A Claim on the Rank of an Injective Map

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§0 Introduction

When I was looking for a subject for my dissertation I was confronted with the question, whether the rank of the derivative of an injection f from $\left(\mathbb{C}^2, 0\right)$ to $\left(\mathbb{C}^3, 0\right)$ could possibly be zero. A quite long standing conjecture says "no". But although the question can be formulated so easily and does not involve any difficult terms, so far one was neither able to prove it nor to find a counterexample. I was caught by the fascination of this question. The main problem in tackeling this conjecture directly is the necessity to find a sensible definition for the set of double points of f, which should be the preimage of the points with more than one preimage. A sensible definition has to commute with the change of the base space, but such a definition will automatically include the points where f is not an immersion although such a point may be the only point in the preimage of its image under f. However, András Némethi claimed in [Nem] that an injection violating the above conjecture would have a very peculiar property, which we define somewhat later as being "bad", and that the "good" injections would not only satisfy the condition $\mathbf{rank}(df(0)) \geq 1$ but, moreover, their images were equisingular families of plane curves. His proof of this assertion was actually framed in a somewhat wider context in that he looked at germs $f : (\mathbb{C}^n, 0) \to \left(\mathbb{C}^{n+1}, 0\right)$ and derived interesting results even in the case when f was only finite and birational. It was my aim to understand his work and to reduce the proof to the case where $f : \left(\mathbb{C}^2, 0\right) \to \left(\mathbb{C}^3, 0\right)$is injective.

When I had worked out his main arguments, there remained some questions which I tried to clarify in an example, namely $f : \left(\mathbb{C}^2, 0\right) \to \left(\mathbb{C}^3, 0\right)$given by $(x, y) \mapsto (x, y^2, y^3 + x^2)$. However, what I found was not the answer I had expected. The example shows in a quite simple way that there has to be a major mistake in the proof, although it does not contradict Némethi's claim itself. To clarify this, let me just summarize the main idea of the proof. The goodness of f (or more precisely, of its image) allows us to choose a suitable coordinate system $(\omega_1, \omega_2, \omega_3)$ of $\left(\mathbb{C}^3, 0\right)$ such that $V = (\omega_1 \circ f)^{-1}(0)$ and $V' = im(f) \cap \{\omega_1 = 0\}$ are two isolated plane curve singularities. Moreover, it shall ensure that either V is smooth or V and V' are isomorphic **(*)** . However, we will see in section 3 that for the above example the coordinate system (z, x, y) is suitable, where (x, y, z) denotes the standard coordinate system in $\left(\mathbb{C}^3, 0\right)$. But then $V = \{x^2 + y^3 = 0\}$ is an A_2–singularity and $V' = \{x^4 - y^3 = 0\}$ is an E_6. Thus, neither of the situations in Némethi's conclusion occurs, which changed my aim considerably. I will now try to show where exactly his argument goes wrong, working along the example already mentioned. For this it is necessary to reproduce Némethi's proof in detail. I will do so in the clarified version (for $f : \left(\mathbb{C}^2, 0\right) \to \left(\mathbb{C}^3, 0\right)$injective) after an introductory section in which we lay the foundations for the constructions and notions which we need (*versal deformation/unfolding; good representative (of a map germ); geometric basis and Dynkin diagram*).

Before I start, I would like to thank my supervisor David Mond for his kind introduction in a field, which was completely new to me, and for the patience, which he had during the many hours in which I besieged him with my questions. I would also like to thank Jan Stevens with whom I had many fruitful discussions concerning my dissertation (although he had to do most of the talking and explaining) and to whom a most valuable construction in 3.2 b. is due. Finally I would like to thank Pia Maas and Jens Pfeiffer for their corrections and criticisms which eliminated many errors and clarified several obscurities.

§1 Basic definitions and results

To get the conclusion that either V is smooth or V and V' are isomorphic, Némethi chooses a geometric basis in $H_1(V_t, \mathbb{Z})$ and tries to embed it into a geometric basis of $H_1(V'_s, \mathbb{Z})$, where V_t and V'_s are Milnor fibers of V and V' respectively. This embedding shall be done in a way which leads to a disconnection of a Dynkin diagram of V', unless V and V' satisfy one of the conditions in **(*)** on page 1. A result of Gabrielov and Lazzeri on the connectedness of Dynkin diagrams of singularities would thus ensure that the conclusion in **(*)** holds. This in turn would easily imply that the rank of $df(0)$ had to be greater than one.
To obtain the embedding of a geometric basis of $H_1(V_t, \mathbb{Z})$ into one of $H_1(V'_s, \mathbb{Z})$ Némethi considers their construction via a miniversal deformation of $(\mathrm{V}, 0)$ and an $\mathcal{R}$–miniversal unfolding of a generator of $(\mathrm{V}', 0)$ respectively. He wants to choose the constructing machinery for the basis in $H_1(V_t, \mathbb{Z})$ in the base space of the miniversal deformation of $(\mathrm{V}, 0)$ in such a way that it can be transported into the base space of an $\mathcal{R}$–miniversal unfolding of $(\mathrm{V}', 0)$ to be part of the constructing machinery there. However, it will turn out, that it is exactly the transport of this machinery that goes wrong — namely it comes down in the wrong place.
To make my dissertation consistent, I will now give the definitions of the notions used above and do the necessary constructions.

1.1 Deformations and unfoldings

In what follows we will only be concerned with deformations and unfoldings of isolated hypersurface singularities, indeed only with isolated plane curve singularities. We therefore restrict ourselves in the following definitions and results to the hypersurface case.

Let $\varphi : (\mathbb{C}^{\mathrm{n}}, 0) \to (\mathbb{C}, 0)$ be a hypersurface germ with an isolated singularity of finite multiplicity μ, i.e. its Milnor number is $\mu = \dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^n,0}/\left(\frac{\partial\varphi}{\partial z_1}, \ldots, \frac{\partial\varphi}{\partial z_n}\right) < \infty$, and let V be equal to $\varphi^{-1}(0)$. Denote furthermore by τ the Tjurina number of $(\varphi^{-1}(0), 0)$, i.e. $\tau = \dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^n,0}/\left(\varphi, \frac{\partial\varphi}{\partial z_1}, \ldots, \frac{\partial\varphi}{\partial z_n}\right) < \infty$.

We remark that the finiteness of μ and τ respectively is equivalent to 0 being an isolated singularity (see e.g. [Lo], prop. 1.2).

Definition 1

a. *A deformation of $(\mathrm{V}, 0)$ is a sequence $(\mathrm{V}, 0) \overset{i}{\hookrightarrow} (\mathcal{X}, 0) \overset{p}{\to} (\mathrm{B}, 0)$ where i denotes the identification of $(\mathrm{V}, 0)$ with the fiber $(\varphi^{-1}(0), 0)$ in $(\mathcal{X}, 0)$, such that the map p is flat, i.e. such that $\mathcal{O}_{\mathcal{X},0}$ becomes a flat $\mathcal{O}_{\mathrm{B},0}$–module via $p^* : \mathcal{O}_{\mathrm{B},0} \to \mathcal{O}_{\mathcal{X},0}$: $g \mapsto g \circ p$.*

b. *A deformation of φ is an analytic function germ $\tilde{\Phi} : (\mathbb{C}^{\mathrm{n}} \times \mathrm{B}, 0) \to (\mathbb{C}, 0)$ such that $\tilde{\Phi}(.,0) = \varphi$.*

c. *An unfolding of φ is an analytic map germ $\Phi : (\mathbb{C}^{\mathrm{n}} \times \mathrm{B}, 0) \to (\mathbb{C} \times \mathrm{B}, 0)$ such that there exists a deformation $\tilde{\Phi} : (\mathbb{C}^{\mathrm{n}} \times \mathrm{B}, 0) \to (\mathbb{C}, 0)$ of φ which satisfies $\Phi(x,\lambda) = \left(\tilde{\Phi}(x,\lambda), \lambda\right)$.*

In each of the above cases the space germ $(\mathrm{B}, 0)$ *is called the* <u>*base space*</u> *of the deformation or unfolding respectively.*

Remark 1
In the sequel we will not mention the inclusion i in the notion of a deformation of $(\mathrm{V}, 0)$ explicitly, assuming i is known or $(\mathrm{V}, 0)$ is naturally embedded in $(\mathcal{X}, 0)$. Furthermore, we will restrict ourselves to deformations and unfoldings with smooth base spaces, i.e. $(\mathrm{B}, 0) = (\mathbb{C}^{\mathrm{k}}, 0)$ for some $k \in \mathbb{N}_0$

Definition 2

a. *Two deformations* $p : (\mathcal{X}, 0) \to (\mathbb{C}^{\mathrm{k}}, 0)$ *and* $p' : (\mathcal{X}', 0) \to (\mathbb{C}^{\mathrm{k}}, 0)$ *of* $(\mathrm{V}, 0)$ *over the same base space* $(\mathbb{C}^{\mathrm{k}}, 0)$ *are said to be* <u>*isomorphic*</u> *if there exists a biholomorphic map germ* $h : (\mathcal{X}, 0) \to (\mathcal{X}', 0)$ *such that* $p' \circ h = p$ *and* $h_{|V}$ *is the identity.*

b. *Given two unfoldings* Φ *and* Φ' *of* φ *over the same base space* $(\mathbb{C}^{\mathrm{k}}, 0)$.

   (i) Φ *and* Φ' *are said to be* <u>$\mathcal{K}$*–isomorphic*</u> *if there exists a biholomorphic map germ* $h : (\mathbb{C}^n \times \mathbb{C}^k, 0) \to (\mathbb{C}^n \times \mathbb{C}^k, 0)$, *which is itself an unfolding of the identity of* $(\mathbb{C}^{\mathrm{n}}, 0)$, *such that* $\left((\Phi' \circ h)^{-1}(0), 0\right)$ *and* $(\Phi^{-1}(0), 0)$ *are isomorphic as complex spaces, i.e.* $\mathcal{O}_{\mathbb{C}^n \times \mathbb{C}^k, 0}/(\Phi' \circ h) \cong \mathcal{O}_{\mathbb{C}^n \times \mathbb{C}^k, 0}/(\Phi)$.

   (ii) Φ *and* Φ' *are said to be* <u>$\mathcal{R}$*–isomorphic*</u> *(*$\hat{=}$ *right isomorphic) if there exists a biholomorphic map germ* $h : (\mathbb{C}^n \times \mathbb{C}^k, 0) \to (\mathbb{C}^{\mathrm{n}} \times \mathbb{C}^{\mathrm{k}}, 0)$, *which is an unfolding of the identity of* $(\mathbb{C}^{\mathrm{n}}, 0)$, *such that* $\Phi' \circ h = \Phi$.

Remark 2

a. If two unfoldings are $\mathcal{R}$–isomorphic then they are certainly $\mathcal{K}$–isomorphic.

b. If $\Phi : (\mathbb{C}^n \times \mathbb{C}^k, 0) \to (\mathbb{C} \times \mathbb{C}^k, 0)$ is an unfolding of φ, then Φ and the restriction $\Phi_| : \left(\Phi^{-1}\left(0 \times \mathbb{C}^{\mathrm{k}}\right), 0\right) \to \left(0 \times \mathbb{C}^{\mathrm{k}}, 0\right)$ are deformations of $(\mathrm{V}, 0)$. For this note that φ is flat since it defines an hypersurface, and the flatness of φ implies the flatness of Φ (so in particular Φ itself is a deformation of $(\mathrm{V}, 0)$). But hence $\Phi_|$ is flat since flatness is preserved by base change (see e.g. [Mat], page 46).

c. If two unfoldings Φ and Φ' of φ are $\mathcal{K}$–isomorphic then they are isomorphic as deformations of $(\mathrm{V}, 0)$ and so are $\Phi_|$ and $\Phi'_|$.

Definition 3

a. *Given two deformations* $p : (\mathcal{X}, 0) \to (\mathbb{C}^{\mathrm{k}}, 0)$ *and* $p' : (\mathcal{X}', 0) \to (\mathbb{C}^{\mathrm{l}}, 0)$ *of* $(\mathrm{V}, 0)$. *We say* p' *is* <u>*induced*</u> *by* p *via some base change* $b : (\mathbb{C}^{\mathrm{l}}, 0) \to (\mathbb{C}^{\mathrm{k}}, 0)$ *if* $(\mathcal{X}', 0)$ *is the pullback of* $(\mathcal{X}, 0)$ *under* b, *and* p' *is the restriction of the canonical projection onto* $(\mathbb{C}^{\mathrm{l}}, 0)$, *i.e.* $\mathcal{X}' = \left\{(x, \lambda) \in (\mathcal{X} \times \mathbb{C}^{\mathrm{l}}, 0) \mid b(\lambda) = p(x)\right\}$ *and* $p'(x,\lambda) = \lambda$ *for* $(x, \lambda) \in (\mathcal{X}', 0)$.

b. *Similarly, given two unfoldings of* φ, *say* $\Phi : (\mathbb{C}^n \times \mathbb{C}^k, 0) \to (\mathbb{C} \times \mathbb{C}^k, 0)$ *and* $\Phi' : (\mathcal{Y}, 0) \to (\mathbb{C} \times \mathbb{C}^l, 0)$, *we say* Φ' *is* <u>*induced*</u> *by* Φ *via some base change* $b : (\mathbb{C} \times \mathbb{C}^l, 0) \to (\mathbb{C} \times \mathbb{C}^k, 0)$ *if* $(\mathcal{Y}, 0)$ *is the pullback of* $(\mathbb{C}^n \times \mathbb{C}^k, 0)$ *under* b *and*

Φ' is the restriction of the canonical projection onto $\left(\mathbb{C}\times\mathbb{C}^l, 0\right)$,
i.e. $\mathcal{Y} = \left\{(x,\lambda)\in\left(\left(\mathbb{C}^n\times\mathbb{C}^k\right)\times\left(\mathbb{C}\times\mathbb{C}^l\right), 0\right) \mid b(\lambda) = \Phi(x)\right\}$ *and* $\Phi'(x,\lambda) = \lambda$ *for* $(x,\lambda)\in(\mathcal{Y}, 0)$.

Definition 4

a. *A deformation p of* $(\mathrm{V}, 0)$ *is said to be* <u>*versal*</u> *if every deformation of* $(\mathrm{V}, 0)$ *is isomorphic to one induced by p via a base change.*
If furthermore the dimension of the base space is minimal (among those of versal deformations) then the deformation is said to be <u>*miniversal*</u>.

b. *An unfolding Φ of φ is said to be $\mathcal{K}$–versal or $\mathcal{R}$–versal respectively if every unfolding of φ is $\mathcal{K}$–isomorphic or $\mathcal{R}$–isomorphic respectively to one induced by Φ via a base change.*
If furthermore the dimension of the base space is minimal (among those of the $\mathcal{K}$–/$\mathcal{R}$–versal unfoldings) then the unfolding is said to be <u>*$\mathcal{K}$–miniversal*</u> *or* <u>*$\mathcal{R}$–miniversal*</u> *respectively.*

Remark 3

a. ($\mathcal{K}$–/$\mathcal{R}$–)miniversal deformations/ unfoldings of $(\mathrm{V}, 0)$ or φ respectively exist always and any two of them are ($\mathcal{K}$–/$\mathcal{R}$–)isomorphic.
The dimension of the base space of a miniversal deformation of $(\mathrm{V}, 0)$ or a $\mathcal{K}$–miniversal unfolding of φ is the <u>Tjurina number</u>

$$\tau = \dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^n,0}/\left(\varphi, \frac{\partial\varphi}{\partial z_1}, \ldots, \frac{\partial\varphi}{\partial z_n}\right).$$

One can indeed identify the base space[1] with $\mathcal{O}_{\mathbb{C}^n,0}/\left(\varphi, \frac{\partial\varphi}{\partial z_1}, \ldots, \frac{\partial\varphi}{\partial z_n}\right)$.
The dimension of the base space of an $\mathcal{R}$–miniversal unfolding of φ is the <u>Milnor number</u>

$$\mu = \dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^n,0}/\left(\frac{\partial\varphi}{\partial z_1}, \ldots, \frac{\partial\varphi}{\partial z_n}\right).$$

Again one can identify the base space[2] with $\mathcal{O}_{\mathbb{C}^n,0}/\left(\frac{\partial\varphi}{\partial z_1}, \ldots, \frac{\partial\varphi}{\partial z_n}\right)$. (see e.g. [Tei], §4)

b. If $p : (\mathcal{X}, 0) \to (\mathbb{C}^\tau, 0)$ is a miniversal deformation of $(\mathrm{V}, 0)$, then $(\mathcal{X}, 0)$ is smooth. (see e.g. [Voh], Bemerkung 2.3.10)

c. If b is some base change such that a given deformation of $(\mathrm{V}, 0)$/unfolding of φ is induced isomorphic via b to a ($\mathcal{K}$–/$\mathcal{R}$–)miniversal one, then $db(0)$ is uniquely determined, although b itself is not. (We call $db(0)$ the <u>reduced Kodaira-Spencer map</u> of the given deformation/unfolding).

d. If $\Phi : (\mathbb{C}^n \times \mathbb{C}^\tau, 0) \to (\mathbb{C}\times\mathbb{C}^\tau, 0)$ is a $\mathcal{K}$–miniversal unfolding of φ and if we denote $\Phi^{-1}\left(0\times\mathbb{C}^\tau\right)$ by $\mathcal{X}$, then $\Phi_| : (\mathcal{X}, 0) \to (0\times\mathbb{C}^\tau, 0)$ is a miniversal deformation of $(\mathrm{V}, 0)$. (see e.g. [Tei], prop. 4.5.1 and remark 4.5.2)

[1] or more precisely the tangentspace at 0
[2] or more precisely the tangentspace at 0

e. If $\Phi : \left(\mathbb{C}^n \times \mathbb{C}^k, 0\right) \to \left(\mathbb{C} \times \mathbb{C}^k, 0\right)$ is an $\mathcal{R}$–miniversal unfolding of φ, then it is $\mathcal{K}$–versal.
If Φ is $\mathcal{K}$–versal, then it is versal as deformation of $(\mathrm{V}, 0)$, since $\Phi_{|\left(\Phi^{-1}\left(0 \times \mathbb{C}^k\right), 0\right)}$ is a versal deformation of $(\mathrm{V}, 0)$.

f. The Tjurina number of $(\mathrm{V}, 0)$ is less than or equal to its Milnor number, i.e. $\tau \leq \mu$. (see e.g. [L-S])

Since miniversal deformations and unfoldings play an extremly important role in what follows we will briefly outline how they are constructed.

Theorem 1

a. *Let* $e_1, \ldots, e_\tau \in \mathcal{O}_{\mathbb{C}^n, 0}$ *be such that their images in* $\mathbf{T}^1_\varphi = \mathcal{O}_{\mathbb{C}^n, 0} / \left(\varphi, \frac{\partial \varphi}{\partial z_1}, \ldots, \frac{\partial \varphi}{\partial z_n}\right)$ *form a basis of* $\mathbf{T}^1_\varphi$. *Here we may choose* e_1 *to be identically one. If we now define* $\Phi : (\mathbb{C}^n \times \mathbb{C}^\tau, 0) \to (\mathbb{C} \times \mathbb{C}^\tau, 0)$ *by* $\Phi(z,\lambda) = \left(\varphi(z) + \sum_{i=1}^{\tau} \lambda_i e_i(z), \lambda\right)$, *then* Φ *is a* $\mathcal{K}$*–miniversal unfolding of* φ.
By remark 1 d. we get with $\mathcal{X} = \left\{(x, \lambda) \in (\mathbb{C}^n \times \mathbb{C}^\tau, 0) \mid \varphi(z) + \sum_{i=1}^{\tau} \lambda_i e_i(z) = 0\right\}$ *that* $p : (\mathcal{X}, 0) \to (\mathbb{C}^\tau, 0) : (z, \lambda) \mapsto \lambda$ *is a miniversal deformation of* $(\mathrm{V}, 0)$.

b. *Let* $g_1, \ldots, g_\mu \in \mathcal{O}_{\mathbb{C}^n, 0}$ *be such that their images in* $\mathbf{R}^1_\varphi = \mathcal{O}_{\mathbb{C}^n, 0} / \left(\frac{\partial \varphi}{\partial z_1}, \ldots, \frac{\partial \varphi}{\partial z_n}\right)$ *form a basis of* $\mathbf{R}^1_\varphi$. *Again we may choose* g_1 *to be identically one. Similarly as above define* $\Phi : (\mathbb{C}^n \times \mathbb{C}^\mu, 0) \to (\mathbb{C} \times \mathbb{C}^\mu, 0)$ *by* $\Phi(z,\sigma) = \left(\varphi(z) + \sum_{i=1}^{\mu} \sigma_i g_i(z), \sigma\right)$, *and get that* Φ *is an* $\mathcal{R}$*–miniversal unfolding of* φ.

(see e.g. [Tei], cor. 2.5.7)

1.2 Good representatives

Apart from the notion of an $\mathcal{R}$–miniversal unfolding of φ, the construction of the Dynkin diagram corresponding to the singularity of φ at zero requires the existence of a well behaved fibration of $(\mathrm{V}, 0)$, the so called Milnor fibration. Before we actually construct the fibration let us give the following

Definition 5

Let $\pi : M \to B$ *be an analytic map of complex manifolds. We call* π *a* <u>*smooth locally trivial fibration*</u> *with fiber* F *if the following condition is satisfied:*

$\forall b \in B\ \exists \mathcal{U}(b) \subseteq B$ *open neighbourhood of* b *and a diffeomorphism* $\varphi_{\mathcal{U}} : \varphi^{-1}(\mathcal{U}) \to \mathcal{U} \times F$ *such that* $\pi_{|\varphi^{-1}(\mathcal{U})} = pr_1 \circ \varphi_{\mathcal{U}}$

where pr_1 *denotes the canonical projection from* $\mathcal{U} \times F$ *onto* $\mathcal{U}$.
This means that the fibers of π *are all isomorphic to* F *and that this fact matches locally smoothly.*

In order to construct the Milnor fibration of φ we have to fix representatives of germs of φ or some unfolding of φ. It is not sufficient to fix just any representative, but we have to make sure that our choice is "good" in the following sense:

Construction 1

Let $g : \left(\mathbb{C}^{\mathrm{n+k}}, 0\right) \to \left(\mathbb{C} \times \mathbb{C}^k, 0\right)$ define an ICIS (isolated complete intersection singularity), i.e. 0 is an isolated singularity and $\dim(g^{-1}(0)) = n-1$. Now let $g : U \to D$ be any representative of the germ and denote $g^{-1}(0)$ by X_0. Given any real-analytic function $r : U \to [0,\infty)$ with $r^{-1}(0) = \{0\}$ (e.g. the square of the Euclidean Norm). Then according to Looijenga (see [Lo], (2.4)) there exists an $\varepsilon > 0$ such that $X_0 \cap r^{-1}(\varepsilon)$ (the link of X_0) is smooth (i.e. the intersection is transverse) and compact, and that $r_{|X_0\setminus\{0\}}$ has no critical value in $(0,\varepsilon]$ (i.e. $X_0 \pitchfork r^{-1}(\varepsilon')\ \forall \varepsilon' \in (0,\varepsilon]$). This enables us to find a ball B around 0 in D such that the restriction of g to $U \cap r^{-1}(\varepsilon)$ is a local submersion along $g^{-1}(B) \cap r^{-1}(\varepsilon)$ (i.e. $g^{-1}(b) \pitchfork r^{-1}(\varepsilon)\ \forall b \in B$). If we denote $g^{-1}(B) \cap r^{-1}[0,\varepsilon)$ by X then $g_| : X \to B$ will be called a "good" representative of the ICIS g. (see [Lo], (2.7))

Let us just summarize some of the properties of a good representative, which we will need later.

Theorem 2

Let $\overline{X}$ denote $g^{-1}(B) \cap r^{-1}[0,\varepsilon]$ and let ∂X be $g^{-1}(B) \cap r^{-1}(\varepsilon)$.

Then:

(i) $g_| : \overline{X} \to B$ is proper.

(ii) The critical locus C_g of g in X is analytic in X and the restriction of g to C_g is finite ($\hat{=}$ proper with finite fibers).

(iii) The discriminant locus $D_g = g(C_g)$ is analytic in B and $\dim(D_g) = \dim(C_g) = k$ unless D_g is void.

Definition and Theorem 6

Let $\Phi : \left(\mathbb{C}^n \times \mathbb{C}^k, 0\right) \to \left(\mathbb{C} \times \mathbb{C}^k, 0\right)$ be an unfolding of φ. Choose a good representative $\Phi : X \to B$ of Φ and denote by D_Φ its discriminant locus.

Then:

(i) $X' = X \setminus \Phi^{-1}(D_\Phi)$ and $B' = B \setminus D_\Phi$ are smooth.

(ii) $\Phi_| : X' \to B'$ is a smooth locally trivial fibration with fiber $\Phi^{-1}(0) \cong \varphi^{-1}(0) = V$.

(iii) For any $s \in B'$ the fiber $V_s = \Phi^{-1}(s)$ has the homotopy type of a bouquet of μ spheres of dimension n-1, where μ is the Milnor number of $(\mathrm{V}, 0)$.

We call the fibration in (ii) the Milnor fibration of $(\mathrm{V}, 0)$ and V_s a Milnor fiber of $(\mathrm{V}, 0)$.

Remark 4

a. Note first of all that an unfolding of the isolated hypersurface singularity φ defines an ICIS, namely $(\mathrm{V}, 0)$ itself, which enables us to choose a good representative.

b. The above theorem does not only hold for unfoldings of isolated hypersurface singularities but for general ICIS with the Milnor number of $(\mathrm{V}, 0)$ replaced by the one of the ICIS. In particular it holds for deformations of $(\mathrm{V}, 0)$. (see e.g. [Hamm] or [Lo])

c. The definition of "the" Milnor fiber V_s does neither depend on the choice of the unfolding nor on the choice of the good representative. One could for instance use $\varphi : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ as an unfolding of itself. (see e.g. [Lo], (2.9), or [A-G-V], 10.3.1)

1.3 Geometric bases and Dynkin diagrams

The theorem on the Milnor fibration tells us that the only non trivial reduced homology group of a Milnor fiber V_s of $(V, 0)$ is indeed $\tilde{H}_{n-1}(V_s, \mathbb{Z})$, which is a free group of rank μ. If we supply $\tilde{H}_{n-1}(V_s, \mathbb{Z})$ with the bilinear form $\langle ., . \rangle$ obtained by the intersection number of two cycles, i.e. $\langle \gamma, \delta \rangle$ = intersection number of γ and δ, then $\left(\tilde{H}_{n-1}(V_s, \mathbb{Z}), \langle ., . \rangle\right)$is the so called Milnor lattice of $(V, 0)$. (see e.g. [Ebel], page 2)

Our aim is now, roughly speaking, to choose a suitable basis for $\tilde{H}_{n-1}(V_s, \mathbb{Z})$ such that the graph induced by the intersection matrix contains valuable information about our singularity. We will call this graph the Dynkin diagram of $(V, 0)$ and a suitable basis geometric or distinguished.

There are different approaches to the construction of these bases. Husein–Zade works in his paper [Hus] with a morsification $\tilde{\varphi}$ of φ which gives rise to a split of the critical point 0 into μ critical points $\{z_1, \ldots, z_\mu\} \subset \mathbb{C}$ with distinct values, all of type A_1. Arnold [A-G-V] chooses a generic line l in the base space of an $\mathcal{R}$–miniversal unfolding of φ, which then intersects the bifurcation set $\Sigma = \{\sigma \in \mathbb{C}^\mu \mid (0, \sigma) \in D_\Phi \cap (\mathbb{C} \times \{\sigma\})\}$ transversally in μ points $\{\sigma_1, \ldots, \sigma_\mu\}$. Here Φ is chosen to be a good representative of the miniversal deformation in theorem 1 and D_Φ denotes its discriminant. For each $1 \le i \le \mu$ $(0, \sigma_i)$ is a critical value of Φ, and $\Phi^{-1}(0,\sigma_i)$ has exactly one singular point of type A_1.

In both situations one chooses paths u_i from some chosen base point z in $\mathbb{C}$or σ in l respectively to the intersection points z_i or σ_i respectively such that they have neither any intersection with each other apart from the base point nor any selfintersection. Near the z_i or σ_i respectively the inverse image of a point under $\tilde{\varphi}$ or Φ respectively contains in a natural way a real (n-1)–sphere. This sphere can be pulled back into the corresponding level sets $\tilde{\varphi}^{-1}(z)$ or $\Phi^{-1}(0,\sigma)$ respectively along the paths u_i. The homology class of that sphere in $\tilde{H}_{n-1}(\tilde{\varphi}^{-1}(z), \mathbb{Z})$ or $\tilde{H}_{n-1}(\Phi^{-1}(0,\sigma), \mathbb{Z})$ respectively will be called a vanishing cycle Δ_i corresponding to z_i or σ_i respectively. The set $\{\Delta_1, \ldots, \Delta_\mu\}$ forms a basis of $\tilde{H}_{n-1}$ and with respect to a suitable ordering of the z_i or σ_i respectively in the first place we get a geometric or distinguished basis.

We will, however, use a slightly different approach, along the ideas in [Ebel] (see also [Lam] and [Lo]). Again, the difference will mainly occur in the way we find μ suitable points which we then link to some base point.

Construction 2

Let $\Phi : (\mathbb{C}^n \times \mathbb{C}^\mu, 0) \to (\mathbb{C} \times \mathbb{C}^\mu, 0)$ be an $\mathcal{R}$–miniversal unfolding of φ as in theorem 1, i.e. consider $g_1 \equiv 1, g_2, \ldots, g_\mu \in \mathcal{O}_{\mathbb{C}^n,0}$ such that their images form a basis of $\mathbf{R}^1_\varphi$ and define $\Phi(z,\sigma) = (\varphi(z) + \sum_{i=1}^{\mu} \sigma_i g_i(z), \sigma)$. Denote by $\pi : \mathbb{C}^n \times \mathbb{C}^\mu \to \mathbb{C}^\mu$ the canonical projection onto $\mathbb{C}^\mu$. If we denote by $(D_\Phi, 0)$ the germ of the discriminant locus of Φ we see that $(\mathbb{C} \times \{0\}^\mu) \cap D_\Phi$ is $\{0\}$.

For this, let us look at the derivative of Φ *at a point* $(z,0)$ *in* $\Phi^{-1}(\mathbb{C}\times\{0\}^{\mu})$*:*

$$d\Phi_{(z,0)} = \begin{pmatrix} \frac{\partial\varphi}{\partial z}(z) & g(z) \\ 0 & I_\mu \end{pmatrix} \qquad \text{with } g=(g_1,\dots,g_\mu).$$

Hence: $\Phi(z,0)=(\varphi(z),0)\in D_\Phi \Leftrightarrow \varphi(z)\in D_\varphi=\{0\}$. *(see also [Lo], page 68 and page 76)*
According to Looijenga we may choose a good representative $\Phi: X\to D\times S$ *of* Φ *with* $D\subseteq\mathbb{C}$ *and* $S\subseteq\mathbb{C}^\mu$. *Denote by* D_Φ *the discriminant of* Φ. *For any* $\lambda\in S$, *set* $l_\lambda = D\times\{\lambda\}$, *and thus we have* $l_\lambda\cap D_\Phi=\{(x_1,\lambda),\dots,(x_\mu,\lambda)\}$, *where* $\mu=$ *Milnor number of* $(\mathrm{V},0)$ *is also equal to the multiplicity* $m_0(D_\Phi)$ *of the discriminant of* Φ.
We choose now any $x\in D\setminus\{x_1,\dots,x_\mu\}$ *as base point and set* $s=(x,\lambda)$, $s_i=(x_i,\lambda)$ *for* $1\le i\le\mu$. *Let us recall that* $V_s=\Phi^{-1}(s)$ *is a Milnor fiber of* $(\mathrm{V},0)$ *and that* $\tilde{H}_{n-1}(V_s,\mathbb{Z})\cong\mathbb{Z}^\mu$. *Under these circumstances* $V_{s_i}=\Phi^{-1}(s_i)$ *has exactly one singular point* (z_i,λ) *of type* A_1. *This implies that there exists a small neighbourhood* U_i *of* (z_i,λ) *in* $\pi^{-1}(\lambda)=\mathbb{C}^n\times\{\lambda\}$ *and local coordinates* $(v_1,\dots,v_n)$ *on* U_i *such that* $\Phi(v_1,\dots,v_n)=s_i+\sum_{k=1}^n v_k^2$. *We may then fix a small disc* D_i *of radius* ρ *in* l_λ, *centered at* s_i, *such that* $\Phi_| : X_i\setminus V_{s_i}\to D_i\setminus\{s_i\}$ *is a Milnor fibration, where* $X_i=U_i\cap\Phi^{-1}(D_i)$.
We then choose paths $u_i:[0,1]\to l_\lambda$ *from* z *to* z_i *in* l_λ *simultaneously for all* $1\le i\le\mu$ *such that the* u_i *have neither any intersection with each other apart from* z *nor any selfintersection, and that, furthermore,* u_i *passes through the boundary* ∂D_i *of* D_i *at the point* $s_i+\rho$ *at the time* $\theta_i\in[0,1]$.

Let us now introduce some further notation: $V_{s_i+\rho}=\Phi^{-1}(s_i+\rho)$, $F_i=V_{s_i+\rho}\cap U_i$, $S_i=\{(v_1,\dots,v_n)\in F_i \mid \text{all } v_k \text{ are real}\}=\{(v_1,\dots,v_n)\in U_i\mid \sum_{k=1}^n v_k^2=\rho,\ \text{all } v_k \text{ real}\}$ *and* $\gamma_i=u_i[0,\theta_i]$.

Lamotke shows in [Lam] 5.5 that there is a natural way to identify S_i *with the zero section of the tangent bundle* $\mathbf{T}S^{n-1}$ *of the real (n-1)–sphere* S^{n-1} *via some real analytic diffeomorphism from* F_i *to* $\mathbf{T}S^{n-1}$. *Moreover, he shows in 6.2 that* S_i *can be transported along* u_i *into* V_s *in the following sense:*

The inverse image $\Phi^{-1}(\gamma_i)$ *fibers trivially, which ensures the existence of an embedding* $j: V_{s_i+\rho}\times\gamma_i\to\pi^{-1}(\lambda)$ *with the following property:*

$j(V_{s_i+\rho}\times\gamma_i)=\Phi^{-1}(\gamma_i)$, $j((z,\lambda),s_i+\rho)=(z,\lambda)$ *and* $\Phi\circ j((z,\lambda),u_i(t))=u_i(t)$ *for* $(z,\lambda)\in V_{s_i+\rho}$ *and* $t\in[0,\theta_i]$.

But now the image of $S_i\times\{s\}$ *under* j *is an embedded (n-1)–sphere in* V_s. *We denote its homology class in* $\tilde{H}_{n-1}(V_s,\mathbb{Z})$ *by* Δ_i *and call it a* <u>cycle</u> <u>vanishing</u> <u>along</u> u_i *or, in a shorter form, a* <u>vanishing</u> <u>cycle</u>. *The system* $(\Delta_1,\dots,\Delta_\mu)$ *forms a basis of* $\tilde{H}_{n-1}(V_s,\mathbb{Z})$.
Let us now suppose that x *was a point on the boundary* ∂D *of* D *and that we have ordered the system* $(s_1,\dots,s_\mu)$ *in such a way that the paths* u_i *are numbered in the order in which they start in* $s=(x,\lambda)$, *where we count clockwise and start at the "boundary" of the "disc"* $l_\lambda=D\times\{\lambda\}$ [3]. *This basis of vanishing cycles coming from such a system of paths* $(u_1,\dots,u_\mu)$ *is called* <u>geometric</u> *or* <u>distinguished</u> *(for* $(\mathrm{V},0)$*). (see [Ebel], 1.4, or [Hus],1.2.3)*

Remark 5

a. In order to construct the geometric basis we had to make quite a few choices. But fortunately, once the base point s is fixed, the homotopy type of the paths u_i in l_λ, with the condition of not passing through any other critical value of Φ than

[3] If x has been chosen in ∂D the paths u_i map obviously to $l_\lambda\cup\{s\}$

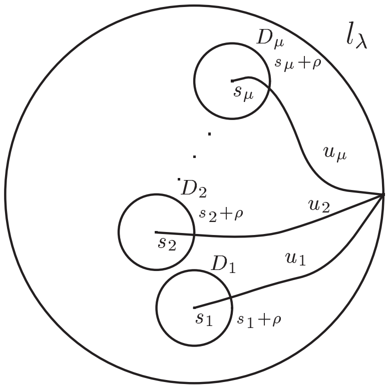

Figure 1

s_i and having no selfintersection, determines the vanishing cycle Δ_i uniquely up to the orientation, which is given by the choice of an orientation of $j\,(S_i \times \{s\})$. This, however, is enough to ensure that the properties of the Dynkin diagram, that we will construct next, are as good as required. Moreover, the construction of the Dynkin diagram is independent of the choice of base points s and λ, and one could also have replaced the line $\mathbb{C} \times \{\lambda\}$ by any other generic[4] line near to zero in $\mathbb{C} \times \mathbb{C}^\mu$, and then have chosen a suitable representative of Φ with a similar splitting in the base space (see e.g. [Ebel], page 2 and (1.5)).

b. *We claim that we even can replace a generic line l in $\mathbb{C} \times \mathbb{C}^\mu$ by a smooth curve C_t with the property (P), construct a system of μ vanishing cycles as above and get indeed a geometric basis, i.e. get a basis that can already be constructed in a generic line.*
A smooth curve C_t, with t sufficiently small has the property (P) if C_t is a member of a 1-parameter family $\{C_\theta\}_{\theta \in (\mathbb{C},o)}$ of smooth curves in $\mathbb{C} \times \mathbb{C}^\mu$ such that:

- *(i)* $0 \in C_0$
- *(ii) The intersection multiplicity $m_0\,(C_0, D_\Phi)$ of C_0 and D_Φ equals the multiplicity $m_0\,(D_\Phi)$ of D_Φ which is just the Milnor number μ of* (V, o)*.*
- *(iii) C_θ intersects D_Φ transversally for any* $0 < \theta \ll 1$*.*
- *(iv) C_θ intersected with the boundary ∂D_Φ of D_Φ is the empty set $\emptyset$.*

Proof: Some generic line l will intersect some of the C_θ in a point s, not contained in D_Φ. (For this note that D_Φ is a hypersurface, the family has dimension two and is by (iii) and (iv) in a certain sense in general position w.r.t. D_Φ, as is any generic line.) The property (P) implies that C_t intersects D_Φ in μ regular points.

[4]Generic means "parallel to a line through the origin, which is not contained in the tangent cone of the discriminant of Φ".

Each of these points can be joined to s by a continous path in $\left(\bigcup_{\theta\in(\mathbb{C},0)} C_\theta\right) \setminus D_\Phi$. Each of these paths gives rise to a vanishing cycle as constructed above. The fact that the set of all vanishing cycles (obtained in such a way) is a single orbit under the action of the monodromy group (unless we are in the trivial case of an A_1–singularity) implies that the set of all vanishing cycles can already be constructed by the paths in an arbitrary generic line (e.g. $\mathbb{C} \times \{\lambda\}$). (see e.g. [Lo], page 118, or [Ebel], page 10) Since the vanishing cycles derived from paths joining the μ points in $D_\Phi^{reg} \cap C_t$ to s form certainly a basis of $\tilde{H}_{n-1}(V_s, \mathbb{Z})$, this completes the proof of the claim.

□

c. Instead of using an $\mathcal{R}$–miniversal unfolding of φ to construct a geometric basis we could equally well have worked with a miniversal deformation p of $(\mathrm{V}, 0)$ (see e.g. [Lo], (7.4)) using the fact that the discriminant D_p of p has multiplicity $m_0(D_\Phi)$ equal to the Milnor number μ of $(\mathrm{V}, 0)$ (see e.g. [Tei], prop. 5.5.2).

We will now define the Dynkin diagram of $(\mathrm{V}, 0)$:

Definition 7
With the same notation as in construction 2 suppose that $(\Delta_1, \ldots, \Delta_\mu)$ is a geometric basis in $\tilde{H}_{n-1}(V_s, \mathbb{Z})$. The Dynkin diagram of $(\mathrm{V}, 0)$ corresponding to $(\Delta_1, \ldots, \Delta_\mu)$ is the graph with μ vertices $\{e_1, \ldots, e_\mu\}$ where e_i is joint to e_j $(i \neq j)$ by $|\langle\Delta_i, \Delta_j\rangle|$ edges. (If $\langle\Delta_i, \Delta_j\rangle$ is negative, one uses a dotted line for the edge.) Here $\langle.,.\rangle$ denotes the bilinear form on $\tilde{H}_{n-1}(V_s, \mathbb{Z})$ given by the intersection number of two cycles.

Remark 6

a. The Dynkin diagram constructed in this way is certainly dependent on the choice of the above geometric basis. But any two Dynkin diagrams obtained from geometric bases are strongly equivalent in the sense that they can be obtained from each other by a finite number of quite simple operations on the corresponding geometric bases, namely changing the orientation of a basis element or applying a standard generator of the braid group to the basis (see e.g. [Hus], thm. 2.2.3). A similar result is also true if one allows a somewhat wider range of bases, the so called weakly distinguished bases for the construction of a Dynkin diagram. For further information on this see [Ebel], 1.4 and 1.5.

b. The equivalence of the Dynkin diagrams is sufficient to ensure that such a graph determines some important invariants of the singularity in question, namely its Milnor lattice, its set of vanishing cycles, its monodromy group and also the conjugacy class of the generic monodromy operator (see e.g. [Ebel], 1.5). This implies that the Dynkin diagram of a singularity is a useful tool in the classification of singularities; especially since the Dynkin diagram can be defined for a much wider class than only the hypersurface singularities. However, we will only use one particular property of "the" Dynkin diagram of $(\mathrm{V}, 0)$, which will be formulated in the next theorem.

c. Before we move on, I would just like to remark that there is also a slightly different definition of Dynkin diagrams of (V, o), related to the choice of its "generator" φ, which might seem more convenient in some respects; namely the connection to classical Dynkin diagrams and Weyl groups. The monodromy group will turn out to be a group of reflections. However, the results obtained so far and in the next theorem still hold in this version.
Replace $\varphi : (\mathbb{C}^{\mathrm{n}}, o) \to (\mathbb{C}, o)$ by $\tilde{\varphi} : (\mathbb{C}^{n} \times \mathbb{C}^{m}, o) \to (\mathbb{C}, o)$ with $\tilde{\varphi}(z,x) = \varphi(z) + x_1^2 + \ldots + x_m^2$, then we get the following:
   (i) $(\tilde{\varphi}^{-1}(0), o)$ is still an isolated hypersurface singularity.

   (ii) The Milnor numbers of $\varphi^{-1}(0)$ and $\tilde{\varphi}^{-1}(0)$ are equal.

   (iii) The Tjurina numbers of $\varphi^{-1}(0)$ and $\tilde{\varphi}^{-1}(0)$ are equal.

   (iv) There exists a geometric basis $\left\{\tilde{\Delta}_1, \ldots, \tilde{\Delta}_\mu\right\}$ for $(\tilde{\varphi}^{-1}(0), o)$ such that

$$\mid \langle \tilde{\Delta}_i, \tilde{\Delta}_j \rangle \mid = \mid \langle \Delta_i, \Delta_j \rangle \mid \text{ for } i \neq j.$$

Moreover, if we choose $m \in \mathbb{N}_0$ such that $m + n \equiv 3 \pmod 4$ we have that $\langle \tilde{\Delta}_i, \tilde{\Delta}_i \rangle = -2$ for $1 \leq i \leq \mu$ and the monodromy group is indeed generated by the reflections $a \mapsto a + \langle a, \tilde{\Delta}_i \rangle \tilde{\Delta}_i$ on the orthogonal complement of $\tilde{\Delta}_i$ with $i = 1, \ldots, \mu$, which is then an easy consequence of the Picard-Lefschetz formulae (see e.g. [Hus], 2.3, and [Lo], (7.17)).

Let us now state the theorem of Gabrielov on the connectedness of the Dynkin diagram which will turn out to be a key result in Némehtis proof.

Theorem 3 (Gabrielov)
Any Dynkin diagram of (V, o) *is connected,*
i.e. if any two disjoint subsets of a geometric basis for (V, o) *are mutually orthogonal to each other, then one of these subsets is empty.*

In order to proof this result, Gabrielov constructs first a geometric basis along the ideas of Arnold [Arn], as outlined above, and shows the independence of the Dynkin diagram of the chosen paths. He then uses an indecomposable covering of the bifurcation diagram Σ and the connectedness of the Dynkin diagram A_2, corresponding to $\varphi : \left(\mathbb{C}^2, o\right) \to (\mathbb{C}, o){:}(x, y) \mapsto (x^2 + y^3)$, to which any hypersurface singularity (indeed every ICIS) with Milnor number $\mu > 1$ deforms (in the sense of [Lo], (7.15)), to conclude.
Again the connectedness result holds also for Dynkin diagrams derived from a weakly distinguished basis.

§2 Némethi's claim and proof

We have now all the tools needed to go along the lines of Némethi's proof. Let us therefore now define what it means for $f : \left(\mathbb{C}^2, 0\right) \to \left(\mathbb{C}^3, 0\right)$(or rather its image) to be "good".

Definition 8

a. *Let $F : \left(\mathbb{C}^3, 0\right) \to (\mathbb{C}, 0)$ be a hypersurface germ. We call F a* <u>*good*</u> *germ if there exists a coordinate system $(\omega_1, \omega_2, \omega_3)$ in $\left(\mathbb{C}^3, 0\right)$ such that the following conditions are satisfied:*

   (i) $F_| : (\{\omega_1 = 0\}, 0) \to (\mathbb{C}, 0)$ defines an isolated plane curve singularity, i.e. $V' = F^{-1}(0) \cap \{\omega_1 = 0\}$ is an isolated plane curve singularity.

   (ii) $\frac{\partial F}{\partial \omega_1} \notin \left(\omega_1, \frac{\partial F}{\partial \omega_2}, \frac{\partial F}{\partial \omega_3}\right) \mathcal{O}_{\mathbb{C}^3,0}$

b. *A hypersurface germ $(\mathrm{X}, 0) \subseteq \left(\mathbb{C}^3, 0\right)$ is said to be* <u>*good*</u> *if there exists a good hypersurface germ $F : \left(\mathbb{C}^3, 0\right) \to (\mathbb{C}, 0)$ representing $(\mathrm{X}, 0)$, i.e. $(\mathrm{X}, 0) = (F^{-1}(0), 0)$. Otherwise $(\mathrm{X}, 0)$ is said to be bad, i.e. for any representative F and any coordinate system $(\omega_1, \omega_2, \omega_3)$ satisfying (i) we have $\frac{\partial F}{\partial \omega_1} \in \left(\omega_1, \frac{\partial F}{\partial \omega_2}, \frac{\partial F}{\partial \omega_3}\right) \mathcal{O}_{\mathbb{C}^3,0}$.*

Lemma 1
Let $F : \left(\mathbb{C}^3, 0\right) \to (\mathbb{C}, 0)$ be a good hypersurface germ. Then there exists a coordinate system $(\omega_1, \omega_2, \omega_3)$ in $\left(\mathbb{C}^3, 0\right)$ such that the following holds:

   (i) $F_| : (\{\omega_1 = 0\}, 0) \to (\mathbb{C}, 0)$ defines an isolated plane curve singularity, i.e. $V' = F^{-1}(0) \cap \{\omega_1 = 0\}$ is such.

   (ii) $\frac{\partial F}{\partial \omega_1} \notin \left(\omega_1, \frac{\partial F}{\partial \omega_2}, \frac{\partial F}{\partial \omega_3}\right) \mathcal{O}_{\mathbb{C}^3,0}$

   (iii) $X \cap \{\omega_1 = 0\} \cap \left\{\frac{\partial F}{\partial \omega_1} = 0\right\} \subseteq \{0\}$ with $X = F^{-1}(0)$

<u>Proof:</u> Since $(\mathrm{X}, 0)$ is a good hypersurface germ, we find a generator F and a system of coordinates $\{\omega_1', \omega_2', \omega_3'\}$ satisfying (i) and (ii). Given $z, z' \in \mathbb{C}$, we get with $\omega_1 = \omega_1'$, $\omega_2 = \omega_2' + z\omega_1'$ and $\omega_3 = \omega_3' + z'\omega_1'$ that

$$\begin{aligned}
\frac{\partial F}{\partial \omega_1} &= \frac{\partial F}{\partial \omega_1'}\frac{\partial \omega_1'}{\partial \omega_1} + \frac{\partial F}{\partial \omega_2'}\frac{\partial \omega_2'}{\partial \omega_1} + \frac{\partial F}{\partial \omega_3'}\frac{\partial \omega_3'}{\partial \omega_1} \\
&= \frac{\partial F}{\partial \omega_1'}\frac{\partial \omega_1}{\partial \omega_1} + \frac{\partial F}{\partial \omega_2'}\frac{\partial (\omega_2 - z\omega_1)}{\partial \omega_1} + \frac{\partial F}{\partial \omega_3'}\frac{\partial (\omega_3 - z'\omega_1)}{\partial \omega_1} \\
&= \frac{\partial F}{\partial \omega_1'} - z\frac{\partial F}{\partial \omega_2'} - z'\frac{\partial F}{\partial \omega_3'}.
\end{aligned}$$

This means that such a change of coordinates allows us to add to $\frac{\partial F}{\partial \omega_1'}$ any linear combination of the remaining partial derivatives of F, which by (i) cannot both be zero, and hence to achieve (iii) without interfering in (i) and (ii).

□

We state now Némethi's claim and reproduce his proof in the case $n = 2$.

Theorem 4
Let $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ *be an injective map germ such that* $(\mathrm{X}, 0) = (\mathrm{im}(\mathrm{f}), 0)$ *is good. Then we have* $\mathbf{rank}(df(0)) \geq 1$, *and, moreover,* $(\mathrm{X}, 0)$ *is an equisingular family of isolated plane curve singularities over the smooth base space* $(\mathbb{C}, 0)$.

Proof:

0. Introductory remark
Since f is injective, it is certainly finite. This implies that $X = \mathrm{im}(f)$ is a hypersurface (see [Nar], page 71, prop. 5), and thus, our hypothesis makes sense. Furthermore, $(\mathrm{X}, 0)$ is irreducible since $(\mathbb{C}^2, 0)$ is so and f is injective.
Before we now start with the proof, let us give a list of the contents of the proof.

   1. Some notation
   (I) Proof that the $\mathbf{rank}(df(0))$ is greater than 0.
   2. V is an isolated plane curve singularity, $\psi_{|\varphi^{-1}(\lambda)}$ is injective and $V \cap C_\psi = \{0\}$.
   3. V not regular.
   4. Miniversal deformation p of $(\mathrm{V}, 0)$ and base change b.
   5. Definition of S.
   6. The image q of p as a deformation of $(\mathrm{V}', 0)$ and good representatives of p and q.
   7. The $\mathcal{R}$–miniversal unfolding Θ of ψ, its restriction $\Theta_|$ and the base changes b_2 & b_3.
   8. The non-triviality of $db_2(0)$.
   9. The construction of a geometric basis $\{\Delta_1, \ldots, \Delta_\mu\}$ for $(\mathrm{V}, 0)$.
   10. An important remark on the constructing machinery.
   11. The transport of the geometric basis $\{\Delta_1, \ldots, \Delta_\mu\}$.
   12. The Dynkin diagram of $(\mathrm{V}', 0)$.
   13. $\psi_| : V \to V'$ is an isomorphism.
   14. $df(0)$ has rank greater than 0.
   (II) Proof that $(\mathrm{X}, 0)$ is an equisingular family of isolated plane curve singularities.
   15. X_{sing} has dimension one.
   16. $(\mathrm{X}, 0)$ is equisingular.

1. Some notation:
We will denote coordinate systems in $(\mathbb{C}^3, 0)$ by $(\omega_1, \omega_2, \omega_3)$. Given such a coordinate system we define $\varphi : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ and $\psi : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ by $\varphi = \omega_1 \circ f$ and $\psi = (\omega_2 \circ f, \omega_3 \circ f)$ respectively. We denote the zero locus $\varphi^{-1}(0)$ of φ by V and its image $\psi(V)$ under ψ by V'. Via the identification of $(\mathbb{C}^2, 0)$ with $\{\omega_1 = 0\}$

and the resulting embedding of V' into $(\mathbb{C}^3, 0)$ we identifiy V' with $X \cap \{\omega_1 = 0\}$. We define μ and μ' to be the Milnor numbers of $(V, 0)$ and $(V', 0)$ respectively and τ to be the Tjurina number of $(V, 0)$.
If α is any map (germ), C_α will denote the critical locus of α and $D_\alpha = \alpha(C_\alpha)$ its discriminant. Similarly, for any analytic set A, A_{sing} will be the singular locus of A and A_{reg} its complement in A.

(I) **Claim: rank**$(df(0)) \geq 1$

Choose $(\omega_1, \omega_2, \omega_3)$ such that lemma 1 is satisfied.

2. **Claim:**

a. $V = \varphi^{-1}(0)$ *is an isolated plane curve singularity.*

b. The restriction of ψ *to* $\varphi^{-1}(\lambda)$ *for any* $\lambda \in (\mathbb{C}, 0)$ *is injective.*

c. $V \cap C_\psi = \{0\}$

Proof:

a. By lemma 1 (i): V' has an isolated singularity at 0.

$\Leftrightarrow$ If $x \in V' \setminus \{0\}$, then (X, x) is regular and $\{\omega_1 = 0\} \pitchfork X$ at x.
$\Leftrightarrow$ If $x \in V' \setminus \{0\}$, then (X, x) is regular and $T_x X + T_x\{\omega_1 = 0\} = T_x\mathbb{C}^2$.
$\Leftrightarrow$ If $x \in V' \setminus \{0\}$, then (X, x) is regular and $df_z\left(T_z\mathbb{C}^2\right) + T_x\{\omega_1 = 0\} = T_x\mathbb{C}^2$ for $z \in f^{-1}(x)$.
$\Leftrightarrow$ If $x \in V' \setminus \{0\}$, then (X, x) is regular and $f \pitchfork \{\omega_1 = 0\}$ at x.
$\Leftrightarrow$ If $x \in V' \setminus \{0\}$, then $(f^{-1}(\{\omega_1 = 0\}), f^{-1}(x))$ is regular.
$\Leftrightarrow$ $f^{-1}(\{\omega_1 = 0\}) = V$ has an isolated singularity at 0.

b. Suppose x, $y \in \varphi^{-1}(\lambda)$ such that $\psi(x) = \psi(y)$. Then:

$$(\omega_1, \omega_2, \omega_3)(f(x)) = (\lambda, \psi(x)) = (\lambda, \psi(y)) = (\omega_1, \omega_2, \omega_3)(f(y))$$

and hence $f(x) = f(y)$. Since f is injective this implies that $x = y$ and thus $\psi_{|\varphi^{-1}(\lambda)}$ is injective.

c. Given $x \in V \cap C_\psi$.

$$\begin{array}{lcll} \text{Then :} & x \in V & \Rightarrow & 0 = \varphi(x) = \omega_1(f(x)) \\ & & \Rightarrow & f(x) \in X \cap \{\omega_1 = 0\} \\ \text{and} & x \in C_\psi & \Rightarrow & \det(d\psi(x)) = 0 \\ & & \Rightarrow & 0 = c(x)\det(d\psi(x)) = \frac{\partial F}{\partial \omega_1}(f(x)) \\ & & & \text{where } c(x) \text{ is the conductor of } f^{-1}(X_{sing}) \\ & & \Rightarrow & f(x) \in \left\{\frac{\partial F}{\partial \omega_1} = 0\right\} \end{array}$$

Hence by lemma 1 (iii) $f(x) = 0$ and thus by the injectivity of f, $x = 0$. The other inclusion is clear.

□

3. **Claim:** *W.l.o.g. we may suppose that V is not regular.*

Proof: If V is regular, then the $\mathbf{rank}(d\varphi(0)) = 1$ and thus $\mathbf{rank}(df(0)) \geq 1$, since $d\varphi(0) = (d\omega_1(f(0)))_{1\times 3}(df(0))_{3\times 2}$.

□

4. As in theorem 1 we choose $e_1 \equiv 1, e_2, \ldots, e_\tau$ in $\mathcal{O}_{\mathbb{C}^2,0}$ such that their images in $\mathbf{T}^1_\varphi$ form a basis of $\mathbf{T}^1_\varphi$, set $\mathcal{X} = \left\{(z,\lambda) \in \left(\mathbb{C}^2 \times \mathbb{C}^\tau, 0\right) \mid \varphi(z) + \sum_{i=1}^{\tau} \lambda_i e_i(z) = 0\right\}$ and get that $(\mathrm{V},0) \hookrightarrow (\mathcal{X},0) \xrightarrow{p} (\mathbb{C}^\tau,0)$ is a miniversal deformation of $(\mathrm{V},0)$, where p denotes the canonical projection onto $(\mathbb{C}^\tau,0)$.

Claim: *If we define a map germ $b : (\mathbb{C},0) \to (\mathbb{C}^\tau,0)$ by $b(t) = (t,0,\ldots,0)$, then $\varphi : \left(\mathbb{C}^2,0\right) \to (\mathbb{C},0)$ is isomorphic to the deformation of $(\mathrm{V},0)$ induced from p via the base change b.*

Proof: Define $b^*\mathcal{X} = \{((y,\lambda),t) \in (\mathcal{X} \times \mathbb{C},0) \mid \lambda = p(z,\lambda) = b(t) = (t,0,\ldots,0)\}$.

Thus, the induced deformation is given by $pr_2 : (\mathrm{b}^*\mathcal{X},0) \to (\mathbb{C},0)$ where pr_2 is the canonical projection onto $(\mathbb{C},0)$. We define a map germ from $\left(\mathbb{C}^2,0\right)$ to $(\mathrm{b}^*\mathcal{X},0)$ by $z \mapsto ((z,(-\varphi(z),0,\ldots,0)),-\varphi(z))$. This map germ is obviously biholomorphic and leaves the embedded $(\mathrm{V},0)$ invariant. Hence it is an isomorphism.

□

Next we choose a good representative $p : \mathcal{X} \to B$ of the miniversal deformation $p : (\mathcal{X},0) \to (\mathbb{C}^\tau,0)$ of $(\mathrm{V},0)$ in the sense of construction 1. For $\lambda \in B$ we will denote $p^{-1}(\lambda)$ by V_λ. Let us recall that for $\lambda \in B \setminus D_p$, V_λ is a Milnor fiber of $(\mathrm{V},0)$ (according to remark 4, a.) and that for $\lambda \in (D_p)_{reg}$, V_λ has exactly one singular point of type A_1. (see e.g. [Tei], 5.4.2).

5. In part 9. we are going to construct a geometric basis for $(\mathrm{V},0)$ via a generic line in B. However, to "ensure" that this basis embeds in one for $(\mathrm{V}',0)$, Némethi restricts the set in which he chooses the line to a two–dimensional linear subspace of B, with some suitable properties.

Claim: *There exists a two–dimensional linear subspace of $\mathbb{C}^\tau$ such that its intersection S with B has the following properties:*

a. $b(\mathbb{C}) = \{(t,0,\ldots,0) \in B \mid t \in \mathbb{C}\} \subset S$

b. $((D_p \cap S) \setminus \{0\}) \subseteq (D_p)_{reg}$

c. $S \pitchfork (D_p)_{reg}$

d. If $\lambda \in ((D_p \cap S) \setminus \{0\})$ and (z_λ,λ) denotes the corresponding A_1–singularity in V_λ,
then: (i) $z_\lambda \notin C_\psi$
and (ii) whenever $(z,\lambda) \in V_\lambda$ with $z \neq z_\lambda$, then $\psi(z) \neq \psi(z_\lambda)$.
(i.e. $\psi(z_\lambda)$ is neither a critical value of ψ nor a genuine double point.)

Proof: Let $\mathcal{N}$ be $\left\{\lambda \in (D_p)_{reg} \mid \text{the } A_1\text{–singularity } (z_\lambda,\lambda) \text{ satisfies (i) and (ii)}\right\}$. This is obviously a Zariski open subset of D_p. Let us for the moment suppose

that $\mathcal{N} \neq \emptyset$ holds. Then $\mathcal{N}$ is open and dense in D_p and thus its complement R in D_p has dimension strictly less than $\dim(D_p) = \tau - 1$. Furthermore, $b(\mathbb{C})$ intersects R in 0. This enables us to choose a linear complement L of $b(\mathbb{C}) = \langle(1, 0, \ldots, 0)\rangle_{\mathbb{C}}$ in $\mathbb{C}^\tau$ such that $\dim(L \cap R) \leq \dim(L) - 2 = \tau - 3$ holds. Hence $\dim(L \cap R)$ is not complementary to $\dim(b(\mathbb{C})) = 1$ and we may choose some $v \in L$ such that $\langle v, (1, 0, \ldots, 0)\rangle_{\mathbb{C}} \cap R$ is 0. Replacing v possibly by an arbitrarily small perturbed $u \in L$ we can achieve that $\langle u, (1, 0, \ldots, 0)\rangle_{\mathbb{C}}$ is transverse to $(D_p)_{reg}$ and still $\langle u, (1, 0, \ldots, 0)\rangle_{\mathbb{C}} \cap R$ equals 0. This plane satisfies the requirements of the claim.
It remains to show $\mathcal{N} \neq \emptyset$:
We consider the family $\bigcup_{\lambda \in B} V_\lambda$ of plane curves with $V_0 = V$. Since $V \cap C_\psi$ equals $\{0\}$, we have that for sufficiently small $\lambda \in B$ the intersection $V_\lambda \cap C_\psi$ is still finite. Similarly, the injectivity of $\psi_{|V}$ implies that for small $\lambda \in B$ only finitely many points are not in the injectivity domain of φ, i.e. they map to multiple points. Thus, we may suppose that our representative $p : \mathcal{X} \to B$, or more precisely B, was sufficiently small in the first place to ensure this for all $\lambda \in B$.
Consider now the following extended deformation of $(\mathrm{V}, 0)$:

$$p' : (\mathcal{X}', 0) \to \left(\mathbb{C}^\tau \times \mathbb{C}^{2\times 2}, 0\right)$$

with

$$\mathcal{X}' = \left\{(z, \lambda, A) \in \left(\mathbb{C}^2 \times \mathbb{C}^\tau \times \mathbb{C}^{2\times 2}, 0\right) \mid \varphi(z+Az) + \textstyle\sum_{i=1}^{\tau} \lambda_i e_i(z+Az) = 0\right\}$$

and the restriction of the canonical projection p' onto $\left(\mathbb{C}^\tau \times \mathbb{C}^{2\times 2}, 0\right)$, and set

$$\mathcal{X}_{\lambda,A} = p'^{-1}(\lambda,A) = \{(z, \lambda, A) \in \mathcal{X}' \mid (z + Az, \lambda) \in V_\lambda\}.$$

Suppose $p' : \mathcal{X}' \to B'$ is a good representative of the germ p'. Given $\bar{\lambda} \in (D_p)_{reg}$ such that $\left(\bar{\lambda}, 0\right) \in B'$ then we can find an $\bar{A} \in \mathbb{C}^{2\times 2}$ arbitrarily near to 0 such that

(α') $\left(\bar{\lambda}, \bar{A}\right) \in (D_{p'})_{reg}$ and $\mathcal{X}_{\bar{\lambda},\bar{A}}$ has a single A_1–singularity $\left(\bar{z}, \bar{\lambda}, \bar{A}\right)$ and

(β') this $\bar{z}$ fulfills now (i) and (ii), i.e. $\bar{z} \notin C_\psi$, but in the injectivity domain of ψ.

(Note for this that the choice of A implies a slight perturbation of the coordinates in $(\mathbb{C}^\tau, 0)$ which suffices to omit the finite number of bad points.)
Restricting p' to $\mathcal{X}'' = \left\{(z, \lambda) \in \left(\mathbb{C}^2 \times \mathbb{C}^\tau, 0\right) \mid \varphi((I+\bar{A})z) + \sum_{i=1}^{\tau} \lambda_i e_i((I+\bar{A})z) = 0\right\}$ and recalling that $\bar{A}$ arbitrarily close to 0 implies the invertibility of $I + \bar{A}$, we get that $p'' = p'_{|} : (\mathcal{X}'', 0) \to (\mathbb{C}^\tau, 0)$ is a deformation of $(\mathrm{V}, 0)$ which is isomorphic to the original deformation $p : (\mathcal{X}, 0) \to (\mathbb{C}^\tau, 0)$ over the same base space and is hence miniversal itself. The isomorphism is given by

$$(\mathcal{X}, 0) \to (\mathcal{X}'', 0) : (z, \lambda) \mapsto \left(\left(I + \bar{A}\right)^{-1} z, \lambda\right).$$

Moreover, if we fix a good representative $p'' : \mathcal{X}'' \to B''$ of p'' with $\bar{\lambda} \in B''$, then we get:

(α'') $\bar{\lambda} \in \left(D_{p''}\right)_{reg}$ and $(p'')^{-1}(\bar{\lambda})$ has a single A_1–singularity $\left(\bar{z}, \bar{\lambda}\right)$ and

(β'') this $\bar{z}$ fulfills now (i) and (ii).

Thus, if we replace the germ $p : (\mathcal{X}, 0) \to (\mathbb{C}^\tau, 0)$ by $p'' : (\mathcal{X}'', 0) \to (\mathbb{C}^\tau, 0)$, i.e. replacing $e_1, \ldots, e_\tau$ in <u>4.</u> by $e_1 \circ \left(I + \bar{A}\right) \equiv 1, e_2 \circ \left(I + \bar{A}\right), \ldots, e_\tau \circ \left(I + \bar{A}\right)$, we may suppose that $\mathcal{N}$ is not empty.

□

We will, however, stick to p as the notation of our miniversal deformation of $(\mathrm{V}, 0)$, assuming we have made the right choice for the basis in the first place. Denote by γ_1 and γ_2 the canonical coordinates on S, given by

$$\gamma_1(\alpha u + \beta(1, 0, \ldots, 0)) = \alpha$$

and

$$\gamma_2(\alpha u + \beta(1, 0, \ldots, 0)) = \beta,$$

and denote by p_S the restriction of p to S. For $\lambda = (\gamma_1, \gamma_2) \in S$ we will again set $V_\lambda = (p_S)^{-1}(\gamma_1, \gamma_2)$.

<u>6.</u> Now we have the tools, which are necessary to construct a geometric basis for $(\mathrm{V}, 0)$, and next we will collect those tools needed for the construction of the shuttle which should carry it into a geometric basis for $(\mathrm{V}', 0)$. To get the geometric basis for $(\mathrm{V}', 0)$ we will work with an $\mathcal{R}$–miniversal unfolding of $h = F_{|\{\omega_1 = 0\}} : \left(\mathbb{C}^2, 0\right) \to (\mathbb{C}, 0)$, the generator of $(\mathrm{V}', 0)$, and it will be necessary to get a connection between this unfolding and the defining equation $\varphi : \left(\mathbb{C}^2, 0\right) \to (\mathbb{C}, 0)$ of $(\mathrm{V}, 0)$. For this we construct an auxiliary deformation of $(\mathrm{V}', 0)$, namely the image $q : (\mathcal{Y}, 0) \to (\mathbb{C}^\tau, 0)$ of $p : (\mathcal{X}, 0) \to (\mathbb{C}^\tau, 0)$ under $\psi \times id$.

Claim: *After possibly shrinking $\mathcal{X}$ and B we have that $q : \mathcal{Y} = (\psi \times id)(\mathcal{X}) \to B$ is a good representative of the deformation*

$$q : \left(\mathcal{Y} = \left\{(\psi(z), \lambda) \in \mathbb{C}^2 \times \mathbb{C}^\tau \mid (z, \lambda) \in (\mathcal{X}, 0)\right\}, 0\right) \to (\mathbb{C}^\tau, 0)$$

of $(\mathrm{V}', 0)$ where q is again the restriction of the canonical projection onto $(\mathbb{C}^\tau, 0)$.

<u>Proof:</u> Suppose $\psi \times id$ denotes a good representative of the map germ $\psi \times id$ from $\mathbb{C}^2 \times \mathbb{C}^\tau$ to $\mathbb{C}^2 \times \mathbb{C}^\tau$, and $\mathcal{X}$ is sufficiently small such that it is contained in the domain of $\psi \times id$.

Define	$r : \mathcal{X} \to [0, \infty)$	by
	$r(z,\lambda) = \Vert \psi(z) \Vert^2 + \Vert id(\lambda) \Vert^2$	
and	$\bar{r} : \mathcal{Y} = (\psi \times id)(\mathcal{X}) \to [0, \infty)$	by
	$\bar{r}(z,\lambda) = \Vert z \Vert^2 + \Vert \lambda \Vert^2$.	

Then construction 1 tells us that there exist $\varepsilon_1, \varepsilon_2 > 0$ such that $p^{-1}(0) \cap r^{-1}(\varepsilon_1)$ is compact and $p^{-1}(0) \pitchfork r^{-1}(\varepsilon')$ for all $0 < \varepsilon' \leq \varepsilon_1$, and similarly $q^{-1}(0) \cap \bar{r}^{-1}(\varepsilon_2)$ is compact and $q^{-1}(0) \pitchfork \bar{r}^{-1}(\varepsilon')$ for all $0 < \varepsilon' \leq \varepsilon_2$. Replacing ε_1 and ε_2 by their minimum ε, we get, moreover, that there exist balls B_1 and B_2 around 0 in B such

that $p^{-1}(\lambda) \pitchfork r^{-1}(\varepsilon)$ for all $\lambda \in B_1$ and $q^{-1}(\lambda) \pitchfork \bar{r}^{-1}(\varepsilon)$ for all $\lambda \in B_2$. Redefining B to be the intersection of B_1 and B_2, $\mathcal{X}$ to be the intersection of $p^{-1}(B)$ with $r^{-1}([0,\varepsilon))$ and $\mathcal{Y}$ to be the intersection of $q^{-1}(B)$ with $\bar{r}^{-1}([0,\varepsilon))$, we get that the restrictions $p : \mathcal{X} \to B$ and $q : \mathcal{Y} \to B$ are good representatives.
Moreover:

$$(\psi \times id)(\mathcal{X}) = \{(\psi(z), \lambda) \mid \lambda \in B,\ \bar{r}(\psi(z)) = r(z) < \varepsilon,\ \varphi(z) + \textstyle\sum_{i=1}^{\tau} \lambda_i e_i(z) = 0\}$$

That $q : (\mathcal{Y}, 0) \to (\mathbb{C}^{\tau}, 0)$ is actually a flat deformation of $(\mathrm{V}', 0)$ follows from the fact that $(\mathrm{V}', 0)$ is an isolated plane curve singularity and hence the restriction of q to $(q^{-1}(0), 0) = (\mathrm{V}', 0)$ is flat. (see e.g. [Tei], remark 4.5.2)

$\square$

7. Let us now consider an $\mathcal{R}$–miniversal unfolding $\Theta : (\mathbb{C}^2 \times \mathbb{C}^{\mu'}, 0) \to (\mathbb{C} \times \mathbb{C}^{\mu'}, 0)$ of $h = F_{|\{\omega_1 = 0\}} : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ as in theorem 1, (i.e. fix $g_1 \equiv 1, g_2, \ldots, g_{\mu'}$ in $\mathcal{O}_{\mathbb{C}^2,0}$ such that they project to a basis of

$$\mathbf{R}_h^1 = \mathcal{O}_{\mathbb{C}^2,0} / \left(\frac{\partial h}{\partial \omega_2}, \frac{\partial h}{\partial \omega_3} \right) \cong \mathcal{O}_{\mathbb{C}^3,0} / \left(\omega_1, \frac{\partial F}{\partial \omega_2}, \frac{\partial F}{\partial \omega_3} \right).$$

Define Θ by $\Theta(z,\sigma) = \left(h(z) + \sum_{i=1}^{\mu'} \sigma_i g_i(z), \sigma \right)$ and choose a good representative $\Theta : H \to D \times T$ as in construction 2 with $D \subseteq \mathbb{C}$ and $T \subseteq \mathbb{C}^{\mu'}$. Restricting Θ as in remark 3 e. gives a good representative of the deformation

$$\bar{\Theta} = \Theta_{|} : \left(\bar{\mathrm{H}} = \left\{ \varphi(z) + \textstyle\sum_{i=1}^{\mu'} \sigma_i g_i(z) = 0 \right\}, 0 \right) \to \left(\mathbb{C}^{\mu'}, 0 \right)$$

of $(\mathrm{V}', 0)$. The $\mathcal{R}$–versality of Θ ensures that Θ and $\bar{\Theta}$ are versal deformations of $(\mathrm{V}', 0)$ (see remark 3 e.).
Thus, there exists a map germ $b_1 : (\mathbb{C}^{\tau}, 0) \to (\mathbb{C}^{\mu'}, 0)$ such that the deformation $q : (\mathcal{Y}, 0) \to (\mathbb{C}^{\tau}, 0)$ is isomorphic to one induced from $\bar{\Theta} : (\bar{H}, 0) \to (\mathbb{C}^{\mu'}, 0)$ via b_1, and, moreover, we may assume that the representative $q : \mathcal{Y} \to B$ is small enough to get a representative $b_1 : B \to T$ of the germ b_1.

Claim: *$\varphi : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ is isomorphic to the pullback of $\bar{\Theta}$ under the base change $b_2 = b_1 \circ b : (\mathbb{C}, 0) \to (\mathbb{C}^{\mu'}, 0)$.*

Proof: It obviously suffices to show that $\varphi : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ is isomorphic to the pullback

$$pr_2 : (\mathrm{b}^*\mathcal{Y} = \{(z, \lambda, t) \in (\mathcal{Y} \times \mathbb{C}, 0) \mid \lambda = q(z,\lambda) = b(t) = (t, 0, \ldots, 0)\}, 0) \to (\mathbb{C}, 0)$$

of $q : (\mathcal{Y}, 0) \to (\mathbb{C}^{\tau}, 0)$ under b, where pr_2 is once more the projection onto $(\mathbb{C}, 0)$. Let us recall that $(z, (t, 0, \ldots, 0)) \in (\mathcal{Y}, 0)$ is equivalent to the existence of an $(x, (t, 0, \ldots, 0)) \in (\mathcal{X}, 0)$ with $\varphi(x) + t = 0$, i.e. $x \in \varphi^{-1}(-t)$. Since ψ restricted to $\varphi^{-1}(-t)$ is injective by 4., we have that the x corresponding to z is uniquely determined, and, furthermore, the map germ

$$\left(\mathbb{C}^2, 0\right) \longrightarrow (\mathrm{b}^*\mathcal{Y}, 0) : x \mapsto ((\psi(x), (-\varphi(x), 0, \ldots, 0)) - \varphi(x))$$

is an isomorphism, mapping $(\mathrm{V},0)$ to the embedded $(\mathrm{V}' = \psi(V),0)$.

□

If $i : (\mathbb{C}^{\mu'},0) \to (\mathbb{C}\times\mathbb{C}^{\mu'},0)$ denotes the natural embedding of the base space of $\bar{\Theta}$ into the base space of Θ, we define $b_3 : (\mathbb{C}^{\tau},0) \to (\mathbb{C}\times\mathbb{C}^{\mu'},0)$ by $b_3 = i\circ b_1$ and denote by $\tilde{b}_3$ the map germ or some representative respectively between the total spaces induced by the base change b_3. b_3 and $\tilde{b}_4 = (\psi\times id)\circ\tilde{b}_3$ provide us with the main means of transport in question.

8. The main reason, why b_3 seems suitable for this purpose, is the next

Claim: *The reduced Kodaira-Spencer map*

$$db_{2(0)} : \mathrm{T}_0(\mathbb{C}) = \mathbb{C} \to \mathrm{T}_0(T) \cong \mathrm{R}^1_h \cong \mathcal{O}_{\mathbb{C}^3,0}/\left(\omega_1, \tfrac{\partial F}{\partial\omega_2}\tfrac{\partial F}{\partial\omega_3}\right)$$

is not trivial, i.e. $(db_{2(0)})(1) \neq 0$.

Proof: First we note the following:

$$\begin{aligned}
(b_2)^*\bar{H} \;\equiv\; b^*\mathcal{Y} \;&\equiv\; \{(\psi(z),t) \mid (z,(t,0,\ldots,0)) \in (\mathcal{X},0),\ t\in(\mathbb{C},0)\}\\
&=\; \{(\psi(z),t) \mid t\in(\mathbb{C},0),\ \varphi(z)=t\}\\
&=\; \{(\omega_2,\omega_3,t) \mid t\in(\mathbb{C},0),\ F(t,\omega_2,\omega_3)=0\}
\end{aligned}$$

Since $\bar{H}$ equals $\left\{(\omega_2,\omega_3,\sigma)\in\left(\mathbb{C}^2\times\mathbb{C}^{\mu'},0\right) \mid h(\omega_2,\omega_3) + \sum_{i=1}^{\mu'}\sigma_i g_i(\omega_2,\omega_3) = 0\right\}$ we have that $F(t,\omega_2,\omega_3)$ has the form

$$F(t,\omega_2,\omega_3) = h(\omega_2,\omega_3) + \textstyle\sum_{i=1}^{\mu'}\sigma_i(t)g_i(\omega_2,\omega_3)$$

with $\sigma_i(.) \in \mathbb{C}\{t\}$. But thus we get $b_2(t) = (\sigma_1(t),\ldots,\sigma_{\mu'}(t))$ for $t\in(\mathbb{C},0)$ and

$$\begin{aligned}
db_{2(0)} :\ \mathrm{T}_0(\mathbb{C}) = \mathbb{C} \;&\longrightarrow\; \mathrm{T}_0(T) = \mathrm{R}^1_h\\
1 \;&\longmapsto\; \left(\tfrac{\partial\sigma_1}{\partial t},\ldots,\tfrac{\partial\sigma_{\mu'}}{\partial t}\right)(0)\\
&\equiv \textstyle\sum_{i=1}^{\mu'}\tfrac{\partial\sigma_i}{\partial t}(0)g_i(\omega_2,\omega_3)\ \left(\mathrm{mod}\ \left(\omega_1,\tfrac{dF}{d\omega_2},\tfrac{dF}{d\omega_3}\right)\right)\\
&= \frac{\partial F(t,\omega_2,\omega_3)}{\partial t}_{|t=0} = \frac{\partial F}{\partial\omega_1}(0,\omega_2,\omega_3).
\end{aligned}$$

Now we use the "goodness" of $(\mathrm{X},0)$, namely that by the choice of the coordinate system $(\omega_1,\omega_2,\omega_3)$ $\frac{\partial F}{\partial\omega_1} \notin \left(\omega_1,\frac{\partial F}{\partial\omega_2},\frac{\partial F}{\partial\omega_3}\right)\mathcal{O}_{\mathbb{C}^3,0}$, to conclude that $(db_{2(0)})(1)$ is not 0 in $\mathrm{T}_0(T) = \mathrm{R}^1_h$.

□

The non-triviality of the reduced Kodaira-Spencer map $db_{2(0)}$ implies that the embedding of $(\mathbb{C},0)$ in $\left(\mathbb{C}\times\mathbb{C}^{\mu'},0\right)$ as $(i\circ b_2)(\mathbb{C}) = b_3(b(\mathbb{C})) = b_3(\gamma_1 = 0)$ is a smooth curve. Némethi wants to use a small perturbation of this curve for the construction of the geometric basis for $(\mathrm{V}',0)$.

9. Construction of the geometric basis for $(\mathrm{V}, 0)$:
First of all we remark, that, since $b_3(\gamma_1 = 0) = b_3(\langle(1,0,\ldots,0)\rangle_{\mathbb{C}})$ is a smooth curve in the base space $D \times T$ of the $\mathcal{R}$–miniversal unfolding Θ, $b_3(\ell)$ will also be a smooth curve in $D \times T$ for any sufficiently small perturbation ℓ of $\{\gamma_1 = 0\}$. Let ℓ be such a perturbation of $\{\gamma_1 = 0\} = \langle(1,0,\ldots,0)\rangle_{\mathbb{C}}$ in S which intersects $\{\gamma_1 = 0\}$ in $\lambda_0 = (0, t_0) \notin D_p$ (e.g. $\ell : \gamma_1 + \varepsilon\gamma_2 = \eta,\ 0 < \eta \ll \varepsilon \ll 1$ & $t_0 = -\eta/\varepsilon$). This is possible by 5.a.
According to [Le], 3.5 and 3.6.4, the intersection multiplicity of $\{\gamma_1 = 0\} = \langle(1,0,\ldots,0)\rangle_{\mathbb{C}}$ with D_p is just the Milnor number μ of $(\mathrm{V}, 0)$ (see also [Tei], 5.5.2 prop. and proof). This together with 5. b. and c. ensures that ℓ intersects D_p in μ regular points $\{\lambda_1, \ldots, \lambda_\mu\}$, and the corresponding fibers V_{λ_i} have a single singularity (z_i, λ_i) of type A_1 with $z_i \notin C_\psi$ but z_i in the injectivity domain of ψ (by 5.d.). Since $\lambda_0 \notin D_p$, V_{λ_0} is a Milnor fiber, and hence, using remark 5 c., we may construct a geometric basis $(\Delta_1, \ldots, \Delta_\mu)$ in $H_1(V_{\lambda_0}, \mathbb{Z})$ for $(\mathrm{V}, 0)$ along lines similar to those outlined in construction 2.

10. By $\delta_i : S^1 \to V_{\lambda_0}$ we will denote the cycle whose homology class $[\delta_i]$ in $H_1(V_{\lambda_0}, \mathbb{Z})$ is Δ_i. In 2. we proved that ψ is injective on $\varphi^{-1}(t_0)$. Thus $\tilde{b}_4 = (\psi \times id) \circ \tilde{b}_3$ is injective on $V_{\lambda_0} = \{(z, \lambda_0) \mid z \in \varphi^{-1}(t_0)\}$; in particular $\delta_i(S^1)$ is in the injectivity domain of $\tilde{b}_4$.
Moreover, z_i is in the injectivity domain of ψ and not in C_ψ by 5.d.(ii). Thus, in fibers sufficiently close to (z_i, λ_i) the vanishing cycles derived from the path u_i joining λ_0 to λ_i will be in the injectivity domain of $\tilde{b}_4$ and not in its critical locus. As seen in the proof of 5. the number of points in each fiber V_λ, $\lambda \in B$, which violate the injectivity of ψ or lie in its critical locus are finite in number, and Némethi claims that therefore the pull back of the vanishing cycles along the u_i can be organized in such a way that they lie in the injectivity domain of $\tilde{b}_4$ and not in its critical locus for the remaining fibers as well.[5]

11. Transport of the basis $(\Delta_1, \ldots, \Delta_\mu)$ "into a basis for $(\mathrm{V}', 0)$":
Let V'_{λ_0} denote the image of V_{λ_0} in H under $\tilde{b}_4$, i.e. $V'_{\lambda_0} = \tilde{b}_4(V_{\lambda_0}) = \Theta^{-1}(b_{3(\lambda_0)})$, and ℓ' denote the image of ℓ under the base change b_3, i.e. $\ell' = b_3(\ell)$, which is a smooth curve in $D \times T$ (see 9.).
By 5.d.(i) $z_i \notin C_\psi$, and thus $\tilde{b}_4$ locally is a diffeomorphism at (z_i, λ_i). This, and the fact that (z_i, λ_i) is in the injectivity domain of $\tilde{b}_4$ (see 5.d.(ii)), ensures, that the image $(\bar{z}_i, \sigma_i) = \tilde{b}_{4(z_i, \lambda_i)}$ of (z_i, λ_i) in H is isolated in the set

$$\{(z, \sigma) \in \Theta(\ell') \mid (z, \sigma) \text{ is singular in } \Theta^{-1}(\Theta_{(z,\sigma)})\}$$

and it is indeed an A_1–singularity in

$$\Theta^{-1}(\Theta_{(\bar{z}_i, \sigma_i)}) = \Theta^{-1}(b_{3(\lambda_i)}) = \tilde{b}_4(V_{\lambda_i}).$$

[5] This last argument is not sufficient, as we will see later, and could lead to problems in a general example, although the problem can be solved in the example which we give. But even there it is anything but clear.

Némethi uses the arguments in <u>10.</u> to carry the whole constructing machinery of $(\Delta_1, \ldots, \Delta_\mu)$ over to ℓ', and thus to achieve μ vanishing cycles in $\left(V'_{\lambda_0}\right)_{reg}$ corresponding to the μ A_1–singularities $(\bar{z}_i, \sigma_i)$ over $(0, \sigma_i) = b_{3(\lambda_i)} \in D_\Theta$.[6]
Unfortunately, at the moment we can neither be sure that V'_{λ_0} is a Milnor fiber of V' nor that ℓ' is in general position (i.e. suitable for the construction of a geometric basis for $(V', 0)$). V'_{λ_0} might have several singularities $\{P_1, \ldots, P_k\}$ itself.

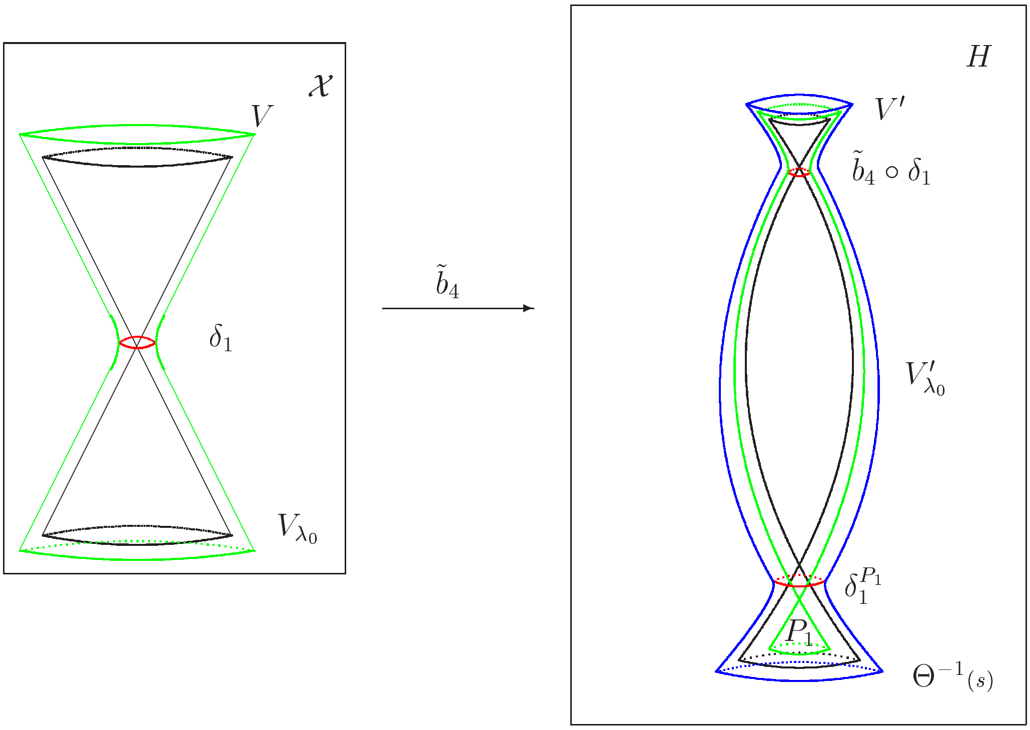

Figure 2

However, if μ_i is the Milnor number of $\left(V'_{\lambda_0}, P_i\right)$, then

$$\mu' = \mu + \sum_{i=1}^{k} \mu_i.$$

Moreover, by the product decomposition theorem[7], D_Θ has at $(0, \sigma_i) = b_{3(\lambda_i)}$ the discriminant D^i_Θ of the A_1–singularity $(\bar{z}_i, \sigma_i)$ as irreducible component and ℓ'

[6] As already remarked, we cannot be sure that <u>10.</u> holds in general, but in our example we will be able to get these vanishing cycles.

[7] see [Tei], thm. 4.8.2 and cor. 4.8.3, for the version for miniversal deformations; an analogue for versal deformations holds also

intersects D^i_Θ transversally at $(0, \sigma_i)$.
According to Némethi we may now choose[8] an arbitrarily small perturbation ℓ'' of ℓ' in $D \times T$ which intersects D_Θ in μ' regular points, and since ℓ'' is an arbitrarily small perturbation of $b_3(\gamma_1 = 0)$ they can be embedded in a familiy of smooth curves each of which intersects D_Θ transversally (apart from $b_3(\gamma_1 = 0)$) in μ' regular points. This shall ensure that ℓ'' is suitable for the construction of a geometric basis for $(\mathrm{V}', 0)$.[9]
Furthermore, since ℓ' is transverse to D^i_Θ at $(0, \sigma_i)$ and may be embedded with ℓ'' in a family of smooth curves all of which are transverse to D_Θ (apart from possibly ℓ'), the $(\bar{z}_i, \sigma_i)$ correspond to μ A_1–singularities over ℓ'', and the constructing machinery for the corresponding vanishing cycles in V_{λ_0} can be carried into ℓ'' via this family, once we have fixed a base point $s = (t, \sigma) \in \ell'' \setminus D_\Theta$. This in turn gives us μ vanishing cycles $\{\bar{\Delta}_1, \ldots, \bar{\Delta}_\mu\}$ in $H_1(\Theta^{-1}(s), \mathbb{Z})$ and we have completed the transport of the geometric basis $(\Delta_1, \ldots, \Delta_\mu)$.

12. It still remains to show that the results of 11.[10] would lead to a disconnection of the Dynkin diagram of $(\mathrm{V}', 0)$ unless the sum $\sum_{i=1}^k \mu_i$ is zero, which means that V'_{λ_0} is smooth and was therefore a Milnor fiber of V' in the first place. To achieve this we will complete $\{\bar{\Delta}_1, \ldots, \bar{\Delta}_\mu\}$ to a geometric basis by vanishing cycles $\{\bar{\Delta}^{P_i}_1, \ldots, \bar{\Delta}^{P_i}_{\mu_i}\}$ derived from the singularities P_i of V'_{λ_0}.

    Claim: *If $\mathcal{D}_V$ denotes the Dynkin diagram of $(\mathrm{V}, 0)$ corresponding to $(\Delta_1, \ldots, \Delta_\mu)$ and $\mathcal{D}_{P_i}$ the Dynkin diagram of (V'_{λ_0}, P_i) corresponding to $(\bar{\Delta}^{P_i}_1, \ldots, \bar{\Delta}^{P_i}_{\mu_i})$, then there is a Dynkin diagram $\mathcal{D}$ of $(\mathrm{V}', 0)$ corresponding to $(\bar{\Delta}_1, \ldots, \bar{\Delta}_\mu, \bar{\Delta}^{P_1}_1, \ldots, \bar{\Delta}^{P_k}_{\mu_k})$ which is just the disjoint union of $\mathcal{D}_V$ with the $\mathcal{D}_{P_i}$, i.e. $\langle \bar{\Delta}_j, \bar{\Delta}^{P_i}_m \rangle = 0$ for $j = 1, \ldots, \mu$, $i = 1, \ldots, k$, $m = 1, \ldots, \mu_i$ and $\langle \bar{\Delta}^{P_j}_n, \bar{\Delta}^{P_i}_m \rangle = 0$ for $j \neq i$, $n = 1, \ldots, \mu_j$, $m = 1, \ldots, \mu_i$.*

    Proof: Since the image of $\delta_i(S^1)$ for $i = 1, \ldots, \mu$ in V'_{λ_0} induced by the base change b_3 is contained in $(V'_{\lambda_0})_{reg}$, we may find an open set W in V'_{λ_0}, whose closure is still contained in $(V'_{\lambda_0})_{reg}$ and which contains the vanishing cycles induced by the δ_i in $(V'_{\lambda_0})_{reg}$. This W may be transported to a subset W_0 of $\Theta^{-1}(s)$, containing the cycles defining $\{\bar{\Delta}_1, \ldots, \bar{\Delta}_\mu\}$, via the family connecting ℓ' and ℓ''.
    Moreover, for each of the singularities P_i of V'_{λ_0} we get a subset W_i in $\Theta^{-1}(s)$ containing the μ_i cycles, which define $\{\bar{\Delta}^{P_i}_1, \ldots, \bar{\Delta}^{P_i}_{\mu_i}\}$ and having an empty intersection with W_j, $0 \leq j \leq k$, $j \neq i$. This proves the claim.

□

13. **Claim:** *$\psi_| : V \to V'$ is an isomorphism of analytic germs.*

[8] This is the main source of our trouble, and it seems that we do not have this choice.
[9] To achieve this Némethi applies an argument which is similar to the one in remark 5 b. but does neither contain any information about what happens on the boundary of $D \times T$ (he works with germs) nor about the intersection multiplicity of the curve through the origin with the discriminant. We will come later to the problems which result from this.
[10] We suppose now that 11. holds - as well as any previous result.

Proof: Recall that by 3. V is not regular and hence μ is not zero. The result 12. would contradict theorem 3 (on the connectedness of the Dynkin diagram), unless the singularities $P_1, \ldots, P_k$ do not exist. This proves that the Milnor numbers μ and μ' of $(\mathrm{V}, 0)$ and $(\mathrm{V}', 0)$ respectively are equal.
The injectivity of $\psi_{|V}$ implies that the numbers r and r' of irreducible components of $(\mathrm{V}, 0)$ and $(\mathrm{V}', 0)$ coincide as well.
By [Mil], thm. 10.5, or [B-G], 1.2.1, the delta invariant δ of a plane curve singularity is just $\delta = \frac{1}{2}(\mu + r - 1)$, and thus the delta invariants of $(\mathrm{V}, 0)$ and $(\mathrm{V}', 0)$ are also equal. Since V and V' are isolated plane curve singularities, this suffices to ensure that $\psi_| : V \to V'$ is an isomorphism.

□

14. **Claim: rank**$(df(0)) \geq 1$

Proof: $(\mathrm{V}', 0)$ is an isolated plane curve singularity and therefore not both x and y can project to zero in $\mathcal{O}_{\mathrm{V}',0}$. W.l.o.g. we may assume that the image of x in $\mathcal{O}_{\mathrm{V}',0}$ is not zero. Since by 13. $\psi_|^* : \mathcal{O}_{\mathrm{V}',0} \to \mathcal{O}_{\mathrm{V},0}$ is an isomorphism, $\psi_|^*(x)$ cannot be contained in $\mathcal{M}^2 = \langle x^k y^l \mid k + l \geq 2 \rangle_{\mathbb{C}}$, where $\mathcal{M} = (x, y)\mathcal{O}_{\mathbb{C}^2,0}$ denotes the maximal ideal of $\mathcal{O}_{\mathbb{C}^2,0}$. But then $\psi_|^*(x) = x \circ \psi = \omega_2 \circ f \notin \mathcal{M}^2$. Therefore the Taylor expansion of $\omega_2 \circ f$ contains a summand of degree less than or equal to one. Again by 13., $\psi_|^*(x)$ is not constant in $\mathcal{O}_{\mathrm{V},0}$, since $\psi_|^*(1)$ is so. Thus, the Taylor expansion of $\psi_|^*(x) = \omega_2 \circ f$ contains indeed a summand of degree equal to one. But then $d\omega_2 \circ f(0)$ is not zero and we have

$$0 \neq d\omega_2 \circ f(0) = \big(d\omega_2(f(0))\big)_{1\times 3} (df(0))_{3\times 2},$$

and hence $df(0)$ cannot be zero, or equivalently **rank**$(df(0)) \geq 1$.

□

This "completes" the first part of our proof, and it only remains to show that X is an equisingular family of isolated plane curve singularities.

(II) **Claim:** $(\mathrm{X}, 0)$ *is an equisingular family of isolated plane curve singularities.*

If $(\mathrm{X}, 0)$ is smooth, there is nothing to show, and hence we may as well suppose that $(\mathrm{X}, 0)$ is not smooth.

15. **Claim:** $\dim(X_{sing}) = 1$

Proof: Suppose $(\mathrm{X}, 0)$ is not smooth. Since $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ is injective, f is certainly finite and birational.
The finiteness of f implies that (via $f^* : \mathcal{O}_{\mathrm{X},0} \to \mathcal{O}_{\mathbb{C}^2,0}$) $\mathcal{O}_{\mathbb{C}^2,0}$ is a finite $\mathcal{O}_{\mathrm{X},0}$–module, and thus integral over $\mathcal{O}_{\mathrm{X},0}$. Therefore we have

$$\mathcal{O}_{\mathbb{C}^2,0} \subseteq \mathrm{Intclo}_{\mathcal{Q}(\mathcal{O}_{\mathbb{C}^2,0})}(\mathcal{O}_{\mathrm{X},0}) \subseteq \mathrm{Intclo}_{\mathcal{Q}(\mathcal{O}_{\mathbb{C}^2,0})}(\mathcal{O}_{\mathbb{C}^2,0}) = \mathcal{O}_{\mathbb{C}^2,0},$$

where the second inclusion follows from the fact that f^* embeds $\mathcal{O}_{\mathrm{X},0}$ in $\mathcal{O}_{\mathbb{C}^2,0}$ and the last equality comes from the normality of $(\mathbb{C}^2, 0)$.

The birationality of f tells us now that the quotient field $\mathcal{Q}(\mathcal{O}_{\mathrm{X},0})$ is identified with the quotient field $\mathcal{Q}(\mathcal{O}_{\mathbb{C}^2,0})$ via the isomorphism f^* which sends $g/h \in \mathcal{Q}(\mathcal{O}_{\mathrm{X},0})$ to $g \circ f/h \circ f \in \mathcal{Q}(\mathcal{O}_{\mathbb{C}^2,0})$. Thus, $\mathcal{O}_{\mathbb{C}^2,0}$ is the normalization of $\mathcal{O}_{\mathrm{X},0}$, or equivalently $f : \left(\mathbb{C}^2, 0\right) \to (\mathrm{X}, 0)$ is the normalization of $(\mathrm{X}, 0)$.
The germ $(\mathrm{X}, 0)$ is not normal, since otherwise $f_| : f^{-1}(X_{reg}) \to X_{reg}$ could be biholomorphically extended to a map $\tilde{f} : \left(\mathbb{C}^2, 0\right) \to (\mathrm{X}, 0)$ contradicting the non–smoothness of $(\mathrm{X}, 0)$ (see [Whi], page 256, thm. 3A). But $(\mathrm{X}, 0)$ is Cohen–Macaulay, as it is a hypersurface germ, and hence singular in codimension at most one (see e.g. [Voh], 1.2.4), which then implies that $\dim(X_{sing})$ is indeed one.

□

16. **Claim:** $(\mathrm{X}, 0)$ *is an equisingular family of isolated plane curve singularities.*

Proof: Since the rank of $df(0)$ is positive, we may choose a coordinate system $(\omega_1, \omega_2, \omega_3)$ in $\left(\mathbb{C}^3, 0\right)$ with ω_1 a linear form. Using the notation in 1. we have that V is smooth and its Milnor number μ is zero.
Moreover, since f is the normalization of X and the preimage V_t of each V_t' is smooth, we get that f is a simultaneous normalization of the V_t', $t \in (\mathbb{C}, 0)$. Thus, $\bigcup_{t\in(\mathbb{C},0)} V_t'$ is a δ–invariant family of plane curves. Since the number of irreducible components of V_t' is also constant, namely one by the injectivity of f and the irreducibility of V_t, we get that the Milnor number of the V_t' is constant (see [B-G], 1.2.1).
By 15., $V_t' = \psi(\varphi^{-1}(t))$, $t \in (\mathbb{C}, 0)$, must have at least one singularity. But V_t' cannot have more than one singularity, since its Milnor number is equal to the sum of the Milnor numbers of its singularities, and therefore more than one singularity would actually lead to a disconnection of the Dynkin diagram.
But a family of plane curves with isolated singularity and constant Milnor number is equisingular (see e.g. [B-G], 5.3.1).

□

This "completes" the proof of theorem 4.

□

§3 Counterexample

3.1 Localizing the problem

In the proof of theorem 4 we tried to make our arguments as rigorously as possible. By doing so we have already localized the areas where the proof could possibly go wrong, namely in the parts 10., 11. and 12. Although the arguments in the proof of 12. are quite vague, they seem very much to be correct - in particular since similar results have already been used in earlier papers to show that for an isolated hypersurface singularity, where after a small perturbation the Milnor numbers of the new singularities add up to the original Milnor number, the perturbed analytic space has again only one singularity. This leaves us with 10. and 11.

Before actually giving the example and showing that, at least in this case, it is not part 10. that causes the trouble, let us explain which problems we have to expect from 10. and 11. respectively, and what could cause them.

In 10. we tried to show that it was possible to choose the paths u_i in the line ℓ and the corresponding vanishing cycles in the fibers of the deformation p over the u_i such that none of the vanishing cycles contained a critical point of $\psi \times id$ or a point in the preimage of a genuine double point of this map. Now, why is this important?

In order to construct the vanishing cycles in V'_{λ_0}, coming from those in V_{λ_0}, we transport the *whole machinery* from $p_| : p^{-1}(\ell) \to \ell$ to $\Theta_| : \Theta^{-1}(\ell') \to \ell'$. This means the new paths are obtained from the u_i by composition with the base change b_3 and the new cycles are obtained by mapping the old ones by $\tilde{b}_4 = (\psi \times id) \circ \tilde{b}_3$. If we now have a cycle over some point in the path u_i which meets either the critical locus of $\psi \times id$ or the complement of its injectivity domain, its image under $\tilde{b}_4$ will have a singularity. Thus, the construction will no longer be suitable to obtain vanishing cycles in V'_{λ_0}, since for this none of the cycles during the pullback may have a singularity. Moreover, one cannot argue, that a small perturbation of this cycle would remove it from its singularity. For this we would also have to move nearby cycles slightly and then one of them would have a singularity (since 10. is assumed to be wrong).

And why is the argument in 10. not sufficient?

Our example will show that we may expect the bad points in $p^{-1}(\ell)$, namely those where $\psi \times id$ does not behave well, to form a subvariety of real dimension two, although in each fiber there will only be finitely many of them. The family of vanishing cycles which we have to choose over the u_i has also real dimension two and has to fit together smoothly. Thus, it is no problem to perturb each cycle on its own so that it no longer meets any bad point, but since the space $p^{-1}(\ell)$ has only real dimension four, it is not at all clear that we may perturb all cycles simultaneously such that neither of them meets a bad point but the family remains smooth. We will indeed show that for an unsuitable choice of the paths u_i it really can go wrong (see 3.2 b. claim 16).

In 11. we want to choose a small perturbation ℓ'' of ℓ' which shall intersect the discriminant D_Θ in μ' regular points, μ of which are connected to the special critical values in ℓ', giving rise to the vanishing cycles in $\left(V'_{\lambda_0}\right)_{reg}$. Furthermore, this line ℓ'' shall then be suitable for the construction of a basis of vanishing cycles. As we have seen in remark 5 b. this would be satisfied if we could embed ℓ'' into a family of curves with a suitable

curve ℓ''' (with the correct intersection multiplicity with D_Θ) through the origin such that each member of this family apart from ℓ''' intersected D_Θ transversally and if we could keep the family fixed where it meets the boundary of the base space $D \times T$ of Θ. To recover the vanishing cycles from ℓ' we would more or less have to get the same for the family of curves connecting ℓ'' to ℓ'. The problem which we face in our example is that the whole of S gets mapped into the discriminant D_Θ by the base change b_3. This means not only that ℓ' lies completely in D_Θ, but $b_3(\gamma_1 = 0)$ does so as well. So neither can we decide whether $b_3(\gamma_1 = 0)$ would be a suitable curve ℓ''' through zero, as far as its intersection multiplicity with D_Θ is concerned, nor may we even dream of joining either $b_3(\gamma_1 = 0)$ or ℓ' with any curve intersecting D_Θ in μ' points such that the joining family is fixed on the boundary of $D \times T$. This does not yet show that we cannot find a suitable small perturbation ℓ'', but it says the argument which Némethi wants to use (see [Nem], <u>13.</u>, page 11), does not work. Moreover, it makes it very unlikely that we are able to find such an ℓ''. And our example shows that this must be the place where the proof goes wrong.
What could happen?
There are different possibilities and we will try to list them:

1. Any small perturbation will intersect in more (or less) than μ' points and thus the set of cycles which we construct need not contain a basis for $H_1(V'_s, \mathbb{Z})$, even if they generate it.
   (Short: find ℓ'', but not with μ' intersection points)

2. A perturbation intersecting in μ' regular points is not a <u>small</u> perturbation — in the sense that we loose the intersection with D^i_Θ, and thus the vanishing cycles coming from V_{λ_0}.
   (Short: find ℓ'' with μ' intersection points, but loose the cycles $\Delta_1, \ldots, \Delta_\mu$)

3. Even though we find a "small" perturbation ℓ'' intersecting in μ' regular points this curve is not suitable to construct a geometric basis for $H_1(V'_s, \mathbb{Z})$.
   (Short: find ℓ'' with μ' intersection points and where $\Delta_1, \ldots, \Delta_\mu$ are preserved, but do not get a basis)

Even in the quite simple example which we have chosen, it seems to be too difficult to decide which of these possibilities describes the situation best or whether all of them do, since the discriminant space of the E_6 has complex dimension five.

3.2 The example $(x, y) \mapsto \left(x, y^2, y^3 + x^2\right)$

We will split the following calculations into two parts. The first part will show that $f : \left(\mathbb{C}^2, 0\right) \to \left(\mathbb{C}^3, 0\right)$ satisfies the hypotheses of theorem 4 but that V and V' do not relate to each other in the way claimed by Némethi. In the second part we will work through the example along the lines of the proof of theorem 4. My aim is it, to show that up to part <u>10.</u> included everything works as stated in the proof, although in part <u>10.</u> there can occur problems if we choose the wrong paths. Recalling then that the parts <u>12.</u> up to <u>16.</u> are correct, we have shown that the error lies in part <u>11.</u> and thus in fact we *may not choose* a suitable small perturbation ℓ'' of ℓ'.

a. **Claim:** *Given the situation described in the proof of theorem 4 we do not necessarily have that μ equals 0 or μ' as claimed in part* <u>15.</u> *of the proof.*

<u>Proof:</u> Here we choose our example slightly more general than the one mentioned in the introduction. Let $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ be given by $(x, y) \mapsto (x, y^2, y^3 + x^k)$, $k \geq 2$. Each of these germs can be derived from

$$(\mathbb{C}^2, 0) \longrightarrow (\mathbb{C}^3, 0) : (x, y) \mapsto (x, y^2, y^3),$$

the ordinary cuspidal edge, by a suitable coordinate change. We will now show that f satisfies the hypotheses of theorem 4 and we will determine a generator for $X = im(f)$ and a suitable coordinate system for $(\mathbb{C}^3, 0)$.

Claim 1 *f is injective.*

<u>Proof:</u> Let (x, y), $(x', y') \in (\mathbb{C}^3, 0)$ with $f(x,y) = f(x',y')$ be given.

$$\begin{array}{rlllll} \text{Then we have} & : & x = x', & y^2 = y'^2 & \& & y^3 + x^k = y'^3 + x'^k \\ & \Rightarrow & x = x', & y^2 = y'^2 & \& & y^3 = y'^3 \\ & \Rightarrow & x = x', & \& \; y = y'. & & \end{array}$$

Thus f is injective.

$\square$

Claim 2 *$F : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ with $F(x,y,z) = (z - x^k)^2 - y^3$ is a generator of $X = im(f)$, i.e. $X = F^{-1}(0)$.*

<u>Proof:</u> $(F \circ f)(x,y) = (y^3 + x^k - x^k)^2 - (y^2)^3 = 0$ hence $F^{-1}(0) \supseteq X$, and the other inclusion is also clear.

$\square$

Claim 3 *$F_| : \{z = 0\} \to \mathbb{C}$ defines an isolated plane curve singularity.*

<u>Proof:</u> We have $F_|(x,y) = x^{2k} - y^3$. Thus $(dF_|)(x,y) = (2kx^{2k-1}, 3y^2) = (0, 0)$ if and only if $(x, y) = (0, 0)$.

$\square$

Claim 4 *$\frac{\partial F}{\partial z} \notin \left(z, \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}\right)$ and $X \cap \{z = 0\} \cap \{\frac{\partial F}{\partial z} = 0\} = \{0\}$.*

<u>Proof:</u>

$$\begin{array}{rclcl} \frac{\partial F}{\partial x} & = & -2kx^{k-1}(z - x^k) & \equiv & x^{2k-1} \pmod{z} \\ \frac{\partial F}{\partial y} & = & -3y^2 & \equiv & y^2 \pmod{z} \\ \frac{\partial F}{\partial z} & = & 2(z - x^k) & \equiv & x^k \pmod{z} \end{array}$$

Thus for $k \geq 2$ we have $\frac{\partial F}{\partial z} \equiv x^k \notin (z, y^2, x^{2k-1}) = \left(z, \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}\right)$.

Moreover, let $(x, y, z) \in X \cap \{z = 0\} = \big\{x^{2k} - y^3 = 0\big\}$, then $0 = \frac{\partial F}{\partial z}(x,y,z) = 2(z - x^k) = -2x^k$ implies that x is zero and thus y is zero as well. Therefore we get $X \cap \{z = 0\} \cap \big\{\frac{\partial F}{\partial z} = 0\big\} = \{0\}$.

$\square$

But this altogether proves that $X = im(f)$ is a good hypersurface germ, and therefore f suits the hypotheses of theorem 4 with F as generator of X and the coordinate system (z, x, y) (satisfying lemma 1.3).
However, with the notation of the theorem, $(\omega_1, \omega_2, \omega_3)$ replaced by (z, x, y), we get the following:

Claim 5 $V = \big\{y^3 + x^k = 0\big\}$ *and* $\mu = 2(k - 1)$.

Proof:

$$V = (z \circ f)^{-1}(0) = \big\{y^3 + x^k = 0\big\}$$

and

$$d\varphi_{(x,y)} = d(z \circ f)_{(x,y)} = \big(kx^{k-1}, 3y^2\big).$$

Thus:

$$\mu = \dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^2,0}/(d\varphi) = \dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^2,0}/\big(x^{k-1}, y^2\big) = 2(k - 1).$$

$\square$

Claim 6 $V' = X \cap \{z = 0\} = \big\{(x, y) \mid x^{2k} - y^3 = 0\big\}$ *and* $\mu' = 2(2k - 1)$

Proof:

$$X \cap \{z = 0\} = \Big(F_{|\{z\,=\,0\}}\Big)^{-1}(0) = \big\{(x, y) \mid x^{2k} - y^3 = 0\big\}$$

and

$$\Big(dF_{|\{z\,=\,0\}}\Big)_{(x,y)} = \big(2kx^{2k-1}, -3y^2\big).$$

Thus:

$$\mu' = \dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^2,0}/\big(dF_{|}\big) = \dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^2,0}/\big(x^{2k-1}, y^2\big) = 2(2k - 1).$$

$\square$

We are assuming that $k \geq 2$. Therefore μ is neither zero nor equal to μ'. This shows that the proof of theorem 4 cannot be correct.

$\square$

b. In order to show where the proof actually goes wrong, we will proceed with this example along the lines of the proof. First of all we will describe the miniversal deformation of $(\mathrm{V}, 0)$ and the $\mathcal{R}$–miniversal unfolding of $h = F_{|\{z\,=\,0\}}$ explicitly.

Claim 7 *$e_1 \equiv 1$, $e_2 \equiv y$, $e_3 \equiv x$, $e_4 \equiv xy, \dots,$ $e_{2k-3} \equiv x^{k-2}$, $e_{2k-2} \equiv x^{k-2}y$ project to a basis of $\mathrm{T}^1_\varphi = \mathcal{O}_{\mathbb{C}^2,0}/\left(x^{k-1}, y^2\right)$ and*

$$p : (\mathcal{X}, o) \to \left(\mathbb{C}^{2(\mathrm{k}-1)}, o\right) : (z, \lambda) \mapsto \lambda$$

is a miniversal deformation of (V, o), where

$$\mathcal{X} = \left\{(x, y, \lambda) \in \left(\mathbb{C}^2 \times \mathbb{C}^{2(k-1)}, o\right) \mid x^k + y^3 + \textstyle\sum_{i=0}^{k-2} \left(\lambda_{2i+1}x^i + \lambda_{2i+2}x^i y\right) = 0\right\}$$

In particular: $\tau = \mu$.

<u>Proof:</u>

$$\tau = \dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^2,0}/(x^k + y^3, x^{k-1}, y^2) = \dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^2,0}/(x^{k-1}, y^2) = \mu$$

Thus: $\mathrm{T}^1_\varphi = \mathrm{R}^1_\varphi = \mathcal{O}_{\mathbb{C}^2,0}/\left(x^{k-1}, y^2\right)$ and $(e_1, \dots, e_{2k-2})$ is obviously a basis. Then use theorem 1.

□

Claim 8 *$g_1 \equiv 1$, $g_2 \equiv y$, $g_3 \equiv x$, $g_4 \equiv xy, \dots,$ $e_{4k-3} \equiv x^{2k-2}$, $e_{4k-2} \equiv x^{2k-2}y$ project to a basis of $\mathrm{R}^1_h = \mathcal{O}_{\mathbb{C}^2,0}/\left(x^{2k-1}, y^2\right)$ and*

$$\Theta : \left(\mathbb{C}^2 \times \mathbb{C}^{2(2k-1)}, o\right) \to \left(\mathbb{C} \times \mathbb{C}^{2(2k-1)}, o\right)$$

with

$$(x, y, \sigma) \mapsto x^{2k} - y^3 + \sum_{i=0}^{2k-2} \left(\sigma_{2i+1}x^i + \sigma_{2i+2}x^i y\right)$$

is an $\mathcal{R}$–miniversal unfolding of $h = F_{|}\{z = 0\}$

<u>Proof:</u> $(g_1, \dots, g_{4k-2})$ is obviously a basis. Then use theorem 1.

□

In the following, we will restrict to the simplest case, $k = 2$, in order to keep the calculations managable. In this case we have $\mu = \tau = 2$ and $\mu' = 6$.
Next we will calculate the discriminant of p and see that the whole base space of p satisfies the requirements of S in part <u>5.</u> of the proof.

Claim 9 (α) $\mathcal{X}_{sing} = \{(0, y, 2y^3, -3y^2) \mid y \in (\mathbb{C}, o)\}$
(β) $D_p = \{(2t^3, -3t^2) \mid t \in (\mathbb{C}, o)\}$

<u>Proof:</u> Define $\Phi : \left(\mathbb{C}^2 \times \mathbb{C}^2, o\right) \to (\mathbb{C}, o) : (x, y, \lambda_1, \lambda_2) \mapsto x^2 + y^3 + \lambda_1 + \lambda_2 y$.
Then: $\mathcal{X} = \Phi^{-1}(o)$, $\frac{\partial \Phi}{\partial x} = 2x$ and $\frac{\partial \Phi}{\partial y} = 3y^2 + \lambda_2$.
And these partial derivatives determine $\mathcal{X}_{sing}$. Thus we get

$$\mathcal{X}_{sing} = \left\{(0, y, \lambda_1, \lambda_2) \mid y^2 = -\tfrac{1}{3}\lambda_2\right\} \cap \mathcal{X} = \{(0, y, 2y^3, -3y^2) \mid y \in (\mathbb{C}, o)\}$$

and

$$D_p = p(\mathcal{X}_{sing}) = \{(2y^3, -3y^2) \mid y \in (\mathbb{C}, 0)\}.$$[11]

$\square$

Claim 10 *(α) $(D_p)_{reg} = D_p \setminus \{0\}$*

(β) $p^{-1}(\lambda)$ has (for $\lambda \in (D_p)_{reg}$) exactly one singular point $(x_\lambda, y_\lambda, \lambda)$ of type A_1 with $(x_\lambda, y_\lambda, \lambda) = (0, t, 2t^3, -3t^2)$ and $t \neq 0$ uniquely determined.

(γ) $(x_\lambda, y_\lambda) \notin C_\psi$

(δ) (x_λ, y_λ) is in the injectivity domain of $\psi = (x \circ f, y \circ f) = (x, y^2)$

Proof:

(α) Clear, since D_p is a cusp.

(β) Let $\lambda = (\lambda_1, \lambda_2) \in (D_p)_{reg}$ be given. There exists a unique $t \neq 0$ with $\lambda_1 = 2t^3$ and $\lambda_2 = -3t^2$.

$$\begin{aligned} p^{-1}(\lambda) &\equiv \{(x,y) \mid x^2 + y^3 + 2t^3 - 3t^2 y = 0\} \\ &= \{(x,y) \mid x^2 + (y+2t)(y-t)^2 = 0\} \end{aligned}$$

And $d(x^2 + y^3 + 2t^3 - 3t^2 y) = (2x, 3y^2 - 3t^2) = (0,0)$

$\Leftrightarrow \quad x = 0 \quad \& \quad y^2 = t^2$

$\Leftrightarrow \quad x = 0 \quad \& \quad y = t,$

since $x^2 + y^3 + 2t^3 - 3t^2 y = 0$ has to be satisfied!

Thus, $(0, t, 2t^3, -3t^2)$ is the only singular point of $p^{-1}(\lambda)$. In order to find the type of the singularity, we have to look at the defining equation $x^2 + (y+2t)(y-t)^2 = 0$. For y close to t, $(y+2t)$ is nearly constant and non–zero. With $\eta = y - t$ and $0 \neq c \approx y + 2t$ we have the equivalent equation: $x^2 + c\eta^2 = 0$. Thus, the singularity is of type A_1. (For the Milnor number calculate the dimension of $\mathcal{O}_{\mathbb{C}^2,(0,t)}/(x, y^2 - t^2) = \mathcal{O}_{\mathbb{C},0}/(\eta(\eta + 2t)) = 1$).

(γ) $d\psi(x,y) = (1, 2y)$ has rank less than 2 if and only if $y = 0$.

Therefore: $C_\psi = \{y = 0\}$,

and thus: $(x_\lambda, y_\lambda) \in C_\psi \Leftrightarrow y_\lambda = 0 \Leftrightarrow \lambda = (2y_\lambda^3, -3y_\lambda^2) = (0,0)$.

(δ) If $(x,y) \in p^{-1}(\lambda)$ with $\psi(x,y) = \psi(x_\lambda, y_\lambda)$, then we get $x = x_\lambda = 0$ and $y^2 = y_\lambda^2$. Again $y = -y_\lambda$ contradicts the defining equation of $p^{-1}(\lambda)$, since $\lambda = (2y_\lambda^3, -3y_\lambda^2)$.

Hence: $(x,y) = (x_\lambda, y_\lambda)$.

$\square$

[11] Another way to see this is the following: We may identify our miniversal deformation with

$$(\mathbb{C}^2 \times \mathbb{C}, 0) \xrightarrow{p'} (\mathbb{C} \times \mathbb{C}, 0) : (x, y, \lambda_2) \mapsto \left(x^2 + y^3 + \lambda_2 y, \lambda_2\right)$$

and

$$dp' = \begin{pmatrix} 2x & 3y^2 + \lambda_2 & y \\ 0 & 0 & 1 \end{pmatrix}.$$

Thus the rank of dp' is not maximal exactly if $2x = 0$ and $3y^2 + \lambda_2 = 0$ is fulfilled.

Let us just recall that the map germ $b : (\mathbb{C}, 0) \to \left(\mathbb{C}^2, 0\right)$ from 6. is given by $t \mapsto (t, 0)$, and thus, $b(\mathbb{C}) \cap D_p = 0$.
We will now have a look at the image of $p : (\mathcal{X}, 0) \to \left(\mathbb{C}^2, 0\right)$ under $\psi \times id$ and its embedding in $\bar{\Theta} : (\bar{H}, 0) \to (\mathbb{C}^{\mu'}, 0)$.

Claim 11 *(α)* $\mathcal{Y} = (\psi \times id)(\mathcal{X}) = \left\{(x, y, \lambda_1, \lambda_2) \mid \left(x^2 + \lambda_1\right)^2 = y(y + \lambda_2)^2\right\}$

(β) b_1:
$$\begin{array}{rcl} \left(\mathbb{C}^2, 0\right) & \to & \left(\mathbb{C}^6, 0\right) \\ (\lambda_1, \lambda_2) & \mapsto & \left(\lambda_1^2 + \frac{2}{27}\lambda_2^3, \frac{1}{3}\lambda_2^2, 0, 0, 2\lambda_1, 0\right) \end{array}$$

is a base change such that $q : (\mathcal{Y}, 0) \to \left(\mathbb{C}^2, 0\right)$ is induced isomorphic to

$$\bar{\Theta} : \left(\bar{\mathrm{H}}, 0\right) \to \left(\mathbb{C}^6, 0\right) : (x, y, \sigma) \mapsto \sigma$$

where $\bar{H}$ is given by

$$\bar{H} = \Theta^{-1}(0) = \{x^4 - y^3 + \sigma_1 + \sigma_2 y + \sigma_3 x + \sigma_4 xy + \sigma_5 x^2 + \sigma_6 x^2 y = 0\}$$

(γ) The reduced Kodaira-Spencer map $db_{2(0)} : \mathbb{C} \to \mathbb{C}^6$ is not trivial.

Proof:

(α) Let $(x, y, \lambda) \in \mathcal{X}$, i.e. $x^2 + y^3 + \lambda_1 + \lambda_2 y = 0$.
Then we have $\left(x^2 + \lambda_1\right)^2 = y^2\left(y^2 + \lambda_2\right)^2$.
Thus $\psi_{(x,y)} = (x, y^2) = (x, \bar{y})$ satisfies the relation $\left(x^2 + \lambda_1\right)^2 = \bar{y}(\bar{y} + \lambda_2)^2$.
And: $(\psi \times id)(\mathcal{X}) = \left\{\left(x^2 + \lambda_1\right)^2 = y(y + \lambda_2)^2\right\}$.

(β) $b_1^*\left(\bar{H}\right)$

$$\begin{aligned}
&= \left\{(x, y, \sigma, \lambda) \in \left(\bar{\mathrm{H}} \times \mathbb{C}^2, 0\right) \mid b_{1(\lambda)} = \bar{\Theta}_{(x,y,\sigma)} = \sigma\right\} \\
&= \Big\{(x, y, \sigma, \lambda) \in \left(\mathbb{C}^2 \times \mathbb{C}^6 \times \mathbb{C}^2, 0\right) \mid \\
&\qquad x^4 - y^3 + \sigma_1 + \sigma_2 y + \sigma_3 x + \sigma_4 xy + \sigma_5 x^2 + \sigma_6 x^2 y = 0 \\
&\qquad \text{and } (\sigma_1, \ldots, \sigma_6) = \left(\lambda_1^2 + \tfrac{2}{27}\lambda_2^3, \tfrac{1}{3}\lambda_2^2, 0, 0, 2\lambda_1, 0\right)\Big\} \\
&\equiv \left\{(x, y, \lambda) \in \left(\mathbb{C}^2 \times \mathbb{C}^2, 0\right) \mid x^4 - y^3 + \lambda_1^2 + \tfrac{2}{27}\lambda_2^3 + \tfrac{1}{3}\lambda_2^2 y + 2\lambda_1 x^2 = 0\right\} \\
&= \left\{(x, y, \lambda) \in \left(\mathbb{C}^2 \times \mathbb{C}^2, 0\right) \mid \left(x^2 + \lambda_1\right)^2 - \left(y - \tfrac{2}{3}\lambda_2\right)\left(y + \tfrac{1}{3}\lambda_2\right)^2 = 0\right\}
\end{aligned}$$

Thus $(b_1)^*\left(\bar{H}\right) \longrightarrow \mathcal{Y}$:$(x, y, \lambda) \mapsto \left(x, y - \frac{2}{3}\lambda_2, \lambda\right)$ is the appropriate isomorphism with

$$\begin{tikzcd}
(\mathcal{Y}, 0) \arrow[r, "\cong"] \arrow[dr, "q"'] & b_1^*\left(\bar{H}\right) \arrow[r] \arrow[d, "pr_2"] & \left(\bar{\mathrm{H}}, 0\right) \arrow[d, "\bar{\Theta}"] \\
& \left(\mathbb{C}^2, 0\right) \arrow[r, "b_1"] & \left(\mathbb{C}^6, 0\right)
\end{tikzcd}$$

(γ) The base change b_2 is given by $b_{2(t)} = (t^2, 0, 0, 0, -2t, 0)$ and therefore $db_{2(0)} = (0, 0, 0, 0, -2, 0)$ is not trivial.

$\square$

Thus, we get the base change $b_3 : (\mathbb{C}^2, 0) \to (\mathbb{C} \times \mathbb{C}^6, 0)$ by $b_{3(\lambda_1,\lambda_2)} = (0, b_{1(\lambda_1,\lambda_2)})$ and we can show that $b_3(\mathbb{C}^2)$ is completely contained in the discriminant of D_Θ.

Claim 12 $D_\Theta \cap b_3(\mathbb{C}^2, 0) = D_\Theta \cap (\{0\} \times b_1(\mathbb{C}^2, 0))$
$= \{(0, x^4 - 2y^3, 3y^2, 0, 0, -2x^2, 0) \mid x, y \in (\mathbb{C}, 0)\} = b_3(\mathbb{C}^2, 0)$

Proof: In order to calculate the discriminant D_Θ we need to know the critical locus C_Θ. Θ is given by

$$\Theta_{(x,y,\sigma)} = (x^4 - y^3 + \sigma_1 + \sigma_2 y + \sigma_3 x + \sigma_4 xy + \sigma_5 x^2 + \sigma_6 x^2 y, \sigma)$$

Therefore $\frac{\partial\Theta}{\partial x}$ and $\frac{\partial\Theta}{\partial y}$ determine whether or not a point (x, y, σ) is in C_Θ:

$$\begin{aligned}\frac{\partial\Theta}{\partial x} &= (4x^3 + \sigma_3 + \sigma_4 y + 2\sigma_5 x + 2\sigma_6 xy, 0)\\ \frac{\partial\Theta}{\partial y} &= (-3y^2 + \sigma_2 + \sigma_4 x + \sigma_6 x^2, 0)\end{aligned}$$

Given $(x, y, \sigma) \in C_\Theta \cap \Theta^{-1}(b_3(\mathbb{C})) = C_\Theta \cap \tilde{b}_4(\mathcal{X})$ we have the following:

$$\text{by claim 11:} \quad \sigma_3 = \sigma_4 = \sigma_5 = 0 \tag{1}$$

$$\text{by (1) and } \Theta_{(x,y,\sigma)}{=}(0,\sigma)\text{:} \quad x^4 - y^3 + \sigma_1 + \sigma_2 y + \sigma_5 x^2 = 0 \tag{2}$$

By claim 11 σ_1, σ_2 and σ_5 satisfy the equation:

$$\left(\sigma_1 - \left(\tfrac{1}{2}\sigma_5\right)^2\right)^2 - \frac{4}{27}\sigma_2^3 = 0 \tag{3}$$

And since (x, y, σ) is a critical point of Θ we have

$$\frac{\partial\Theta}{\partial x}(x,y,\sigma) = 0 \tag{4}$$

$$\frac{\partial\Theta}{\partial y}(x,y,\sigma) = 0 \tag{5}$$

Now, (4) and (1) imply that $4x^3 + 2\sigma_5 x = 0$ and hence

$$\text{either } x = 0 \text{ or } \sigma_5 = -2x^2. \tag{6}$$

Similarly, (5) and (1) give $-3y^2 + \sigma_2 = 0$ and thus

$$\sigma_2 = 3y^2. \tag{7}$$

- If $x = 0$, we get from (2) and (7):

$$\sigma_1 = -2y^3 \tag{8}$$

Furthermore, (3),(7) and (8) imply:

$$0 = \left(-2y^3 - \tfrac{1}{4}\sigma_5^2\right)^2 - \frac{4}{27}\left(3y^2\right)^3 = \sigma_5^2\left(y^3 + \tfrac{1}{16}\sigma_5^2\right)$$

And thus either $\sigma_5 = 0$ or $\sigma_5^2 = -16y^3$.
Therefore we get

$$\Theta_{(x,y,\sigma)} = \begin{cases} (0, -2y^3, 3y^2, 0, 0, 0, 0) \\ (0, -2y^3, 3y^2, 0, 0, \sigma_5, 0),\ \sigma_5^2 = -16y^3 \end{cases} \tag{9}$$

- If $x \neq 0$, we get from (2),(6) and (7):

$$\sigma_1 = x^4 - 2y^3 \tag{10}$$

Furthermore, for any $x, y \in (\mathbb{C}, 0)$ $\sigma_1 = x^4 - 2y^3$, $\sigma_2 = 3y^2$ & $\sigma_5 = -2x^2$ fulfill the equation (2).
Therefore we get

$$\Theta_{(x,y,\sigma)} = (0, x^4 - 2y^3, 3y^2, 0, 0, -2x^2, 0). \tag{11}$$

(9) and (11) imply that $b_3\left(\mathbb{C}^2, 0\right) \subseteq D_\Theta$: namely, given $\lambda = (\lambda_1, \lambda_2) \in \left(\mathbb{C}^2, 0\right)$ choose some $(x, y) \in \left(\mathbb{C}^2, 0\right)$ such that $x^2 = -\lambda_1$ and $y = -\frac{1}{3}\lambda_2$.

$$\begin{array}{llll} \text{Then} & b_{3(\lambda_1,\lambda_2)} & = & \left(0, \lambda_1^2 + \frac{2}{27}\lambda_2^3, \frac{1}{3}\lambda_2^2, 0, 0, 2\lambda_1, 0\right) \\ & & = & (0, x^4 - 2y^3, 3y^2, 0, 0, -2x^2, 0) \\ & & = & \Theta_{(x,y,x^4-2y^3,3y^2,0,0,-2x^2,0)} \end{array}$$

$$\begin{array}{llllll} \text{and} & \frac{\partial\Theta}{\partial x}(x,y,x^4-2y^3,3y^2,0,0,-2x^2,0) & = & 4x^3 - 4x^3 & = & 0 \\ & \frac{\partial\Theta}{\partial y}(x,y,x^4-2y^3,3y^2,0,0,-2x^2,0) & = & -3y^2 + 3y^2 & = & 0 \end{array}$$

Thus $(x, y, x^4 - 2y^3, 3y^2, 0, 0, -2x^2, 0) \in C_\Theta$ and therefore $b_{3(\lambda_1,\lambda_2)} \in D_\Theta$.

□

We have now seen, what happens to the base spaces. Let us check the total spaces and fix the result in the following diagram.

$$(x,y) \longrightarrow (x,y,-\varphi(x,y),0)$$
$$(x,y,\lambda_1,\lambda_2) \longrightarrow \big(x,y^2,\lambda_1,\lambda_2\big) \longrightarrow \big(x,y^2+\tfrac{2}{3}\lambda_2,\lambda_1,\lambda_2\big)$$
$$\longrightarrow \big(x,y^2+\tfrac{2}{3}\lambda_2,\lambda_1^2+\tfrac{2}{27}\lambda_2^3,\tfrac{1}{3}\lambda_2^2,0,0,2\lambda_1,0\big)$$
$$\longrightarrow \big(x,y^2+\tfrac{2}{3}\lambda_2,\lambda_1^2+\tfrac{2}{27}\lambda_2^3,\tfrac{1}{3}\lambda_2^2,0,0,2\lambda_1,0\big)$$

$$\begin{tikzcd}
\mathbb{C}^2 \arrow[r, "id\times-\varphi\times 0"'] \arrow[d, "\varphi=z\circ f"] & \mathcal{X} \arrow[r, "\psi\times id"'] \arrow[d, "p"] & \mathcal{Y} \arrow[r, "\cong"'] \arrow[d, "q"] & b^*\bar{H} \arrow[r, "id\times b_1"'] \arrow[d, "pr_2"] & \bar{H} \arrow[r, "="'] \arrow[d, "\bar{\Theta}"] & H=\mathbb{C}^2\times\mathbb{C}^6 \arrow[d, "\Theta"] \\
\mathbb{C} \arrow[r, "b=-id\times 0"] & \mathbb{C}^2 \arrow[r, "="] & \mathbb{C}^2 \arrow[r, "="] & \mathbb{C}^2 \arrow[r, "b_1"] & \mathbb{C}^6 \arrow[r, "0\times id"] & \mathbb{C}\times\mathbb{C}^6
\end{tikzcd}$$

$$\longrightarrow \big(0,\lambda_1^2+\tfrac{2}{27}\lambda_2^3,\tfrac{1}{3}\lambda_2^2,0,0,2\lambda_1,0\big)$$
$$(\lambda_1,\lambda_2) \longrightarrow (\lambda_1,\lambda_2) \longrightarrow (\lambda_1,\lambda_2) \longrightarrow \big(\lambda_1^2+\tfrac{2}{27}\lambda_2^3,\tfrac{1}{3}\lambda_2^2,0,0,2\lambda_1,0\big)$$
$$t \longrightarrow (-t,0)$$

Figure 3

We will now slightly deviate from the strict lines of Némethi's proof to show that <u>10.</u> is not violated, and we will later explain why this alteration does not matter. Instead of choosing a line ℓ' in $\mathbb{C}^2$ which intersects the line $\{\lambda_2 = 0\}$ we choose a line L parallel to $\{\lambda_2 = 0\}$ to construct a basis of vanishing cycles for (V, o). Again the reason for this is to keep the calculations as simple as possible.

Claim 13 *(i) Given* $\lambda = (2t^3, -3t^2) \in (D_p)_{reg}$.
Then $(x_\lambda, y_\lambda, \lambda) = (0, t, 2t^3, -3t^2)$ *is the only singularity (of type* A_1*) in* $p^{-1}(\lambda)$ *and*

$$\tilde{b}_{4(x_\lambda,y_\lambda,\lambda)} = \tilde{b}_{3(0,t^2,2t^3,-3t^2)} = (0, -t^2, 2t^6, 3t^4, 0, 0, 4t^3, 0)$$

(ii) Given the line $L = \{\lambda_2 = a\} \subseteq \big(\mathbb{C}^2, o\big)$, $a \neq 0$.
We have:

$$D_p \cap L = \{(2t_a^3, a), (-2t_a^3, a) \mid t_a \text{ fixed such that } a = -3t_a^2\},$$

$$b_3(D_p \cap L) = \Big\{\big(0, -\tfrac{2}{27}a^3, \tfrac{1}{3}a^2, 0, 0, \pm 4t_a^3, 0\big) \mid a = -3t_a^2\Big\}$$

and

$$\tilde{b}_4(0, \pm t_a, \pm 2t_a^3, a) = \big(0, \tfrac{1}{3}a, -\tfrac{2}{27}a^3, \tfrac{1}{3}a^2, 0, 0, \pm 4t_a^3, 0\big).$$

(iii) Let $L' = b_3(L) = \Big\{\big(0, t^2 + \tfrac{2}{27}a^3, \tfrac{1}{3}a^2, 0, 0, 2t, 0\big) \mid t \in (\mathbb{C}, o)\Big\}$, *then* $\Theta^{-1}(L') = \tilde{b}_4(p^{-1}(L))$

$$\begin{aligned}
&= \Big\{\big(x, y, t^2 + \tfrac{2}{27}a^3, \tfrac{1}{3}a^2, 0, 0, 2t, 0\big) \mid t \in (\mathbb{C}, o), \\
&\qquad (x^2 + t)^2 - \big(y - \tfrac{2}{3}a\big)\big(y + \tfrac{1}{3}a\big)^2 = 0\Big\} \\
&= \Big\{\big(x, y, t^2 + \tfrac{2}{27}a^3, \tfrac{1}{3}a^2, 0, 0, 2t, 0\big) \mid t \in (\mathbb{C}, o), \\
&\qquad x^4 - y^3 + t^2 + \tfrac{2}{27}a^3 + \tfrac{1}{3}a^2 y + 2tx^2 = 0\Big\}
\end{aligned}$$

Proof: Clear from claim 10, claim 11 and figure 3.

□

Now we calculate the set of singular points in the fibers over L'.

Claim 14 $\mathcal{L}_S$

$$\begin{aligned} &= \{(x,y,\sigma) \in \Theta^{-1}(L') \mid (x,y,\sigma) \text{ singular in } \Theta^{-1}(\Theta_{(x,y,\sigma)})\} \\ &= \left\{\left(0, \tfrac{1}{3}a, -\tfrac{2}{27}a^3, \tfrac{1}{3}a^2, 0, 0, \pm 4t_a^3, 0\right)\right\} \\ &\quad \cup \left\{\left(x_t, -\tfrac{1}{3}a, t^2 + \tfrac{2}{27}a^3, \tfrac{1}{3}a^2, 0, 0, 2t, 0\right) \mid t \in (\mathbb{C}, o), t = -x_t^2\right\} \end{aligned}$$

Proof: Given $(x,y,\sigma) \in \mathcal{L}_S$. By $L' \subseteq \left(\{0\} \times \mathbb{C}^6, o\right)$ we have $\Theta_{(x,y,\sigma)} = (0,\sigma)$ and there exists a $t = t_{(\sigma)} \in (\mathbb{C}, o)$ with

$$\sigma = \left(t^2 + \tfrac{2}{27}a^3, \tfrac{1}{3}a^2, 0, 0, 2t, 0\right).$$

We are interested in the singularities of

$$\begin{aligned} \Theta^{-1}{}_{(0,\sigma)} = \Big\{&\left(x, y, t^2 + \tfrac{2}{27}a^3, \tfrac{1}{3}a^2, 0, 0, 2t, 0\right) \mid t = t_{(\sigma)}, \\ &(x^4 - y^3) + \left(t^2 + \tfrac{2}{27}a^3\right) + \tfrac{1}{3}a^2 y + 2tx^2 = 0\Big\} \end{aligned}$$

To calculate these singularities we have to investigate the derivative of

$$g_{(x,y)} = x^4 - y^3 + 2tx^2 + \frac{1}{3}a^2 y + \frac{2}{27}a^3 + t^2.$$

We get

$$\begin{aligned} \tfrac{\partial g}{\partial x} &= 4x^3 + 4tx &= 0 \Leftrightarrow\ & x = 0 \text{ or } -x^2 = t \\ \tfrac{\partial g}{\partial y} &= -3y^2 + \tfrac{1}{3}a^2 &= 0 \Leftrightarrow\ & y^2 = \tfrac{1}{9}a^2 = \left(\tfrac{1}{3}a\right)^2 \end{aligned}$$

- If $\underline{x = 0}$, then $\underline{y = -\frac{1}{3}a}$ will imply that $0 = g_{(x,y)} = t^2$.
  Thus $y = -\frac{1}{3}a$ is possible only for $t = 0$, in which case we get

  $$(x,y,\sigma) = \left(0, -\tfrac{1}{3}a, \tfrac{2}{27}a^3, \tfrac{1}{3}a^2, 0, 0, 0, 0\right).$$

  For $\underline{y = \frac{1}{3}a}$ we have $0 = g_{(x,y)} = t^2 + \frac{4}{27}a^3$.
  Thus $y = \frac{1}{3}a$ is possible only for $t^2 = -\frac{4}{27}a^3 = 4t_a^6$, in which case we get

  $$(x,y,\sigma) = \left(0, \tfrac{1}{3}a, -\tfrac{2}{27}a^3, \tfrac{1}{3}a^2, 0, 0, \pm 4t_a^3, 0\right).$$

- If $x \neq 0$, then choose an x_t such that $-x_t^2 = t$, thus $\underline{x = \pm x_t}$.
  If $\underline{y = -\frac{1}{3}a}$, we have $0 = g_{(x_t,y)} = (x_t^2 + t)^2 + \left(y - \frac{2}{3}a\right)\left(y + \frac{1}{3}a\right)^2$, which is satisfied for all $t \in (\mathbb{C}, o)$.
  Thus we get

  $$\left\{\left(\pm x_t, -\tfrac{1}{3}a, t^2 + \tfrac{2}{27}a^3, \tfrac{1}{3}a^2, 0, 0, 2t, 0\right) \mid t \in (\mathbb{C}, o)\right\},$$

  which is a <u>curve</u> in $\mathcal{L}_S$. Whereas for $y = \frac{1}{3}a$ we get $0 = g_{(x_t,y)} = \frac{4}{27}a^3$, which is a contradiction. So $y = \frac{1}{3}a$ is not possible.

Thus $\mathcal{L}_S$ is as stated above.

$\square$

Remark 7
For $t = t(\sigma) = \pm 2t_a^3$ we get $x_t^2 = \mp 2t_a^3$ or $x_t^4 = -\frac{4}{27}a^3 \neq 0$ and with $\sigma = \left(-\frac{2}{27}a^3, \frac{1}{3}a^2, 0, 0, \pm 4t_a^3, 0\right)$ the fiber $\Theta^{-1}(0,\sigma)$ has three singular points, namely

$$\left(x, -\tfrac{1}{3}a, \sigma\right) \text{ with } x \in \{0, x^2 = -t\},$$

whereas all other fibers over L' have only two singularities.
In particular: For $\lambda = (\pm 2t_a^3, a)$ and $(x_\lambda, y_\lambda, \lambda) = (0, \pm t_a, \lambda)$ we have that

$$\tilde{b}_4(x_\lambda, y_\lambda, \lambda) = \left(0, \tfrac{1}{3}a, -\tfrac{2}{27}a^3, \tfrac{1}{3}a^2, 0, 0, \pm 4t_a^2, 0\right)$$

are isolated in $\mathcal{L}_S$.
To see the type of these singularities, we look at

$$\mathcal{O}_{\mathbb{C}^2,\left(0,\frac{1}{3}a\right)}/\left(\tfrac{\partial g}{\partial x}, \tfrac{\partial g}{\partial y}\right) = \mathcal{O}_{\mathbb{C}^2,\left(0,\frac{1}{3}a\right)}/\left(x(x^2+2t), \left(y - \tfrac{1}{3}a\right)\left(y + \tfrac{1}{3}a\right)\right) \cong \mathcal{O}_{\mathbb{C}^2,0}/(x, \eta),$$

where $\eta = y - \frac{1}{3}a$.
Thus, the Milnor number of the singularity is 1 and its type is A_1.

The aim of part 10. in the proof of theorem 4 is to ensure that none of the vanishing cycles needed in the construction over the line ℓ (or L respectively) gets mapped to a cycle with singularity. We thus have to make sure that we can choose our paths in ℓ (or L respectively) and the cycles over these paths such that the cycles do not meet the preimage of $\mathcal{L}_S$ in $\mathcal{X}$ in any other point than the singularities $(x_\lambda, y_\lambda, \lambda)$ over ℓ (or L respectively). We will call that part of the preimage of $\mathcal{L}_S$ in $\mathcal{X}$ under $\tilde{b}_4$ $\mathcal{X}_{bad}$.

Claim 15 *The preimage of the curve of singularities*

$$\left\{\left(x_t, -\tfrac{1}{3}a, t^2 + \tfrac{2}{27}a^3, \tfrac{1}{3}a^2, 0, 0, 2t, 0\right) \mid t \in (\mathbb{C}, o),\ x_t^2 = -t\right\}$$

under the map $\tilde{b}_4$ *is*

$$\mathcal{X}_{bad} = \{(x_t, \bar{y}_t, t, a) \mid \bar{y}_t = -a,\ t \in (\mathbb{C}, o)\}$$

and

$$\tilde{b}_4^{-1}\left(0, \tfrac{1}{3}a, -\tfrac{2}{27}a^3, \tfrac{1}{3}a^2, 0, 0, \pm 4t_a^3, 0\right) = \left(0, \pm t_a, \pm 2t_a^3, a\right).$$

Proof: Given $\left(x_t, -\frac{1}{3}a, t^2 + \frac{2}{27}a^3, \frac{1}{3}a^2, 0, 0, 2t, 0\right)$ for $t \in (\mathbb{C}, o)$ denote by $(\bar{x}_t, \bar{y}_t, \lambda_t)$ a point in its preimage under $\tilde{b}_4$ (which is indeed one of two points). We know from claim 11 that $\lambda_t = (t, a)$.
Furthermore, $\bar{x}_t = x_t$ and $\bar{y}_t^2 + \frac{2}{3}a = -\frac{1}{3}a$, which means $\bar{y}_t^2$=-a

$\square$

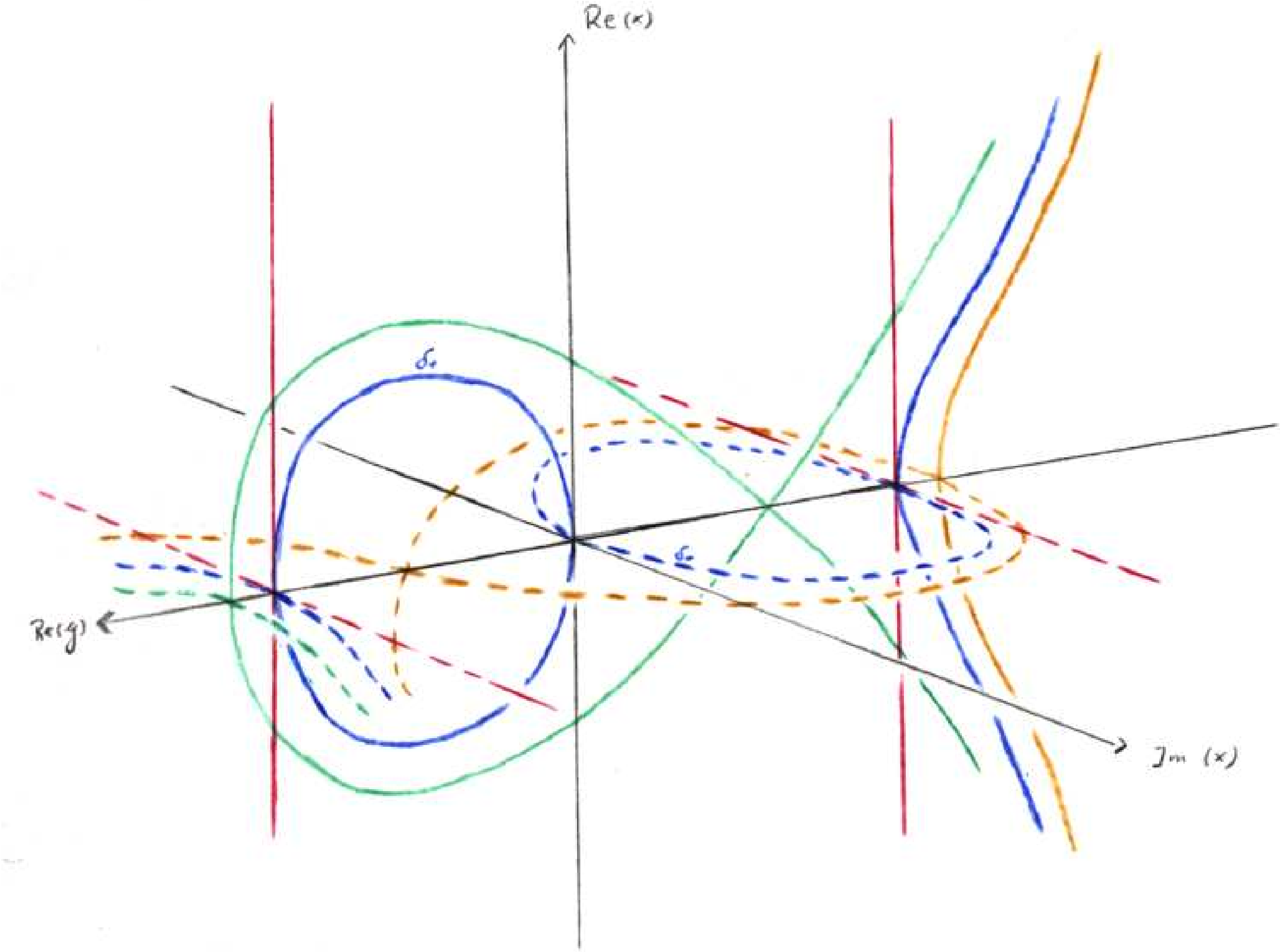

Figure 4

Remark 8
If we assume that a is a negative real constant of small absolute value, we have:
$t_a = \frac{1}{\sqrt{3}}\sqrt{\mid a \mid}$

$$\left(\tilde{b}_4\right)^{-1}\left(\mathcal{L}_S \setminus \tilde{b}_{4(0,t_a,\pm 2t_a^3,a)}\right) = \left\{\left(x_t, \pm\sqrt{\mid a \mid}, t, a\right) \mid x_t^2 = -t,\, t \in (\mathbb{C}, 0)\right\} = \mathcal{X}_{bad}.$$

Claim 16 *For this choice of a it is possible to find paths u_+ and u_-, joining a basepoint λ_0 in L to the critical values $\lambda_+ = (2t_a^3, a)$ and $\lambda_- = (-2t_a^3, a)$ respectively, and to find corresponding smooth families of cycles in $p^{-1}(L)$ but which do not meet $\mathcal{X}_{bad}$.*

<u>Proof:</u> We are interested in the intersection of $\mathcal{X}_{bad}$ with $p^{-1}(u\,[0,1])$ where $u = u_+$ or $u = u_-$ are the paths joining some base point λ_0 in L to the singular points

$$\lambda_+ = (2t_a^3, a) = \left(2\left(\frac{\mid a \mid}{3}\right)^{\frac{3}{2}}, a\right) \text{ and } \lambda_- = (-2t_a^3, a) = \left(-2\left(\frac{\mid a \mid}{3}\right)^{\frac{3}{2}}, a\right).$$

We have to show that we may simultaneously choose a cycle in each $u(\xi)$ such that the family of cycles, defined by this, is smooth and none of the cycles actually meets $\mathcal{X}_{bad}$. In order to see what happens, it is quite instructive to choose the paths u_+ and u_- to lie completely in $\Gamma = [-2t_a^3, 2t_a^3] \times \{a\}$, although they will not be suitable for our final construction.
If we now project $p^{-1}(\Gamma)$ to the first two coordinates (x, y) and omit the imaginary part of y, we get figure 4, which is a slight modification of the picture in [B-K], page 164.

Dotted curves denote the intersection of a certain level set with the Im(x)-Re(y) plane, whereas full lines correspond to the intersection with the Re(x)-Re(y) plane. The green curves come from the singular level set $p^{-1}(\lambda_-)$ and the brown ones from the singular level set $p^{-1}(\lambda_+)$. The blue curves correspond to the non–singular level set $p^{-1}(t,a)$ with $t = 0$. Varying t between $-2t_a^3 = -2\left(\frac{\mid a \mid}{3}\right)^{\frac{3}{2}}$ and $2t_a^3 = 2\left(\frac{\mid a \mid}{3}\right)^{\frac{3}{2}}$ causes a contraction or expansion of δ_+ or δ_-.
We have chosen $t = 0$, since that is exactly the value where the cycles δ_+ and δ_- meet the projection of $\mathcal{X}_{bad}$ in one point, as does their non–projected preimage w.r.t. $\mathcal{X}_{bad}$. The projected $\mathcal{X}_{bad}$ consists of two curves, each with two branches, in the form of a cross. These curves are exactly

$$\left(\mathbb{R} \times \left\{-\sqrt{\mid a \mid}, \sqrt{\mid a \mid}\right\}\right) \cup \left(i\mathbb{R} \times \left\{-\sqrt{\mid a \mid}, \sqrt{\mid a \mid}\right\}\right),$$

since $p^{-1}(\Gamma) \cap \mathcal{X}_{bad} = \left\{(x, y, t, a) \mid (t, a) \in \Gamma,\, y = \pm\sqrt{\mid a \mid},\, x^2 = -t\right\}$, and they are drawn in red colour.
This picture gives a sufficiently good indication how $p^{-1}(t,a)$ with $(t, a) \in \Gamma$ looks like.

Since we want to choose u_+ and u_- in Γ, we will have that the point $(0, y)$is a member of at least one of the two paths, and w.l.o.g. we may assume that $(0, a)$ lies on u_+. If we now want to construct the family of vanishing cycles over u_+ such that their projection lies in the Re(x)-Re(y) plane, the non–projected preimage of δ_+ will be one of the cycles, and it will intersect $\mathcal{X}_{bad}$ in $\left(0, \sqrt{\mid a \mid}, 0, a\right)$.
To avoid this, one could think it would be sufficient to move δ_+ slightly off the Re(x)-Re(y) plane in the Im(x) direction so that it no longer intersects the red cross. However, this enforces us to move very close fibers in the same way. If we now look at the intersection point $\left(0, -\sqrt{\mid a \mid}\right)$ of δ_+ with the Im(x)-Re(y) plane, then we see that moving a cycle in the Im(x) direction implies an increasing of the Re(y)–par of this intersection point. Thus, some other cycle (corresponding to some (t, a) on u_+ with $0 < t \ll 2t_a^3$) will have Re(y)–part $\sqrt{\mid a \mid}$ and therefore meet $\mathcal{X}_{bad}$. Pushing δ_+ and nearby cycles quite far off the Re(x)-Re(y) plane corresponds to moving the trouble point (t, a) closer to the critical point (t_a^3, a). This shows that the choice of u_+ and u_- in Γ will not work. But let us have a closer look at why this is the case and suppose again w.l.o.g. that $(0, a)$ lies on u_+.
Since we may assume that the trouble occurs in a fiber $p^{-1}(t,a)$ with t close to zero and for t sufficiently small, the fiber $p^{-1}(t,a)$ may be locally flattened around the "two" points of $\mathcal{X}_{bad} \cap p^{-1}(t,a) \cap \left(\mathbb{C} \times \left\{\sqrt{\mid a \mid}\right\} \times \mathbb{C}^2\right)$. This leads to the family of pictures in figure 5a.

The arrows indicate, in which direction the two points move as t increases. Thus, we have two points coming together on the Re(x)–axis, collapsing in a single point and spreading out again in the Im(x) direction. (see figure 5b)

This crossing point makes it impossible to find a suitable family of cycles with the restriction of u_+ and u_- to Γ.
We therefore do the following construction, which is due to Jan Stevens:
Choose $0 < \varepsilon \ll \mid a \mid$ and set

$$\Gamma' = \left(\left[-2\left(\frac{\mid a \mid}{3}\right)^{\frac{3}{2}}, -\varepsilon \right] \cup \{\varepsilon e^{i\alpha} \mid \alpha \in [0, \pi]\} \cup \left[\varepsilon, 2\left(\frac{\mid a \mid}{3}\right)^{\frac{3}{2}} \right] \right) \times \{a\}.$$

Choose the base point λ_0 in L to be $(-\varepsilon, a)$ and the paths u_+ and u_- in Γ' in the obvious way, i.e. such that $u_+([0,1]) = \left(\{\varepsilon e^{i\alpha} \mid \alpha \in [0, \pi]\} \cup \left[\varepsilon, 2\left(\frac{\mid a \mid}{3}\right)^{\frac{3}{2}} \right] \right) \times \{a\}$ and $u_-([0,1]) = \left[-2\left(\frac{\mid a \mid}{3}\right)^{\frac{3}{2}}, -\varepsilon \right] \times \{a\}$.
Thus, u_- causes no problems. We may choose the cycles over u_- such that their projection lies in the Im(x)-Re(y) plane. They will not meet $\mathcal{X}_{bad}$.
It remains now to show that such a choice of a smooth family of cycles over u_+ is also possible. For the part where u_+ is $\left[\varepsilon, 2\left(\frac{\mid a \mid}{3}\right)^{\frac{3}{2}} \right] \times \{a\}$ we may choose

the cycles such that their projection lies in the Re(x)-Re(y) plane as suggested by figure 4. Again they will not meet $\mathcal{X}_{bad}$. Thus, we only have to find the cycles over $\{\varepsilon e^{i\alpha} \mid \alpha \in [0,\pi]\} \times \{a\}$ such that they fit smoothly together, avoid $\mathcal{X}_{bad}$, and for $\alpha = 0$ come smoothly together with those whose projection lies in the Re(x)-Re(y) plane.
Again we have to worry about the points in $\mathcal{X}_{bad} \cap p^{-1}(\varepsilon e^{i\alpha}) \cap \left(\mathbb{C} \times \left\{\sqrt{\mid a \mid}\right\} \times \mathbb{C}^2\right)$ $= Q_\alpha$. But keeping in mind that the projection of $\mathcal{X}_{bad} \cap \left(\mathbb{C} \times \left\{\sqrt{\mid a \mid}\right\} \times \mathbb{C}^2\right)$ gives the picture in figure 6, we get for $p^{-1}(\varepsilon e^{i\alpha})$, α running from π back to 0, the family of pictures in the figures 7a and 7b, after a local flattening around Q_α.

Given a fixed fiber, it is clear that, once we are away from $\mathcal{X}_{bad}$, we may deform our cycle however we want. So the question that remains is whether we can choose the family of cycles locally at Q_α such that it fits smoothly with the other family. But this problem can be solved by the family of pictures in the figures 8a and 8b. This proves the claim.

□

Remark 9
Claim 16 ensures that the construction of vanishing cycles over the paths $b_3 \circ u_+$ and $b_3 \circ u_-$ in $\Theta^{-1}(L')$ is possible, since the transported families of cycles do not meet the singular curve $\tilde{b}_4(\mathcal{X}_{bad})$. The vanishing cycles constructed in this way lie in $\left(V'_{\lambda_0}\right)_{reg} = \left(\tilde{b}_4(V_{\lambda_0})\right)_{reg}$.
However, we have still to explain why it does not matter that our line L does not intersect $\{\lambda_2 = 0\}$, and that λ_0 is not contained in $\{\lambda_2 = 0\}$. For this, we recall that the choice of $\ell = \{\varepsilon\lambda_1 + \lambda_2 = a\}$ with $0 < a \ll \varepsilon \ll 1$ was made in order to ensure that a construction like the one in claim 16 should be possible. So at least its spirit is not violated.
Furthermore, we notice that both, ℓ and L, are small perturbations of $\{\lambda_2 = 0\}$ and thus their images ℓ' and L' are small perturbations of each other. Hence, any small perturbation ℓ'' of ℓ' will also be a small perturbation of L, and it is indeed such a perturbation ℓ'' which is needed for the construction of the geometric basis for $(\mathrm{V}', 0)$ in the proof of theorem 4.
Finally, one could think that the fiber $V'_{\left(-\varepsilon,\, a\right)} = \Theta^{-1}(b_{3(-\varepsilon,a)})$ was worse than the fiber $V'_{\left(\frac{\varepsilon}{a},\, 0\right)} = \Theta^{-1}(b_{3(\frac{\varepsilon}{a},0)})$, or equivalently $V_{\left(-\varepsilon,\, a\right)}$ was worse than $V_{\left(\frac{\varepsilon}{a},\, 0\right)}$. But this is indeed not the case. Both $V'_{\left(-\varepsilon,\, a\right)}$ and $V'_{\left(\frac{\varepsilon}{a},\, 0\right)}$ have two singularities of type A_2 and they come from two points in $V_{\left(-\varepsilon,\, a\right)}$ and $V_{\left(\frac{\varepsilon}{a},\, 0\right)}$ respectively, where $\psi \times id$ is not immersive.
Thus, it does not matter that we worked with $L = \{\lambda_2 = a\}$ instead of $\ell = \{\varepsilon\lambda_1 + \lambda_2 = a\}$ to keep the calculations simple.

This finally proves that the main error lies in 11. where Némethi assumes that we *may choose a small perturbation* of ℓ' which suits.

Claim 17 *We cannot perturb L' (or ℓ') to get ℓ'' by a "small" perturbation such that ℓ'' intersects D_Θ transversally in 6 points giving rise to a geometric basis.*

Proof: Suppose such a perturbation ℓ'' exists. The adjective "small" implies that two of these 6 intersection points contribute the vanishing cycles coming from $V_{\lambda_0} = p^{-1}(\lambda_0)$ via $(\Theta(b_{3(\lambda_0)}))_{reg} = (V'_{\lambda_0})_{reg}$.
Since the constructing machinery for these cycles is such that during the pullback the cycles do not meet the additional singularities in any fiber, they may actually be separated from the vanishing cycles coming from those additional singularities in the way described in 12. Thus, we get a disconnected Dynkin diagram, which leads to a contradiction. Therefore ℓ'' cannot exist.

□

We will close this section by looking at what the different fibers V_{λ_0}, V'_{λ_0} and $\Theta^{-1}(s)$ "are" – and how the vanishing cycles lie in them.
V_{λ_0} is a Milnor fiber of an ordinary cusp and thus has the homotopy type of two circles. Since the cusp has only one irreducible component, we may identify V_{λ_0} with a torus with one hole and the vanishing cycles δ_+ and δ_- may be taken as indicated in figure 9. Similarly, we come to figure 10a for $\Theta^{-1}(s)$ since $\Theta^{-1}(s)$ is the Milnor fiber of an E_6–singularity. Choosing a geometric basis in $\Theta^{-1}(s)$ in the way indicated in figure 10a, there are two different ways to see, that V'_{λ_0} must more or less look like as shown in figure 11:

- V'_{λ_0} is the image of V_{λ_0} under an injection with two non–immersive points.
- The two A_2–singularities of V'_{λ_0} give rise to non–intersecting vanishing cycles when going from V'_{λ_0} to $\Theta^{-1}(s)$; and thus, we may regain V'_{λ_0} from $\Theta^{-1}(s)$ by contracting the cycles δ_1 to δ_4 simultaneously.

The first way also suggests that the vanishing cycles induced from V_{λ_0} lie in V'_{λ_0} as shown in figure 11.

If Némethi's construction in part 11. of the proof of theorem 4 was possible, we would get the situation in figure 10b and thus the Dynkin diagram E_6 had to be equivalent to the disjoint union of three A_2's. This is not the case. In 3.1 we gave three different suggestions 1.-3. of what could possibly happen instead. If any small perturbation intersects in the wrong number of points (1.) we would get a situation as described in figure 10c. Suggestion 2. that we loose δ_+ and δ_- would give us figure 10a. And finally number 3. is represented in figure 10d.
As we already mentioned, we cannot decide which of these pictures describes the situation best. Indeed, it seems very likely that the "correct" choice of the deformation ℓ'' could lead to any of the figures 10a, 10c and 10d and that just figure 10b cannot be achieved by any small perturbation at all.

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Figures

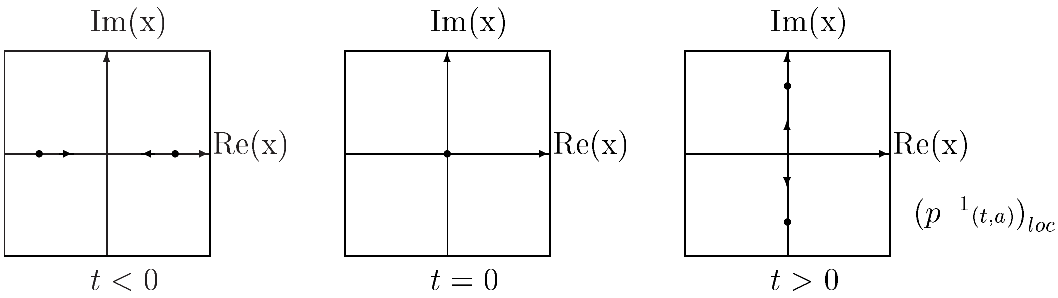

Figure 5a

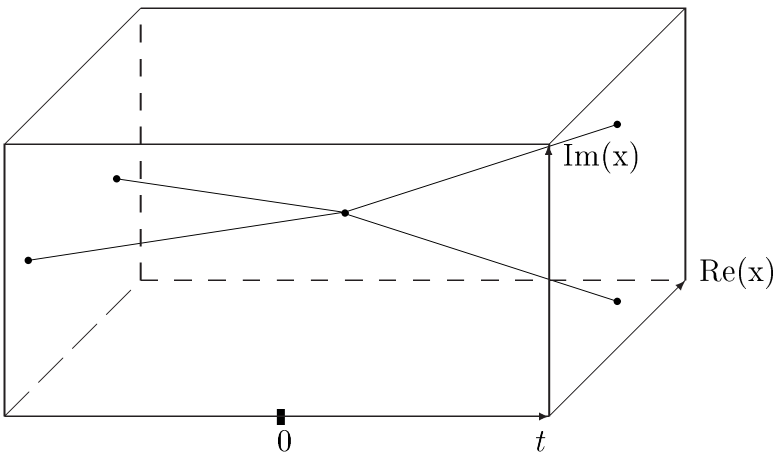

Figure 5b

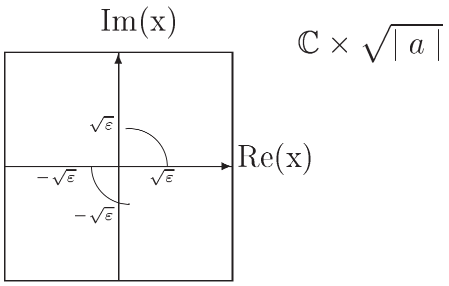

Figure 6

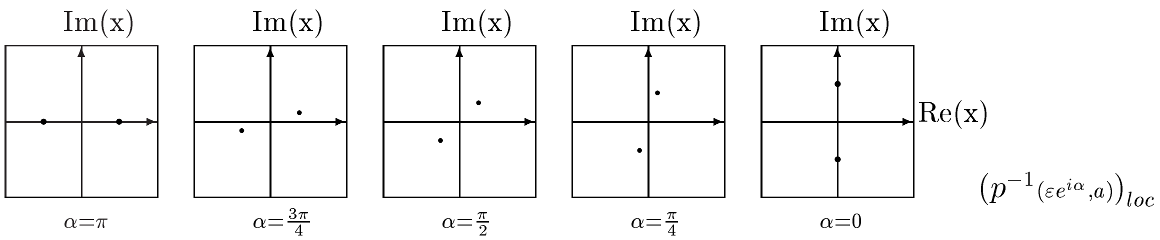

Figure 7a

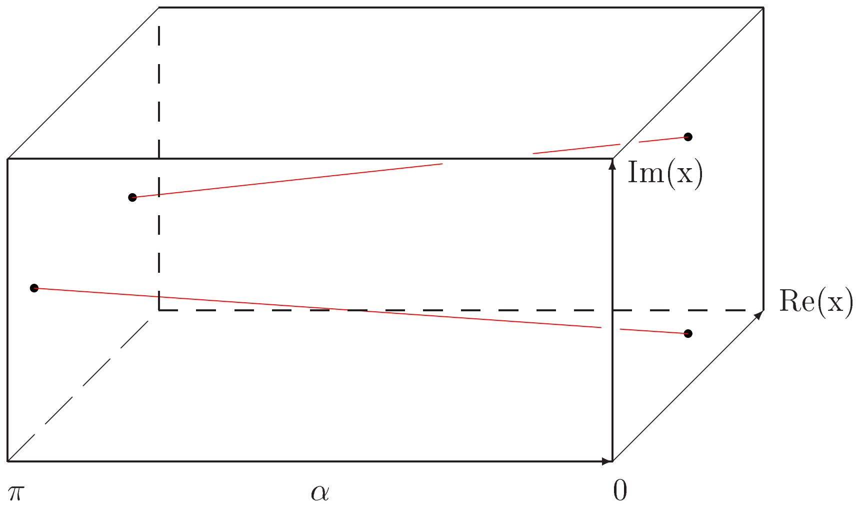

Figure 7b

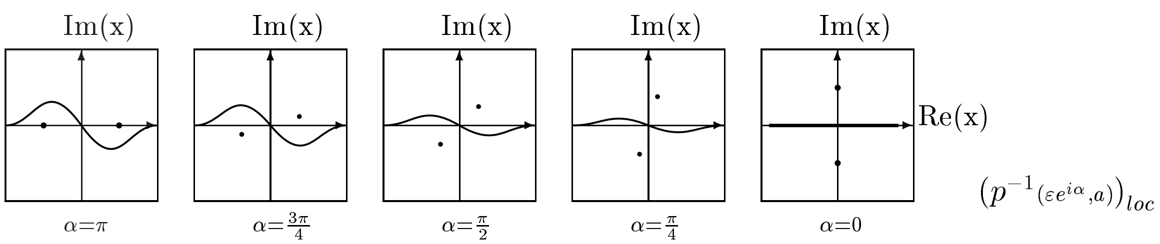

Figure 8a

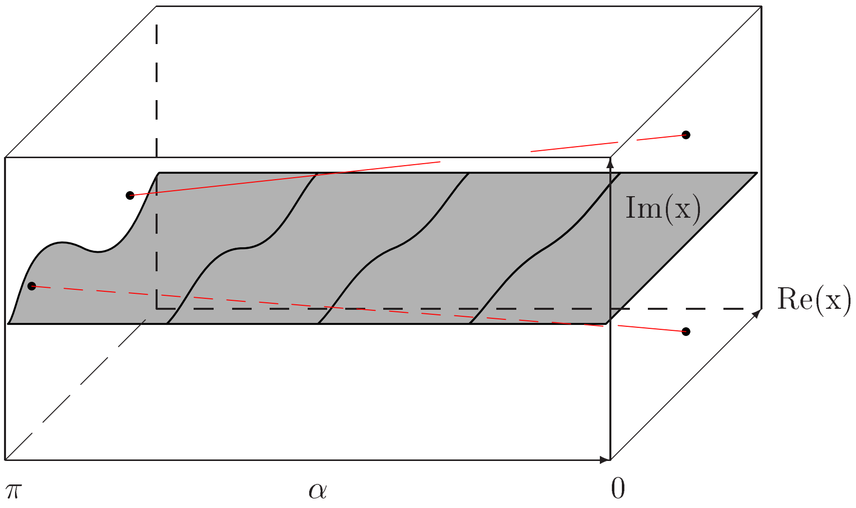

Figure 8b

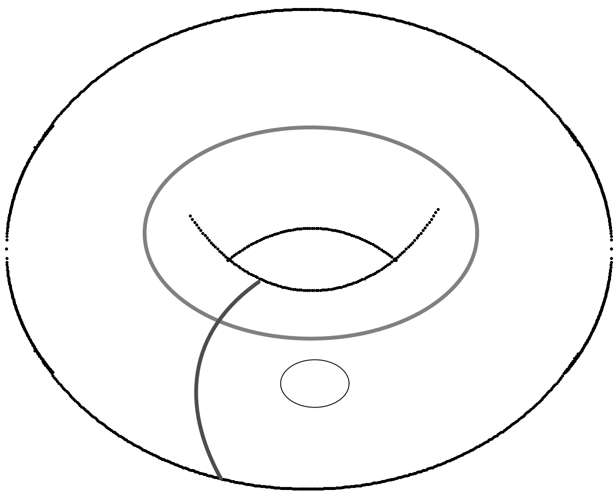

Figure 9

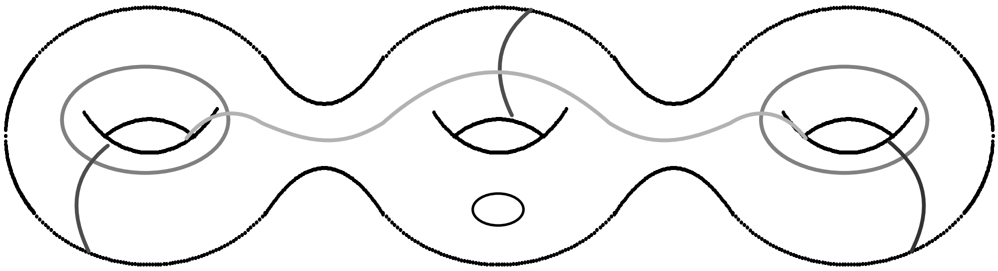

Figure 10a

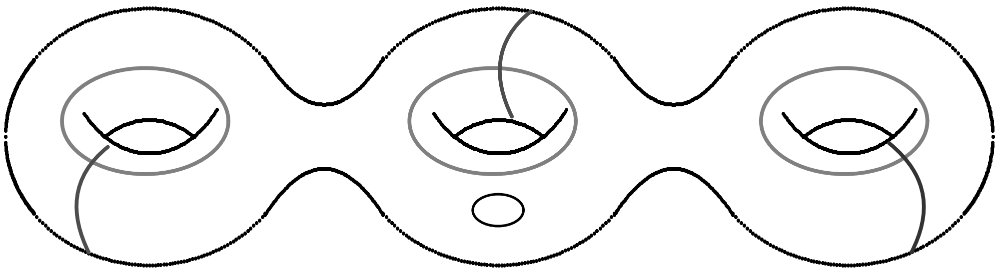

Figure 10b

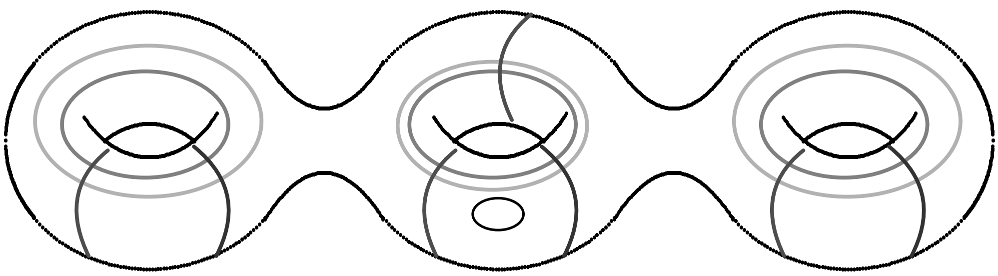

Figure 10c

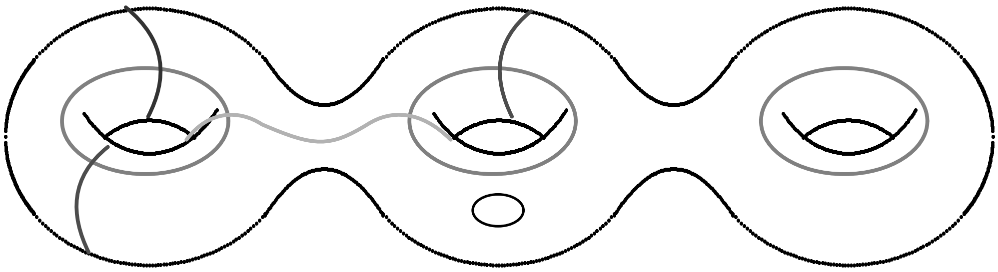

Figure 10d

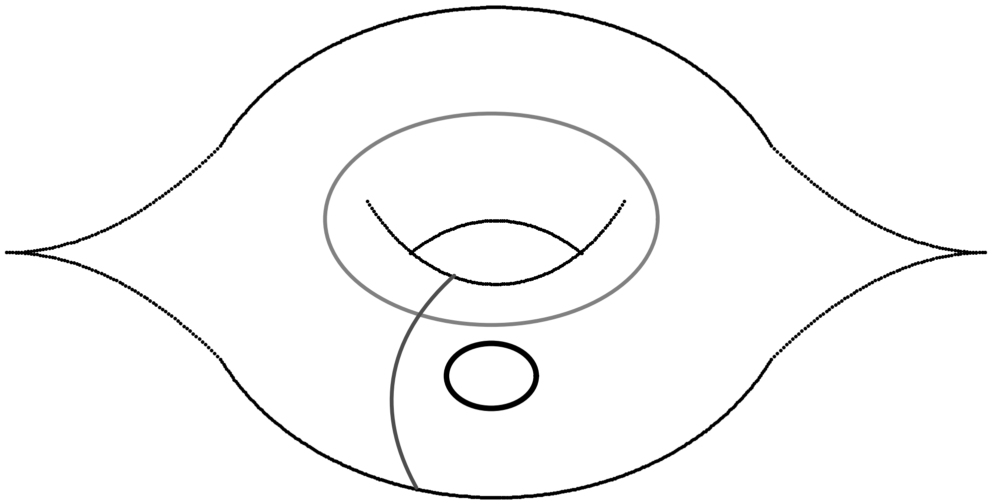

Figure 11