $p \in \mathrm{in}_{(-1,w)}(G)|_{t=1}$, $\overline{\mathrm{in}_{(-1,w)}(G)|}_{t=1}$ is a standard basis of $\overline{\mathrm{in}_{(-1,w)}(\pi^{-1}I)|}_{t=1}$ with respect to the restriction of $>$. $\square$

Example 3.10
Consider the preimage $\pi^{-1}I \trianglelefteq \mathbb{Z}[\![t]\!][x,y,z]$ of the ideal $I = \langle 2y+x, z^2+y^2\rangle \trianglelefteq \mathbb{Q}_2[x,y,z]$ and the two weight vectors $w = (1,3,7), v = (1,10,5) \in \mathbb{R}^3$. Fix a lexicographical tiebreaker $>$ with $x > y > z > 1 > t$.
The initially reduced standard basis of $\pi^{-1}I$ under $>_{(-1,w)}$ and $>_{(-1,v)}$ are the following two sets respectively (initial forms underlined):

$$G_{(-1,w)} = \{\underline{2} - t, \underline{ty} + x, \underline{z^2} + y^2\},\; G_{(-1,v)} = \{\underline{2} - t, \underline{ty} + x, \underline{xy} - tz^2, \underline{t^2z^2} + x^2, \underline{y^2} + z^2\},$$

yielding the following Gröbner basis of $\mathrm{in}_{\nu,w}(I)$ and $\mathrm{in}_{\nu,v}(I)$ under $>$:

$$\mathcal{G}_w = \{y, z^2\}, \quad \mathcal{G}_v = \{y, xy, z^2, y^2\}.$$

One immediately sees that both initial ideals coincide, meaning that the two Gröbner cones $C_{(-1,w)}(\pi^{-1}I)$ and $C_{(-1,v)}(\pi^{-1}I)$ are mapped to the same Gröbner polytope $C_{\nu_2,w}(I) = C_{\nu_2,v}(I)$.

Remark 3.11 (homogenization and dehomogenization)
A lot of effort has been put into developing algorithms for computing Gröbner cones $C_w(I)$ for x-homogeneous ideals $I \trianglelefteq R[\![t]\!][x]$ and weight vectors $w \in \mathbb{R}_{<0} \times \mathbb{R}^n$ in [20] which terminate in finite time in case I can be generated by elements in $R[t,x]$, avoiding the necessity to use homogenization and dehomogenization techniques as described in [4, Lemma 1.1], which are known to refine the Gröbner fan structure in general.
The most prominent phenomenon showing the refinement is the non-regular Gröbner fan in [13, Theorem 1]. Note that Gröbner fans of homogeneous ideals are known to be regular, as they are the normal fans of the state polytopes [25, Theorem 2.5]. The non-regular Gröbner fan $\Sigma(I)$ arises from the inhomogeneous ideal

$$I := \langle x_1x_3x_4 + x_1^2x_3 - x_1x_2, x_1x_4^2 - x_3, x_1x_4^4 + x_1x_3 \rangle \trianglelefteq \mathbb{Q}[x_1, \ldots, x_4]$$

and hence is restricted to the positive orthant $\mathbb{R}^4_{\geq 0}$. However, once homogenized it yields a regular Gröbner fan $\Sigma(I^h)$ living in $\mathbb{R}^5$, whose restriction to $\{0\} \times \mathbb{R}^4_{\geq 0}$ refines $\Sigma(I)$.

Remark 3.12 (p-adic Gröbner bases)
A Gröbner basis of an ideal $I \trianglelefteq K[x]$ over valued fields with respect to a weight vector $w \in \mathbb{R}^n$ is by [19, Section 2.4] a finite generating set whose initial forms generate the initial ideal $\mathrm{in}_{\nu,w}(I)$. Observe that Corollary 3.9 implies that such a Gröbner basis can be computed by projecting an initially reduced standard basis of $\pi^{-1}I \trianglelefteq R[\![t]\!][x]$ under the monomial ordering $>_w$ via π to $K[x]$.
Figure 5 shows timings of the Macaulay2 Package `GroebnerValuations` from Andrew Chan [10, 7], a toy-implementation of a p-adic Matrix-F5 algorithm by Tristan Vaccon in Sage [23, 28] and the standard basis engine of Singular over integers under mixed orderings [9]. The examples are:

Cyclic(n): In $\mathbb{Q}_2[x_0, \ldots, x_n]$, the cyclic ideal in the variables $x_1, \ldots, x_n$, homogenized using the variable x_0, and weight vector $(1, \ldots, 1)$.

Katsura(n): In $\mathbb{Q}_2[x_0, \ldots, x_n]$, the Katsura ideal in the variables $x_1, \ldots, x_n$, homogenized using the variable x_0, and weight vector $(1, \ldots, 1)$.

Chan: In $\mathbb{Q}_3[x_0, \ldots, x_n]$, the ideal $\langle 2x_1^2 + 3x_1x_2 + 24x_3x_4, 8x_1^3 + x_2x_3x_4 + 18x_3^2x_4 \rangle$ and weight vector $(-1, -11, -3, -19)$ taken from [6, Chapter 3.6].

All computations were aborted after exceeding either 1 CPU day or 16 GB memory. Note that the computations in Sage were done up to a finite precision of p^{50} and that the correctness of the result could only be verified for the examples for which either Macaulay2 or Singular finished.

4. Computation of tropical varieties

In this section, we present an algorithm for computing the tropical variety of an x-homogeneous ideal $I \trianglelefteq R[\![t]\!][x]$, provided it is pure and connected in codimension one,

Examples	Macaulay2	Sage	Singular
Cyclic(4)	1	10	1
Cyclic(5)	-	-	1
Cyclic(6)	-	-	-
Katsura(3)	1	1	1
Katsura(4)	-	10	1
Katsura(5)	-	190	1
Katsura(6)	-	2900	-
Chan	1	4	-

Figure 5. Timings in seconds unless aborted

as is the case for all preimages of ideals in $K[x]$ under π. All algorithms in this section are straight-forward modification of the techniques developed by Bogart, Jensen, Speyer, Sturmfels and Thomas for tropical varieties of homogeneous polynomial ideals over ground fields with trivial valuation, which is why proofs are omitted and instead references to [4] are added.
Before we begin, we quickly note that the computation of tropical hypersurfaces is simple:

Algorithm 4.1 (TropHypersurface, [4, Algorithm 4.3])
Input: $g = \sum_{\beta,\alpha} c_{\alpha,\beta} \cdot t^\beta x^\alpha$, $g \neq 0$.
Output: Δ, collection of maximal dimensional cones in $\mathbb{R}_{\leq 0} \times \mathbb{R}^n$ such that

$$\mathcal{T}(g) := \mathcal{T}(\langle g \rangle) = \bigcup_{\sigma\in\Delta} \sigma.$$

1: Construct the finite set of exponent vectors with minimal entry in t,

$$\Lambda := \left\{ (\beta,\alpha) \in \mathbb{N} \times \mathbb{N}^n \;\middle|\; \begin{array}{c} \alpha \in \mathbb{N}^n \text{ with } c_{\alpha,\beta'} \neq 0 \text{ for some } \beta' \in \mathbb{N} \\ \beta = \min\{\beta' \in \mathbb{N} \mid c_{\alpha,\beta'} \neq 0\} \end{array} \right\}.$$

2: Construct the normal fan of its convex hull

$$\Delta := \mathrm{NormalFan}(\mathrm{Conv}(\Lambda)).$$

3: **return** $\{\sigma \in \Delta \mid \sigma \cap \mathbb{R}_{<0} \times \mathbb{R}^n \neq \emptyset \text{ and } \dim(\sigma) = n\}$.

The computation of general tropical varieties on the other hand works in three steps:

(1) Finding a first maximal Gröbner cone $C_w(I) \subseteq \mathcal{T}(I)$, Alg. 4.7.
(2) Given $C_u(I) \subseteq \mathcal{T}(I)$ of codimension one, describe $\mathcal{T}(I)$ around $C_u(I)$, Alg. 4.13.
(3) Given $C_w(I) \subseteq \mathcal{T}(I)$ maximal, compute an adjacent $C_v(I) \subseteq \mathcal{T}(I)$, Alg. 4.2.

of which (3) is a generalisation of the well-known *flip* of Gröbner bases, which we will simply cite from [20] without going into any algorithmic details:

Algorithm 4.2 (Flip, [20, Algorithm 5.5])

Input: $(G, H, v, >_w)$, where

- $>_w$ a weighted monomial ordering with weight vector $w \in \mathbb{R}_{<0} \times \mathbb{R}^n$,
- v an outer normal vector of $C_w(I)$,
- $G = \{g_1, \ldots, g_k\} \subseteq I$ an initially reduced standard basis of an x-homogeneous ideal I w.r.t. $>_w$,
- $H = \{h_1, \ldots, h_k\}$ with $h_i = \mathrm{in}_w(g_i)$.

Output: $(G', >_{w'})$, where

- $C_{w'}(I)$ adjacent to $C_w(I)$ in direction v, i.e. $C_{w'}(I) = C_{w+\varepsilon\cdot v}(I)$ for $\varepsilon > 0$ sufficiently small,
- $G' \subseteq I$ an initially reduced standard basis w.r.t. $>_{w'}$.

To show how to find a first maximal dimensional Gröbner cone on $\mathcal{T}(I)$, we need to introduce the homogeneity space, since the starting cone algorithm works inductively over the codimension of it, and we have to recall the lift of standard bases, which we will again cite from [20] without going into any algorithmic details. The latter allows us to lift a standard basis of an initial ideal into a standard basis of the original ideal, useful for avoiding unnecessary standard basis computations.

Definition 4.3 (homogeneity space)
Given an x-homogeneous ideal $I \trianglelefteq R[\![t]\!][x]$, we define the *homogeneity space* of I (or of $\mathcal{T}(I)$) to be the intersection of all its lower Gröbner cones, i.e. Gröbner cones of the form $C_w(I)$ for some $w \in \mathbb{R}_{<0} \times \mathbb{R}^n$,

$$C_0(I) := \bigcap_{w \in \mathbb{R}_{<0} \times \mathbb{R}^n} C_w(I).$$

Example 4.4
Note that our definition of homogeneity space $C_0(I)$ differs from the natural lineality space $C_0(I)$ of tropical varieties over fields with trivial valuation. In general, our $C_0(I)$ is neither a linear subspace nor is it the set of all vectors with respect to whom the ideal is weighted homogeneous. Consider the principal ideal

$$I = \langle (1+t)\cdot x + (1+t)\cdot y \rangle \trianglelefteq \mathbb{Z}[\![t]\!][x, y],$$

whose Gröbner Fan splits the weight space $\mathbb{R}_{\leq 0} \times \mathbb{R}^2$ into two maximal cones, see Figure 6, and whose homogeneity space is given by

$$C_0(I) = \{(w_t, w_x, w_y) \in \mathbb{R}_{\leq 0} \times \mathbb{R}^n \mid w_x = w_y\}.$$

Clearly, $C_0(I)$ is no subspace and we have $(-1, 0, 0) \in C_0(I)$ despite the ideal not being weighted homogeneous with respect to it. This effect is caused by the terms tx and ty in the generator, which do not appear in any initial form and hence have no influence on $C_0(I)$, yet still exist and thus prevent I from being weighted homogeneous with respect to any weight vector in the interior of $C_0(I)$.

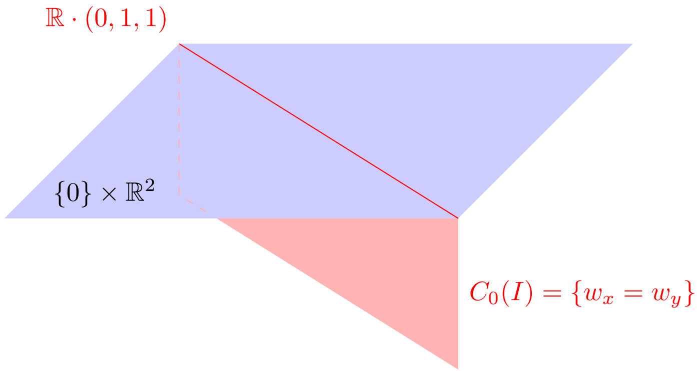


FIGURE 6. $C_0(\langle(1+t)\cdot x+(1+t)\cdot y\rangle)$

We follow up our observation in Example 4.4 with the following Lemma, which shows that the homogeneity space behaves properly in the case which is of interest to us:

Lemma 4.5
Let $I \trianglelefteq R[\![t]\!][x]$ be an x-homogeneous ideal and $w \in \mathbb{R}_{<0}\times\mathbb{R}^n$ a weight vector. Then

$$C_0(\mathrm{in}_w(I)) = \overline{\{v\in\mathbb{R}_{<0}\times\mathbb{R}^n \mid \mathrm{in}_v\,\mathrm{in}_w(I)=\mathrm{in}_w(I)\}} = \mathrm{Lin}(C_w(I))\cap(\mathbb{R}_{\leq 0}\times\mathbb{R}^n).$$

Proof. The second equality follows directly from the perturbation of initial ideals, i.e. it follows from the fact that for any $v\in\mathbb{R}_{<0}\times\mathbb{R}^n$ we have $\mathrm{in}_v\,\mathrm{in}_w(I)=\mathrm{in}_{w+\varepsilon\cdot v}(I)$ for $\varepsilon>0$ sufficiently small [20, Proposition 5.4]. It remains to show the first equality. The $\supseteq$ inclusion can be shown in a similar fashion: Suppose $v\in\mathbb{R}_{<0}\times\mathbb{R}^n$ such that $\mathrm{in}_v(\mathrm{in}_w(I))=\mathrm{in}_w(I)$. Then for any $u\in\mathbb{R}_{<0}\times\mathbb{R}^n$ we have

$$\mathrm{in}_{v+\varepsilon\cdot u}(\mathrm{in}_w(I))=\mathrm{in}_u(\mathrm{in}_v(\mathrm{in}_w(I)))=\mathrm{in}_u(\mathrm{in}_w(I)),$$

showing that $v+\varepsilon\cdot u\in C_u(\mathrm{in}_w(I))$ for any $\varepsilon>0$ sufficiently small. As $C_u(\mathrm{in}_w(I))$ is closed by definition, this implies $v\in C_u(\mathrm{in}_w(I))$. This shows that v is contained in every lower Gröbner cone of $\mathrm{in}_w(I)$, and hence also in their intersection $C_0(\mathrm{in}_w(I))$. For the $\subseteq$ inclusion, consider $v\in C_0(\mathrm{in}_w(I))\cap(\mathbb{R}_{<0}\times\mathbb{R}^n)$, so that $v\in C_u(\mathrm{in}_w(I))$ for all $u\in\mathbb{R}_{<0}\times\mathbb{R}^n$. In particular, $v\in C_w(\mathrm{in}_w(I))$ which is the middle set by definition. ∎

Algorithm 4.6 (Lift, [20, Algorithm 5.2])

Input: $(H', >', H, G, >)$, where

- $>$ a weighted t-local monomial ordering on $\mathrm{Mon}(t, x)$ with weight vector in $\mathbb{R}_{<0} \times \mathbb{R}^n$,
- $G = \{g_1, \ldots, g_k\} \subseteq I$ an initially reduced standard basis of an x-homogeneous ideal I w.r.t. $>$,
- $H = \{h_1, \ldots, h_k\}$ with $h_i = \mathrm{in}_w(g_i)$ for some $w \in C_>(I)$ with $w_0 < 0$,
- $>'$ a t-local monomial ordering such that $w \in C_>(I) \cap C_{>'}(I)$,
- $H' \subseteq \mathrm{in}_w(I)$ a weighted homogeneous standard basis w.r.t. $>'$.

Output: $G' \subseteq I$, an initially reduced standard basis of I w.r.t. $>'$.

Algorithm 4.7 (TropStartingCone, [4, Algorithm 4.12])

Input: $(G, >_w)$, where G is an initially reduced standard basis of an x-homogeneous ideal I with respect a weighted ordering $>_w$, $w \in \mathbb{R}_{<0} \times \mathbb{R}^n$.

Output: $(C_{w'}(I), G', >_{w'})$, where $C_{w'}(I) \subseteq \mathcal{T}(I)$ maximal dimensional and G' an initially reduced standard basis of I with respect to the weighted ordering $>_{w'}$.

if $\dim(I) = \dim(C_0(I))$ **then return** $(C_0(I), G, >)$
Find a weight vector $w \in (\mathcal{T}(I) \setminus C_0(I)) \cap (\mathbb{R}_{<0} \times \mathbb{R}^n)$.
Compute an initially reduced standard basis G'' of I with respect to $>_w$.
Set $H'' := \{\mathrm{in}_w(g) \mid g \in G''\}$.
Rerun

$$(C_{w_0'}(I), G_0', >_0') = \mathrm{TropStartingCone}(H'', >_w).$$

Let $>'$ be the weighted ordering with weight vector w and tiebreaker $>_0'$.
Lift G_0' to an initially reduced standard basis G' of I:

$$G' = \mathrm{Lift}(G_0', >', H'', G'', >_w).$$

Construct the corresponding Gröbner cone $C_{w'}(I) := C(H', G', >')$.
return $(C_{w'}(I), G', >')$

Example 4.8
Let $I \trianglelefteq \mathbb{Z}[\![t]\!][x_1, \ldots, x_4]$ be the preimage from Example 2.8,

$$I = \langle 3 - t, 2x_1^2 + 3x_1x_2 + 24x_3x_4, 8x_1^3 + x_2x_3x_4 + 18x_3^2x_4 \rangle.$$

A short calculation reveals that $\dim(\mathcal{T}(I)) = \dim(I) = 3 > 1 = \dim(C_0(I))$ with

$$C_0(I) = \mathbb{R} \cdot (0, 1, 1, 1, 1).$$

Picking $w := (-2, -1, 1, 5, -5) \in \mathcal{T}(I)$, the initial ideal $\mathrm{in}_w(I)$ is generated by

$$\{3, \quad tx_3x_4 - tx_1x_2 + x_1^2, \quad tx_1x_2^2 - x_1^2x_2 - t^3x_1x_2x_3 + t^2x_1^2x_3\}.$$

Another short calculation reveals $\dim(\mathrm{in}_w(I)) = 3 > 2 = \dim(C_0(\mathrm{in}_w(I)))$ with

$$C_0(\mathrm{in}_w(I)) = \mathbb{R}_{\geq 0} \cdot w + \mathbb{R} \cdot (0, 1, 1, 1, 1).$$

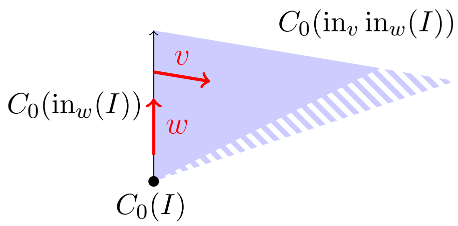
Picking $v := (6, -11, 11, -1, 1) \in \mathcal{T}(I)$, the initial ideal $\mathrm{in}_v\,\mathrm{in}_w(I)$ is generated by

$$\{3, \quad tx_3x_4 - tx_1x_2, \quad x_2x_3x_4 - t^2x_1x_2x_3\}.$$

And since $\dim(\mathrm{in}_v\,\mathrm{in}_w(I)) = 3 = \dim(C_0(\mathrm{in}_v\,\mathrm{in}_w(I)))$ with

$$C_0(\mathrm{in}_v\,\mathrm{in}_w(I)) = \mathbb{R}_{\geq 0} \cdot (-1, 1, -1, 1, -1) + \mathbb{R} \cdot (0, 1, 0, 0, 1) + \mathbb{R} \cdot (0, 0, 1, 1, 0),$$

the recursions end.



$$1 = \dim(C_0(I)) < \dim(C_0(\mathrm{in}_w(I))) < \dim(C_0(\mathrm{in}_v\,\mathrm{in}_w(I))) = \dim \mathcal{T}(I) = 3$$

FIGURE 7. computing a tropical starting cone recursively

This shows that $w + \varepsilon \cdot v \in \mathcal{T}(I)$ for $\varepsilon > 0$ sufficiently small. Together with the one-dimensional $C_0(I)$, this determines a maximal, three-dimensional Gröbner cone in our tropical variety, see Figure 7.

Two centrals tools necessary to describe the tropical variety around one of its codimension one cells are generic weight vectors and tropical witnesses.

Definition 4.9 (multiweights and generic weights)
Given weight vectors $w \in \mathbb{R}_{<0} \times \mathbb{R}^n$ and $v_1, \ldots, v_d \in \mathbb{R} \times \mathbb{R}^n$, we define the *initial form* of an element $g \in R[\![t]\!][x]$ with respect to the multidegree $(w, v_1, \ldots, v_d)$ to be

$$\mathrm{in}_{(w,v_1,\ldots,v_d)}(g) = \mathrm{in}_{v_d} \ldots \mathrm{in}_{v_1}\,\mathrm{in}_w(g),$$

and we define the *initial ideal* of $I \trianglelefteq R[\![t]\!][x]$ with respect to $(w, v_1, \ldots, v_d)$ to be

$$\mathrm{in}_{(w,v_1,\ldots,v_d)}(I) = \mathrm{in}_{v_d} \cdots \mathrm{in}_{v_1}\,\mathrm{in}_w(I) = \langle \mathrm{in}_{(w,v_1,\ldots,v_d)}(g) \mid g \in I \rangle.$$

Also, still fixing the lexicographical ordering $>$ with $x_1 > \ldots > x_n > 1 > t$ from Definition 3.6, we define the multiweighted ordering $>_{(w,v_1,\ldots,v_k)}$ to be

$$t^\beta \cdot x^\alpha >_{(w,v_1,\ldots,v_k)} t^\delta \cdot x^\gamma \quad \Longleftrightarrow \quad \text{either :}$$

- $w \cdot (\beta, \alpha) > w \cdot (\delta, \gamma)$ or
- $w \cdot (\beta, \alpha) = w \cdot (\delta, \gamma)$ and there exists an $1 \leq l \leq d$ with $v_i \cdot (\beta, \alpha) = v_i \cdot (\delta, \gamma)$ for all $1 \leq i < l$ and $v_l \cdot (\beta, \alpha) > v_l \cdot (\delta, \gamma)$ or
- $w \cdot (\beta, \alpha) = w \cdot (\delta, \gamma)$ and $v_i \cdot (\beta, \alpha) = v_i \cdot (\delta, \gamma)$ for all $1 \leq i \leq d$ and $t^\beta \cdot x^\alpha > t^\delta \cdot x^\gamma$.

Moreover, given a polyhedral cone $\sigma \subseteq \mathbb{R}_{\leq 0} \times \mathbb{R}^n$ of dimension d with $\sigma \nsubseteq \{0\} \times \mathbb{R}^n$ and a point $w \in \mathrm{relint}(\sigma)$ (note that $w \in \mathbb{R}_{<0} \times \mathbb{R}^n$ necessarily), we call a weight vector $u \in \sigma$ *generic around* w, if for all open neighbourhoods U around w there exists a weight vector $u' \in U \cap \sigma$ not lying on any Gröbner cone of dimension lower than d such that $\mathrm{in}_{u'}(\pi^{-1}I) = \mathrm{in}_u(\pi^{-1}I)$.

Algorithm 4.10 ($\mathrm{in}_{\sigma,w}(G)$, generic initial ideal around a weight)

Input: (σ, w, G), where

(1) $\sigma \subseteq \mathbb{R}_{\leq 0} \times \mathbb{R}^n$, a polyhedral cone with $\sigma \nsubseteq \{0\} \times \mathbb{R}^n$,
(2) $w \in \mathrm{relint}(\sigma)$ a relative interior point,
(3) $G \subseteq I$ a generating set of an x-homogeneous ideal $I \trianglelefteq R[\![t]\!][x]$.

Output: $(H', G', >') = \mathrm{in}_{(\sigma,w)}(G)$, where

(1) $>' := >_u$ for a weight vector $u \in \sigma$ generic around w,
(2) G' an initially reduced standard basis of I with respect to $>'$,
(3) $H' = \{\mathrm{in}_u(g) \mid g \in G'\}$.

1: Choose a basis $v_1, \ldots, v_d$ of the linear span of σ.
2: Pick a t-local monomial ordering $>$ on $\mathrm{Mon}(t, x)$.
3: Compute an initially reduced standard basis G' of $I = \langle G \rangle$ w.r.t. $>_{(w,v_1,\ldots,v_d)}$.
4: Set $H' := \{\mathrm{in}_{(w,v_1,\ldots,v_d)}(g) \mid g \in G'\}$
5: **return** $(H', G', >_{(w,v_1,\ldots,v_d)})$.

Definition 4.11
Let $I \trianglelefteq R[\![t]\!][x]$ and let $u \in \mathbb{R}_{<0} \times \mathbb{R}^n$ be such that $C_u(I) \nsubseteq \mathcal{T}(I)$. We call an element $f \in I$ a *tropical witness* of $C_u(I)$ if $\mathrm{in}_v(f)$ is a monomial for all $v \in \mathrm{Relint}(C_u(I))$.

Algorithm 4.12 (TropWitness, [4, Algorithm 4.7])

Input: $(m, H, G, >)$, where

(1) $>_w$ a weighted monomial ordering for some $w \in \mathbb{R}_{<0} \times \mathbb{R}^n$,
(2) $G = \{g_1, \ldots, g_k\}$ an initially reduced standard basis of an x-homogeneous ideal $I \trianglelefteq R[\![t]\!][x]$ with respect to $>_w$,
(3) $H = \{h_1, \ldots, h_k\}$ with $h_i = \mathrm{in}_w(g_i)$,
(4) $m \in \mathrm{in}_w(I)$ a monomial.

Output: $f \in I$, a tropical witness of $C_w(I)$.

1: Compute a standard representation $m = q_1 \cdot h_1 + \ldots + q_k \cdot h_k$, i.e. no term of $q_i \cdot h_i$ lies in $\langle \mathrm{LM}_>(h_j) \mid j < i \rangle$ for all $1 \leq i \leq k$.
2: **return** $q_1 \cdot g_1 + \ldots + q_k \cdot g_k$.

Algorithm 4.13 (TropStar, [4, Algorithm 4.8])

Input: G, the generating set of an x-homogeneous ideal $I \trianglelefteq R[\![t]\!][x]$ with $\dim \mathcal{T}(I) = \dim C_0(I) + 1$.

Output: Δ, a collection of maximal dimensional polyhedral cones in $\mathbb{R}_{\leq 0} \times \mathbb{R}^n$ covering $\mathcal{T}(I)$.

1: Compute the common refinement of all tropical hypersurfaces, throwing away cones in $\{0\} \times \mathbb{R}^n$,
$$\Delta := \{\sigma \in \textstyle\bigwedge_{g\in G} \mathrm{TropHypersurface}(g) \mid \sigma \cap \mathbb{R}_{<0} \times \mathbb{R}^n \neq \emptyset\}.$$
2: Set $L := \Delta$.
3: **while** $L \neq \emptyset$ **do**
4: Pick $\sigma \in L$ maximal and $w \in \mathrm{relint}(\sigma)$.
5: Compute an initial ideal with respect to a weight $w \in \sigma$ generic around w:
$$(H', G', >') = \mathrm{in}_{(\sigma,w)}(G).$$
6: **if** $\mathrm{in}_u(I) = \langle H' \rangle$ contains a monomial $s \neq 0$ **then**
7: Compute a tropical witness $g := \mathrm{TropicalWitness}(s, H', G', >')$.
8: Set
$$G := G \cup \{g\}, \quad \Delta := \Delta \wedge \mathcal{T}(g), \quad L := L \wedge \mathcal{T}(g).$$
9: **continue**
10: Suppose $w = (w_t, w_x) \in \mathbb{R}_{<0} \times \mathbb{R}^n$, set $w_{\mathrm{neg}} := (w_t, -w_x) \in \mathbb{R}_{<0} \times \mathbb{R}^n$.
11: **if** $w_{\mathrm{neg}} \in \sigma$ **then**
12: Redo Steps 5 to 9 with $w := w_{\mathrm{neg}}$.
13: Set $L := L \setminus \{\sigma\}$.
14: **return** Δ

Example 4.14
Consider the ideal $I \trianglelefteq \mathbb{Z}[\![t]\!][x_1, \dots, x_4]$ generated by
$$g_0 := 3, \quad g_1 := tx_3x_4 - tx_1x_2 + x_1^2, \quad g_2 := tx_1x_2^2 - x_1^2x_2 - t^3x_1x_2x_3 + t^2x_1^2x_3,$$
which is 3-dimensional with $C_0(I) = \mathrm{Cone}((-2,-1,1,5,-5)) + \mathbb{R} \cdot (0,1,1,1,1)$. Figure 8 illustrates the combinatorial structure of $\mathcal{T}(g_0) \cap \mathcal{T}(g_1) \cap \mathcal{T}(g_2)$.

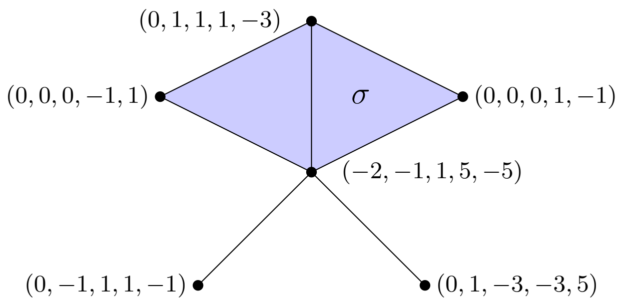


FIGURE 8. combinatorial structure of $Trop(g_0) \cap \mathcal{T}(g_1) \cap \mathcal{T}(g_2)$

One cone σ that can be seen to be contained in the intersection is
$$\sigma := \mathrm{Cone}(\underbrace{(0,0,0,-1,1)}_{=:w_1}, \underbrace{(0,1,1,1,-3)}_{=:w_2}) + C_0(I),$$

because we have

$$g_1 = \underbrace{tx_3x_4 \overbrace{-tx_1x_2 + x_1^2}^{\mathrm{in}_{w_2}(g_1)}}_{\mathrm{in}_{w_1}(g_1)}, \quad g_2 = \underbrace{\overbrace{tx_1x_2^2 - x_1^2x_2}^{\mathrm{in}_{w_1}(g_2)} - t^3x_1x_2x_3 + t^2x_1^2x_3}_{\mathrm{in}_{w_2}(g_2)},$$

so that for any weight $w \in \sigma$, $\mathrm{in}_w(g_1)$ contains at least the binomial $-tx_1x_2 + x_1^2$ and $\mathrm{in}_w(g_2)$ contains at least the binomial $tx_1x_2 - x_1^2x_2$. In particular, neither are monomials. However, it can be shown that, for

$$g_3 := tx_2x_3x_4 + t^2x_1^2x_3 - t^3x_1x_2x_3 \in I \quad \text{and} \quad w := (-1,1,2,2,0) \in \sigma,$$

$\mathrm{in}_w(g_3) = tx_2x_3x_4$ is a monomial, which implies that $\sigma \not\subseteq \mathcal{T}(I)$ (not that we would have expected otherwise considering $\dim(\sigma) = 4 > 3 = \dim(\mathcal{T}(I))$). Figure 9 illustrates the combinatorial structure of $\bigcap_{i=0}^{3} \mathcal{T}(g_i)$, red highlighting all weights that have been eliminated through the intersection with $\mathcal{T}(g_3)$.

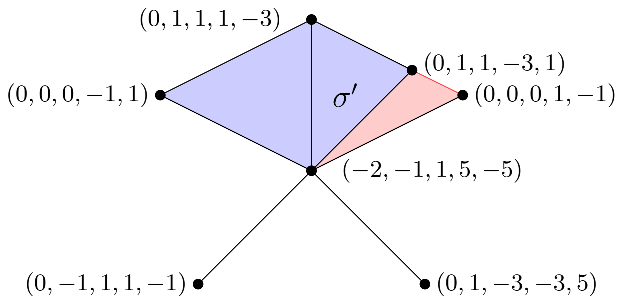


FIGURE 9. combinatorial structure of $\bigcap_{i=0}^{3} \mathcal{T}(g_i)$

Continuing, the following cone can be seen to be contained in the intersection of the tropical varieties of $g_0, \ldots, g_3$,

$$\sigma' := \mathrm{Cone}(\underbrace{(0,1,1,-3,1)}_{=:w_1}, \underbrace{(0,1,1,1,-3)}_{=:w_2}) + C_0(I),$$

since

$$g_1 = tx_3x_4 \overbrace{-tx_1x_2 + x_1^2}^{\mathrm{in}_{w_1}(g_1),\, \mathrm{in}_{w_2}(g_1)}, \quad g_2 = \underbrace{\overbrace{tx_1x_2^2 - x_1^2x_2}^{\mathrm{in}_{w_1}(g_2)} - t^3x_1x_2x_3 + t^2x_1^2x_3}_{\mathrm{in}_{w_2}(g_2)},$$

$$g_3 = \underbrace{tx_2x_3x_4 \overbrace{+t^2x_1^2x_3 - t^3x_1x_2x_3}^{\mathrm{in}_{w_3}(g_3)}}_{\mathrm{in}_{w_1}(g_3)}.$$

However, setting

$$g_4 := tx_2x_3x_4 - t^3x_3^2x_4 \in I \quad \text{and} \quad w' := (-1,3,4,5,0) \in \sigma',$$

$\mathrm{in}_{w'}(g_4) = tx_2x_3x_4$ is a monomial. Hence, we have again $\sigma' \nsubseteq \mathcal{T}(I)$ and Figure 10 illustrates the combinatorial structure of $\bigcap_{i=0}^4 \mathcal{T}(g_i)$. Further calculations will yield that indeed $\mathcal{T}(I) = \bigcap_{i=0}^4 \mathcal{T}(g_i)$.

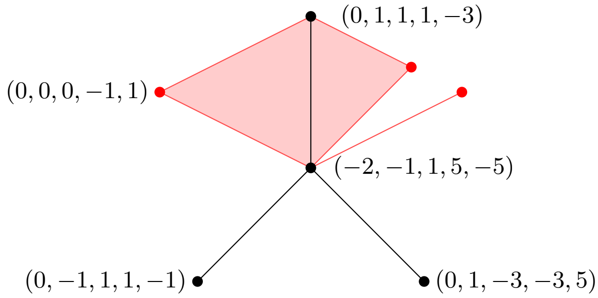


FIGURE 10. combinatorial structure of $\bigcap_{i=0}^4 \mathcal{T}(g_i)$

Tropical varieties with one-codimensional homogeneity space are important as they describe general tropical varieties locally around a codimension one cone.

Example 4.15
Consider again the ideal $I \trianglelefteq \mathbb{Z}[\![t]\!][x_1, \ldots, x_4]$ from Example 2.8 and 4.8 generated by

$$3 - t, \quad 8tx_3x_4 + tx_1x_2 + 2x_1^2, \quad tx_1x_2^2 + 2x_1^2x_2 + 2t^3x_1x_2x_3 + 4t^2x_1^2x_3 - 64tx_1^3.$$

For any weight vector inside $\mathcal{T}(I)$, say $w = (-2, -1, 1, 5, -5)$, $\mathcal{T}(\mathrm{in}_w(I))$ describes $\mathcal{T}(I)$ locally around w, see Figure 11. In particular, if w lies on a Gröbner cone of codimension 1, we have

$$\dim C_0(\mathrm{in}_w(I)) \overset{\text{Lem.}}{\underset{4.5}{=}} \dim C_w(I) = \dim \mathcal{T}(I) - 1 = \dim \mathcal{T}(\mathrm{in}_w(I)) - 1,$$

which allows us to compute $\mathcal{T}(\mathrm{in}_w(I))$ using Algorithm 4.13.

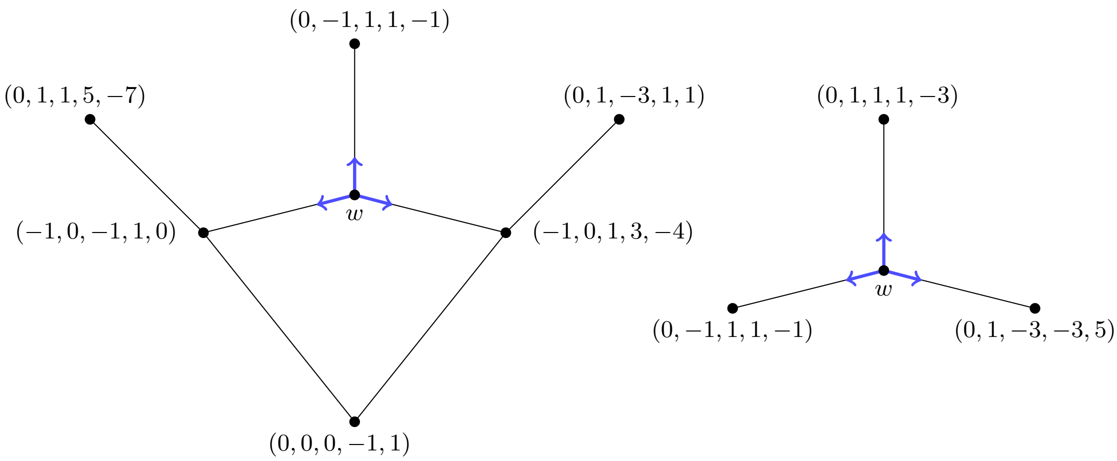


FIGURE 11. $\mathcal{T}(I)$ and $\mathcal{T}(\mathrm{in}_w(I))$

Combining Algorithms 4.7, 4.13 and 4.2, we obtain an algorithm to compute the tropical variety of a general ideal, provided it is pure and connected in codimension one.

Algorithm 4.16 (Trop, [4, Algorithm 4.11])

Input: $(G_{\text{input}}, >_{\text{input}})$, where for an x-homogeneous ideal $I \trianglelefteq R[\![t]\!][x]$ with $\mathcal{T}(I)$ pure and connected in codimension one:

- $>_{\text{input}}$ is a weighted monomial ordering,
- G_{input} an initially reduced standard basis of I with respect to $>_{\text{input}}$.

Output: $\Delta = \{C_w(I) \mid C_w(I) \in \mathcal{T}(I) \text{ maximal}\}$, so that

$$\mathcal{T}(I) = \textstyle\bigcup_{C_w(I)\in\Delta} C_w(I).$$

Compute a starting cone

$$(C_w(I), G, >) = \text{TropStartingCone}(G_{\text{input}}, >_{\text{input}}).$$

Initialize $\Delta := \{C_w(I)\}$.

Initialize a working list $L := \{(G, >, C_w(I))\}$.

while $L \neq \emptyset$ **do**

  Pick $(G, >, C_w(I)) \in L$.

  for all facets $\tau \leq C_w(I)$, $\tau \nsubseteq \{0\} \times \mathbb{R}^n$ **do**

    Compute a relative interior point $u \in \tau$.

    Set $H := \{\text{in}_u(g) \mid g \in G\}$.

    Compute the tropical star

$$\Delta_{\text{star}} = \text{TropStar}(H).$$

    for $\theta \in \Delta_{\text{star}}$ **do**

      Compute a relative interior point $v \in \theta$.

      if $C_{u+\varepsilon\cdot v}(I) \notin \Delta$ for $\varepsilon > 0$ sufficiently small **then**

        Flip the standard basis to the adjacent ordering

$$(G', >') := \text{Flip}(G, H, v, >).$$

        Set $H' := \{\text{in}_{(u,v)}(g) \mid g \in G'\}$.

        Construct the adjacent Gröbner cone

$$C_{w'}(I) := C(H', G', >').$$

        Set

$$\Delta := \Delta \cup \{C_{w'}(I)\} \quad \text{and} \quad L := L \cup \{(G', >', C_{w'}(I))\}.$$

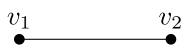
  Set $L := L \setminus \{(G, >, C_w(I))\}$.

return Δ.

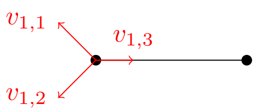
Example 4.17 (tropical traversal)
For a visual example of Algorithm 4.16 at work, consider the 3-dimensional ideal

$$\begin{aligned} I &= \langle 4x^2 + xy + 16y^2 + xz + 8z^2, 2-t\rangle \\ &= \langle \underbrace{t^2x^2 + xy + t^4y^2 + xz + t^3z^2}_{=:g}, 2-t\rangle \in \mathbb{Z}[\![t]\!][x,y,z]. \end{aligned}$$

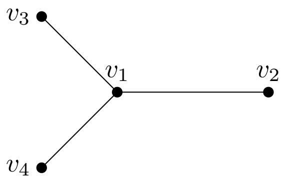
As $\mathrm{in}_w(2-t) = 2$ for all $w \in \mathbb{R}_{<0} \times \mathbb{R}^3$, it suffices to solely focus on g. For the starting cone, we begin with weight vector $w = (-3,-10,1,0) \in \mathbb{R}_{<0} \times \mathbb{R}^3$, since $\mathrm{in}_w(g) = xy + t^3z^2$ is no monomial. In fact, its initial form is binomial, hence the only weight vectors v such that $\mathrm{in}_{w+\varepsilon v}(g)$ is no monomial are the v such that $\mathrm{in}_{w+\varepsilon v}(g) = \mathrm{in}_w(g)$, or in other words $v \in C_w(I)$. This shows that $C_w(I)$ is a maximal cone in the tropical variety.



Note that all Gröbner cones are invariant under translation by $(0,1,1,1)$. Hence the 3-dimensional Gröbner cone $C_w(I)$ is spanned by two rays, which are generated by $v_1 = (-2,-7,1,0)$ and $v_2 = (-1,-3,0,0)$ respectively. This can be seen from their respective initial forms, which gain one additional term compared to $\mathrm{in}_w(g)$, $\mathrm{in}_{v_1}(g) = xy + t^4y^2 + t^3z^2$ and $\mathrm{in}_{v_2}(g) = xy + xz + t^3z^2$. We have thus finished computing a starting cone and identified its two facets, which we need to traverse.

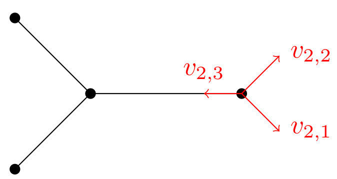


If we pick one of the facets, say the one generated by v_1, we see that its tropical star consists of three rays. One ray points in the direction $v_{1,3} = (0,0,-2,-1)$ so that $\mathrm{in}_{v_1+\varepsilon\cdot v_{1,3}}(g) = xy + t^3z^2 = \mathrm{in}_w(g)$, which undoubtedly points into our starting cone. Another ray points in the direction $v_{1,2} = (0,0,1,1)$ so that $\mathrm{in}_{v_1+\varepsilon\cdot v_{1,2}}(g) = t^4y^2 + t^3z^2$. The last ray points in the direction $v_{1,1} = (0,0,0,-1)$ so that $\mathrm{in}_{v_1+\varepsilon\cdot v_{1,1}}(g) = xy + t^4y^2$.



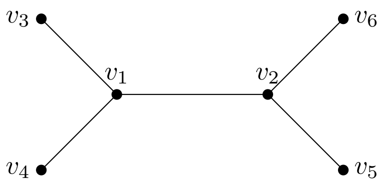
Continuing with direction $v_{1,2} = (0,0,1,1)$, to whose side lies the closure of equivalence class such that $\mathrm{in}_{w'}(g) = t^4y^2 + t^3z^2$, we see that the other ray of the maximal Gröbner cone is generated by $v_3 = (0,0,1,1)$ with $\mathrm{in}_{v_3}(g) = t^4y^2 + t^3z^2$. The ray lies on the boundary of the maximal Gröbner cone because it lies on the boundary of the lower halfspace.

Continuing with the direction $v_{1,1} = (0,0,0,-1)$, which is the closure of the equivalence class such that $\mathrm{in}_{w'}(g) = xy + t^4y^2$, we get that the other ray of the maximal Gröbner cone is $v_4 = (0,0,0,-1)$ with $\mathrm{in}_{v_4}(g) = t^2x^2 + xy + t^4y^2$.



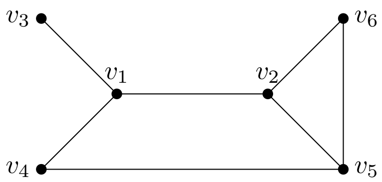
Because both v_3 and v_4 lie on the boundary of the lower halfspace, the only facet left to traverse is the one generated by v_2. The tropical star around v_2 consists of three rays. One ray points in the direction of $v_{2,1} = (0,1,0,0)$ so that

$\mathrm{in}_{v_2+\varepsilon\cdot v_{2,1}}(g) = xy + xz$. Another ray points in the direction of $v_{2,2} = (0,0,-1,0)$ so that $\mathrm{in}_{v_2+\varepsilon\cdot v_{2,2}}(g) = xz + t^3z^2$. The final ray points in the direction of $v_{2,3} = (0,0,2,1)$ so that $\mathrm{in}_{v_2+\varepsilon\cdot v_{2,3}}(g) = xy + t^3z^2 = \mathrm{in}_w(g)$, this is the vector pointing into our starting cone.



Continuing in the direction of $v_{2,1}$, the other ray of the maximal Gröbner cone is generated by $v_5 = (-1,2,0,0)$ as $\mathrm{in}_{v_5}(g) = t^2x^2 + xy + xz$. And continuing in the direction of $v_{2,2}$, the other ray is generated by $v_6 := (0,0,-1,0)$ as $\mathrm{in}_{v_6}(g) = t^2x^2 + xz + t^3z^2$.

Because v_6 lies on the boundary of the lower half-space, v_5 generates the only facet left to traverse. A quick glance at the initial forms imply that it is connected to the facets generated by v_4 and v_6, as it has two terms in common with each of them.



We obtain that $\mathcal{T}(I)$ is covered by a polyhedral fans which, modulo the homogeneity space $\mathbb{R}\cdot(0,1,1,1)$, has 6 rays, of which the ones generated by v_1, v_2, v_5 lie in the interior of the lower halfspace $\mathbb{R}_{\leq 0}\times\mathbb{R}^n$, while the ones generated by v_3, v_4, v_6 lie on its boundary.

The 6 rays are pairwise connected via 7 edges. The edges connecting (v_1,v_3), (v_1,v_4), (v_2,v_6) and (v_4,v_5) intersect the boundary in codimension one, while the cones connecting (v_1,v_2) and (v_2,v_5) intersect the boundary in codimension 2, which has to be the homogeneity space.

Example 4.18 (dependency on the valuation)
Consider the ideal from Example 2.16, $I := \langle x_1 - 2x_2 + 3x_3, 3x_2 - 4x_3 + 5x_4\rangle \trianglelefteq \mathbb{Q}[x]$. Figure 12 shows its tropical varieties for all possible valuations on $\mathbb{Q}$. Regardless of the valuation, all tropical varieties share the same recession fan, as was proven by Gubler [11]. The latter is also necessarily the tropical variety under the trivial valuation. Note that for p sufficiently large, the tropical varieties under ν_p coincides with the tropical variety under the trivial valuation. This is because p is simply too large for $p-t$ to matter in any of our standard basis calculations. These p are referred to as *good primes* while other p are referred to as *bad primes* in the theory of modular techniques [2].

Example 4.19 (independency of the valuation, Singular output)
Consider the following ideal of Grassmann-Plücker relations for Grass(2, 5),

$$\begin{aligned} I := \langle & x_1x_5 - x_0x_7 - x_2x_4, x_1x_6 - x_0x_8 - x_3x_4, x_2x_6 - x_0x_9 - x_3x_5, \\ & x_2x_8 - x_1x_9 - x_3x_7, x_5x_8 - x_4x_9 - x_6x_7\rangle \trianglelefteq \mathbb{Q}[x_0,\dots,x_9]. \end{aligned}$$

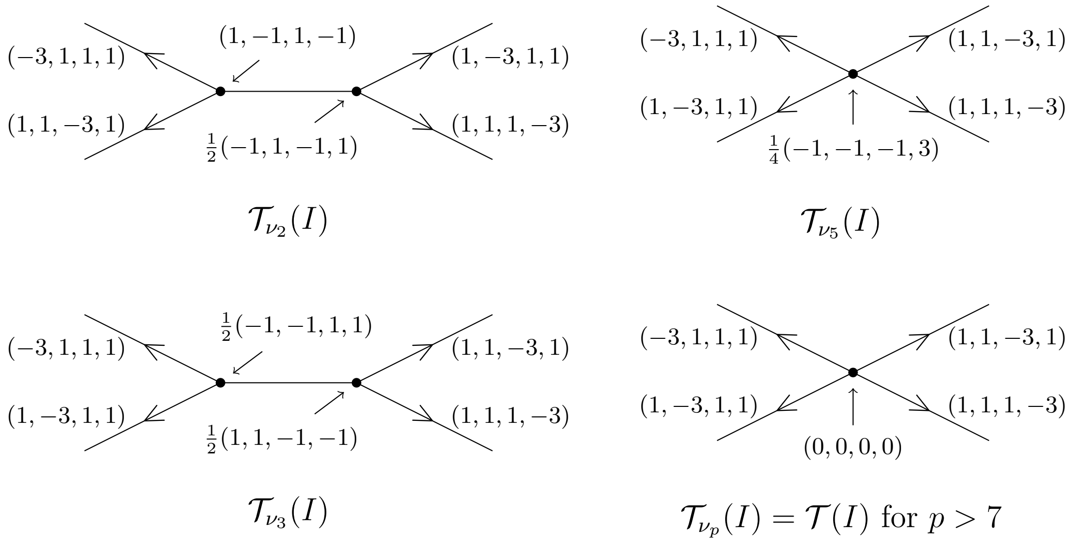


FIGURE 12. $\mathcal{T}_\nu(I)$ for various p-adic and the trivial valuations.

Unlike Example 4.18, its tropical variety does not seem to dependent on the choice of valuation, which is not surprising as Speyer and Sturmfels showed that it is characteristic-free [24, Theorem 7.1]. In this case, the computations under the p-adic valuation are mathematically equivalent to the computations under the trivial valuation, though the practical timings under the p-adic valuation are slightly slower due to a constant overhead of a more general framework.

Figure 13 shows a shortened output of SINGULAR when computing its tropical variety with respect to the 2-adic valuation. It describes a polyhedral fan whose intersection with the affine hyperplane $\{-1\} \times \mathbb{R}^{10}$ yields again a polyhedral fan: The ray #0 represents the 5-dimensional lineality space of $\mathcal{T}_{\nu_2}(I)$, while the maximal cones {`0 i j`} represent polyhedral cones in $\mathcal{T}_{\nu_2}(I)$ spanned by the lineality space and rays #i, #j. Note that, from a perspective of $\mathbb{R}^n = \{-1\} \times \mathbb{R}^n$, all data is given in homogenized coordinates, which is why the f-Vector shown is slightly distorted by lower-dimensional cones at infinity.

Figure 14 illustrates the combinatorial structure of Δ. Each vertex represents a ray of Δ, while each edge represents a maximal cone of Δ. The graph shown should be thought of as lying on a sphere S^2, on which the colored edges connect with their counterpart on the other side.

5. OPTIMIZATIONS FOR NON-TRIVIAL VALUATIONS

Up till now, all algorithms for computing $\mathcal{T}_\nu(I)$ via $\mathcal{T}(\pi^{-1}I)$ appear to be strictly worse than computing $\mathcal{T}(I)$, as we are working with an inhomogeneous ideal $\pi^{-1}I$ over a coefficient ring R instead of a homogeneous ideal I over a coefficient field K. In this section, however, we consider simple optimizations for the traversal, which suggest that working under a nontrivial valuation need not necessarily be slower than working under a trivial valuation.

```
SINGULAR                                                  /
A Computer Algebra System for Polynomial Computations    /   Version 4.1.0
                                                        0<
by: W. Decker, G.-M. Greuel, G. Pfister, H. Schoenemann \   Dec 2016
FB Mathematik der Universitaet, D-67653 Kaiserslautern   \
> LIB "gfanlib.so";
> printlevel = 1;
> ring r=0,(a,b,c,d,e,f,g,h,i,j),dp;
> ideal I = bf-ah-ce, bg-ai-de, cg-aj-df, ci-bj-dh, fi-ej-gh;
> tropicalVariety(I,number(2));
cones finished: 1 cones in working list: 4
[...] information on the state of the traversal because printlevel=1 was set
cones finished: 14 cones in working list: 1
cones finished: 15 cones in working list: 0
```

```
_application PolyhedralFan
_version 2.2
_type PolyhedralFan

AMBIENT_DIM
11

DIM
8

LINEALITY_DIM
5

RAYS
-1 0 0 0 0 0 0 0 0 0 0 # 0
0 -3 1 1 1 1 1 1 -1 -1 -1# 1
0 -1 -1 1 1 -1 1 1 1 1 -3# 2
0 -1 1 -1 1 1 -1 1 1 -3 1# 3
0 -1 1 1 -1 1 1 -1 -3 1 1# 4
0 1 -3 1 1 1 -1 -1 1 1 -1# 5
0 1 -1 -1 1 1 1 -3 -1 1 1# 6
0 1 -1 1 -1 1 -3 1 1 -1 1# 7
0 1 1 -3 1 -1 1 -1 1 -1 1# 8
0 1 1 -1 -1 -3 1 1 1 1 -1# 9
0 1 1 1 -3 -1 -1 1 -1 1 1# 10

LINEALITY_SPACE
0 -1 0 0 0 0 0 0 1 1 1   # 0
0 0 -1 0 0 0 1 1 0 0 1   # 1
0 0 0 1 0 0 1 0 1 0 1    # 2
0 0 0 0 1 0 0 1 0 1 1    # 3
0 0 0 0 0 -1 -1 -1 -1 -1 -1# 4

F_VECTOR
1 11 25 15

MAXIMAL_CONES
{0 1 2}# Dimension 8
{0 1 3}
{0 1 4}
{0 2 5}
{0 2 9}
{0 3 7}
{0 4 6}
{0 3 8}
{0 4 10}
{0 5 6}
{0 5 7}
{0 6 8}
{0 7 10}
{0 8 9}
{0 9 10}
```

FIGURE 13. SINGULAR output for the Grassmann-Plücker ideal

The main algebraic bottlenecks in the computation of tropical varieties are:

(1) computing generic initial ideals, Algorithm 4.10 ,
(2) checking generic initial ideals for monomials in Algorithm 4.13,
(3) the flip of standard bases, Algorithm 4.2,

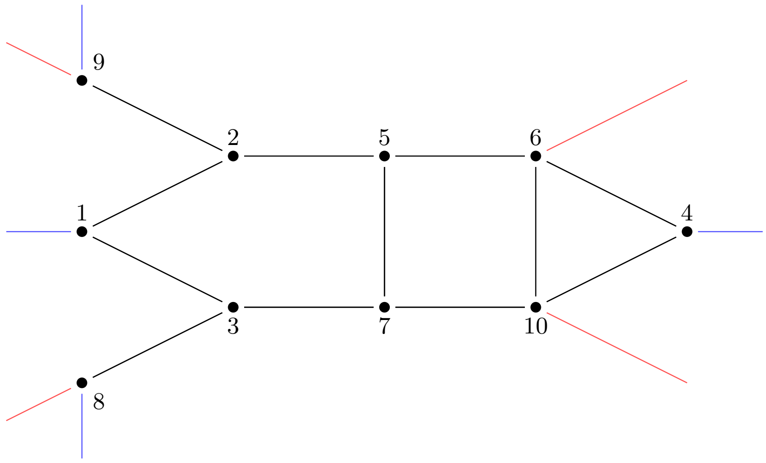


FIGURE 14. tropical variety of the Grassmann-Plücker ideal

all of which require at least one standard basis computation, which is the reason for the bottleneck. However, from Algorithm 4.16, they are never called on the actual input ideal, they are exclusively called on its initial ideals instead. This can be exploited, should the input ideal of Algorithm 4.16 be of the form $\pi^{-1}I$ for some $I \trianglelefteq K[x]$. Lemma 5.4 then shows that many computations can actally be done over the residue field $\mathfrak{K}$.

Convention 5.1
Let $I \trianglelefteq K[x]$ be a homogeneous ideal and fix an initial ideal $J := \operatorname{in}_{(-1,w)}(\pi^{-1}I) \trianglelefteq R[t,x]$ of its preimage as well as the corresponding monomial ordering $>_{(-1,w)}$. Note that necessarily $p \in J$.

Lemma 5.2 (quasi-homogeneity of J)
There exists a positive weight vector $u \in (\mathbb{R}_{>0})^{n+1}$ such that J is weighted homogeneous with respect to it.

Proof. Because J is weighted homogeneous with respect to $w \in \mathbb{R}_{<0} \times \mathbb{R}^n$ and x-homogeneous, it is, picking $k \in \mathbb{N}$ sufficiently high, also weighted homogeneous with respect to $k \cdot (0, 1, \dots, 1) + w \in (\mathbb{R}_{>0})^{n+1}$. □

Definition 5.3
We call an element $g = \sum_{\beta,\alpha} c_{\beta,\alpha} \cdot t^\beta x^\alpha \in R[t,x]$ a *canonical representative* of its residue class $\overline{g} \in \mathfrak{K}[t,x]$, if

$$c_{\beta,\alpha} = 0 \iff \overline{c}_{\beta,\alpha} = \overline{0} \qquad \text{and} \qquad c_{\beta,\alpha} = 1 \iff \overline{c}_{\beta,\alpha} = \overline{1}.$$

Lemma 5.4 (standard bases of J)
Let $\{\overline{g}_1, \dots, \overline{g}_k\}$ be a monic standard basis of $\overline{J}$ with respect to $>_{(-1,w)}$. Then $\{g_0, g_1, \dots, g_k\}$ is a standard basis of J with respect to $>_{(-1,w)}$, where $g_0 = p$ and $g_1, \dots, g_k$ are canonical representatives of their residue classes.

Proof. Let $G = \{g_0, \ldots, g_k\}$. Since $p \in J$, it is clear that $\langle G \rangle \subseteq J$ and therefore $\langle \mathrm{LT}_>(g) \mid g \in G \rangle \subseteq \mathrm{LT}_>(J)$. For the converse, consider a term $s = c \cdot t^\beta x^\alpha \in \mathrm{LT}_>(J)$. Now if $p \mid c$, then $s \in \langle \mathrm{LT}_>(g) \mid g \in G \rangle$, since $p \in G$ and $\mathrm{LT}_>(p) = p$. And if $p \nmid c$, we may use $p \in \mathrm{LT}_>(J)$ to normalize s, and get $t^\beta x^\alpha \in \mathrm{LT}_>(J)$. Thus $t^\beta x^\alpha \in \mathrm{LT}_>(\overline{J})$, and hence there is a $\overline{g_i}$ such that $\mathrm{LM}_>(\overline{g}_i) \mid t^\beta x^\alpha$. Since all $\overline{g}_i$ were chosen to be monic, this implies $\mathrm{LT}_>(\overline{g}_i) \mid t^\beta x^\alpha$, and because all g_i were chosen to be canonical representatives, this implies $\mathrm{LT}_>(g_i) \mid s$. $\square$

This article was dedicated to show how $\mathcal{T}_\nu(I)$ can be computed via $\mathcal{T}(\pi^{-1}I)$, however until now we have not addressed how to compute the preimage $\pi^{-1}I$ in the first place. We will therefore end the article with two results: The first will show that $\pi^{-1}I$ can be obtained by a saturation. The second will allow us get around computing the saturation.

Lemma 5.5
Let $I \trianglelefteq K[x]$ be an ideal, and let $G = \{g_1, \ldots, g_k\} \subseteq I \cap \mathcal{O}_K[x]$ be a generating set over the valuation ring. Since $\pi : R[\![t]\!][x] \to \mathcal{O}_K[x]$ is surjective, there exist $g_1', \ldots, g_k' \in R[\![t]\!][x]$ such that $\pi(g_i') = g_i \in R[x]$. Then

$$\pi^{-1}I = \Big(\langle g_1', \ldots, g_k' \rangle + \langle p - t \rangle\Big) : p^\infty \trianglelefteq R[\![t]\!][x].$$

Proof. $\pi^{-1}I \supseteq (\langle g_1', \ldots, g_k' \rangle + \langle p - t \rangle) : p^\infty$ is obvious, as $p - t$ is mapped to 0 and p is invertible in K.
For the converse inclusion, let $f \in \pi^{-1}I$. Then there are $q_1, \ldots, q_k \in K[x]$ such that

$$\pi(f) = q_1 \cdot g_1 + \ldots + q_k \cdot g_k \in K[x],$$

which means that for a sufficiently high power $l \in \mathbb{N}$ we have

$$p^l \cdot \pi(f) = \underbrace{p^l q_1}_{\in \mathcal{O}_K[x]} \cdot g_1 + \ldots + \underbrace{p^l q_k}_{\in \mathcal{O}_K[x]} \cdot g_k \in \mathcal{O}_K[x].$$

Since the map $\pi : R[\![t]\!][x] \to \mathcal{O}_K[x]$ is surjective, there exist $q_1', \ldots, q_k' \in R[\![t]\!][x]$ such that

$$p^l \cdot \pi(f) = \pi(q_1' \cdot g_1' + \ldots + q_k' \cdot g_k'),$$

or rather

$$p^l \cdot f - q_1' \cdot g_1' + \ldots + q_k' \cdot g_k' \in \ker(\pi) = \langle p - t \rangle.$$

Thus $p^l \cdot f \in \langle g_1', \ldots, g_k' \rangle + \langle p - t \rangle$, and hence

$$f \in (\langle g_1', \ldots, g_k' \rangle + \langle p - t \rangle) : p^\infty. \qquad \square$$

Proposition 5.6
Let $I \trianglelefteq K[x]$ be an ideal, and let $G = \{g_1', \ldots, g_k'\} \subseteq \pi^{-1}I$ such that $I = \langle \pi(g_1'), \ldots, \pi(g_k') \rangle$. Then

$$\mathcal{T}(\pi^{-1}I) = \mathcal{T}(\langle g_1', \ldots, g_k' \rangle + \langle p - t \rangle).$$

Proof. By Lemma 5.5, we have

$$\pi^{-1}I = \Big(\underbrace{\langle g_1',\dots,g_k'\rangle + \langle p-t\rangle}_{=:I'}\Big) : p^\infty \trianglelefteq R[\![t]\!][x].$$

Consider a weight vector $w \in \mathbb{R}_{<0} \times \mathbb{R}^n$ and suppose $\mathrm{in}_w(I')$ contains a monomial $t^\beta x^\alpha$. By Algorithm 4.12, there exists a witness $f \in I'$ with $\mathrm{in}_w(f) = t^\beta x^\alpha$. However since $I' \subseteq \pi^{-1}I$, $\mathrm{in}_w(\pi^{-1}I)$ then contains the monomial $t^\beta x^\alpha$ as well.
Now suppose $\mathrm{in}_w(\pi^{-1}I)$ contains a monomial $t^\beta x^\alpha$. By Algorithm 4.12, there exists a witness $f \in \pi^{-1}I$ with $\mathrm{in}_w(f) = t^\beta x^\alpha$. Let $l \in \mathbb{N}$ be sufficiently high such that $p^l \cdot f \in I'$. Now since $p-t \in I'$, this implies $t^l \cdot f \in I'$ and $\mathrm{in}_w(I')$ then contains the monomial $\mathrm{in}_w(t^l \cdot f) = t^{\beta+l}x^\alpha$. □

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