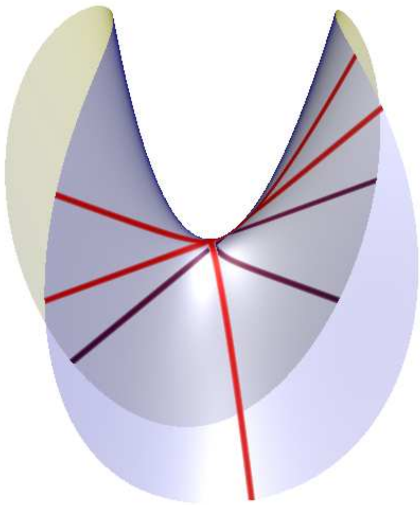
# Families of Curves with Prescribed Singularities



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To my parents
Hildegard and
Eduard Keilen

The picture on the titlepage shows a singular curve on an affine chart of the Hirzebruch surface $\mathbb{F}_0 = \mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$ embedded into $\mathbb{P}^3_{\mathbb{C}}$ as a quadric.

# Preface

The study of families of curves with prescribed singularities has a long tradition. Its foundations were laid by Plücker, Severi, Segre, and Zariski at the beginning of the 20th century. Leading to interesting results with applications in singularity theory and in the topology of complex algebraic curves and surfaces it has attained the continuous attraction of algebraic geometers since then.

Throughout this thesis we examine the varieties $V^{irr}_{|D|}(\mathcal{S}_1,\ldots,\mathcal{S}_r)$ of irreducible reduced curves in a fixed linear system $|D|$ on a smooth projective surface $\Sigma$ over $\mathbb{C}$ having precisely $r$ singular points of types $\mathcal{S}_1,\ldots,\mathcal{S}_r$ – for a more precise definition we refer to Chapter I. We are mainly interested in the following three questions:

(a) Is $V^{irr}_{|D|}(\mathcal{S}_1,\ldots,\mathcal{S}_r)$ non-empty?

(b) Is $V^{irr}_{|D|}(\mathcal{S}_1,\ldots,\mathcal{S}_r)$ T-smooth, that is smooth of the expected dimension?

(c) Is $V^{irr}_{|D|}(\mathcal{S}_1,\ldots,\mathcal{S}_r)$ irreducible?

That the dimension be the expected one means that the dimension of $|D|$ drops for each imposed singularity type $\mathcal{S}_i$ exactly by the number of conditions imposed by $\mathcal{S}_i$ – e. g. a node imposes one condition, a cusp two.

The simplest possible case of nodal plane curves was more or less completely answered by Severi in the early 20th century. He showed that $V^{irr}_{|dH|}(rA_1)$, where $H$ is a line in $\mathbb{P}^2_{\mathbb{C}}$, is non-empty if and only if

$$0 \leq r \leq \frac{(d-1)\cdot(d-2)}{2}.$$

Moreover, he showed that $V^{irr}_{|dH|}(rA_1)$ is T-smooth whenever it is non-empty, and he claimed that the variety is always irreducible. Harris proved this claim, which had become known as the Severi Conjecture by then, in 1985 (cf. [**Har85b**]). Considering more complicated singularities we may no longer expect such complete answers. Hirano provides in [**Hir92**] a series of examples of irreducible cuspidal plane curves of degree $d = 2\cdot 3^k$, $k\in\mathbb{N}$, imposing more than $\frac{d(d-3)}{2}$ conditions on $|dH|$ – that means in particular, we may hardly expect to be able to realize all smaller quantities of cusps on an irreducible curve of degree $d$. Moreover, we see that $V^{irr}_{|dH|}(rA_2)$ does not necessarily have the expected dimension – examples of this behaviour were already known to Segre (cf. [**Seg29**]). In 1974 Jonathan Wahl (cf. [**Wah74b**]) showed that $V^{irr}_{|104\cdot H|}(3636\cdot A_1, 900\cdot A_2)$ is non-reduced and hence singular. However, its reduction is smooth. The first example, where also the reduction is singular, is due to Luengo. In [**Lue87a**] he shows that the plane curve $C$ given by $x^9+z\big(xz^3+y^4\big)^2$ has a single singular point of simple type $A_{35}$ and that

$V^{irr}_{|9\cdot H|}(A_{35})$ is non-smooth, but reduced at $C$. Thus also the smoothness will fail in general. And finally, already Zariski (cf. [**Zar35**]) knew that $V^{irr}_{|6\cdot H|}(6\cdot A_2)$ consists of two connected components.

The best we may expect thus is to find numerical conditions, depending on the divisor $D$ and certain invariants of the singularities, which imply either of the properties in question. In order to see that the conditions are of the right kind - we then call them *asymptotically proper* -, they should not be too far from necessary conditions respectively they should be nearly fulfilled for series of counterexamples. Let us make this last statement a bit more precise. Suppose that $D = dL$ for some fixed divisor $L$ and $d \geq 0$. We are looking for conditions of the kind

$$\sum_{i=1}^{r} \alpha(\mathcal{S}_i) < p(d),$$

where $\alpha$ is some invariant of topological respectively analytical singularity types and $p \in \mathbb{R}[x]$ is some polynomial, neither depending on $d$ nor on the $\mathcal{S}_i$. We say that the condition is *asymptotically proper*, if there is a necessary condition with the same invariants and a polynomial of the same degree. If instead we find an infinite series of examples not having the desired property, where, however, the above inequality is reversed for the same invariants and some other polynomial of the same degree, we say that at least for the involved subclass of singularity types, the condition is *asymptotically proper*. (See also [**Los98**] Section 4.1.)

While the study of nodal and cuspidal curves has a long tradition, the consideration of families of more complicated singularities needed a suitable description of the tangent space of the family at a point, giving a concrete meaning to "the number of conditions imposed by a singularity type", that is to the expected dimension of the family. Greuel and Karras in [**GrK89**] in the analytical case, Greuel and Lossen in [**GrL96**] in the topological case identify the tangent spaces basically with the global sections of the ideal sheaves of certain zero-dimensional schemes associated to the singularity types (see Defintion I.2.7 and Remark I.2.15). This approach - in combination with a Viro gluing type method in the existence case - allows to reduce the existence, T-smoothness and irreducibility problem to the vanishing of certain cohomology groups. Various efforts in this direction culminate in asymptotically proper conditions for the existence (cf. [**GLS98c**]) and conditions for the T-smoothness and irreducibility, which are better than any previously known ones (cf. [**GLS00**]). Due to known examples the conditions for the T-smoothness are even asymptotically proper for simple singularities and ordinary multiple points. For an overview on the state of the art in the case of the plane curves we refer to [**GrS99**].

While the previous investigations mainly considered curves in $\mathbb{P}^2_{\mathbb{C}}$, we study curves on arbitrary smooth projective surfaces and derive results on various families of surfaces, in many cases the first results in that direction known on these surfaces at all.

In Chapter III we study the question of the *existence* of curves and the main condition (cf. Corollary III.2.5), which we derive in the case of topological singularity types, is of the form

$$\sum_{i=1}^{r} \delta(\mathcal{S}_i) \leq \alpha D^2 + \beta D.K + \gamma,$$

and for analytical singularity types it is of the form

$$\sum_{i=1}^{r} \mu(\mathcal{S}_i) \leq \alpha D^2 + \beta D.K + \gamma,$$

for some fixed divisor $K$ and some absolute constants $\alpha$, $\beta$ and $\gamma$, where $\delta(\mathcal{S}_i)$ is the delta-invariant of $\mathcal{S}_i$ and $\mu(\mathcal{S}_i)$ is its Milnor number. The corresponding necessary condition

$$\sum_{i=1}^{r} \mu(\mathcal{S}_i) \leq 2 \cdot \sum_{i=1}^{r} \delta(\mathcal{S}_i) \leq D^2 + D.K_\Sigma + 2$$

is of the same asymptotical behaviour.

Also for the main condition

$$\left( \sum_{i=1}^{r} \big(\tau^*(\mathcal{S}_i) + 1\big) + \tfrac{D.K_\Sigma}{2} \right)^2 < \left( r + \tfrac{K_\Sigma^2}{4} \right)^2 \cdot D^2,$$

which we get in Chapter IV for the *T-smoothness* (cf. Theorem IV.1.1) we cannot expect to do better in general – here for topological singularity types $\tau^*(\mathcal{S}_i) = \tau^{es}(\mathcal{S}_i)$ is the codimension of the equisingular stratum of $\mathcal{S}_i$ in its semiuniversal deformation, and for analytical types $\tau^*(\mathcal{S}_i) = \tau(\mathcal{S}_i)$, the Tjurina number of $\mathcal{S}_i$. In the case of simple singularities on plane curves the mentioned examples in [**GLS00**] show that the conditions are asymptotically proper. However, already for ordinary multiple points this is no longer the case, as is shown there as well. Still, our results apply to surfaces which have not been considered before, and even in the well studied case of of nodal curves on quintics in $\mathbb{P}^3_{\mathbb{C}}$ we get the same sharp results as Chiantini and Sernesi (cf. [**ChS97**]).

We do not know wether our main condition (cf. Theorems V.2.1-V.2.4)

$$\sum_{i=1}^{r} \big(\tau^*(\mathcal{S}_i) + 2\big)^2 < \gamma \cdot (D - K_\Sigma)^2$$

in the study of the *irreducibility* in Chapter V shows a similar properness, since neither sufficient conditions of this form nor counterexamples with the same asymptotical behaviour are known – here, $\gamma$ is some constant, possibly depending on $D$. And we do not even expect them to be of the right kind. However, the results which we get are for most of the considered surfaces the only known ones, and even in the well studied plane case they are asymptotically of the same quality than the best previously known conditions.

All our results somehow rely on the vanishing or non-vanishing of certain cohomology groups. The results of Chapter IV and of Chapter V could be formulated completely as a vanishing theorem; and although we derive the existence of curves with prescribed topological or analytical singularities by gluing these into a suitable given curve, we derive the existence of that curve (cf. Theorem III.1.2) again with the aid of a vanishing theorem. The latter is a generalisation of a vanishing theorem of Geng Xu (cf. [**Xu95**]) and thus of the Kawamata–Viehweg Vanishing Theorem . Chapter II is devoted to its proof.

In Chapter I we introduce the notions used throughout the thesis and state several important facts which are well known. A compact and profound description of the introduced objects and cited results can be found in the thesis of Christoph Lossen [**Los98**], and for the convenience of the reader we usually refer to this thesis as well as to the original sources.

Apart from the introduction, each chapter of the thesis consists of one or two sections describing the main results possibly followed by a section containing essential technical details of the proofs. At the end of each chapter we examine the derived conditions on several classes of surfaces:

(a) $\mathbb{P}^2_{\mathbb{C}}$,

(b) geometrically ruled surfaces,

(c) product-surfaces,

(d) surfaces in $\mathbb{P}^3_{\mathbb{C}}$, and

(e) K3-surfaces.

We have chosen these partly due to their important role in the classification of surfaces, and partly since they all have advantages in their own which make it possible to keep control on the numerical conditions. In the appendix we gather a number of facts which we suppose are well known, but for which we nevertheless could not find a suitable reference. In particular, we give an overview of the properties of the studied surfaces, which we need for the examinations.

Throughout a fixed chapter the theorems, lemmata, definitions and remarks are numbered by sections, e. g. within Section 1 of Chapter V the Theorem 1.1 is followed by Corollary 1.2, while in Section 2 we start again with Theorem 2.1. The same applies to the equations. There is, however, one exception from this rule. In the example sections we reproduce certain statements (equations, theorems, ...) in special cases, and instead of using new numbers we rather keep the old ones adding the letter of the corresponding subsection, e. g. in the example section of Chapter V we have Theorem 2.1a with equation (2.2a) in Subsection 4.a, which is just the appropriate form of Theorem 2.1

in Chapter V. Whenever we refer to a statement within the same chapter, we just cite it by its number. If we, however, refer to it in some other chapter, we add the number of the chapter, e. g. we refer to Theorem 1.1 of Chapter IV within Chapter IV as Theorem 1.1, while in the preface we would cite it as Theorem IV.1.1.

Chapter II and Chapter III are a joint work with Ilya Tyomkin (Tel Aviv University) and have been accepted for publication in the Transactions of the American Mathematical Society. Chapter IV is a slight modification of results already published by Gert-Martin Greuel (Universität Kaiserslautern), Christoph Lossen (Universität Kaiserslautern) and Eugenii Shustin (Tel Aviv University) in [**GLS97**], and a slightly stronger version may be found in [**GLS05**]. Finally, Chapter V extends an approach via Bogomolov unstability of vector bundles used by Gert-Martin Greuel, Christoph Lossen and Eugenii Shustin in the plane case (cf. [**GLS98b**]).

## Acknowledgements

I would like to express my thanks to my supervisor, Gert-Martin Greuel, for supporting me in many ways during the last years, and I would also like to thank Christoph Lossen, Eugenii Shustin and Ilya Tyomkin for their cooperation while working on this thesis. I owe a special thank to Bernd Kreußler for many fruitful conversations widening my understanding of algebraic geometry. Finally, I have the pleasure to thank my parents, Hildegard and Eduard Keilen, for their love and care which accompanied me all my life, and Hannah Markwig for her love and for her patience.

## Post Scriptum

This is an extended version of the original thesis. Section I.3, Section IV.3 and Section IV.4 have been added, Chapter V has largely been rewritten and improved. Moreover, a number of misprints and minor errors have been corrected.

# Contents

CHAPTER I

# Introduction

## 1. General Assumptions and Notations

Throughout this thesis $\Sigma$ will denote a smooth projective surface over $\mathbb{C}$.

Given distinct points $z_1, \ldots, z_r \in \Sigma$, we denote by $\pi : \mathrm{Bl}_{\underline{z}}(\Sigma) = \widetilde{\Sigma} \to \Sigma$ the blow up of $\Sigma$ in $\underline{z} = (z_1, \ldots, z_r)$, and the exceptional divisors $\pi^* z_i$ will be denoted by $E_i$. We shall write $\widetilde{C} = \mathrm{Bl}_{\underline{z}}(C)$ for the strict transform of a curve $C \subset \Sigma$.

For any smooth projective surface $\Sigma$ we will denote by $\mathrm{Div}(\Sigma)$ the group of divisors on $\Sigma$ and by $K_\Sigma$ its canonical divisor. If $D$ is any divisor on $\Sigma$, $\mathcal{O}_\Sigma(D)$ shall be a corresponding invertible sheaf. A curve $C \subset \Sigma$ will be an effective (non-zero) divisor, that is a one-dimensional locally principal scheme, not necessarily reduced; however, when we talk of an "irreducible curve" this shall include that the curve is reduced. $|D| = |D|_l = \mathbb{P}\big(H^0(\Sigma, \mathcal{O}_\Sigma(D))\big)$ denotes the system of curves linearly equivalent to $D$, while we use the notation $|D|_a$ for the system of curves algebraically equivalent to $D$ (cf. **[Har77]** Ex. V.1.7), that is the reduction of the connected component of $\mathrm{Hilb}_\Sigma$, the Hilbert scheme of all curves on $\Sigma$, containing any curve algebraically equivalent to $D$ (cf. **[Mum66]** Chapter 15).[1] We write $\mathrm{Pic}(\Sigma)$ for the Picard group of $\Sigma$, that is $\mathrm{Div}(\Sigma)$ modulo linear equivalence (denoted by $\sim_l$), $\mathrm{NS}(\Sigma)$ for the Néron–Severi group, that is $\mathrm{Div}(\Sigma)$ modulo algebraic equivalence (denoted by $\sim_a$), and $\mathrm{Num}(\Sigma)$ for $\mathrm{Div}(\Sigma)$ modulo numerical equivalence (denoted by $\sim_n$). Given a reduced curve $C \subset \Sigma$ we write $p_a(C)$ for its arithmetical genus and $g(C)$ for the geometrical one.

Given any scheme $X$ and any coherent sheaf $\mathcal{F}$ on $X$, we will often write $H^\nu(\mathcal{F})$ instead of $H^\nu(X, \mathcal{F})$ when no ambiguity can arise. Moreover, if $\mathcal{F} = \mathcal{O}_X(D)$ is the invertible sheaf corresponding to a divisor $D$, we will usually use the notation $H^\nu(X, D)$ instead of $H^\nu\big(X, \mathcal{O}_X(D)\big)$. Similarly when considering tensor products over the structure sheaf of some scheme $X$ we may sometimes just write $\otimes$ instead of $\otimes_{\mathcal{O}_X}$.

Given any subscheme $X \subset Y$ of a scheme $Y$, we denote by $\mathcal{J}_X = \mathcal{J}_{X/Y}$ the *ideal sheaf* of $X$ in $\mathcal{O}_Y$, and for $z \in Y$ we denote by $\mathcal{O}_{Y,z}$ the local ring of $Y$ at $z$ and by $\mathfrak{m}_{Y,z}$ its maximal ideal, while $\widehat{\mathcal{O}}_{Y,z}$ denotes the $\mathfrak{m}_{Y,z}$-adic completion of $\mathcal{O}_{Y,z}$. If $X$ is zero-dimensional we denote by $\#X$ the number of points in its *support* $\mathrm{supp}(X)$, by $\deg(X) = \sum_{z \in Y} \dim_{\mathbb{C}}(\mathcal{O}_{Y,z}/\mathcal{J}_{X/Y,z})$ its *degree*, and by $\mathrm{mult}(X, z) = \max\big\{m \in \mathbb{N} \mid \mathcal{J}_{X/Y,z} \subseteq \mathfrak{m}_{Y,z}^m\big\}$ its *multiplicity at* $z$.

[1]Note that indeed the reduction of the Hilbert scheme gives the Hilbert scheme $\mathrm{Hilb}_\Sigma^{red}$ of curves on $\Sigma$ over reduced base spaces.

If $X \subset \Sigma$ is a zero-dimensional scheme on $\Sigma$ and $D \in \mathrm{Div}(\Sigma)$, we denote by $\left|\mathcal{J}_{X/\Sigma}(D)\right|_l = \mathbb{P}\Big(H^0\big(\mathcal{J}_{X/\Sigma}(D)\big)\Big)$ the linear system of curves $C$ in $|D|_l$ with $X \subset C$.

If $L \subset \Sigma$ is any reduced curve and $X \subset \Sigma$ a zero-dimensional scheme, we define the *residue scheme* $X : L \subset \Sigma$ *of* $X$ by the ideal sheaf $\mathcal{J}_{X:L/\Sigma} = \mathcal{J}_{X/\Sigma} : \mathcal{J}_{L/\Sigma}$ with stalks

$$\mathcal{J}_{X:L/\Sigma,z} = \mathcal{J}_{X/\Sigma,z} : \mathcal{J}_{L/\Sigma,z},$$

where ":" denotes the ideal quotient. This leads to the definition of the *trace scheme* $X \cap L \subset L$ *of* $X$ via the ideal sheaf $\mathcal{J}_{X\cap L/L}$ given by the exact sequence

$$0 \longrightarrow \mathcal{J}_{X:L/\Sigma}(-L) \xrightarrow{\cdot L} \mathcal{J}_{X/\Sigma} \longrightarrow \mathcal{J}_{X\cap L/L} \longrightarrow 0.$$

Let $Y$ be a Zariski topological space (cf. **[Har77]** Ex. II.3.17). We say a subset $U \subseteq Y$ is *very general* if it is an at most countable intersection of open dense subsets of $Y$. Some statement is said to hold for points $z_1, \ldots, z_r \in Y$ (or $\underline{z} \in Y^r$) *in very general position* if there is a suitable very general subset $U \subseteq Y^r$, contained in the complement of the closed subvariety $\bigcup_{i\neq j}\{\underline{z} \in Y^r \mid z_i = z_j\}$ of $Y^r$, such that the statement holds for all $\underline{z} \in U$.

## 2. Singularity Schemes

This thesis is a generalisation of results described in **[Los98]**, and the zero-dimensional schemes with which we are concerned are examined very carefully in **[Los98]** Chapter 2. We therefore restrict ourselves to a very short description and refer to **[Los98]** for more details.

**2.1 Definition**

(a) Let $f \in \mathcal{O}_{\Sigma,z} \subset \mathbb{C}\{x,y\}$ be given. The germ $(C,z) \subset (\Sigma,z)$ defined by a representative $C = \big\{z' \mid f(z') = 0\big\}$ is called a *plane curve singularity*. The $\mathbb{C}$-algebra $\mathcal{O}_{C,z} := \mathcal{O}_{\Sigma,z}/(f)$ is called the *local ring of* $(C,z)$.

(b) Two plane curve singularities $(C,z)$ and $\big(C',z'\big)$ are said to be *topologically equivalent* if there exists a homeomorphism $\Phi : (\Sigma,z) \to (\Sigma,z')$ such that $\Phi(C) = C'$.[2] We write $(C,z) \sim_t \big(C',z'\big)$.
An equivalence class $\mathcal{S}$ of this equivalence relation is called a *topological singularity type*.

(c) Two plane curve singularities $(C,z)$ and $\big(C',z'\big)$ are said to be *analytically equivalent* (or *contact equivalent*) if there exists an isomorphism $\Phi^\# : \widehat{\mathcal{O}}_{C',z'} \to \widehat{\mathcal{O}}_{C,z}$ of the complete local rings. We write $(C,z) \sim_c \big(C',z'\big)$.
An equivalence class $\mathcal{S}$ of this equivalence relation is called an *analytical singularity type*.

(d) Let $f \in \mathcal{O}_{\Sigma,z}$ and $f' \in \mathcal{O}_{\Sigma,z'}$. We say $f$ and $f'$ are *topologically* respectively *analytically equivalent* if the singularities defined by $f$ and $f'$ are so. We then write $f \sim_t f'$ respectively $f \sim_c f'$.

[2] This means of course that the equality holds for suitable representatives.

**2.2 Remark**

We will need a number of invariants of singularity types, which we are just going to list here. Note that every invariant of the topological type of a singularity is of course also an invariant of its analytical type. Let $(C, z)$ be a reduced plane curve singularity with representative $f \in \mathcal{O}_{\Sigma,z} \subset \mathbb{C}\{x, y\}$.

We refer to Definition 2.7 respectively Remark 2.8 for the definition of $I^{es}(C, z)$, $I^s(C, z)$ and $I^a(C, z)$, and we abreviate *complete intersection ideal* by CI. Denoting by $i(f, g) = \dim_{\mathbb{C}}\{x, y\}/(f, g)$ for $g \in \mathbb{C}\{x, y\}$ the intersection multiplicity of $f$ and $g$, we define for a rational number $\alpha \in [0, 1]$ and for an ideal $I$ in $R = \mathbb{C}\{x, y\}$ with $\left(f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) \subseteq I$

$$\lambda_\alpha(f; I, g) = \frac{\big(\alpha \cdot i(f, g) + (1 - \alpha) \cdot \dim_{\mathbb{C}}(R/I)\big)^2}{i(f, g) - \dim_{\mathbb{C}}(R/I)}$$

and

$$\gamma_\alpha(f; I) = \max\left\{(1 + \alpha)^2 \cdot \dim_{\mathbb{C}}(R/I),\ \lambda_\alpha(f; I, g) \;\middle|\; g \in I,\ i(f, g) \leq 2 \cdot \dim_{\mathbb{C}}(R/I)\right\}.$$

(a) The following are known to be invariants of the topological type $\mathcal{S}$ of the plane curve singularity $(C, z)$.
   (1) $\mathrm{mult}(\mathcal{S}) = \mathrm{mult}(C, z) = \mathrm{ord}(f)$ is the *multiplicity* of $\mathcal{S}$.
   (2) $\mu(\mathcal{S}) = \mu(C, z) = \dim_{\mathbb{C}} \mathbb{C}\{x, y\}/\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$ is the *Milnor number* of $\mathcal{S}$.
   (3) $\tau^{es}(\mathcal{S}) = \tau^{es}(C, z)$ is the codimension of the $\mu$-constant stratum in the semiuniversal deformation of $(C, z)$.
   (4) $\kappa(\mathcal{S}) = \kappa(C, z) = i(f, g)$, with $g$ a generic polar curve of $f$.
   (5) $\nu^s(\mathcal{S}) = \nu^s(C, z)$ shall be the minimal integer $m$ such that $\mathfrak{m}_{\Sigma,z}^{m+1} \subseteq I^s(C, z)$ and is called the *topological deformation determinacy*.
   (6) $\delta(\mathcal{S}) = \delta(C, z) = \dim_{\mathbb{C}}\left(n_*\mathcal{O}_{\widetilde{C},z}/\mathcal{O}_{C,z}\right)$ is the *delta invariant* of $\mathcal{S}$, where $n : \left(\widetilde{C}, z\right) \to (C, z)$ is a normalisation of $(C, z)$.
   (7) $r(\mathcal{S}) = r(C, z)$ is the number of branches of $(C, z)$.
   (8) $\tau^{es}_{ci}(C, z) = \max\left\{ \dim_{\mathbb{C}}(\mathbb{C}\{x, y\}/I) \;\middle|\; I^{es}(C, z) \subseteq I \text{ a CI}\right\}$ and $\tau^{es}_{ci}(\mathcal{S}) = \max\{\tau^{es}_{ci}(C, z) \mid (C, z) \text{ some representative of } \mathcal{S}\}$.
   (9) $\gamma^{es}_\alpha(C, z) = \max\left\{\gamma_\alpha(f; I) \;\middle|\; I^{es}(C, z) \subseteq I \text{ a CI}\right\}$ and $\gamma^{es}_\alpha(\mathcal{S}) = \max\{\gamma^{es}_\alpha(C, z) \mid (C, z) \text{ some representative of } \mathcal{S}\}$.

(b) For the analytical type $\mathcal{S}$ of $(C, z)$ we have some additional invariants:
   (1) $\tau(\mathcal{S}) = \tau(C, z) = \dim_{\mathbb{C}} \mathbb{C}\{x, y\}/\left(f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$ is the *Tjurina number*.
   (2) $\nu^a(\mathcal{S}) = \nu^a(C, z)$ shall be the minimal integer $m$ such that $\mathfrak{m}_{\Sigma,z}^{m+1} \subseteq I^a(C, z)$ and is called the *analytical deformation determinacy*.
   (3) $\tau_{ci}(\mathcal{S}) = \tau^{es}_{ci}(C, z) = \max\left\{ \dim_{\mathbb{C}}(\mathbb{C}\{x, y\}/I) \;\middle|\; \left(f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) \subseteq I \text{ a CI}\right\}$.
   (4) $\gamma^{ea}_\alpha(\mathcal{S}) = \gamma^{es}_\alpha(C, z) = \max\left\{\gamma_\alpha(f; I) \;\middle|\; \left(f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) \subseteq I \text{ a CI}\right\}$

(5) $\mathrm{mod}(\mathcal{S}) = \mathrm{mod}(C,z)$, the *modality* of the singularity $\mathcal{S}$. (Cf. [**AGZV85**] p. 184.)

Throughout the thesis we will very often consider topological and analytical singularity types at the same time. We therefore introduce the notation $\tau^*(\mathcal{S})$, $\tau^*_{ci}(\mathcal{S})$, $\nu^*(\mathcal{S})$, and $\gamma^*_\alpha(\mathcal{S})$, where for a topological singularity type $\tau^*(\mathcal{S}) = \tau^{es}(\mathcal{S})$, $\tau^*_{ci}(\mathcal{S}) = \tau^{es}_{ci}(\mathcal{S})$, $\nu^*(\mathcal{S}) = \nu^s(\mathcal{S})$, and $\gamma^*_\alpha(\mathcal{S}) = \gamma^{es}_\alpha(\mathcal{S})$, while for an analytical singularity type $\tau^*(\mathcal{S}) = \tau(\mathcal{S})$, $\tau^*_{ci}(\mathcal{S}) = \tau_{ci}(\mathcal{S})$, $\nu^*(\mathcal{S}) = \nu^a(\mathcal{S})$, and $\gamma^*_\alpha(\mathcal{S}) = \gamma^{ea}_\alpha(\mathcal{S})$.

**2.3 Remark**

For the convenience of the reader we would like to gather some known relations between the above topological and analytical invariants of the singularity types corresponding to some reduced plane curve singularity $(C,z)$.

(a) $\mu(\mathcal{S}) \leq \mu(\mathcal{S}) + r(\mathcal{S}) - 1 = 2\delta(\mathcal{S})$. (Cf. [**Mil68**] Chapter 10.)

(b) $\delta(\mathcal{S}) \leq \tau^{es}(\mathcal{S})$, since the $\delta$-constant stratum of the semiuniversal deformation of $(C,z)$ contains the $\mu$-constant stratum and since its codimension is just $\delta(\mathcal{S})$ (see also [**DiH88**]).

(c) $\delta(\mathcal{S}) < \tau^{es}(\mathcal{S})$, if $\mathcal{S} \neq A_1$ (see Lemma 3.6).

(d) $\kappa(\mathcal{S}) = \mu(\mathcal{S}) + \mathrm{mult}(\mathcal{S}) - 1$.

(e) $\kappa(\mathcal{S}) \leq 2\delta(\mathcal{S})$. (Cf. [**Los98**] Lemma 5.12.)

(f) $\tau^{es}(\mathcal{S}) = \mu(\mathcal{S}) - \mathrm{mod}(\mathcal{S})$. (Cf. [**AGZV85**] p. 245.)

(g) $\tau^{es}(\mathcal{S}) \leq \tau(\mathcal{S})$, since $\tau(\mathcal{S})$ is the dimension of the base space of the semiuniversal deformation of $(C,z)$.

(h) $\tau(\mathcal{S}) \leq \mu(\mathcal{S})$, by definition.

(i) $\nu^s(\mathcal{S}) \leq \tau^{es}(\mathcal{S})$. (Cf. [**GLS00**] Lemma 1.5.)

(j) $\nu^s(\mathcal{S}) \leq \delta(\mathcal{S})$, if all branches of $(C,z)$ have at least multiplicity three. (Cf. [**GLS00**] Lemma 1.5.)

(k) $\nu^a(\mathcal{S}) \leq \tau(\mathcal{S})$. (Cf. [**GLS00**] Remark 1.9.)

(l) $\nu^s(\mathcal{S}) \leq \nu^a(\mathcal{S})$, since $I^a(C,z) \subseteq I^s(C,z)$ (cf. Definition 2.7).

(m) $(1+\alpha)^2 \cdot \tau^{es}_{ci} \leq \gamma^{es}_\alpha(\mathcal{S}) \leq \left(\tau^{es}_{ci}(\mathcal{S}) + \alpha\right)^2$ (see Lemma 3.4).

(n) $(1+\alpha)^2 \cdot \tau^{ea}_{ci} \leq \gamma^{ea}_\alpha(\mathcal{S}) \leq \left(\tau^{ew}_{ci}(\mathcal{S}) + \alpha\right)^2$ (see Lemma 3.4).

(o) $\gamma^{es}_\alpha(\mathcal{S}) < \gamma^{es}_\beta(\mathcal{S})$, for $0 \leq \alpha < \beta \leq 1$ (see Lemma 3.3).

(p) $\gamma^{ea}_\alpha(\mathcal{S}) < \gamma^{ea}_\beta(\mathcal{S})$, for $0 \leq \alpha < \beta \leq 1$ (see Lemma 3.3).

Combining these results we get:

$$\nu^s(\mathcal{S}) \leq \left\{ \begin{array}{c} \tau^{es}(\mathcal{S}) \\ \nu^a(\mathcal{S}) \end{array} \right\} \leq \tau(\mathcal{S}) \leq \mu(\mathcal{S}) \leq \kappa(\mathcal{S}) \leq 2\delta(\mathcal{S}) \leq 2\tau^{es}(\mathcal{S}) \leq 2\tau(\mathcal{S}).$$

**2.4 Example**

Let $M_m$, $m \geq 2$, be the topological singularity type of an ordinary $m$-fold point, e. g. given by $f = x^m - y^m \in \mathbb{C}\{x,y\}$. Then[3]

$$I^{es}_{fix}(f) = I^s(f) = \langle x,y\rangle^m$$

and

$$I^{es}(f) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right\rangle + \langle x,y\rangle^m.$$

Thus we get:

| mult$(M_m)$ | $\mu(M_m) = \tau(M_m)$ | $\delta(M_m)$ | $\tau^{es}(M_m)$ |
|---|---|---|---|
| $m$ | $(m-1)^2$ | $\frac{m\cdot(m-1)}{2}$ | $\frac{m\cdot(m+1)}{2} - 2$ |
| $\kappa(M_m)$ | mod $(M_m)$ | $\nu^s(M_2)$ | $\nu^s(M_m), m \geq 3$ |
| $m\cdot(m-1)$ | $\frac{(m-2)\cdot(m-3)}{2}$ | $0$ | $m-1$ |
| $\gamma_1^{es}(M_2)$ | $\gamma_1^{es}(M_m), m \geq 3$ | $\gamma_0^{es}(M_2)$ | $\gamma_0^{es}(M_m)$ |
| $4$ | $2m^2$ | $1$ | $2\cdot(m-1)^2$ |
| $\tau_{ci}^{es}(M_2)$ | $\tau_{ci}^{es}(M_m), m \geq 4$ even | $\tau_{ci}^{es}(M_m), m$ odd | $r(M_m)$ |
| $1$ | $\frac{m^2+2m}{4}$ | $\frac{m^2+2m+1}{4}$ | $m$ |

**2.5 Remark**

It is well known that the simple singularities, that is the singularities with modality zero, form two infinite series $A_\mu$, given by $x^{\mu+1} - y^2$ for $\mu \geq 1$, and $D_\mu$, given by $y\cdot\left(x^2 - y^{\mu-2}\right)$ for $\mu \geq 4$, together with three excepitional singularities $E_\mu$ for $\mu = 6, 7, 8$, given by $x^3 - y^4$, $x^3 - xy^3$ and $x^3 - y^5$ respectively. Moreover, they are the only plane curve singularities $\mathcal{S}$ with $\mu(\mathcal{S}) = \tau(\mathcal{S}) \leq 8$.

The first non-simple family of singularities $X_9$, that is, the only one-modular family of singularities of Milnor number 9, is given by $x^4 - y^4 + a\cdot x^2y^2$. The modulus corresponds to the crossratio of the four intersecting lines. Topologically all these singularities are indeed equivalent. Note that, apart from the simple singularities, these are the only singularities $\mathcal{S}$ with $\tau^{es}(\mathcal{S}) \leq 8$, as a view at the classification table in **[AGZV85]** Chapter 15 shows, taking the restrictions on $\tau^{es}(\mathcal{S})$ given in Remark 2.3 into account.

Moreove, from the classification we deduce for an analytical singularity type $\mathcal{S}$ some results which will be useful in Lemma V.3.10.

| mult$(\mathcal{S})$ | mod$(\mathcal{S})$ | $\mu(\mathcal{S})$ |
|---|---|---|
| $2$ | $0$ | $1 \leq \mu(\mathcal{S})$ |

[3]For the definition of $I^{es}(f)$, $I^{es}_{fix}(f)$ and $I^s(f)$ see Definition 2.7.

| $\text{mult}(\mathcal{S})$ | $\text{mod}(\mathcal{S})$ | $\mu(\mathcal{S})$ |
|---|---|---|
| 3 | 0 | $4 \leq \mu(\mathcal{S})$ |
| | 1 | $10 \leq \mu(\mathcal{S}) \leq 14$ |
| | 2 | $16 \leq \mu(\mathcal{S})$ |
| | 3 | $22 \leq \mu(\mathcal{S})$ |
| | $\geq 4$ | $28 \leq \mu(\mathcal{S})$ |
| 4 | 1 | $9 \leq \mu(\mathcal{S}) \leq 12$ |
| | 2 | $15 \leq \mu(\mathcal{S})$ |
| | $\geq 3$ | $22 \leq \mu(\mathcal{S})$ |

And finally:

| $\text{mod}(\mathcal{S})$ | $\mu(\mathcal{S})$ | $\tau(\mathcal{S})$ | $\text{mult}(\mathcal{S})$ |
|---|---|---|---|
| 0 | $1 \leq \mu(\mathcal{S}) = \tau(\mathcal{S}) \leq 8$ | $1 \leq \mu(\mathcal{S}) = \tau(\mathcal{S}) \leq 8$ | $2, 3$ |
| 1 | $9 \leq \mu(\mathcal{S}) \leq 14$ | $9 \leq \tau(\mathcal{S}) \leq 14$ | $3, 4$ |
| 2 | $15 \leq \mu(\mathcal{S})$ | $13 \leq \mu(\mathcal{S}) - \text{mod}(\mathcal{S}) \leq \tau(\mathcal{S})$ | $3, 4$ |
| $\geq 3$ | $16 \leq \mu(\mathcal{S})$ | $\mu(\mathcal{S}) - \text{mod}(\mathcal{S}) \leq \tau(\mathcal{S})$ | $3 \leq \text{mult}(\mathcal{S})$ |

**2.6 Definition**
Given distinct points $z_1, \ldots, z_r \in \Sigma$ and non-negative integers $m_1, \ldots, m_r$ we denote by $X(\underline{m}; \underline{z}) = X(m_1, \ldots, m_r; z_1, \ldots, z_r)$ the zero-dimensional subscheme of $\Sigma$ defined by the ideal sheaf $\mathcal{J}_{X(\underline{m};\underline{z})/\Sigma}$ with stalks

$$\mathcal{J}_{X(\underline{m};\underline{z})/\Sigma,z} = \begin{cases} \mathfrak{m}_{\Sigma,z_i}^{m_i}, & \text{if } z = z_i, i = 1, \ldots, r, \\ \mathcal{O}_{\Sigma,z}, & \text{else.} \end{cases}$$

We call a scheme of the type $X(\underline{m}; \underline{z})$ an *ordinary fat point scheme*.

**2.7 Definition**
Let $C \subset \Sigma$ be a reduced curve.

(a) The scheme $X^{ea}(C)$ in $\Sigma$ is defined via the ideal sheaf $\mathcal{J}_{X^{ea}(C)/\Sigma}$ with stalks

$$\mathcal{J}_{X^{ea}(C)/\Sigma,z} = I^{ea}(C, z) = I^{ea}(f) = \left(f, \tfrac{\partial f}{\partial x}, \tfrac{\partial f}{\partial y}\right) \subseteq \mathcal{O}_{\Sigma,z},$$

where $x, y$ denote local coordinates of $\Sigma$ at $z$ and $f \in \mathcal{O}_{\Sigma,z}$ a local equation of $C$. $I^{ea}(C, z)$ is called the *Tjurina ideal* of the singularity $(C, z)$, and it is of course $\mathcal{O}_{\Sigma,z}$ whenever $z$ is a smooth point of $C$. We call $X^{ea}(C)$ the *equianalytical singularity scheme of* $C$.

(b) We define the zero-dimensional subscheme $X^{es}(C)$ of $\Sigma$ via the ideal sheaf $\mathcal{J}_{X^{es}(C)/\Sigma}$ with stalks

$$\mathcal{J}_{X^{es}(C)/\Sigma,z} = I^{es}(C,z) = \left\{ g \in \mathcal{O}_{\Sigma,z} \;\middle|\; f + \varepsilon g \text{ is equisingular over } \mathbb{C}[\varepsilon]/(\varepsilon^2) \right\},$$

where $f \in \mathcal{O}_{\Sigma,z}$ is a local equation of $C$ at $z$. $I^{es}(C,z)$ is called the *equisingularity ideal* of the singularity $(C,z)$, and it is of course $\mathcal{O}_{\Sigma,z}$ whenever $z$ is a smooth point. $I^{es}(C,z)/I^{ea}(C,z)$ can be identified with the tangent space of the equisingular stratum in the semiuniversal deformation of $(C,z)$ (cf. [**Wah74b**], [**DiH88**], and Definition III.2.1). We call $X^{es}(C)$ the *equisingularity scheme of* $C$.

(c) The scheme $X^{ea}_{fix}(C)$ in $\Sigma$ is defined via the ideal sheaf $\mathcal{J}_{X^{ea}_{fix}(C)/\Sigma}$ with stalks

$$\mathcal{J}_{X^{ea}_{fix}(C)/\Sigma,z} = I^{ea}_{fix}(C,z) = I^{ea}_{fix}(f) = (f) + \mathfrak{m}_{\Sigma,z} \cdot \left( \tfrac{\partial f}{\partial x}, \tfrac{\partial f}{\partial y} \right) \subseteq \mathcal{O}_{\Sigma,z},$$

where $x, y$ denote local coordinates of $\Sigma$ at $z$ and $f \in \mathcal{O}_{\Sigma,z}$ a local equation of $C$. $I^{ea}_{fix}(C,z)$ is of course $\mathcal{O}_{\Sigma,z}$ whenever $z$ is a smooth point of $C$.

(d) We define the zero-dimensional subscheme $X^{es}_{fix}(C)$ of $\Sigma$ via the ideal sheaf $\mathcal{J}_{X^{es}_{fix}(C)/\Sigma}$ with stalks

$$\mathcal{J}_{X^{es}_{fix}(C)/\Sigma,z} = I^{es}_{fix}(C,z) = \left\{ g \in \mathcal{O}_{\Sigma,z} \;\middle|\; \begin{array}{c} f + \varepsilon g \text{ is equisingular over } \mathbb{C}[\varepsilon]/(\varepsilon^2) \\ \text{along the trivial section} \end{array} \right\},$$

where $f \in \mathcal{O}_{\Sigma,z}$ is a local equation of $C$ at $z$. $I^{es}_{fix}(C,z)$ is of course $\mathcal{O}_{\Sigma,z}$ whenever $z$ is a smooth point.

(e) The scheme $X^{a}(C)$ in $\Sigma$ is defined via the ideal sheaf $\mathcal{J}_{X^{a}(C)/\Sigma}$ with stalks

$$\mathcal{J}_{X^{a}(C)/\Sigma,z} = I^{a}(C,z) \subseteq \mathcal{O}_{\Sigma,z},$$

where we refer for the somewhat lengthy definition of $I^{a}(C,z)$ to Remark 2.8. $I^{a}(C,z)$ is called the *analytical singularity ideal* of the singularity $(C,z)$, and it is of course $\mathcal{O}_{\Sigma,z}$ whenever $z$ is a smooth point. We call $X^{a}(C)$ the *analytical singularity scheme of* $C$.

(f) The scheme $X^{s}(C)$ in $\Sigma$ is defined via the ideal sheaf $\mathcal{J}_{X^{s}(C)/\Sigma}$ with stalks

$$\mathcal{J}_{X^{s}(C)/\Sigma,z} = I^{s}(C,z) = \left\{ g \in \mathcal{O}_{\Sigma,z} \;\middle|\; g \text{ goes through the cluster } \mathcal{Cl}\left(C, T^*(C,z)\right) \right\},$$

where $T^*(C,z)$ denotes the essential subtree of the complete embedded resolution tree of $(C,z)$. $I^{s}(C,z)$ is called the *singularity ideal* of the singularity $(C,z)$, and it is of course $\mathcal{O}_{\Sigma,z}$ whenever $z$ is a smooth point. We call $X^{s}(C)$ the *singularity scheme of* $C$. (Cf. [**Los98**] Section 2.2.1.)

Throughout the thesis we will frequently treat topological and analytical singularities at the same time. Whenever we do so, we will write $X^{*}(C)$ for $X^{es}(C)$ respectively for $X^{ea}(C)$, and similarly $X^{*}_{fix}(C)$ for $X^{es}_{fix}(C)$ respectively for $X^{ea}_{fix}(C)$, and $X(C)$ for $X^{s}(C)$ respectively for $X^{a}(C)$.

**2.8 Remark**

Let $(C,z)$ be an isolated plane curve singularity of analytical type $\mathcal{S}$ with representative $f \in \mathcal{O}_{\Sigma,z}$.

(a) A collection of ideals $\mathcal{I}(f) = \{I(g) \mid g \in \mathcal{O}_{\Sigma,z} \;:\; g \sim_c f\}$ is said to be *suitable*, if the ideals $I(g) \in \mathcal{I}$ satisfy the following properties:
   (1) $g \in I(g)$.
   (2) For a generic element in $h \in I(g)$ we have $h \sim_c g$ and $I(h) = I(g)$; more precisely, for any $d \geq 0$ the set of polynomial $h \in I(g) \cap \mathbb{C}[x,y]_{\leq d}$ such that $h \sim_c g$ and $I(h) = I(g)$ is open dense in $I(g) \cap \mathbb{C}[x,y]_{\leq d}$, where $\mathbb{C}[x,y]_{\leq d}$ are the polynomials of degree at most $d$.
   (3) $I(g) = \psi^* I(f)$, if $\psi : \mathcal{O}_{\Sigma,z} \to \mathcal{O}_{\Sigma,z}$ is an isomorphism and $u \in \mathcal{O}_{\Sigma,z}$ is a unit such that $g = \psi^*(u \cdot f)$.
   (4) $\exists\, m > 0 \;:\; I(g) = I\big(\mathrm{jet}_m(g)\big)$, i. e. $I(g)$ is determined by $\mathrm{jet}_m(g)$.

   We note that (3) implies that $I(g)$ only depends on the ideal $\langle g \rangle$, and that the isomorphism class of $I(f)$ is an invariant of $\mathcal{S}$. We call $\deg \mathcal{I}(f) = \dim_{\mathbb{C}} \big(\mathcal{O}_{\Sigma,z}/I(f)\big)$ the *degree* of $\mathcal{I}(f)$. (Cf. **[Los98]** Definition 2.39.)

(b) If we define

$$\widetilde{I}^a(g) = \big\{h \in \mathcal{O}_{\Sigma,z} \mid I^{ea}(h) \subseteq I^{ea}(g)\big\}$$

   for $g \in \mathcal{O}_{\Sigma,z}$ with $g \sim_c f$, then the collection $\widetilde{\mathcal{I}}(f) = \big\{\widetilde{I}^a(g) \;\big|\; g \in \mathcal{O}_{\Sigma,z} \;:\; g \sim_c f\big\}$ is suitable. (Cf. **[Los98]** Definition 2.42.)
   By **[Los98]** Lemma 2.44 we know $\deg \widetilde{\mathcal{I}}(f) \leq 3\tau(\mathcal{S})$.

(c) Among the suitable collections of ideals we choose one with minimal degree, and we call it $\mathcal{I}^a(f)$. The elements of $\mathcal{I}^a(f)$ are parametrised by $G := \{g \in \mathcal{O}_{\Sigma,z} \mid g \sim_c f\}$, and we denote the element corresponding to $g \in G$ by $I^a(g)$. We then define

$$I^a(C,z) = I^a(f).$$

   Note that $\mathcal{I}^a(f)$ exists in view of (b), and that its definition depends only on $\mathcal{S}$. Moreover, $\deg \mathcal{I}^a(f) \leq 3\tau(\mathcal{S})$. (Cf. **[Los98]** Definition 2.40.)

**2.9 Remark**

If $C \subset \Sigma$ is a reduced curve, then $\mathcal{J}_{X^a(C)/\Sigma} \subseteq \mathcal{J}_{X^{ea}_{fix}(C)/\Sigma} \subseteq \mathcal{J}_{X^{ea}(C)/\Sigma}$ and $\mathcal{J}_{X^s(C)/\Sigma} \subseteq \mathcal{J}_{X^{es}_{fix}(C)/\Sigma} \subseteq \mathcal{J}_{X^{es}(C)/\Sigma}$.

In particular, the vanishing of $h^1\big(\Sigma, \mathcal{J}_{X(C)/\Sigma}(D)\big)$ implies the vanishing of $h^1\big(\Sigma, \mathcal{J}_{X^*_{fix}(C)/\Sigma}(D)\big)$, and the vanishing of the latter implies that of $h^1\big(\Sigma, \mathcal{J}_{X^*(C)/\Sigma}(D)\big)$.

To see the first assertion, consider the exact sequence

$$0 \to \mathcal{J}_{X(C)/\Sigma}(D) \to \mathcal{J}_{X^*_{fix}(C)/\Sigma}(D) \to \big(\mathcal{J}_{X^*_{fix}(C)/\Sigma}/\mathcal{J}_{X(C)/\Sigma}\big)(D) \to 0.$$

The long exact cohomology sequence then leads to

$$H^1\big(\Sigma, \mathcal{J}_{X/\Sigma}(D)\big) \to H^1\big(\Sigma, \mathcal{J}_{X^*_{fix}/\Sigma}(D)\big) \to H^1\big(\Sigma, \big(\mathcal{J}_{X^*_{fix}/\Sigma}/\mathcal{J}_{X/\Sigma}\big)(D)\big) = 0,$$

where the latter vanishes since its support is zero-dimensional. This proves the claim. The second assertion follows analogously.

**2.10 Remark**

In [**Los98**] Proposition 2.19 and 2.20 and in Remark 2.40 (see also [**GLS00**]) and 2.41 it is shown that, fixing a point $z \in \Sigma$ and a topological respectively analytical type $\mathcal{S}$, the singularity schemes respectively analytical singularity schemes having the same topological respectively analytical type are parametrised by an irreducible Hilbert scheme, which we are going to denote by $\mathrm{Hilb}_z(\mathcal{S})$. This then leads to an irreducible family

$$\mathrm{Hilb}(\mathcal{S}) = \coprod_{z\in\Sigma} \mathrm{Hilb}_z(\mathcal{S}).$$

For a more careful study of these objects we refer to [**Los98**] Remark 2.21 and Remark 2.41.

In particular, equisingular respectively equianalytical singularities have singularity schemes respectively analytical singularity schemes of the same degree (see also [**GLS98c**] or [**Los98**] Lemma 2.8). The same is of course true, regarding the equisingularity scheme respectively the equianalytical singularity scheme. This leads to the following definition.

**2.11 Definition**

If $C \subset \Sigma$ is a reduced curve such that $z$ is a singular point of topological respectively analytical type $\mathcal{S}$, then we define $\deg\big(X^s(\mathcal{S})\big) = \deg\big(X^s(C), z\big)$, $\deg\big(X^{es}_{fix}(\mathcal{S})\big) = \deg\big(X^{es}_{fix}(C), z\big)$ and $\deg\big(X^{es}(\mathcal{S})\big) = \deg\big(X^{es}(C), z\big)$ respectively $\deg\big(X^a(\mathcal{S})\big) = \deg\big(X^a(C), z\big)$, $\deg\big(X^{ea}_{fix}(\mathcal{S})\big) = \deg\big(X^{ea}_{fix}(C), z\big)$ and $\deg\big(X^{ea}(\mathcal{S})\big) = \deg\big(X^{ea}(C), z\big)$.

To express the degree of the schemes $X^*(\mathcal{S})$ and $X^*_{fix}(\mathcal{S})$ in terms of the invariants from Remark 2.2 for an analytical respectively topological singularity type $\mathcal{S}$ is much simpler, since by definition

$$\deg\big(X^{es}(\mathcal{S})\big) = \tau^{es}(\mathcal{S}) \quad \text{and} \quad \deg\big(X^{ea}(\mathcal{S})\big) = \tau(\mathcal{S}),$$

while $\deg\big(X^*_{fix}(\mathcal{S})\big) = \deg\big(X^*(\mathcal{S})\big) + 2$, and hence

$$\deg\big(X^{es}_{fix}(\mathcal{S})\big) = \tau^{es}(\mathcal{S}) + 2 \quad \text{and} \quad \deg\big(X^{ea}_{fix}(\mathcal{S})\big) = \tau(\mathcal{S}) + 2.$$

**2.12 Remark**

We note that for a topological respectively analytical singularity type $\mathcal{S}$

$$\dim \mathrm{Hilb}_z(\mathcal{S}) = \deg\big(X(\mathcal{S})\big) - \deg\big(X^*(\mathcal{S})\big) - 2$$

for any $z \in \Sigma$, and thus

$$\dim \mathrm{Hilb}(\mathcal{S}) = \deg\big(X(\mathcal{S})\big) - \deg\big(X^*(\mathcal{S})\big).$$

For the convenience of the reader we reproduce the proof from [**GLS05**] Lemma 2.1.49 at the end of this chapter.

**2.13 Remark**

(a) A simple calculation shows (see also [**Los98**] p. 28)

| $\mathcal{S}$ | $\deg\big(X^a(\mathcal{S})\big) = \deg\big(X^s(\mathcal{S})\big)$ | |
|---|---|---|
| $A_\mu$ | $\frac{3}{2}\cdot\mu+2,$ | $\mu$ even, |
| $A_\mu$ | $\frac{3}{2}\cdot\mu+\frac{3}{2},$ | $\mu$ odd, |
| $D_\mu$ | $\frac{3}{2}\cdot\mu,$ | $\mu$ even, |
| $D_\mu$ | $\frac{3}{2}\cdot\mu+\frac{1}{2},$ | $\mu\neq 5$ odd, |
| $D_5$ | $\frac{3}{2}\cdot\mu-\frac{1}{2}=7,$ | $\mu=5,$ |
| $E_6$ | $\frac{3}{2}\cdot\mu-1=8,$ | $\mu=6,$ |
| $E_7$ | $\frac{3}{2}\cdot\mu-\frac{1}{2}=10,$ | $\mu=7,$ |
| $E_6$ | $\frac{3}{2}\cdot\mu-1=11,$ | $\mu=8.$ |

In particular, if $\mathcal{S}$ is a simple singularity, then $\deg\big(X^s(\mathcal{S})\big)\leq\frac{3}{2}\mu(\mathcal{S})+2$, where $\mu(\mathcal{S})$ denotes the Milnor number of $\mathcal{S}$.

(b) If $\mathcal{S}$ is an analytical singularity type, then (cf. [**Los98**] Lemma 2.44)

$$\deg\big(X^a(\mathcal{S})\big)\leq 3\tau(\mathcal{S}), \tag{2.1}$$

where $\tau(\mathcal{S})$ denotes the Tjurina number of $\mathcal{S}$.

(c) Let $\mathcal{S}$ be any topological singularity type, then by [**Los98**] Lemma 2.8

$$\deg\big(X^s(\mathcal{S})\big)=\delta(\mathcal{S})+\sum_{q\in T^*(C,z)} m_q,$$

where $(C,z)$ is a plane curve singularity of type $\mathcal{S}$, $T^*(C,z)$ is the essential subtree of the complete embedded resolution tree of $(C,z)$, $m_q$ is the multiplicity of $(C,z)$ at the infinitely near point $q\in T^*(C,z)$, and $\delta(\mathcal{S})$ the delta invariant of $\mathcal{S}$.
In [**Los98**] p. 103 it is shown that if $\mathcal{S}$ is not a simple singularity type, then

$$\deg\big(X^s(\mathcal{S})\big)\leq\tfrac{3}{2}\cdot\big(\mu(\mathcal{S})-1\big).$$

Combining this with (a) we get for $\mathcal{S}$ arbitrary

$$\deg\big(X^s(\mathcal{S})\big)\leq\tfrac{3}{2}\mu(\mathcal{S})+2. \tag{2.2}$$

**2.14 Definition**

(a) Given topological or analytical singularity types $\mathcal{S}_1,\ldots,\mathcal{S}_r$ and a divisor $D\in\mathrm{Div}(\Sigma)$, we denote by $V=V_{|D|}(\mathcal{S}_1,\ldots,\mathcal{S}_r)$ the locally closed subspace of $|D|_l$ of reduced curves in the linear system $|D|_l$ having precisely $r$ singular points of types $\mathcal{S}_1,\ldots,\mathcal{S}_r$.
By $V^{reg}=V^{reg}_{|D|}(\mathcal{S}_1,\ldots,\mathcal{S}_r)$ we denote the open[4] subset

$$V^{reg}=\big\{C\in V\;\big|\;h^1\big(\Sigma,\mathcal{J}_{X(C)/\Sigma}(D)\big)=0\big\}\subseteq V,$$

[4] See Theorem V.1.1.

where $X(C) = X^s(C)$ respectively $X^a(C)$, and by $V^{fix} = V^{fix}_{|D|}(\mathcal{S}_1, \ldots, \mathcal{S}_r)$ we denote the open subset

$$V^{fix} = \big\{C \in V \;\big|\; h^1\big(\Sigma, \mathcal{J}_{X^*_{fix}(C)/\Sigma}(D)\big) = 0\big\} \subseteq V,$$

where $X^*_{fix}(C) = X^{es}_{fix}(C)$ respectively $X^{ea}_{fix}(C)$
Similarly, we use the notation $V^{irr} = V^{irr}_{|D|}(\mathcal{S}_1, \ldots, \mathcal{S}_r)$ to denote the open subset of irreducible curves in the space $V$, and we set $V^{irr,reg} = V^{irr,reg}_{|D|}(\mathcal{S}_1, \ldots, \mathcal{S}_r) = V^{irr} \cap V^{reg}$ and $V^{irr,fix} = V^{irr,fix}_{|D|}(\mathcal{S}_1, \ldots, \mathcal{S}_r) = V^{irr} \cap V^{fix}$, which are open in $V^{reg}$ respectively $V^{fix}$, and hence in $V$.

If a type $\mathcal{S}$ occurs $k > 1$ times, we rather write $k\mathcal{S}$ than $\mathcal{S}, \overset{k}{\ldots}, \mathcal{S}$.

(b) Analogously, $V_{|D|}(m_1, \ldots, m_r) = V_{|D|}(\underline{m})$ denotes the locally closed subspace of $|D|_l$ of reduced curves having precisely $r$ ordinary singular points of multiplicities $m_1, \ldots, m_r$. (Cf. **[GrS99]** or **[Los98]** 1.3.2)

(c) Let $V = V_{|D|}(\mathcal{S}_1, \ldots, \mathcal{S}_r)$ respectively $V = V_{|D|}(\underline{m})$. We say $V$ is *T-smooth* at $C \in V$ if the germ $(V, C)$ is smooth of the (expected) dimension $\dim |D|_l - \deg\big(X^*(C)\big)$, where $X^*(C) = X^{es}(C)$, $X^*(C) = X^{es}(C)$ or $X^*(C) = X(\underline{m}; \underline{z})$ with $\operatorname{Sing}(C) = \{z_1, \ldots, z_r\}$ respectively.

We call these families of curves *equisingular families of curves*.

**2.15 Remark**
With the notation of Definition 2.14 and by **[Los98]** Proposition 2.1 (see also **[GrK89]**, **[GrL96]**, **[GLS00]**) T-smoothness of $V$ at $C$ is implied by the vanishing of $H^1\big(\Sigma, \mathcal{J}_{X^*(C)/\Sigma}(C)\big)$. This is due to the fact that the tangent space of $V$ at $C$ may be identified with $H^0\big(\Sigma, \mathcal{J}_{X^*(C)/\Sigma}(C)\big)/H^0(\Sigma, \mathcal{O}_\Sigma)$.

In particular, the subvarieties $V^{reg}$, $V^{fix}$, $V^{irr,reg}$ and $V^{irr,fix}$ are T-smooth, that is smooth of expected dimension $h^0\big(\Sigma, \mathcal{O}_\Sigma(C)\big) - \deg\big(X^*(C)\big) - 1$.

**2.16 Definition**
Let $D \in \operatorname{Div}(\Sigma)$ be a divisor, $\mathcal{S}_1, \ldots, \mathcal{S}_r$ distinct topological or analytical singularity types, and $k_1, \ldots, k_r \in \mathbb{N} \setminus \{0\}$.

(a) We denote by $\widetilde{B}$ the irreducible parameter space

$$\widetilde{B} = \widetilde{B}(k_1\mathcal{S}_1, \ldots, k_r\mathcal{S}_r) = \prod_{i=1}^{r} \operatorname{Sym}^{k_i}\big(\operatorname{Hilb}(\mathcal{S}_i)\big),$$

and by $B = B(k_1\mathcal{S}_1, \ldots, k_r\mathcal{S}_r)$ the non-empty open, irreducible and dense, subspace

$$B = \Big\{\big([X_{1,1}, \ldots, X_{1,k_1}], \ldots, [X_{r,1}, \ldots, X_{r,k_r}]\big) \in \widetilde{B} \;\Big|\; \operatorname{supp}(X_{i,j}) \cap \operatorname{supp}(X_{s,t}) = \emptyset \;\; \forall\, 1 \leq i, s \leq r, 1 \leq j \leq k_i, 1 \leq t \leq k_s\Big\}.$$

Note that $\dim(B)$ does not depend on $\Sigma$, more precisely, with the notation of Remark 2.12 we have

$$\dim(B) = \sum_{i=1}^{r} k_i \cdot \Big(\deg\big(X(\mathcal{S}_i)\big) - \deg\big(X^*(\mathcal{S}_i)\big)\Big).$$

(b) Let us set $n = \sum_{i=1}^{r} k_i \deg\big(X(\mathcal{S}_i)\big)$. We then define an injective morphism

$$\psi = \psi(k_1\mathcal{S}_1, \ldots, k_r\mathcal{S}_r) : B(k_1\mathcal{S}_1, \ldots, k_r\mathcal{S}_r) \longrightarrow \text{Hilb}_\Sigma^n$$

$$\big([X_{1,1}, \ldots, X_{1,k_1}], \ldots, [X_{r,1}, \ldots, X_{r,k_r}]\big) \longmapsto \bigcup_{i=1}^{r} \bigcup_{j=1}^{k_i} X_{i,j},$$

where $\text{Hilb}_\Sigma^n$ denotes the smooth connected Hilbert scheme of zero-dimensional schemes of degree $n$ on $\Sigma$ (cf. **[Los98]** Section 1.3.1).

(c) We denote by $\Psi = \Psi_D(k_1\mathcal{S}_1, \ldots, k_r\mathcal{S}_r)$ the fibration of $V_{|D|}(k_1\mathcal{S}_1, \ldots, k_r\mathcal{S}_r)$ induced by $B(k_1\mathcal{S}_1, \ldots, k_r\mathcal{S}_r)$, that is, the morphism $\Psi$ is given by

$$\Psi : V_{|D|}(k_1\mathcal{S}_1, \ldots, k_r\mathcal{S}_r) \longrightarrow B(k_1\mathcal{S}_1, \ldots, k_r\mathcal{S}_r)$$

$$C \longmapsto \big([X_{1,1}, \ldots, X_{1,k_1}], \ldots, [X_{r,1}, \ldots, X_{r,k_r}]\big)$$

where $\text{Sing}(C) = \{z_{i,j} \mid i = 1, \ldots, r, j = 1, \ldots, k_i\}$, $X_{i,j} = X(C, z_{i,j})$ and $(C, z_{i,j}) \cong \mathcal{S}_i$ for all $i = 1, \ldots, r, j = 1, \ldots, k_i$.

(d) Denoting by $m = k_1 + \ldots + k_r$ the number of imposed singularities we define the fibration $\Phi = \Phi_D(k_1\mathcal{S}_1, \ldots, k_r\mathcal{S}_r)$ by

$$\Phi : V_{|D|}(k_1\mathcal{S}_1, \ldots, k_r\mathcal{S}_r) \longrightarrow \text{Sym}^m(\Sigma)$$

$$C \longmapsto \text{Sing}(C),$$

sending a curve $C$ to the unordered tuple of its singular points.

**2.17 Remark**

(a) With the notation of Definition 2.14 and Definition 2.16 note that for $C \in V = V_{|D|}(k_1\mathcal{S}_1, \ldots, k_r\mathcal{S}_r)$ the fibre $\Psi^{-1}\big(\Psi(C)\big)$ is the open dense subset of the linear system $\big|\mathcal{J}_{X(C)/\Sigma}(D)\big|_l$ consisting of the curves $C'$ with $X\big(C'\big) = X(C)$. In particular, the fibres of $\Psi$ restricted to $V^{reg}$ are irreducible, and since for $C \in V^{reg}$ the cohomology group $H^1\big(\Sigma, \mathcal{J}_{X(C)/\Sigma}(D)\big)$ vanishes, they are equidimensional of dimension

$$\begin{aligned} h^0\big(\Sigma, \mathcal{J}_{X(C)/\Sigma}(D)\big) - 1 &= h^0\big(\Sigma, \mathcal{O}_\Sigma(D)\big) - \deg\big(X(C)\big) - 1 \\ &= h^0\big(\Sigma, \mathcal{O}_\Sigma(D)\big) - \sum_{i=1}^{r} k_i \cdot \deg\big(X(\mathcal{S}_i)\big) - 1. \end{aligned}$$

(b) Note that for $C \in V$ the fibre $\Phi^{-1}\big(\Phi(C)\big)$ has at $C$ the tangent space $H^0\big(\Sigma, \mathcal{J}_{X^*_{fix}(C)}(D)\big)/H^0(\mathcal{O}_\Sigma)$, so that

$$\dim\Big(\Phi^{-1}\big(\Phi(C)\big)\Big) \leq h^0\big(\Sigma, \mathcal{J}_{X^*_{fix}(C)}(D)\big) - 1. \tag{2.3}$$

Moreover, suppose that $h^1\big(\Sigma, \mathcal{J}_{X^*_{fix}(C)/\Sigma}(D)\big) = 0$, then the germ of the fibration at $C$

$$(\Phi, C) : (V, C) \to \big(\text{Sym}^m(\Sigma), \text{Sing}(C)\big)$$

is smooth of fibre dimension $h^0\big(\Sigma, \mathcal{J}_{X^*_{fix}(C)}(D)\big) - 1$, i. e. locally at $C$ the morphism $\Phi$ is a projection of the product of the smooth base space with the smooth fibre. This implies in particular, that close to $C$ there is a curve having its singularities in very general position. (Cf. **[Los98]** Proposition 2.1 (e).)

**Proof of Remark 2.12:** Since $\dim \mathrm{Hilb}(\mathcal{S})$ is independent of $\Sigma$, we may as well suppose that $\Sigma = \mathbb{P}^2_{\mathbb{C}}$, and we let $H$ be a line in $\mathbb{P}^2_{\mathbb{C}}$.

For a reduced curve $C \subset \mathbb{P}^2_{\mathbb{C}}$ we set $X(C) = X^s(C)$ and $X^*(C) = X^{es}(C)$ respectively $X(C) = X^a(C)$ and $X^*(C) = X^{ea}(C)$.

Since $\deg\big(X\big) = \deg\big(X(\mathcal{S})\big)$ is independent of $X \in \mathrm{Hilb}(\mathcal{S})$, there is an integer $m > 0$ such that for $d > m$

$$h^1\big(\mathbb{P}^2_{\mathbb{C}}, \mathcal{J}_{X/\mathbb{P}^2_{\mathbb{C}}}(d)\big) = 0$$

for any $X \in \mathrm{Hilb}(\mathcal{S})$.

Let $k > 0$ be the determinacy bound of $\mathcal{S}$, that is, any representative $f \in \mathcal{O}_{\mathbb{P}^2_{\mathbb{C}},z} = \mathbb{C}\{x,y\}$ of $\mathcal{S}$ depends only on the $k$-jet of $f$. Hence, for $d > k$ the morphism

$$\Psi = \Psi_{dH}(\mathcal{S}) : V_{|dH|}(\mathcal{S}) \to B(\mathcal{S}) = \mathrm{Hilb}(\mathcal{S})$$

is surjective.

Let us now fix some $d > \max\{k, m\}$. For each $C \in V_{|dH|}(\mathcal{S})$ the fibre $\Psi^{-1}\big(\Psi(C)\big)$ is the open dense subset of $\big|\mathcal{J}_{X(C)/\mathbb{P}^2_{\mathbb{C}}}(d)\big|$, consisting of curves $C' \in V_{|dH|}(\mathcal{S})$ with $X\big(C'\big) = X(C)$. From the long exact cohomology sequence of

$$0 \to \mathcal{J}_{X(C)/\Sigma} \to \mathcal{O}_{\mathbb{P}^2_{\mathbb{C}}}(d) \to \mathcal{O}_{X(C)} \to 0$$

it follows

$$h^0\big(\mathbb{P}^2_{\mathbb{C}}, \mathcal{J}_{X(C)/\mathbb{P}^2_{\mathbb{C}}}(d)\big) = h^0\big(\mathbb{P}^2_{\mathbb{C}}, \mathcal{O}_{\mathbb{P}^2_{\mathbb{C}}}(d)\big) - \deg\big(X(\mathcal{S})\big).$$

In particular the fibres all have the same dimension

$$\dim \Psi^{-1}\big(\Psi(C)\big) = h^0\big(\mathbb{P}^2_{\mathbb{C}}, \mathcal{O}_{\mathbb{P}^2_{\mathbb{C}}}(d)\big) - \deg\big(X(\mathcal{S})\big) - 1.$$

We therefore get

$$\begin{aligned}\dim \mathrm{Hilb}(\mathcal{S}) = \dim \Psi\big(V_{|dH|}(\mathcal{S})\big) &= \dim\big(V_{|dH|}(\mathcal{S})\big) - \dim \Psi^{-1}\big(\Psi(C)\big)\\ &= \dim\big(V_{|dH|}(\mathcal{S})\big) - h^0\big(\mathbb{P}^2_{\mathbb{C}}, \mathcal{O}_{\mathbb{P}^2_{\mathbb{C}}}(d)\big) + \deg\big(X(\mathcal{S})\big) + 1.\end{aligned}$$

Moreover, by Remark 2.9 we know that also $h^1\big(\mathbb{P}^2_{\mathbb{C}}, \mathcal{J}_{X^*(C)/\mathbb{P}^2_{\mathbb{C}}}(d)\big) = 0$ for any $C \in V_{|dH|}(\mathcal{S})$, and thus in view of Remark 2.15 $V_{|dH|}(\mathcal{S})$ is T-smooth, that is

$$\dim\big(V_{|dH|}(\mathcal{S})\big) = h^0\big(\mathbb{P}^2_{\mathbb{C}}, \mathcal{O}_{\mathbb{P}^2_{\mathbb{C}}}(d)\big) - \deg\big(X^*(\mathcal{S})\big) - 1,$$

which finishes the claim. ☐

## 3. The $\gamma_\alpha$-Invariant

When studying numerical conditions for the T-smoothness of equisingular families of curves, new invariants of analytical respectively topological singularity types turns up. It depends on the degree of a locally complete intersection scheme contained in the equianalytical respectively equisingularity scheme of a representative, and on the intersection multiplicity of a representative of the singularity with a curve germ containing this complete intersection scheme. In the following we will define these invariants, and we will calculate them for several classes of singularities.

Throughout this section we will use the notation $R = \mathbb{C}\{x, y\}$ and $\mathfrak{m} = \langle x, y\rangle \lhd R$.

### 3.a. The Invariants

**3.1 Definition**
Let $f \in \mathfrak{m}$ be a reduced power series, and let $0 \leq \alpha \leq 1$ be a rational number.

If $I$ is a zero-dimensional ideal in Rwith $I^{ea}(f) = \left\langle f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right\rangle \subseteq I$ and $g \in I$, we define

$$\lambda_\alpha(f; I, g) := \frac{\big(\alpha \cdot i(f, g) - (1-\alpha)\cdot \dim_{\mathbb{C}}(R/I)\big)^2}{i(f, g) - \dim_{\mathbb{C}}(R/I)},$$

and

$$\gamma_\alpha(f; I) := \max\left\{(1+\alpha)^2 \cdot \dim_{\mathbb{C}}(R/I),\ \lambda_\alpha(f; I, g) \;\middle|\; g \in I, i(f, g) \leq 2\cdot \dim_{\mathbb{C}}(R/I)\right\},$$

where $i(f, g)$ denotes the intersection multiplicity of $f$ and $g$. Note that by Lemma IV.1.3 $i(f, g) > \dim_{\mathbb{C}}(R/I)$ for all $g \in I$ and $\gamma_\alpha(f; I)$ is thus a well-defined positive rational number.

We then set

$$\gamma_\alpha^{ea}(f) := \max\left\{0,\ \gamma_\alpha(f; I) \;\middle|\; I^{ea}(f) \subseteq I \text{ is a complete intersection ideal}\right\}$$

and

$$\gamma_\alpha^{es}(f) := \max\left\{0,\ \gamma_\alpha(f; I) \;\middle|\; I^{es}(f) \subseteq I \text{ is a complete intersection ideal}\right\}$$

Note that if $f \in \mathfrak{m}\backslash\mathfrak{m}^2$, then $I^{ea}(f) = I^{es}(f) = R$ and there is no zero-dimensional complete intersection ideal containing any of those two, and thus $\gamma_\alpha^{ea}(f) = \gamma_\alpha^{es}(f) = 0$.

The equianalytical up to embedded isomorphism only depends on the analytical type of the singularity, i. e. if $f$ is some representative of a singularity type, $u \in \mathbb{C}\{x, y\}$ a unit and $\phi : \mathbb{C}\{x, y\} \to \mathbb{C}\{x, y\}$ an isomorphism, then $I^{ea}(u \cdot f \circ \phi) = \{g \circ \phi \mid g \in I^{ea}(f)\}$. Thus the following definition makes sense.

**3.2 Definition**
Let $\mathcal{S}$ be an analytical respectively topological singularity type and let $f \in R$ be a representative of $\mathcal{S}$. We then define

$$\gamma_\alpha^{ea}(\mathcal{S}) := \gamma_\alpha^{ea}(f)$$

respectively

$$\gamma^{es}_\alpha(\mathcal{S}) := \max\{\gamma^{es}_\alpha(g) \mid g \text{ is a representative of } \mathcal{S}\}.$$

Recalling that $i(f,g) > \dim_{\mathbb{C}}(R/I)$ in the above situation, we deduce the following lemma.

**3.3 Lemma**
*Let* $f \in \langle x,y\rangle^2 \subset \mathbb{C}\{x,y\}$ *be reduced,* $I^{ea}(f) \subseteq I \subsetneqq \mathbb{C}\{x,y\}$ *be a zero-dimensional ideal and* $0 \leq \alpha < \beta \leq 1$*, then* $\gamma_\alpha(f;I) < \gamma_\beta(f;I)$.

*In particular for any analytical respectively topological singularity type*

$$\gamma^{ea}_\alpha(\mathcal{S}) < \gamma^{ea}_\beta(\mathcal{S}) \qquad \textit{respectively} \qquad \gamma^{es}_\alpha(\mathcal{S}) < \gamma^{es}_\beta(\mathcal{S}).$$

The following lemma is again obvious from the definition of $\gamma_\alpha(f;I)$, once we take into account that $\kappa(f) = i(f,g)$ for a generic polar $g \in I^{es}(f)$ of $f$ and that for fixed value of $d = \dim_{\mathbb{C}}(R/I)$ the function $i \mapsto \frac{(\alpha i+(1-\alpha)\cdot d)^2}{i-d}$ takes its maximum on $[d+1, 2d]$ for the minimal possible value $i = d+1$.

**3.4 Lemma**
*Let* $f \in \mathfrak{m}^2$ *be reduced, and let* $I$ *be a zero-dimensional ideal in* $R$.

(a) *If* $I^{ea}(f) \subseteq I$*, then*

$$(1+\alpha)^2 \cdot \dim_{\mathbb{C}}(R/I) \leq \gamma_\alpha(f;I) \leq \big(\dim_{\mathbb{C}}(R/I) + \alpha\big)^2.$$

(b) *If* $I^{es}(f) \subseteq I$*, then*

$$\frac{\big(\alpha\cdot\kappa(f) + (1-\alpha)\cdot\dim_{\mathbb{C}}(R/I)\big)^2}{\kappa(f) - \dim_{\mathbb{C}}(R/I)} \leq \gamma_\alpha(f;I) \leq \big(\dim_{\mathbb{C}}(R/I) + \alpha\big)^2.$$

*In particular for any analytical respectively topological singularity type*

$$(1+\alpha)^2 \cdot \tau_{ci}(\mathcal{S}) \leq \gamma^{ea}_\alpha(\mathcal{S}) \leq \big(\tau_{ci}(\mathcal{S}) + \alpha\big)^2$$

*respectively*

$$(1+\alpha)^2 \cdot \tau^{es}_{ci}(\mathcal{S}) \leq \gamma^{es}_\alpha(\mathcal{S}) \leq \big(\tau^{es}_{ci}(\mathcal{S}) + \alpha\big)^2.$$

**3.5 Lemma**
*If* $\mathcal{S} \neq A_1$ *is topological or analytical singularity type with representative* $f$ *such that* $I^*(f)$ *is a complete intersection ideal, then*

$$(1+\alpha)^2 \cdot \tau^*(\mathcal{S}) < \gamma^*_\alpha(\mathcal{S}).$$

*In particular, if* $\mathcal{S}$ *is a simple singularity, then*

$$\gamma^*(\mathcal{S}) = \max\big\{\lambda_\alpha(f;I,g) \;\big|\; I^*(f) \subseteq I \textit{ comp. inters. }, g \in I, i(f,g) \leq 2\cdot\dim_{\mathbb{C}}(R/I)\big\}.$$

**Proof:** Since $\tau^*(\mathcal{S}) = \dim_{\mathbb{C}}\big(R/I^*(f)\big)$ Lemma 3.4 gives

$$\gamma^*_\alpha(\mathcal{S}) \geq \frac{\big(\alpha\cdot\kappa(\mathcal{S}) + (1-\alpha)\cdot\tau^*(\mathcal{S})\big)^2}{\kappa(\mathcal{S}) - \tau^*(\mathcal{S})}.$$

If we consider the right-hand side as a function in $\kappa(\mathcal{S})$ it takes its minimum at $2 \cdot \tau^{es}(\mathcal{S})$. However, by Lemma 3.6 $\delta(\mathcal{S}) < \tau^{es}(\mathcal{S}) \leq \tau(\mathcal{S})$, and by **[GLS05]** Lemma 4.44 $\kappa(\mathcal{S}) \leq 2 \cdot \delta(\mathcal{S})$. Therefore,

$$\gamma^*_\alpha(\mathcal{S}) > (1+\alpha)^2 \cdot \tau^*(\mathcal{S}).$$

□

**3.6 Lemma**
*If $\mathcal{S} \neq A_1$ is any topological singularity type, then $\delta(\mathcal{S}) < \tau^{es}(\mathcal{S})$.*

**Proof:** If $(C, z)$ is a representative of $\mathcal{S}$ and if $\mathcal{T}^*(C, z)$ is the essential subtree of the complete embedded resolution tree of $(C, z)$, then

$$\delta(\mathcal{S}) = \sum_{p \in \mathcal{T}^*(C,z)} \frac{\mathrm{mult}_p(C) \cdot (\mathrm{mult}_p(C) - 1)}{2}$$

and

$$\tau^{es}(\mathcal{S}) = \sum_{p \in \mathcal{T}^*(C,z)} \frac{\mathrm{mult}_p(C) \cdot (\mathrm{mult}_p(C) + 1)}{2} - \#\text{ free points in } \mathcal{T}^*(C,z) - 1,$$

where $\mathrm{mult}_p(C)$ denotes the multiplicity of the strict transform of $C$ at $p$ (see **[GLS05]** Chapter II, Proposition I.3.33, and Proposition II.1.35). Setting $\delta_p = 0$ if $p$ is satellite, $\delta_p = 1$ if $p \neq z$ is free, and $\delta_z = 2$, then $\mathrm{mult}_p(C) \geq \delta_p$ and therefore

$$\tau^{es}(\mathcal{S}) = \delta(\mathcal{S}) + \sum_{p \in \mathcal{T}^*(C,z)} \left(\mathrm{mult}_p(C) - \delta_p\right) \geq \delta(\mathcal{S}).$$

Moreover, we have equality if and only if $\mathrm{mult}_z(C) = 2$, $\mathrm{mult}_p(C) = 1$ for all $p \neq z$ and there is no satellite point, but this implies that $\mathcal{S} = A_1$. □

For some classes of singularities we can calculate the $\gamma_\alpha$-invariant concretely, and for some others we can at least give an upper bound, which in general is much better than the one derived from Lemma 3.4. We restrict our attention to the simple singularities and to such singularities which have a semi-quasihomogeneous representative $f \in \mathbb{C}\{x, y\}$ (see Definition 3.27).

**3.7 Proposition**
*Let $\alpha$ be a rational number with $0 \leq \alpha \leq 1$.*

| $\mathcal{S}$ | | $\gamma^{ea}_\alpha(\mathcal{S}) = \gamma^{es}_\alpha(\mathcal{S})$ |
|---|---|---|
| $A_k$, | $k \geq 1$ | $(k+\alpha)^2$ |
| $D_k$, | $4 \leq k \leq 4 + \sqrt{2} \cdot (2+\alpha)$ | $\frac{(k+2\alpha)^2}{2}$ |
| $D_k$, | $k \geq 4 + \sqrt{2} \cdot (2+\alpha)$ | $(k-2+\alpha)^2$ |
| $E_k$, | $k = 6, 7, 8$ | $\frac{(k+2\alpha)^2}{2}$ |

**Proof:** Let $\mathcal{S}_k$ be one of the simple singularity types $A_k$, $D_k$ or $E_k$, and let $f \in \mathbb{C}\{x, y\}$ be a representative of $\mathcal{S}_k$. Note that the Tjurina ideal $I^{ea}(f)$ and the equisingularity ideal $I^{es}(f)$ coincide, and hence so do the $\gamma_\alpha$-invariants, i. e.

$$\gamma_\alpha^{ea}(\mathcal{S}_k) = \gamma_\alpha^{es}(\mathcal{S}_k).$$

Moreover, in the considered cases the Tjurina ideal is indeed a complete intersection ideal with $\dim_{\mathbb{C}}\left(R/I^{ea}(f)\right) = k$, so that in particular the given values are upper bounds for $(1+\alpha)^2 \cdot \dim_{\mathbb{C}}(R/I)$ for any complete intersection ideal $I$ containing the Tjurina ideal. By Lemma 3.4 we know

$$\frac{(\alpha \cdot \kappa(\mathcal{S}_k) + (1-\alpha)\cdot k)^2}{\kappa(\mathcal{S}_k) - k} \leq \gamma_\alpha(\mathcal{S}_k) \leq (k+\alpha)^2.$$

Note that $\kappa(A_k) = k+1$, $\kappa(D_k) = k+2$ and $\kappa(E_k) = k+2$, which in particular gives the result for $\mathcal{S}_k = A_k$. Moreover, it shows that for $\mathcal{S}_k = D_k$ or $\mathcal{S}_k = E_k$ we have

$$\gamma_\alpha(\mathcal{S}_k) \geq \frac{(k+2\alpha)^2}{2}.$$

If we fix a complete intersection ideal $I$ with $I^{ea}(f) \subseteq I$, then

$$\lambda_\alpha(f; I, g) = \frac{(\alpha \cdot i(f,g) + (1-\alpha)\cdot \dim_{\mathbb{C}}(R/I))^2}{i(f,g) - \dim_{\mathbb{C}}(R/I)}$$

with $g \in I$ such that $i(f,g) \leq 2 \cdot \dim_{\mathbb{C}}(R/I)$ is maximal when $i(f,g)$ is minimal. If $i(f,g) - \dim_{\mathbb{C}}(R/I) \geq 2$, then

$$\lambda_\alpha(f; I, g) \leq \frac{(k+2\alpha)^2}{2}.$$

It therefore remains to consider the case where

$$i(f,g) - \dim_{\mathbb{C}}(R/I) = 1 \tag{3.1}$$

for some $I$ and some $g \in I$, and to maximize the possible $\dim_{\mathbb{C}}(R/I)$.

We claim that for $\mathcal{S}_k = D_k$ with $f = x^2y - y^{k-1}$ as representative, $\dim_{\mathbb{C}}(R/I) \leq k-2$, and thus $I = \langle x, y^{k-2}\rangle$ and $g = x$ are suitable with

$$\lambda_\alpha(f; I, x) = (k-2+\alpha)^2,$$

which is greater than $\frac{(k+2\alpha)^2}{2}$ if and only if $k \geq 4 + \sqrt{2}\cdot(2+\alpha)$. Suppose, therefore, $\dim_{\mathbb{C}}(R/I) = k-1$. Then $y^{k-1}, x^3 \in I^{ea}(f) = \langle xy, x^2 - (k-1)\cdot y^{k-2}\rangle \subset I$, the leading ideal $L_{<_{ls}}\left(I^{ea}(f)\right) = \langle x^3, xy, y^{k-2}\rangle \subset L_{<_{ls}}(I)$, and since by Proposition 3.14 $\dim_{\mathbb{C}}(R/I) = \dim_{\mathbb{C}}\left(R/L_{<_{ls}}(I)\right)$, either $L_{<_{ls}}(I) = \langle x^3, xy, y^{k-3}\rangle$ or $L_{<_{ls}}(I) = \langle x^2, xy, y^{k-2}\rangle$. In the first case there is a power series $g \in I$ such that $g \equiv y^{k-3} + ax + bx^2 \pmod I$, and hence $I \ni yg \equiv y^{k-2} \pmod I$, i. e. $y^{k-2} \in I$. But then $x^2 \in I$ and $x^2 \in L_{<_{ls}}(I)$, in contradiction to the assumption. In the second case, similarly, there is a $g \in I$ such that $g \equiv x^2 \pmod I$, and hence $x^2 \in I$ which in turn implies that $y^{k-2} \in I$. Thus $I = \langle x^2, xy, y^{k-2}\rangle$, and $\dim_{\mathbb{C}}(I/\mathfrak{m}I) = 3$ which by Remark 3.21 contradicts the fact that $I$ is a complete intersection.

If $\mathcal{S}_k = E_6$, then $f = x^3 - y^4$ is a representative and $I^{ea}(f) = \langle x^2, y^3\rangle$. Suppose that $\dim_{\mathbb{C}}(R/I) = k - 1 = 5$, then $L_{<_{ds}}(I) = \langle x^2, y^3, xy^2\rangle$ and $H^0_{R/I} = H^0_{R/L_{<_{ds}}(I)}$, in contradiction to Lemma 3.20, since $H^0_{R/L_{<_{ds}}(I)}(2) = 2$ and $H^0_{R/L_{<_{ds}}(I)}(3) = 0$. Thus $\dim_{\mathbb{C}}(R/I) \leq 4$ and $\lambda_\alpha(f; I, g) \leq (4+\alpha)^2 \leq \frac{(6+2\alpha)^2}{2}$.

If $\mathcal{S}_k = E_7$, then $f = x^3 - xy^3$ is a representative and $I^{ea}(f) = \langle 3x^2 - y^3, xy^2\rangle \ni x^3, y^5$. If $\dim_{\mathbb{C}}(R/I) \leq 4$, then $\lambda_\alpha(f; I, g) \leq (4+\alpha)^2 \leq \frac{(7+2\alpha)^2}{2}$, and we are done. It thus remains to exclude the cases where $\dim_{\mathbb{C}}(R/I) \in \{5, 6\}$. For this we note first that if there is a $g \in I$ such that $L_{<_{ls}}(g) = y^2$, then

$$g \equiv y^2 + ax + bx^2 + cxy + dx^2y \pmod{I}, \tag{3.2}$$

and therefore $y^2g \equiv y^4 \pmod{I}$, which implies $y^4 \in I$ and hence $x^2y \in I$. Analogously, if there is a $g \in I$ such that $L_{<_{ls}}(g) = x^2y$, then $g \equiv x^2y \pmod{I}$ and again $x^2y, y^4 \in I$. Suppose now that $\dim_{\mathbb{C}}(R/I) = 6$, then $L_{<_{ls}}(I) = \langle y^2, x^3\rangle$ or $L_{<_{ls}}(I) = \langle y^3, xy^2, x^2y, x^3\rangle$. In both cases we thus have $x^2y, y^4 \in I$. However, in the first case then $x^2y \in L_{<_{ls}}(I)$, in contradiction to the assumption. While in the second case we find $I = \langle xy^2, x^2y, 3x^2 - y^3\rangle$, and $\dim_{\mathbb{C}}(I/\mathfrak{m}I) = 3$ contradicts the fact that $I$ is a complete intersection by Lemma 3.21. Suppose, therefore, that $\dim_{\mathbb{C}}(R/I) = 5$. Then $L_{<_{ls}}(I) = \langle y^2, x^2y, x^3\rangle$, or $L_{<_{ls}}(I) = \langle y^3, xy^2, x^2\rangle$, or $L_{<_{ls}}(I) = \langle y^3, xy, x^3\rangle$. In the first case, we know already that $y^4, x^2y \in I$. Looking once more on (3.2) we consider the cases $a = 0$ and $a \neq 0$. If $a = 0$, then $yg \equiv y^3 \pmod{I}$, and thus $y^3 \in I$, which in turn implies $x^2 \in I$. Similarly, if $a \neq 0$, then $xg \equiv ax^2 \pmod{I}$ implies $x^2 \in I$. But then also $x^2 \in L_{<_{ls}}(I)$, in contradiction to the assumption. In the second case there is a $g \in I$ such that $g \equiv x^2 + ax^2y \pmod{I}$, and thus $yg \equiv x^2y \in I$. But then also $x^2 \in I$ and $y^3 \in I$, so that $I = \langle y^3, xy^2, x^2\rangle$. However, $\dim_{\mathbb{C}}(I/\mathfrak{m}I) = 3$ contradicts again the fact that $I$ is a complete intersection. Finally in the third case there is a $g \in I$ with $g \equiv xy + ax^2 + bx^2y \pmod{I}$, and thus $xg \equiv x^2y \pmod{I}$ implies $x^2y \in I$ and then $xy + ax^2 \in I$. Therefore, $I = \langle xy + ax^2, 3x^2 - y^3\rangle$, and for for $h \in I$ and for generic $b, c \in \mathbb{C}$ we have $i(f, h) \geq i(x, h) + i\big(x^2 - y^3, b\cdot(xy + ax^2) + c\cdot(3x^2 - y^3)\big) \geq 3 + 5 = 8$, in contradiction to (3.1).

Finally, if $\mathcal{S}_k = E_8$ with representative $f = x^3 - y^5$ and $I^{ea}(f) = \langle x^2, y^4\rangle$, we get for $\dim_{\mathbb{C}}(R/I) \leq 5$ that $\lambda_\alpha(f; I, g) \leq (5+\alpha)^2 \leq \frac{(8+2\alpha)^2}{2}$. It therefore remains to exclude the cases $\dim_{\mathbb{C}}(R/I) \in \{6, 7\}$. If $\dim_{\mathbb{C}}(R/I) = 7$ then $L_{<_{ds}}(I) = \langle x^2, y^4, xy^3\rangle$. But then $H^0_{R/L_{<_{ds}}(I)}(3) = 2$ and $H^0_{R/L_{<_{ds}}(I)}(4) = 0$ are in contradiction to Lemma 3.20. And if $\dim_{\mathbb{C}}(R/I) = 6$, then $L_{<_{ls}}(I) = \langle y^3, x^2\rangle$ or $L_{<_{ls}}(I) = \langle y^4, xy^2, x^2\rangle$. In the first case there is some $g \in I$ such that $g \equiv y^3 + ax + bxy + cxy^2 + dxy^3 \pmod{I}$, and thus $xg \equiv xy^3 \pmod{I}$ and $xy^3 \in I$. But then $yg \equiv axy + bxy^2 \pmod{I}$ and hence $axy + bxy^2 \in I$. Since neither $xy \in L_{<_{ls}}(I)$ nor $xy^2 \in L_{<_{ls}}(I)$, we must have $a = 0 = b$. Therefore, $g \equiv y^3 + cxy^2 \pmod{I}$ and $I = \langle x^2, y^3 + cxy^2\rangle$, which for $h \in I$ and $a, b \in \mathbb{C}$ generic gives $i(f, g) \geq i\big(x^3 - y^4, ax^2 + b\cdot(y^3 + cxy^2)\big) \geq 8$, in contradiction to

(3.1). In the second case, there is $g \in I$ such that $g \equiv xy^2 + axy^3 \pmod I$, therefore $yg \equiv xy^3 \pmod I$ and $xy^3 \in I$. But then $xy^2 \in I$ and $I = \langle y^4, xy^2, x^2\rangle$. This, however, is not a complete intersection, since $\dim_{\mathbb{C}}(I/\mathfrak{m}I) = 3$, in contradiction to the assumption.

This finishes the proof.

□

**3.8 Proposition**

*Let $\alpha$ be a rational number with $0 \le \alpha \le 1$, and let $M_k$ denote the topological singularity type of an ordinary $k$-fold point with $k \ge 3$. Then*

$$\gamma_\alpha^{es}(M_k) = 2 \cdot (k - 1 + \alpha)^2.$$

*In particular*

$$\gamma_\alpha^{es}(M_k) > (1+\alpha)^2 \cdot \tau_{ci}^{es}(M_k).$$

**Proof:** Note that for any representative $f$ of $M_k$ we have

$$I^{es}(f) = I^{ea}(f) + \mathfrak{m}^k = \left\langle \frac{\partial f_k}{\partial x}, \frac{\partial f_k}{\partial y} \right\rangle + \mathfrak{m}^k,$$

where $f_k$ is the homogeneous part of degree $k$ of $f$, so that we may assume $f$ to be homogeneous of degree $k$.

If $I$ is a complete intersection ideal with $\mathfrak{m}^k \subset I^{es}(f) \subseteq I$, then by Lemma 3.24

$$\dim_{\mathbb{C}}(R/I) \le \big(k - \operatorname{mult}(I) + 1\big) \cdot \operatorname{mult}(I).$$

We note moreover that for any $g \in I$

$$i(f,g) \ge \operatorname{mult}(f) \cdot \operatorname{mult}(g) \ge k \cdot \operatorname{mult}(I),$$

and that for a fixed $I$ we may attain an upper bound for $\lambda_\alpha(f; I, g)$ by replacing $i(f,g)$ by a lower bound for $i(f,g)$.

Hence, if $\operatorname{mult}(I) \ge 2$, we have

$$\lambda_\alpha(f; I, g) \le \frac{\big(k - (1-\alpha)\cdot(\operatorname{mult}(I) - 1)\big)^2 \cdot \operatorname{mult}(I)^2}{\operatorname{mult}(I) \cdot \big(\operatorname{mult}(I) - 1\big)} \le 2 \cdot (k - 1 + \alpha)^2, \qquad (3.3)$$

while $\dim_{\mathbb{C}}(R/I) \le k-1$ for $\operatorname{mult}(I) = 1$ and the above inequality (3.3) is still satisfied. To see $\dim_{\mathbb{C}}(R/I) \le k-1$ for $\operatorname{mult}(I) = 1$ note that the ideal $I$ contains an element $g$ of order 1 with $g_1 = ax + by$ as homogeneous part of degree 1 and the partial derivatives of $f$; applying a linear change of coordinates we may assume $g_1 = x$ and $f = \prod_{i=1}^k (x - a_i y)$ with pairwise different $a_i$, and we may consider the negative degree lexicographical monomial ordering $>$ giving preference to $y$; if some $a_i = 0$, then $L_>\big(\frac{\partial f}{\partial x}\big) = y^{k-1}$, while otherwise $L_>\big(\frac{\partial f}{\partial y}\big) = y^{k-1}$, so that in any case $\langle x, y^{k-1}\rangle \subseteq L_>(I)$, and by Proposition 3.14 therefore $\dim_{\mathbb{C}}(R/I) = \dim_{\mathbb{C}}\big(R/L_>(I)\big) \le \dim_{\mathbb{C}}(R/\langle x, y^{k-1}\rangle) = k-1$.

Equation (3.3) together with Lemma 3.24 shows

$$\gamma_\alpha^{es}(M_k) \le 2 \cdot (k - 1 + \alpha)^2.$$

On the other hand, considering the representative $f = x^k - y^k$, we have

$$I^{es}(f) = \langle x^{k-1}, y^{k-1}, x^a y^b \mid a + b = k \rangle,$$

and $I = \langle y^{k-1}, x^2 \rangle$ is a complete intersection ideal containing $I^{es}(f)$. Moreover, $i\big(f, x^2\big) = 2k$, $\dim_{\mathbb{C}}(R/I) = 2 \cdot (k-1)$, thus

$$\gamma_\alpha^{es}(M_k) \geq \frac{\big(\alpha \cdot i(f, x^2) + (1-\alpha) \cdot \dim_{\mathbb{C}}(R/I)\big)^2}{i\big(f, x^2\big) - \dim_{\mathbb{C}}(R/I)} = 2 \cdot (k - 1 + \alpha)^2.$$

The in particular part then follows right away from Corollary 3.25. □

**3.9 Proposition**

*Let $\mathcal{S}_{p,q}$ be a singularity type with representative $f \in \mathbb{C}\{x, y\}$ which is semi-quasihomogeneous of* $\mathrm{ord}_{(p,q)}(f) = pq$ *and convenient, where $q > p \geq 3$. Then* $\gamma_\alpha^{es}(\mathcal{S}_{p,q}) \geq \frac{\left(q-(1-\alpha)\cdot\lfloor\frac{q}{p}\rfloor\right)^2}{\lfloor\frac{q}{p}\rfloor} \geq \frac{q\cdot(p-1+\alpha)^2}{p}$ *and:*

| $p, q$ | $\gamma_\alpha^{es}(\mathcal{S}_{p,q})$ |
|---|---|
| $q \geq 39$ | $\leq 3 \cdot (q - 2 + \alpha)^2$ |
| $\frac{q}{p} \in (1, 2)$ | $\leq 3 \cdot (q - 1 + \alpha)^2$ |
| $\frac{q}{p} \in [2, 4)$ | $\leq 2 \cdot (q - 1 + \alpha)^2$ |
| $\frac{q}{p} \in [4, \infty)$ | $\leq (q - 1 + \alpha)^2$ |

**Proof:** To see the claimed lower bound for $\gamma_\alpha^{es}(\mathcal{S}_{p,q})$ recall that (see **[GLS05]** Proposition 4.46)

$$I^{es}(f) = \Big\langle \tfrac{\partial f}{\partial x}, \tfrac{\partial f}{\partial y}, x^\alpha y^\beta \;\Big|\; \alpha p + \beta q \geq pq \Big\rangle. \tag{3.4}$$

In particular, $I^{es}(f) \subseteq \big\langle y, x^{q - \lfloor\frac{q}{p}\rfloor} \big\rangle$, $\dim_{\mathbb{C}}(R/I) = q - \lfloor\frac{q}{p}\rfloor$ and $i(f, y) = q$, which implies the claim.

Let now $I$ be a complete intersection ideal with $I^{es}(f) \subseteq I$. Applying Lemma 3.24 and $d(I) \leq q$, we first of all note that

$$(1 + \alpha)^2 \cdot \dim_{\mathbb{C}}(R/I) \leq \frac{(1 + \alpha)^2 \cdot (q + 1)^2}{4} \leq 2 \cdot (q - 1 + \alpha)^2.$$

Moreover, if $\frac{q}{p} \geq 3$, then

$$(1 + \alpha)^2 \cdot \dim_{\mathbb{C}}(R/I) \leq \frac{(1 + \alpha)^2 \cdot \big(q^2 + 4q + 3\big)}{6} \leq (q - 1 + \alpha)^2.$$

since $\dim_{\mathbb{C}}(R/I) \leq \dim_{\mathbb{C}}\big(R/I^{es}(f)\big) \leq \frac{(p+1)\cdot(q+1)}{2}$ by (3.4).

It therefore suffices to show

$$\lambda_\alpha(f;I,g) \leq \begin{cases} 3\cdot(q-2+\alpha)^2, & \text{if } q \geq 39, \\ 3\cdot(q-1+\alpha)^2, & \text{if } \frac{q}{p} \in (1,2), \\ 2\cdot(q-1+\alpha)^2, & \text{if } \frac{q}{p} \in [2,4), \\ (q-1+\alpha)^2, & \text{if } \frac{q}{p} \in [4,\infty), \end{cases} \tag{3.5}$$

where $g \in I$ with $i(f,g) \leq 2\cdot \dim_{\mathbb{C}}(R/I)$. Recall that

$$\lambda_\alpha(f;I,g) = \frac{\big(\alpha\cdot i(f,g) + (1-\alpha)\cdot \dim_{\mathbb{C}}(R/I)\big)^2}{i(f,g) - \dim_{\mathbb{C}}(R/I)}.$$

Fixing $I$ and considering $\lambda_\alpha(f;I,g)$ as a function in $i(f,g)$, where due to (3.12) the latter takes values between $\dim_{\mathbb{C}}(R/I)+1$ and $2\cdot\dim_{\mathbb{C}}(R/I)$, we note that the function is monotonously decreasing. In order to calculate an upper bound for $\lambda_\alpha(f;I,g)$ we may therefore replace $i(f,g)$ by some lower bound, which still exeeds $\dim_{\mathbb{C}}(R/I)+1$. Having done this we may then replace $\dim_{\mathbb{C}}(R/I)$ by an upper bound in order to find an upper bound for $\lambda(f;I,g)$.

Note that for $q \geq 39$ we have

$$\frac{54}{19}\cdot(q-1+\alpha)^2 \leq 3\cdot(q-2+\alpha)^2. \tag{3.6}$$

Let $L_{(p,q)}(g) = x^A y^B$ be the leading term of $g$ w. r. t. the weighted ordering $<_{(p,q)}$ (see Definition 3.12). By Remark 3.28 we know

$$i(f,g) \geq Ap + Bq. \tag{3.7}$$

Working with this lower bound for $i(f,g)$ we reduce the problem to find suitable upper bounds for $\dim_{\mathbb{C}}(R/I)$. For this purpose we may assume that $L_{(p,q)}(g)$ is minimal, and thus, in particular, $B \leq \text{mult}(I)$.

If $A = 0$, in view of Remark 3.22 we therefore have

$$B = \text{mult}(I) \leq \frac{d(I)+1}{2} \leq \frac{q+1}{2},$$

and thus by Lemma 3.24 then

$$\dim_{\mathbb{C}}(R/I) \leq B\cdot(q-B+1). \tag{3.8}$$

Moreover, for $A = 0$ Lemma 3.30 applies with $h = g$ and we get

$$\dim_{\mathbb{C}}(R/I) \leq B\cdot q - 1 - \sum_{i=1}^{B-1} \left\lfloor \tfrac{qi}{p} \right\rfloor \leq B\cdot q - 1 - \left\lfloor \frac{q}{p} \right\rfloor \cdot \frac{B\cdot(B-1)}{2}. \tag{3.9}$$

Since $x^\alpha y^\beta \in I$ for $\alpha p + \beta q = pq$, we may assume $Ap + Bq \leq pq$. But then, since $\dim_{\mathbb{C}}(R/I) \leq \dim_{\mathbb{C}} R/\langle \frac{\partial f}{\partial y}, g, x^\alpha y^\beta \mid \alpha p + \beta q \geq pq\rangle$, we may apply Lemma 3.31 with $h = \frac{\partial f}{\partial y}$ and $C = p-1$. This gives

$$\dim_{\mathbb{C}}(R/I) \leq Ap + Bq - AB - \sum_{i=1}^{A-1} \left\lfloor \tfrac{pi}{q} \right\rfloor - \sum_{i=1}^{B-1} \left\lfloor \tfrac{qi}{p} \right\rfloor - \min\left\{A, \left\lceil \tfrac{q}{p} \right\rceil\right\}, \tag{3.10}$$

and if $B = 0$ we get in addition

$$\dim_{\mathbb{C}}(R/I) \leq A \cdot (p-1). \tag{3.11}$$

Finally note that by Lemma IV.1.3

$$i(f,g) > \dim_{\mathbb{C}}(R/I). \tag{3.12}$$

Let us now use the inequalities (3.6)-(3.12) to show (3.5). For this we have to consider several cases for possible values of $A$ and $B$.

**Case 1:** $A = 0$, $B \geq 1$.

If $B = 1$, then by (3.9) and (3.12) we have $\lambda_\alpha(f; I, g) \leq (q-1+\alpha)^2$.

We may thus assume that $B \geq 2$. By (3.7) and (3.8)

$$\lambda_\alpha(f; I, g) \leq \frac{B^2 \cdot \big(q - (1-\alpha)\cdot(B-1)\big)^2}{B \cdot (B-1)} \leq 2 \cdot (q-1+\alpha)^2.$$

If, moreover, $\frac{q}{p} \geq 3$, then we may apply (3.9) to find

$$\lambda_\alpha(f; I, g) \leq \frac{B^2 \cdot \big(q - (1-\alpha)\cdot(B-1)\big)^2}{\left\lfloor \frac{q}{p} \right\rfloor \cdot \frac{B\cdot(B-1)}{2} + 1} \leq (q-1+\alpha)^2.$$

Taking (3.6) into account, this proves (3.5) in the case $A = 0$ and $B \geq 1$.

**Case 2:** $A = 1$, $B \geq 1$.

From (3.10) we deduce

$$\dim_{\mathbb{C}}(R/I) \leq B \cdot (q-1) + (p-1) - \left\lfloor \tfrac{q}{p} \right\rfloor \cdot \tfrac{B\cdot(B-1)}{2}.$$

Since $\frac{p-1+\alpha}{q-1+\alpha} \leq \frac{p}{q}$ we thus get

$$\lambda_\alpha(f; I, g) \leq \frac{\left(B + \frac{p-1+\alpha}{q-1+\alpha}\right)^2}{B + \left\lfloor \frac{q}{p} \right\rfloor \cdot \frac{B\cdot(B-1)}{2} + 1} \cdot (q-1+\alpha)^2$$

$$\leq \begin{cases} \frac{(B+\frac{1}{3})^2}{\frac{3B^2}{2} - \frac{B}{2} + 1} \cdot (q-1+\alpha)^2 & \leq (q-1+\alpha)^2, \quad \text{if } \frac{q}{p} \geq 3, \\ \frac{(B+\frac{1}{2})^2}{B^2+1} \cdot (q-1+\alpha)^2 & \leq \frac{5}{4} \cdot (q-1+\alpha)^2, \quad \text{if } \frac{q}{p} \geq 2, \\ 2 \cdot \frac{(B+1)^2}{B^2+B+2} \cdot (q-1+\alpha)^2 & \leq \frac{16}{7} \cdot (q-1+\alpha)^2, \quad \text{if } \frac{q}{p} > 1, \end{cases}$$

Once more we are done, since $\frac{16}{7} \leq \frac{54}{19}$.

**Case 3:** $A \geq 2$, $B \geq 1$.

Note that $\lfloor r\rfloor \geq r-1$ for any rational number $r$, and set $s=\frac{q}{p}$, then by (3.10)

$$\dim_{\mathbb{C}}(R/I) \leq Ap+Bq-(A-1)\cdot(B-1)-\frac{A\cdot(A-1)}{2s}-\frac{s\cdot B\cdot(B-1)}{2}-1-\min\left\{A,\lceil s\rceil\right\}.$$

This amounts to

$$\lambda_\alpha(f;I,g) \leq$$

$$\frac{\left(Ap+Bq-(1-\alpha)\cdot\left((A-1)\cdot(B-1)+\frac{A\cdot(A-1)}{2s}+\frac{s\cdot B\cdot(B-1)}{2}+1+\min\{A,\lceil s\rceil\}\right)\right)^2}{(A-1)\cdot(B-1)+\frac{A\cdot(A-1)}{2s}+\frac{s\cdot B\cdot(B-1)}{2}+3}$$

$$\leq \frac{\big(A\cdot(p-1+\alpha)+B\cdot(q-1+\alpha)\big)^2}{(A-1)\cdot(B-1)+\frac{A\cdot(A-1)}{2s}+\frac{s\cdot B\cdot(B-1)}{2}+3} \leq \varphi(A,B)\cdot(q-1+\alpha)^2,$$

where

$$\varphi(A,B)=\frac{\left(\frac{A}{s}+B\right)^2}{(A-1)\cdot(B-1)+\frac{A\cdot(A-1)}{2s}+\frac{s\cdot B\cdot(B-1)}{2}+3}.$$

For the last inequality we just note again that $\frac{p-1+\alpha}{q-1+\alpha}\leq\frac{p}{q}=\frac{1}{s}$, while for the second inequality a number of different cases has to be considered. We postpone this for a moment.

In order to show (3.5) in the case $A\geq 2$ and $B\geq 1$ it now suffices to show

$$\varphi(A,B)\leq\begin{cases}\frac{54}{19}, & \text{if } s\geq 1,\\ 2, & \text{if } s\geq 2,\\ 1, & \text{if } s\geq 4.\end{cases} \tag{3.13}$$

Elementary calculus shows that for $B\geq 1$ fixed the function $[2,\infty)\to\mathbb{R}:A\mapsto\varphi(A,B)$ takes it maximum at

$$A=\max\left\{2,\frac{16-3B}{2+\frac{1}{s}}\right\}.$$

If $B\leq 3$, then the maximum is attained at $A=\frac{16-3B}{2+\frac{1}{s}}$, and

$$\varphi(A,B)\leq\varphi\left(\frac{16-3B}{2+\frac{1}{s}},B\right)=\frac{8sB-8B+64}{4s^2B-4s^2-4sB+28s-1}.$$

Again elementary calculus shows that the function $B\mapsto\varphi\left(\frac{16-3B}{2+\frac{1}{s}},B\right)$ is monotonously decreasing on $[1,3]$ and, therefore,

$$\varphi(A,B)\leq\varphi\left(\frac{13}{2+\frac{1}{s}},1\right)=\frac{8s+56}{24s-1}=:\psi_1(s).$$

Since also the function $\psi_1$ is monotonously decreasing on $[1,\infty)$ and $\psi_1(1)=\frac{64}{23}\leq\frac{54}{19}$, $\psi_1(2)=\frac{72}{47}\leq 2$ and $\psi_1(4)=\frac{88}{95}\leq 1$ Equation (3.13) follows in this case.

As soon as $B\geq 4$ the maximum for $\varphi(A,B)$ is attained for $A=2$ and

$$\varphi(A,B)\leq\varphi(2,B)=\frac{2\cdot(sB+2)^2}{s^3B^2-s^3B+2s^2B+4s^2+2s}.$$

Once more elementary calculus shows that the function $B \mapsto \varphi(2,B)$ is monotonously decreasing on $[4,\infty)$. Thus

$$\varphi(A,B) \leq \varphi(2,4) = \frac{4\cdot(1+2s)^2}{6s^3+6s^2+s} =: \psi_2(s).$$

Applying elementary calculus again, we find that the function $\psi_2$ is monotonously decreasing on $[1,\infty)$, so that we are done since $\psi_2(1) = \frac{36}{13} \leq \frac{54}{19}$, $\psi_2(2) = \frac{50}{37} \leq 2$ and $\psi_2(4) = \frac{81}{121} \leq 1$.

Let us now come back to proving the missing inequality above. We have to show

$$A + B \leq (A-1)\cdot(B-1) + \frac{A\cdot(A-1)}{2s} + \frac{s\cdot B\cdot(B-1)}{2} + 1 + \min\{A, \lceil s\rceil\},$$

or equivalently

$$\frac{A\cdot(A-1)}{2s} + \frac{s\cdot B\cdot(B-1)}{2} + 2 + \min\{A,\lceil s\rceil\} + AB - 2A - 2B \geq 0.$$

If $B \geq 2$, then $AB \geq 2A$ and $\frac{s\cdot B\cdot(B-1)}{2} + 2 + \min\{A,\lceil s\rceil\} \geq 2B$, so we are done. It remains to consider the case $B = 1$, and we have to show

$$A^2 - A - 2sA + 2s\cdot\min\{A,\lceil s\rceil\} \geq 0.$$

If $A \leq \lceil s\rceil$ or $A = 2$ this is obvious. We may thus suppose that $A > \lceil s\rceil$ and $A \geq 3$. Since $\frac{A^2}{3} \geq A$ it remains to show

$$\frac{2A^2}{3} - 2sA + 2s\cdot\lceil s\rceil \geq 0.$$

For this

$$\frac{2A^2}{3} - 2sA + 2s\cdot\lceil s\rceil \geq \begin{cases} \frac{2A^2}{3} - 2sA & \geq 0, \quad \text{if } A \geq 3s,\\ \frac{2A^2}{3} - \frac{4sA}{3} & \geq 0, \quad \text{if } 2s \leq A \leq 3s,\\ \frac{2A^2}{3} - sA & \geq 0, \quad \text{if } \frac{3s}{2} \leq A \leq 2s,\\ \frac{2A^2}{3} - \frac{2sA}{3} & \geq 0, \quad \text{if } \lceil s\rceil \leq A \leq \frac{3s}{2}.\end{cases}$$

**Case 4:** $A \geq 1$, $B = 0$.

Applying (3.10) and (3.11) we get

$$\lambda_\alpha(f;I,g) \leq \begin{cases} \frac{A^2\cdot(p-1+\alpha)^2}{A} \leq \begin{Bmatrix} \frac{A}{s^2}\cdot(q-1+\alpha)^2 \\ A\cdot(q-2+\alpha)^2 \end{Bmatrix} & \text{for any } A, \quad \text{and}\\ \frac{A^2\cdot(p-1+\alpha)^2}{\sum_{i=1}^{A-1}\lfloor\frac{pi}{q}\rfloor + \min\{A,\lceil\frac{q}{p}\rceil\}} \leq \varphi_{\nu,s}(A)\cdot(q-1+\alpha)^2, & \text{if } A \geq 3,\end{cases}$$

where

$$\varphi_{\nu,s}(A) = \frac{\frac{A^2}{s^2}}{\frac{A\cdot(A-1)}{2s} - (A-1) + \nu} = \frac{2A^2}{sA^2 - (2s^2+s)\cdot A + 2\cdot(\nu+1)\cdot s^2}$$

with $\nu = 2$ for $s \in (1,2]$ and $\nu = 3$ for $s \in (2,\infty)$.

In particular, due to the first two inequalities we may thus assume that

$$A > \begin{cases} 3, & \text{if } q \geq 39, \\ 3s^2, & \text{if } s \in (1,2), \\ 2s^2, & \text{if } s \in [2,4), \\ s^2, & \text{if } s \in [4,\infty). \end{cases}$$

Note that $\varphi_{3,s}(A) \leq 1$ for $s \geq 4$, since

$$A \geq s^2 = \frac{9s^2}{16} + \frac{7s^2}{16} \geq \frac{s\cdot(1+2s)}{2\cdot(s-2)} + \frac{s}{s-2}\cdot\sqrt{s^2-3s+\tfrac{33}{4}}.$$

This gives (3.5) for $s \geq 4$.

If now $s \in (2,4)$, then $\varphi_{3,s}$ is monotonously decreasing on $[2s^2,\infty)$, as is $s \mapsto \varphi_{3,s}(2s^2)$ on $[2,4)$, and thus

$$\varphi_{3,s}(A) \leq \varphi_{3,s}(2s^2) = \frac{4s^2}{2s^3-2s^2-s+4} \leq \frac{8}{5} \leq 2,$$

while for $s=2$ the function $\varphi_{2,2}$ is monotonously decreasing on $[8,\infty)$ and thus $\varphi_{2,2}(A) \leq \frac{16}{9} \leq 2$. This finishes the case $s \in [2,4)$.

Let's now consider the case $s \in (1,2)$ and $q \geq 39$ parallel. Applying elementary calculus, we find that $\varphi_{2,s}$ takes its maximum on $[3,\infty)$ at $A = \frac{12s}{1+2s}$ and is monotonously decreasing on $\left[\frac{12s}{1+2s},\infty\right)$. Moreover, the function $s \mapsto \varphi_{2,s}\left(\frac{12s}{1+2s}\right)$ is monotonously decreasing on $(1,2)$. If $s \geq \frac{7}{6}$, then

$$\varphi_{2,s}(A) \leq \varphi_{2,s}\left(\tfrac{12s}{1+2s}\right) \leq \varphi_{2,\frac{7}{6}}\left(\tfrac{21}{5}\right) = \frac{54}{19}.$$

Due to (3.6) it thus remains to consider the case $s \in \left(1,\frac{7}{6}\right)$ and $A > 3$. If $A \geq 8$, then

$$\varphi_{2,s}(A) \leq \varphi_{2,1}(8) = \frac{64}{23} \leq \frac{54}{19},$$

since the function $s \mapsto \varphi_{2,s}(8)$ is monotonously decreasing on $[1,2)$.

So, we are finally stuck with the case $A \in \{4,5,6,7\}$ and $1 \leq \frac{q}{p} = s \leq \frac{7}{6}$. We want to apply Lemma 3.24. For this we note first that by Lemma 3.32 in our situation $d(I) \leq p+1$ and $A = \text{mult}(I) \leq \frac{p+2}{2}$. But then

$$\dim_{\mathbb{C}}(R/I) \leq A\cdot(p-A+2)$$

and thus,

$$\lambda_\alpha(f;I,g) \leq \frac{A^2\cdot\big(p-(1-\alpha)\cdot(A-2)\big)^2}{A\cdot(A-2)} \leq \frac{A}{(A-2)}\cdot(q-2+\alpha)^2 \leq 2\cdot(q-2+\alpha)^2.$$

This finishes the proof. □

**3.10 Remark**

In the proof of the previous lemma we achieved for almost all cases $\lambda_\alpha(f;I,g) \leq \frac{54}{19}\cdot(q-1+\alpha)^2$, apart from the single case $L_{<_{(p,q)}}(g) = x^3$. The following example shows that indeed in this case we cannot, in general, expect any better coefficient than 3. More precisely, the example shows that the bound

$$3 \cdot (q-2+\alpha)^2$$

is sharp for the family of singularities given by $x^q - y^{q-1}$, $q \geq 39$. A closer investigation should allow to lower the bound on $q$, but we cannot get this for all $q \geq 4$, as the example of $E_6$ and $E_8$ show.

Moreover, we give series of examples for which the bound $(q-1+\alpha)^2$ is sharp, respectively for which $2 \cdot (q-1+\alpha)^2$ is a lower bound.

**3.11 Example**
Throughout these examples $q > p \geq 3$ are integers.

(a) Let $f = x^q - y^{q-1}$, then $\gamma_\alpha^{es}(f) \geq 3 \cdot (q-2+\alpha)^2$. In particular, for $q \geq 39$,

$$\gamma_\alpha^{es}(f) = 3 \cdot (q-2+\alpha)^2.$$

For this we note that $I = \langle x^3, y^{q-2} \rangle$ is a complete intersection ideal in $R$ with $I^{es}(f) = \big\langle x^{q-1}, y^{q-2}, x^\alpha y^\beta \;\big|\; \alpha \cdot (q-1) + \beta q \geq q \cdot (q-1) \big\rangle \subseteq I$, since $2 \cdot (q-1) + (q-3) \cdot q = q^2 - q - 2 < q \cdot (q-1)$ and thus $x^2 y^{q-3} \notin I^{es}(f)$. This also shows that the monomial $x^i y^j$ with $0 \leq i \leq 2$ and $0 \leq j \leq q-3$ form a $\mathbb{C}$-basis of $R/I$, so that $\dim_{\mathbb{C}}(R/I) = 3q - 6$. Since $i\big(f, x^3\big) = 3q - 3$, the claim follows.

(b) Let $\frac{q}{p} < 2$ and $f = x^q - y^p$, then

$$\gamma_\alpha^{es}(f) \geq 2 \cdot (q-1+\alpha)^2.$$

By the assumption on $p$ and $q$ we have $(q-2) \cdot p + q < pq$ and hence $x^{q-2} y \notin I^{es}(f)$. Thus $I^{es}(f) = \big\langle x^{q-1}, y^{p-1}, x^\alpha y^\beta \;\big|\; \alpha p + \beta q \geq pq \big\rangle \subseteq I = \langle y^2, x^{q-1} \rangle$, and we are done since $\dim_{\mathbb{C}}(R/I) = 2q - 2$ and $i\big(f, y^2\big) = 2q$.

(c) Let $f \in \mathbb{C}\{x, y\}$ be convenient, semi-quasihomogeneous of $\mathrm{ord}_{(p,q)}(f) = pq$, and suppose that in $f$ no monomial $x^k y$, $k \leq q-2$, occurs (e. g. $f = x^q - y^p$), then $\gamma_\alpha^{es}(f) \geq (q-1+\alpha)^2$. In particular, if $\frac{q}{p} \geq 4$, then

$$\gamma_\alpha^{es}(f) = (q-1+\alpha)^2.$$

By the assumption, $I^{es}(f) \subseteq I = \langle x^{q-1}, y \rangle$, since $\frac{\partial f}{\partial x} \equiv x^{q-1} \cdot u(x) \pmod{y}$ for a unit $u$ and $\frac{\partial f}{\partial y} \equiv 0 \pmod{\langle y, x^{q-1} \rangle}$. Hence we are done since $\dim_{\mathbb{C}}(R/I) = q - 1$ and $i(f, y) = q$.

(d) Let $f = y^3 - 3x^8 y + 3x^{12}$, then $f$ does not satisfy the assumptions of (c), but still $\gamma_\alpha^{es}(f) = (11+\alpha)^2 = (q-1+\alpha)^2$.
For this note that $I = \langle y - x^4, x^{11} \rangle$ contains $I^{es}(f)$, $\dim_{\mathbb{C}}(R/I) = 11$ and $i\big(f, y - x^4\big) = 12$.

(e) Let $f = 7y^3 + 15x^7 - 21x^5 y$, then $f$ is semi-quasihomogeneous with weights $(p, q) = (3, 7)$ and convenient, but $\gamma_0^{es}(f) \leq 25 < 36 = (q-1)^2$. This shows that $(q-1)^2$ is not a general lower bound for $\gamma_0^{es}(\mathcal{S}_{p,q})$.

We note first that $I^{es}(f) = \langle x^7, y^2 - x^5, x^6 - x^4y\rangle$ is not a complete intersection and $\dim_{\mathbb{C}}\big(R/I^{es}(f)\big) = 11$. Let now $I$ be a complete intersection ideal with $I^{es}(f) \subset I$ and let $h \in I$ such that $L_{<_{(3,7)}}(h) = x^A y^B$ is minimal, in particular, $\mathrm{ord}_{(3,7)}(h) = 3A + 7B$ is minimal. Then $\dim_{\mathbb{C}}(R/I) \leq 10$ and $i(f,g) \geq 3A + 7B$ for all $g \in I$.
If, therefore, $3A + 7B \geq 14$, then

$$\frac{\dim_{\mathbb{C}}(R/I)^2}{i(f,g) - \dim_{\mathbb{C}}(R/I)} \leq 25.$$

We may thus assume that $3A + 7B \leq 13$, in particular $B < 2$. If $B = 0$, and hence $A \leq 4$, then by Lemma 3.31 $\dim_{\mathbb{C}}(R/I) \leq 2A$, so that

$$\frac{\dim_{\mathbb{C}}(R/I)^2}{i(f,g) - \dim_{\mathbb{C}}(R/I)} \leq 4A \leq 16.$$

Similarly, if $B = 1$ and $A = 2$, then by the same Lemma $\dim_{\mathbb{C}}(R/I) \leq 9$ and $i(f,g) \geq 13$, so that

$$\frac{\dim_{\mathbb{C}}(R/I)^2}{i(f,g) - \dim_{\mathbb{C}}(R/I)} \leq \frac{81}{4}.$$

So it remains to consider the case $B = 1$ and $A \in \{0,1\}$. That is $h = x^A y + h'$ with $\mathrm{ord}_{(3,7)}(h') \geq 9 + 3A$. Consider the ideal $J = \big\langle x^\alpha y^\beta \;\big|\; 3\alpha + 7\beta \geq 21\big\rangle \subseteq I$. Then $x^{4-A} \cdot h \equiv x^4 y \pmod{J}$, and thus $x^6 - x^4y \equiv x^6 \pmod{\langle h\rangle + J}$, i. e. $\langle h, x^6 - x^4y\rangle + J = \langle h, x^6\rangle + J$. Moreover, $x^6 \notin \langle h\rangle + J$, so that $\dim_{\mathbb{C}}\big(R/\langle g, x^6 - x^4y\rangle + J\big) \leq 6 + A$. If we can show that $\langle g, x^6 - x^4y\rangle + J \subsetneqq I$, then

$$\frac{\dim_{\mathbb{C}}(R/I)^2}{i(f,g) - \dim_{\mathbb{C}}(R/I)} \leq \frac{(5+A)^2}{3A + 7 - 5 - A} \leq \frac{25}{2}.$$

We are therefore done, once we know that $y^2 - x^5 \notin \langle g, x^6\rangle + J$. Suppose there was a $g$ such that $gh = y^2 - x^5 \pmod{\langle x^6\rangle + J}$. Then $y^2 = L_{<_{(3,7)}}(g) \cdot L_{<_{(3,7)}}(h)$, which in particular means $A = 0$ and $L_{<_{(3,7)}}(h) = L_{<_{(3,7)}}(g) = y$. But then the coefficients of $1$, $x$ and $x^2$ in $h$ and $g$ must be zero, so that $x^5$ cannot occur with a non-zero coefficient in the product. This gives the desired contradiction.

## 3.b. Local Monomial Orderings

Throughout the proofs we will make use of some results from computer algebra concerning properties of local monomial orderings. Let us therefore recall some basic definitions and results.

**3.12 Definition**

A *monomial ordering* is a total ordering $<$ on the set of monomials $\big\{x^\alpha y^\beta \;\big|\; \alpha, \beta \geq 0\big\}$ such that for all $\alpha, \beta, \gamma, \delta, \mu, \nu \geq 0$

$$x^\alpha y^\beta < x^\gamma y^\delta \quad \Longrightarrow \quad x^{\alpha+\mu} y^{\beta+\nu} < x^{\gamma+\mu} y^{\delta+\nu}.$$

A monomial ordering $<$ is called *local* if $1 > x^\alpha y^\beta$ for all $(\alpha, \beta) \neq (0,0)$, and it is a *local degree ordering* if

$$\alpha + \beta > \gamma + \delta \quad \Longrightarrow \quad x^\alpha y^\beta < x^\gamma y^\delta.$$

Finally, if $<$ is any local monomial ordering, then we define the *leading monomial* $L_<(f)$ with respect to $<$ of a power series $f \in R$ to be the maximal monomial $x^\alpha y^\beta$ such that the coefficient of $x^\alpha y^\beta$ in $f$ does not vanish, respectively zero if $f = 0$.

If $I \trianglelefteq R$ is an ideal in $R$, then $L_<(I) = \langle L_<(f) \mid f \in I \rangle$ is called the *leading ideal* of $I$.

We will give now some examples of local monomial orderings which are subsequently used in the proofs.

**3.13 Example**
Let $\alpha, \beta, \delta, \gamma \geq 0$ be integers.

(a) The *negative lexicographical ordering* $<_{ls}$ is defined by the relation

$$x^\alpha y^\beta < x^\gamma y^\delta \quad :\Longleftrightarrow \quad \alpha > \gamma \text{ or } (\alpha = \gamma \text{ and } \beta > \delta).$$

(b) The *negative degree reverse lexicographical ordering* $<_{ds}$ is defined by the relation

$$x^\alpha y^\beta < x^\gamma y^\delta \quad :\Longleftrightarrow \quad \alpha + \beta > \gamma + \delta \text{ or } (\alpha + \beta = \gamma + \delta \text{ and } \beta > \delta).$$

(c) If positive integers $p$ and $q$ are given, then we define the *local weighted degree ordering* $<_{(p,q)}$ with weights $(p, q)$ by the relation

$$x^\alpha y^\beta < x^\gamma y^\delta \quad :\Longleftrightarrow \quad \begin{aligned} &\alpha p + \beta q > \gamma p + \delta q \text{ or} \\ &(\alpha p + \beta q = \gamma p + \delta q \text{ and } \beta < \delta). \end{aligned}$$

We note that $<_{ds}$ is a local degree ordering, while $<_{ls}$ is not and $<_{(p,q)}$ is if and only if $p = q$.

Let us finally recall some useful properties of local orderings (see e. g. **[GrP02]** Corollary 7.5.6 and Proposition 5.5.7).

**3.14 Proposition**
*Let $<$ be any local monomial ordering, and let $I$ be a zero-dimensional ideal.*

(a) *The monomials of $R/L_<(I)$ form a $\mathbb{C}$-basis of $R/I$. In particular*

$$\dim_{\mathbb{C}}(R/I) = \dim_{\mathbb{C}}\big(R/L_<(I)\big).$$

(b) *If $<$ is a degree ordering, then*

$$H^1_{R/I} = H^1_{R/L_<(I)},$$

*that is even the Hilbert Samuel functions of $R/I$ and of $R/L_<(I)$ coincide (see Definition 3.15).*

## 3.c. The Hilbert Samuel Function

A useful tool in the study of the degree of zero-dimensional schemes and their subschemes is the Hilbert Samuel function of the structure sheaf, that is of the corresponding Artinian ring.

**3.15 Definition**
Let $I \lhd R$ be a zero-dimensional ideal.

(a) The function

$$H^1_{R/I} : \mathbb{Z} \to \mathbb{Z} : d \mapsto \begin{cases} \dim_{\mathbb{C}} \big(R/(I + \mathfrak{m}^{d+1})\big), & d \geq 0, \\ 0, & d < 0, \end{cases}$$

is called the *Hilbert Samuel function* of $R/I$.

(b) We define the *slope* of the Hilbert Samuel function of $R/I$ to be the function

$$H^0_{R/I} : \mathbb{N} \to \mathbb{N} : d \mapsto H^1_{R/I}(d) - H^1_{R/I}(d-1).$$

Thus

$$H^0_{R/I}(d) = \dim_{\mathbb{C}} \big(\mathfrak{m}^d/((I \cap \mathfrak{m}^d) + \mathfrak{m}^{d+1})\big),$$

is just the number of linearly independent monomials of degree $d$ in $\mathfrak{m}^d$, which is $d+1$, minus the number of linearly independent monomials of degree $d$ in $\big(I \cap \mathfrak{m}^d\big) + \mathfrak{m}^{d+1}$.
Note that if $\overline{\mathfrak{m}} = \mathfrak{m}/I$ denotes the maximal ideal of $R/I$ and $\mathrm{Gr}_{\mathfrak{m}}(R/I) = \bigoplus_{d \geq 0} \overline{\mathfrak{m}}^d/\overline{\mathfrak{m}}^{d+1}$ the associated graded ring, then

$$H^0_{R/I}(d) = \dim_{\mathbb{C}} \big(\overline{\mathfrak{m}}^d/\overline{\mathfrak{m}}^{d+1}\big)$$

is just the dimension of the graded piece of degree $d$ of $\mathrm{Gr}_{\mathfrak{m}}(R/I)$.

(c) Finally we define the *multiplicity* of $I$ to be

$$\mathrm{mult}(I) := \min\big\{\mathrm{mult}(f) \;\big|\; 0 \neq f \in I\big\},$$

and the *degree bound* of $I$ as

$$d(I) := \min\big\{d \in \mathbb{N} \;\big|\; \mathfrak{m}^d \subseteq I\big\}.$$

Let us gather some straight forward properties of the slope of the Hilbert Samuel function.

**3.16 Lemma**
*Let $J \subseteq I \lhd R$ be zero-dimensional ideals.*

(a) $H^0_{R/I}(d) = d + 1$ *for all* $0 \leq d < \mathrm{mult}(I)$.

(b) $H^0_{R/I}(d) \leq H^0_{R/I}(d-1)$ *for all* $d \geq \mathrm{mult}(I)$.

(c) $H^0_{R/I}(d) \leq \mathrm{mult}(I)$.

(d) $H^0_{R/I}(d) = 0$ *for all* $d \geq d(I)$ *and* $H^0_{R/I} \neq 0$ *for all* $d < d(I)$. *In particular*

$$\dim_{\mathbb{C}}(R/I) = \sum_{d=0}^{d(I)-1} H^0_{R/I}(d).$$

(e) $H^0_{R/I}(d) \leq H^0_{R/J}(d)$ *for all* $d \in \mathbb{N}$.

(f) $d(I)$ *and* $\operatorname{mult}(I)$ *are completely determined by* $H^0_{R/I}$.

**Proof:** For (a) we note that $I \subseteq \mathfrak{m}^d$ for all $d \leq \operatorname{mult}(I)$ and thus $H^0_{R/I}(d) = \dim_{\mathbb{C}}\left(\mathfrak{m}^d/\mathfrak{m}^{d+1}\right) = d+1$ for all $0 \leq d < \operatorname{mult}(I)$.

By definition we see that $H^0_{R/I}(d)$ is just the number of linearly independent monomials of degree $d$ in $\mathfrak{m}^d$, which is $d+1$, minus the number of linearly independent monomials, say $m_1, \ldots, m_r$, of degree $d$ in $\left(I \cap \mathfrak{m}^d\right) + \mathfrak{m}^{d+1}$. We note that then the set

$$\{xm_1, \ldots, xm_r, ym_1, \ldots, ym_r\} \subseteq \mathfrak{m} \cdot \left(\left(I \cap \mathfrak{m}^d\right) + \mathfrak{m}^{d+1}\right) \subseteq \left(I \cap \mathfrak{m}^{d+1}\right) + \mathfrak{m}^{d+2}$$

contains at least $r+1$ linearly independent monomials of degree $d+1$, once $r$ was non-zero. However, for $d = \operatorname{mult}(I)$ and $g = g_d + \text{h.o.t} \in I$ with homogeneous part $g_d \neq 0$ of degree $d$, we have $g_d \in \left(I \cap \mathfrak{m}^d\right) + \mathfrak{m}^{d+1}$, that is, $d = \operatorname{mult}(I)$ is the smallest integer $d$ for which there is a monomial of degree $d$ in $\left(I \cap \mathfrak{m}^d\right) + \mathfrak{m}^{d+1}$. Thus for $d \geq \operatorname{mult}(I) - 1$

$$H^0_{R/I}(d+1) \leq (d+2) - (r+1) = d+1-r = H^0_{R/I}(d),$$

which proves (b), while (c) is an immediate consequence of (a) and (b).

If $d \geq d(I)$, then $H^1_{R/I}(d) = \dim_{\mathbb{C}}(R/I)$ is independent of $d$, and hence $H^0_{R/I}(d) = 0$ for all $d \geq d(I)$. In particular,

$$\sum_{i=0}^{d(I)-1} H^0_{R/I}(d) = H^1_{R/I}(d(I)-1) - H^1_{R/I}(-1) = \dim_{\mathbb{C}}(R/I).$$

Moreover, $\mathfrak{m}^{d(I)-1} + I \neq I = I + \mathfrak{m}^{d(I)}$, so that $H^0_{R/I}\big(d(I)-1\big) \neq 0$, and by (b) then $H^0_{R/I}(d) \neq 0$ for all $d < d(I)$. This proves (d), and (e) and (f) are obvious. □

### 3.17 Remark

Let $<$ be a local degree ordering on $R$, then the Hilbert Samuel functions of $R/I$ and of $R/L_<(I)$ coincide by Proposition 3.14, and hence we have as well

$$H^0_{R/I} = H^0_{R/L_<(I)}, \quad d(I) = d\left(L_<(I)\right), \quad \text{and} \quad \operatorname{mult}(I) = \operatorname{mult}\left(L_<(I)\right),$$

since by the previous lemma the multiplicity and the degree bound only depend on the slope of the Hilbert Samuel function.

### 3.18 Remark

The slope of the Hilbert Samuel function of $R/I$ gives rise to a histogram as the graph of the function $H^0_{R/I}$. By the Lemma 3.16 we know that up to $\operatorname{mult}(I)-1$ the histogram is just a staircase with steps of height one, and from $\operatorname{mult}(I)-1$ on it can only go down, which it eventually will do until it reaches the value zero. This will definitely be the case for $d = d(I)$. This means that we get a histogram of form shown in Figure 1.

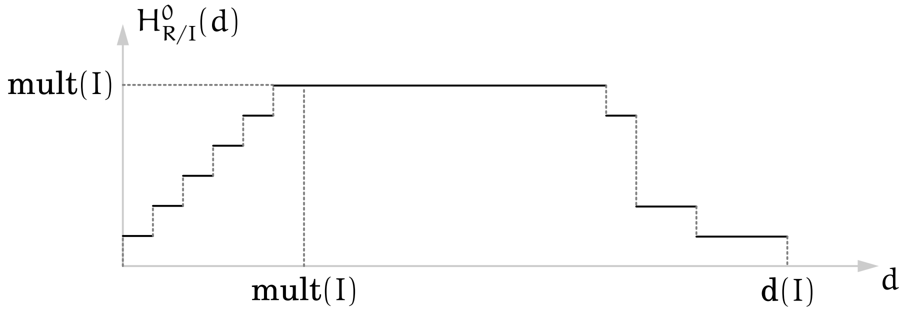


FIGURE 1. The histogram of $H^0_{R/I}$ for a general ideal I.

Note also, that by Lemma 3.16 (a) the area of the histogram is just $\dim_{\mathbb{C}}(R/I)$!

**3.19 Example**

In order to understand the slope of the Hilbert Samuel function better let us consider some examples.

(a) Let $f = x^2 - y^{k+1}$, $k \geq 1$, and let $I = I^{ea}(f) = \langle x, y^k \rangle$ the equisingularity ideal of an $A_k$-singularity. Then $d(I) = k$, $\mathrm{mult}(I) = 1$ and $\dim_{\mathbb{C}}(R/I) = k$.


FIGURE 2. The histogram of $H^0_{R/I}$ for an $A_k$-singularity

(b) Let $f = x^2y - y^{k-1}$, $k \geq 4$, and let $I = I^{ea}(f) = \langle xy, x^2 - y^{k-2} \rangle$ the equisingularity ideal of a $D_k$-singularity. Then $x^3, xy, y^{k-1} \in I$, and thus $\mathfrak{m}^{k-1} \subset I$, which gives $d(I) = k-1$, $\mathrm{mult}(I) = 2$ and $\dim_{\mathbb{C}}(R/I) = k$, which shows that the bound in Lemma 3.24 need not be obtained.

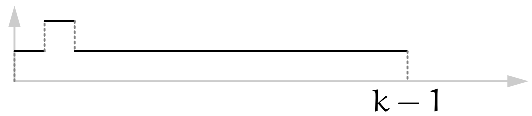


FIGURE 3. The histogram of $H^0_{R/I}$ for an $D_k$-singularity

(c) Let $f = x^3 - y^4$ and let $I = I^{ea}(f) = \langle x^2, y^3 \rangle$ the equisingularity ideal of a $E_7$-singularity. Then $d(I) = 4$, $\mathrm{mult}(I) = 2$ and $\dim_{\mathbb{C}}(R/I) = 6$.
Let $f = x^3 - xy^3$ and let $I = I^{ea}(f) = \langle 3x^2 - y^3, xy^2 \rangle$ the equisingularity ideal of a $E_7$-singularity. Then $x^3, xy^2, y^5 \in I$, and thus $\mathfrak{m}^5 \subset I$, which gives $d(I) = 5$, $\mathrm{mult}(I) = 2$ and $\dim_{\mathbb{C}}(R/I) = 7$.

Let $f = x^3 - y^5$ and let $I = I^{ea}(f) = \langle x^2, y^4 \rangle$ the equisingularity ideal of a $E_7$-singularity. Then $d(I) = 5$, $\text{mult}(I) = 2$ and $\dim_{\mathbb{C}}(R/I) = 6$.

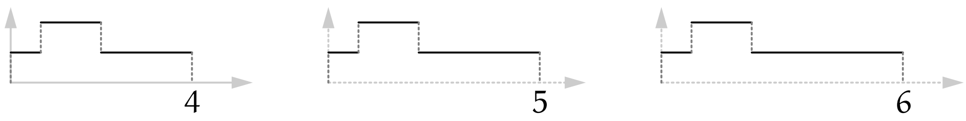


FIGURE 4. The histogram of $H^0_{R/I}$ for $E_6$, $E_7$ and $E_8$.

(d) Let $I = \langle x^3, x^2y, y^3 \rangle$, then $d(I) = 4$, $\text{mult}(I) = 3$ and $\dim_{\mathbb{C}}(R/I) = 7$.

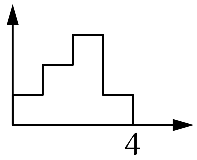


FIGURE 5. The histogram of $H^0_{R/I}$ for $I = \langle x^3, x^2y, y^3 \rangle$.

The following result providing a lower bound for the minimal number of generators of a zero-dimensional ideal in $R$ is due to A. Iarrobino.

**3.20 Lemma**
*Let $I \lhd R$ be a zero-dimensional ideal. Then $I$ cannot be generated by less than $1 + \sup_{d \geq \text{mult}(I)} \left\{ H^0_{R/I}(d-1) - H^0_{R/I}(d) \right\}$ elements.*

*In particular, if $I$ is a complete intersection ideal then for $d \geq \nu(I)$*

$$H^0_{R/I}(d-1) - 1 \leq H^0_{R/I}(d) \leq H^0_{R/I}(d-1).$$

**Proof:** See **[Iar77]** Theorem 4.3 or **[Bri77]** Proposition III.2.1. □

Moreover, by the Lemma of Nakayama and Proposition 3.14 we can compute the minimal number of generators for a zero-dimensional ideal exactly.

**3.21 Lemma**
*Let $I \lhd R$ be zero-dimensional ideal and let $<$ denote any local ordering on $R$. Then the minimal number of generators of $I$ is*

$$\dim_{\mathbb{C}}(I/\mathfrak{m}I) = \dim_{\mathbb{C}} \big(R/L_<(I)\big) - \dim_{\mathbb{C}} \big(R/L_<(\mathfrak{m}I)\big).$$

**3.22 Remark**
If we apply Lemma 3.20 to a zero-dimensional complete intersection ideal $I \lhd R$, i. e. an ideal which is generated by two elements, then we know that the histogram of $H^0_{R/I}$ will be as shown in Figure 6; that is, up to the value $d = \text{mult}(I)$ the histogram of $H^0_{R/I}$ is an ascending staircase with steps of height and length one, then it remains constant for a while, and finally it is

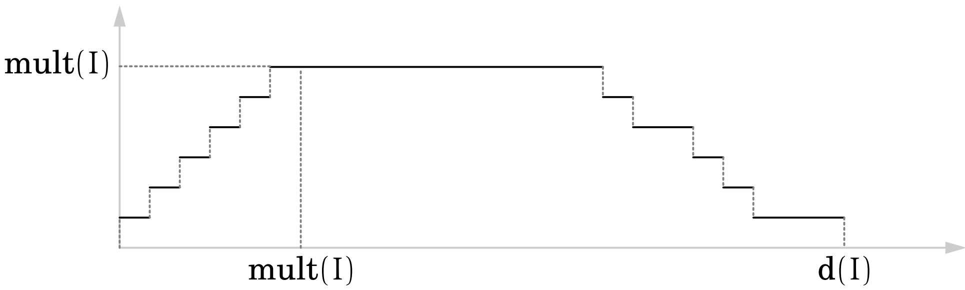


FIGURE 6. The histogram of $H^0_{R/I}$ for a complete intersection.

a descending staircase again with steps of height one, but a possibly longer length. In particular we see that

$$\mathrm{mult}(I) \le \begin{cases} \frac{d(I)+1}{2}, & \text{if } d(I) \text{ is odd,} \\ \frac{d(I)}{2}, & \text{if } d(I) \text{ is even.} \end{cases} \tag{3.14}$$

**3.23 Example**
Let $I = \mathfrak{m}^k$ for $k \ge 1$. Then $d(I) = \mathrm{mult}(I) = k$ and $\dim_{\mathbb{C}}(R/I) = \binom{k+1}{2}$.

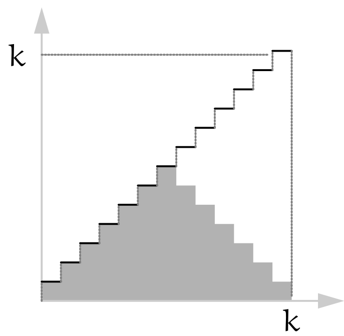


FIGURE 7. The histogram of $H^0_{R/\mathfrak{m}^k}$. The shaded region is the maximal possible value of $\dim_{\mathbb{C}}(R/I)$ for a complete intersection ideal $I$ containing $\mathfrak{m}^k$.

**3.24 Lemma**
*Let $I \lhd R$ be a zero-dimensional complete intersection ideal, then*

$$\dim_{\mathbb{C}}(R/I) \le \big(d(I) - \mathrm{mult}(I) + 1\big) \cdot \mathrm{mult}(I).$$

*In particular*

$$\dim_{\mathbb{C}}(R/I) \le \begin{cases} \frac{(d(I)+1)^2}{4}, & \textit{if } d(I) \textit{ odd,} \\ \frac{d(I)^2+2d(I)}{4}, & \textit{if } d(I) \textit{ even.} \end{cases}$$

**Proof:** By Remark 3.18 we have to find an upper bound for the area $A$ of the histogram of $H^0_{R/I}$. This area would be maximal, if in the descending part the steps had all length one, i. e. if the histogram was as shown in Figure 8. Since the two shaded regions have the same area, we get

$$A \le \big(d(I) - \mathrm{mult}(I) + 1\big) \cdot \mathrm{mult}(I).$$

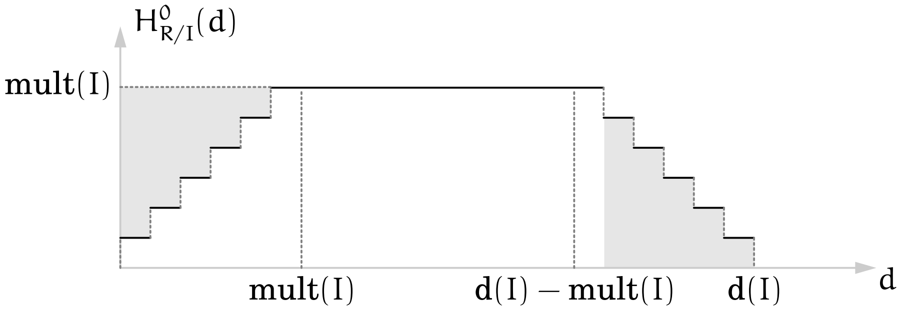


FIGURE 8. Maximal possible area.

Consider now the function

$$\varphi : \left[\operatorname{mult}(I), \tfrac{d(I)+1}{2}\right] \to \mathbb{R} : x \mapsto \big(d(I) - x + 1\big) \cdot x,$$

then this function is monotonously increasing, which finishes the proof in view of Equation (3.14). □

**3.25 Corollary**

*For an ordinary* $m$*-fold point* $M_m$ *we have*

$$\tau^{es}_{ci}(M_m) = \begin{cases} \frac{(m+1)^2}{4}, & \textit{if } m \geq 3 \textit{ odd}, \\ \frac{m^2+2m}{4}, & \textit{if } m \geq 4 \textit{ even}, \\ 1, & \textit{if } m = 2. \end{cases}$$

**Proof:** Let $f$ be a representative of $M_m$. Then

$$I^{es}(f) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial x} \right\rangle + \mathfrak{m}^m,$$

and as in the proof of Proposition 3.8 we may assume that $f$ is a homogeneous of degree $m$.

In particular, if $m = 2$, then $I^{es}(f) = \mathfrak{m}$ is a complete intersection and $\tau^{es}_{ci}(M_2) = 1$. We may therefore assume that $m \geq 3$.

For any complete intersection ideal $I$ with $\mathfrak{m}^m \subset I^{es}(f) \subseteq I$ we automatically have $d(I) \leq m$, and by Lemma 3.24

$$\tau^{es}_{ci}(f) \leq \begin{cases} \frac{(m+1)^2}{4}, & \text{if } m \text{ odd}, \\ \frac{m^2+2m}{4}, & \text{if } m \geq 4 \text{ even}. \end{cases}$$

Consider now the representative $f = x^m - y^m$. If $m = 2k$ is even, then the ideal $I = \langle x^k, y^{k+1} \rangle$ is a complete intersection with $I^{es}(f) \subset I$ and

$$\tau^{es}_{ci}(f) \geq \dim_{\mathbb{C}}(R/I) = k^2 + k = \frac{m^2 + 2m}{4}.$$

Similarly, if $m = 2k - 1$ is odd, then the ideal $I = \langle x^k, y^k \rangle$ is a complete intersection with $I^{es}(f) \subset I$ and

$$\tau^{es}_{ci}(f) \geq \dim_{\mathbb{C}}(R/I) = k^2 = \frac{m^2 + 2m + 1}{4}.$$

□

**3.26 Remark**
Let $I \lhd R$ be any zero-dimensional ideal, not necessarily a complete intersection, then still

$$\dim_{\mathbb{C}}(R/I) \leq \left(d(I) - \frac{\mathrm{mult}(I)-1}{2}\right) \cdot \mathrm{mult}(I).$$

**Proof:** The proof is the same as for the complete intersection ideal, just that we cannot ensure that the histogram goes down to zero at $d(I)$ with steps of size one. The dimension is thus bounded by the region of the histogram in Figure 9. □

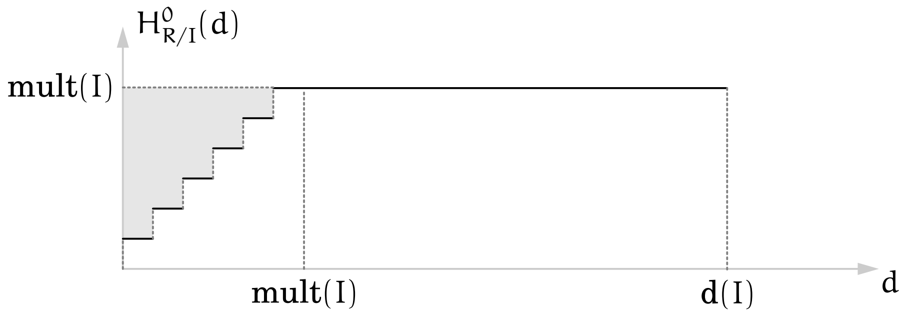


FIGURE 9. Maximal possible area.

### 3.d. Semi-Quasihomogeneous Singularities

**3.27 Definition**
A non-zero polynomial of the form $f = \sum_{\alpha \cdot p + \beta \cdot q = d} a_{\alpha,\beta} x^\alpha y^\beta$ is called *quasihomogeneous* of $(p,q)$-degree $d$. Thus the Newton polygon of a quasihomogeneous polynomial has just one side of slope $-\frac{p}{q}$.

A quasihomogeneous polynomial is said to be *non-degenerate* if it is reduced, that is if it has no multiple factors, and it is said to be *convenient* if $\frac{d}{p}, \frac{d}{q} \in \mathbb{Z}$ and $a_{\frac{d}{p},0}$ and $a_{0,\frac{d}{q}}$ are non-zero, that is if the Newton polygon meets the $x$-axis and the $y$-axis.

If $f = f_0 + f_1$ with $f_0$ quasihomogeneous of $(p,q)$-degree $d$ and for any monomial $x^\alpha y^\beta$ occurring in $f_1$ with a non-negative coefficient we have $\alpha \cdot p + \beta \cdot q > d$, we say that $f$ is of $(p,q)$-*order* $d$, and we call $f_0$ the $(p,q)$-*leading form* of $f$ and denote it by $\mathrm{lead}_{(p,q)}(f)$. We denote the $(p,q)$-order of $f$ by $\mathrm{ord}_{(p,q)}(f)$.

A power series $f \in R$ is said to be *semi-quasihomogeneous* with respect to the weights $(p,q)$ if the $(p,q)$-leading form is non-degenerate.

**3.28 Remark**
Let $f \in R$ with $\deg_{(p,q)}(f) = pq$ and let $f_0$ denote its $(p,q)$-leading form.

(a) If $\gcd(p,q) = r$, then $f_0$ has $r$ factors of the form $a_i x^{\frac{q}{r}} - b_i y^{\frac{p}{r}}$, $i = 1, \ldots, r$. If, moreover, $f_0$ is non-degenerate, then these will all be irreducible and pairwise different, i. e. not scalar multiples of each other.

(b) If $f$ is irreducible, then $f_0$ has only one irreducible factor, possibly of higher multiplicity.

(c) If $f_0$ is non-degenerate, then $f$ has $r = \gcd(p,q)$ branches $f_1, \ldots, f_r$, which are all semi-quasihomogeneous with irreducible $(p,q)$-leading form $a_i x^{\frac{q}{r}} - b_i y^{\frac{p}{r}}$ for pairwise distinct points $(a_i : b_i) \in \mathbb{P}^1_{\mathbb{C}}$, $i = 1, \ldots, r$. In particular, the characteristic exponents of $f_i$ are $\frac{q}{r}$ and $\frac{p}{r}$ for all $i = 1, \ldots, r$, and thus $f_i$ admits a parametrisation of the form

$$\big(x_i(t), y_i(t)\big) = \Big(\alpha_i t^{\frac{p}{r}} + \text{h.o.t}, \beta_i t^{\frac{q}{r}} + \text{h.o.t}\Big).$$

(d) If $f_0$ is non-degenerate, i. e. $f$ is semi-quasihomogeneous, and $g \in R$, then

$$i(f,g) \geq \mathbf{ord}_{(p,q)}(g).$$

**Proof:** (a) If $\alpha p + \beta q = pq$, then $p \mid \beta q$ and hence $p \mid \beta r$, so that $\beta \cdot \frac{r}{p}$ is a natural number. Similarly $\alpha \cdot \frac{r}{q}$ is a natural number. We may therefore consider the transformation

$$f_0\big(x^{\frac{r}{q}}, y^{\frac{r}{p}}\big) \in \mathbb{C}[x,y]_r$$

which is a homogeneous polynomial of degree $r$. Thus $f_0\big(x^{\frac{r}{q}}, y^{\frac{r}{p}}\big)$ factors in $r$ linear factors $a_i x - b_i y$, $i = 1, \ldots, r$, so that $f_0$ factors as

$$f_0 = \prod_{i=1}^{r} \big(a_i x^{\frac{q}{r}} - b_i y^{\frac{p}{r}}\big). \tag{3.15}$$

Since $\gcd\left(\frac{p}{r}, \frac{q}{r}\right) = 1$, the factors $a_i x^{\frac{q}{r}} - b_i y^{\frac{p}{r}}$ are irreducible once neither $a_i$ nor $b_i$ is zero.

If $f_0$ is non-degenerate, then the irreducible factors of $f_0$ are pairwise distinct. So, $a_i = 0$ implies $r = p$ and still $a_i x^{\frac{q}{r}} - b_i y^{\frac{p}{r}} = b_i y$ irreducible, while $b_i = 0$ similarly gives $r = q$ and $a_i x^{\frac{q}{r}} - b_i y^{\frac{p}{r}} = a_i x$ irreducible. Thus, in any case the factors in (3.15) are irreducible and, hence, pairwise distinct.

(b) With the notation from Lemma 3.29 and the factorisation of $f_0$ from (3.15) we get

$$g = \frac{\prod_{i=1}^{r} a_i u^{\frac{bq}{r}} v^{pqr^2} - b_i u^{\frac{ap}{r}} v^{pqr^2}}{u^{ap} v^{pq}} = \prod_{i=1}^{r} (a_i u - b_i).$$

By assumption $f$ is irreducible, hence according to Lemma 3.29 $g$ has at most one, possibly repeated, zero. But thus the factors of $f_0$ all coincide – up to scalar multiple.

(c) The first assertion is an immediate consequence from (a) and (b), while the in particular part follows by Puiseux expansion.

(d) Let $g_0$ be the $(p,q)$-leading form of $g$. Using the notation from (c) we have

$$\begin{aligned} i(f,g) &= \sum_{i=1}^{r} i(f_i,g) = \sum_{i=1}^{r} \textbf{ord}\left(g(x_i(t),y_i(t))\right) \\ &= \sum_{i=1}^{r} \textbf{ord}\left(g_0\big(\alpha_i t^{\frac{p}{r}}, \beta_i t^{\frac{q}{r}}\big) + \text{h.o.t}\right) \geq \sum_{i=1}^{r} \frac{\textbf{ord}_{(p,q)}(g)}{r} = \textbf{ord}_{(p,q)}(g). \end{aligned}$$

□

### 3.29 Lemma

*Let* $f \in R$ *with* $\deg_{(p,q)}(f) = pq$ *and let* $f_0$ *denote its* $(p,q)$*-leading form. Let* $r = \gcd(p,q)$ *and* $a, b \geq 0$ *such that* $qb - pa = r$*. Finally set*

$$g = \frac{f_0\big(u^b v^{\frac{p}{r}}, u^a v^{\frac{q}{r}}\big)}{u^{ap} v^{\frac{pq}{r}}} \in \mathbb{C}[u].$$

*Then the number of different zeros of* $g$ *is a lower bound for the number of branches of* $f$.

**Proof:** See **[BrK86]** Remark on p. 480. □

The following investigations are crucial for the proof of Proposition 3.9.

### 3.30 Lemma

*Let* $f \in R$ *be convenient semi-quasihomogeneous with leading form* $f_0$ *and* $\textbf{ord}_{(p,q)}(f) = pq$*, let* $I = \left\langle x^\alpha y^\beta \;\middle|\; \alpha p + \beta q \geq pq \right\rangle$*, and let* $h \in R$*. Then*

$$\dim_{\mathbb{C}} R/\langle h \rangle + I^{es}(f) < \dim_{\mathbb{C}} R/\langle h \rangle + I.$$

*In particular, if* $L_{(p,q)}(h) = y^B$ *with* $B \leq p$*, then*

$$\dim_{\mathbb{C}} R/\langle h \rangle + I^{es}(f) \leq Bq - 1 - \sum_{i=1}^{B-1} \left\lfloor \tfrac{qi}{p} \right\rfloor.$$

**Proof:** We recall that

$$I^{es}(f) = \left\langle \tfrac{\partial f}{\partial x}, \tfrac{\partial f}{\partial y} \right\rangle + I.$$

It suffices to show that

$$I^{es}(f) \not\subseteq \langle h \rangle + I,$$

which is the same as showing that not both $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ belong to $\langle h \rangle + I$.

Suppose the contrary, that is, there are $h_x, h_y \in R$ such that

$$\tfrac{\partial f}{\partial x} \equiv h_x \cdot h \ (\text{mod } I) \quad \text{and} \quad \tfrac{\partial f}{\partial y} \equiv h_y \cdot h \ (\text{mod } I).$$

We note that

$$\textbf{lead}_{(p,q)}\left(\tfrac{\partial f}{\partial x}\right) = \tfrac{\partial f_0}{\partial x} \quad \text{and} \quad \textbf{lead}_{(p,q)}\left(\tfrac{\partial f}{\partial y}\right) = \tfrac{\partial f_0}{\partial y},$$

and none of the monomials involved is contained in $I$. Therefore

$$\textbf{lead}_{(p,q)}(h_x) \cdot \textbf{lead}_{(p,q)}(h) = \tfrac{\partial f_0}{\partial x} \quad \text{and} \quad \textbf{lead}_{(p,q)}(h_y) \cdot \textbf{lead}_{(p,q)}(h) = \tfrac{\partial f_0}{\partial y},$$

which in particular implies that $\frac{\partial f_0}{\partial x}$ and $\frac{\partial f_0}{\partial y}$ have a common factor. This, however, is then a multiple factor of the quasihomogenous polynomial $f_0$, in contradiction to $f$ being semi-quasihomogeneous.

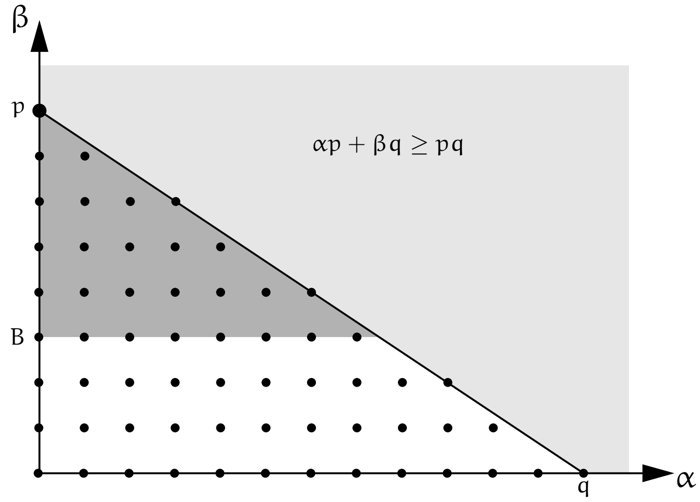


FIGURE 10. A Basis of $R/\langle h\rangle + I$.

For the in particular part we note that by Proposition 3.14

$$\dim_{\mathbb{C}} R/\langle h\rangle + I = \dim_{\mathbb{C}} R/L_{<_{(p,q)}}\big(\langle h\rangle + I\big) \le \dim_{\mathbb{C}} R/\langle y^B\rangle + I,$$

and the monomials $x^\alpha y^\beta$ with $\alpha p + \beta q < pq$ and $\beta < B$ form a $\mathbb{C}$-basis of the latter vector space. (See also Figure 10.) Hence,

$$\dim_{\mathbb{C}} R/\langle h\rangle + I \le \sum_{i=0}^{B-1} \left\lceil q - \tfrac{qi}{p}\right\rceil = Bq - \sum_{i=1}^{B-1} \left\lfloor q - \tfrac{qi}{p}\right\rfloor.$$

□

**3.31 Lemma**
*Let* $g, h \in R$ *such that* $L_{(p,q)}(g) = x^A y^B$ *and* $L_{(p,q)}(h) = y^C$*, and let* $J = \langle x^A y^B, y^C, x^\alpha y^\beta \mid \alpha p + \beta q \ge pq\rangle$ *and* $J' = \langle g, h, x^\alpha y^\beta \mid \alpha p + \beta q \ge pq\rangle$*. Then*

$$\dim_{\mathbb{C}} R/J' \le \dim_{\mathbb{C}} R/J,$$

*and if* $Ap + Bq \le pq$ *and* $B \le C \le p$*, then*

$$\dim_{\mathbb{C}} R/J = Ap + Bq - AB - \sum_{i=1}^{A-1} \left\lfloor \tfrac{pi}{q}\right\rfloor - \sum_{i=1}^{B-1} \left\lfloor \tfrac{qi}{p}\right\rfloor - \sum_{i=C}^{p-1} \min\left\{A, \left\lceil q - \tfrac{Cq}{p}\right\rceil\right\}.$$

*Moreover, if* $B = 0$*, then* $\dim_{\mathbb{C}} R/J \le A \cdot C$*.*

**Proof:** By Proposition 3.14

$$\dim_{\mathbb{C}} R/J' \le \dim_{\mathbb{C}} R/L_{<_{(p,q)}}(J') \le \dim_{\mathbb{C}} R/J.$$

Let $I = \langle x^\alpha y^\beta \mid \alpha p + \beta q \geq pq \rangle$. Then the monomials $x^\alpha y^\beta$ with $(\alpha,\beta) \in \Lambda = \{(\alpha,\beta) \in \mathbb{N} \times \mathbb{N} \mid \alpha p + \beta q < pq\}$ form a basis of $R/I$. Moreover, the monomials $x^\alpha y^\beta$ with $(\alpha,\beta) \in \Lambda_1 \cup \Lambda_2$ are a basis of $J/I$, where

$$\Lambda_1 = \{(\alpha,\beta) \in \Lambda \mid \alpha \geq A \text{ and } \beta \geq B\}$$

and

$$\Lambda_2 = \{(\alpha,\beta) \in \Lambda \setminus \Lambda_1 \mid \beta \geq C\}.$$

(See also Figure 11.) This gives rise to the above values for $\dim_{\mathbb{C}} R/J$.

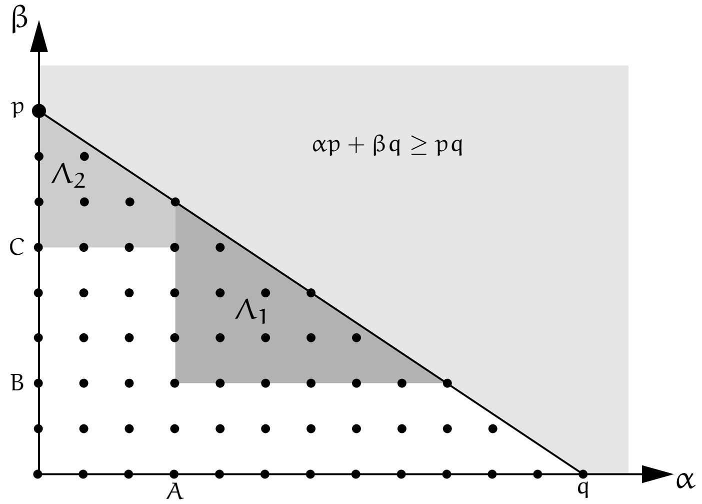


FIGURE 11. A Basis of $R/J$.

□

**3.32 Lemma**

*Let $q > p$ be such that $\frac{q}{p} < \frac{d}{d-1}$ for some integer $d \geq 2$, and let $0 \leq A \leq d$.*

(a) *If $L_{(p,q)}(g) = x^A$, then $L_{<_{ds}}(g) = x^A$.*

(b) $\mathfrak{m}^{p+1} \subseteq \langle x^A, y^{p-1}, x^\alpha y^\beta \mid \alpha p + \beta q \geq pq \rangle$.

(c) *If $I$ is an ideal such that $g, h, x^\alpha y^\beta \in I$ for $\alpha p + \beta q \geq pq$ and where $L_{<_{(p,q)}}(g) = x^A$ and $L_{<_{(p,q)}}(h) = y^{p-1}$, then $d(I) \leq p+1$.*
*Moreover, if $L_{<_{(p,q)}}(g)$ is minimal among the leading monomials of elements in $I$ w. r. t. $<_{(p,q)}$, then $\mathrm{mult}(I) = A$.*

**Proof:** It suffices to consider the case $A = d$, since this implies the other cases. Note that by assumption $d \leq p$.

(a) Since $x^d$ is less than any monomial of degree at least $d$ with respect to $<_{ds}$, we have to show that in $g$ no monomial of degree less than $d$ can occur with a non-zero coefficient. $x^d$ being the leading monomial of $g$ with respect to $<_{(p,q)}$, it suffices to show that $\alpha + \beta < d$ implies $\alpha p + \beta q < dp$, or alternatively, since $\frac{q}{p} < \frac{d}{d-1}$,

$$\alpha + \beta \cdot \frac{d}{d-1} \leq d.$$

For $\alpha + \beta < d$ the left hand side of this inequality will be maximal for $\alpha = 0$ and $\beta = d - 1$, and thus the inequality is satisfied.

(b) We only have to show that $x^\gamma y^{p+1-\gamma} \in \langle x^d, y^{p-1}, x^\alpha y^\beta \mid \alpha p + \beta q \geq pq \rangle$ for $\gamma = 3, \ldots, d-1$, since the remaining generators of $\mathfrak{m}^{p+1}$ definitely are. However, by assumption $\frac{q}{p} < \frac{d}{d-1} \leq \frac{\gamma}{\gamma-1}$, and thus $\gamma \cdot p + (p+1-\gamma) \cdot q \geq pq$.

(c) By the assumption on $I$ we deduce form (a) and (b) that $d\left(L_{<_{ds}}(I)\right) \leq p+1$. However, by Remark 3.17 $d(I) = d\left(L_{<_{ds}}(I)\right)$, which proves the first assertion.
Suppose now that $\text{mult}(I) < d$, i. e. there is an $f \in I$ such that $\text{mult}(f) \leq d-1$. The considerations for (a) show that then $L_{<_{(p,q)}}(f) < x^d$ in contradiction to the assumption.

□

CHAPTER II

# A Vanishing Theorem

In Chapter III we deduce the existence of an irreducible curve with prescribed topological or analytical singularities from the existence of an irreducible curve $C$ with ordinary multiple points $z_1, \ldots, z_r$ of certain multiplicities $m_1, \ldots, m_r$ which has the T-smoothness property, that is such that $h^1\big(\Sigma, \mathcal{J}_{X(\underline{m};\underline{z})/\Sigma}(D)\big) = 0$. If we want to derive numerical conditions for the existence in terms of invariants of the singularities and the divisor $D$ from this theorem, we need to have such numerical conditions for the vanishing of $H^1\big(\Sigma, \mathcal{J}_{X(\underline{m};\underline{z})/\Sigma}(D)\big)$. Section 1 is devoted to the proof of a suitable vanishing theorem. We generalise an approach used by Geng Xu in [**Xu95**] for the plane case. The basic idea is to give conditions such that $\big(\pi^*D - \sum_{i=1}^r m_iE_i\big) - K_{\widetilde{\Sigma}}$ is big and nef, where $\pi : \mathrm{Bl}_{\underline{z}}(\Sigma) = \widetilde{\Sigma} \to \Sigma$ denotes the blow up of $\Sigma$ in $z_1, \ldots, z_r$, and then to apply the Kawamata–Viehweg Vanishing Theorem.

## 1. The Vanishing Theorem

**1.1 Theorem**

*Let* $m_1 \geq \ldots \geq m_r \geq 0$ *be non-negative integers,* $\alpha \in \mathbb{R}$ *with* $\alpha > 1$, $k_\alpha = \max\big\{n \in \mathbb{N} \;\big|\; n < \frac{\alpha}{\alpha-1}\big\}$ *and let* $D \in \mathrm{Div}(\Sigma)$ *be a divisor satisfying the following three conditions*[1]

$$(D - K_\Sigma)^2 \geq \max\left\{\alpha \cdot \sum_{i=1}^r (m_i + 1)^2, (k_\alpha \cdot m_1 + k_\alpha)^2\right\}, \tag{1.1}$$

(1.2) $(D - K_\Sigma).B \geq k_\alpha \cdot (m_1 + 1)$ *for any irreducible curve* $B$ *with* $B^2 = 0$ *and* $\dim |B|_a > 0$*, and*

(1.3) $D - K_\Sigma$ *is nef.*

*Then for* $z_1, \ldots, z_r \in \Sigma$ *in very general position and* $\nu > 0$

$$H^\nu\left(\mathrm{Bl}_{\underline{z}}(\Sigma), \pi^*D - \sum_{i=1}^r m_iE_i\right) = 0.$$

*In particular,*

$$H^\nu\big(\Sigma, \mathcal{J}_{X(\underline{m};\underline{z})/\Sigma}(D)\big) = 0.$$

**Proof:** By the Kawamata–Viehweg Vanishing Theorem (cf. [**Kaw82**] and [**Vie82**]) it suffices to show that $A = \big(\pi^*D - \sum_{i=1}^r m_iE_i\big) - K_{\widetilde{\Sigma}}$ is big and nef, i. e. we have to show:

[1]The proof uses Kawamata–Viehweg vanishing which needs characteristic zero for the ground field.

(a) $A^2 > 0$, and

(b) $A.B' \geq 0$ for any irreducible curve $B'$ in $\widetilde{\Sigma} = \mathrm{Bl}_{\underline{z}}(\Sigma)$.

Note that $A = \pi^*(D - K_\Sigma) - \sum_{i=1}^r (m_i + 1)E_i$, and thus by Hypothesis (1.4) we have

$$A^2 = (D - K_\Sigma)^2 - \sum_{i=1}^r (m_i + 1)^2 > 0,$$

which gives condition (a).

For condition (b) we observe that an irreducible curve $B'$ on $\widetilde{\Sigma}$ is either the strict transform of an irreducible curve $B$ in $\Sigma$ or is one of the exceptional curves $E_i$. In the latter case we have

$$A.B' = A.E_i = m_i + 1 > 0.$$

We may, therefore, assume that $B' = \widetilde{B}$ is the strict transform of an irreducible curve $B$ on $\Sigma$ having multiplicity $\mathrm{mult}_{z_i}(B) = n_i$ at $z_i$, $i = 1, \ldots, r$. Then

$$A.B' = (D - K_\Sigma).B - \sum_{i=1}^r (m_i + 1)n_i,$$

and thus condition (b) is equivalent to

(b') $(D - K_\Sigma).B \geq \sum_{i=1}^r (m_i + 1)n_i$.

Since $\underline{z}$ is in very general position Lemma 2.1 applies in view of Corollary A.3. Using the Hodge Index Theorem E.4, Hypothesis (1.4), Lemma 2.1, and the Cauchy-Schwarz Inequality we get the following sequence of inequalities:

$$\begin{aligned}
\big((D - K_\Sigma).B\big)^2 &\geq (D - K_\Sigma)^2 \cdot B^2 \geq \alpha \cdot \left(\textstyle\sum_{i=1}^r (m_i + 1)^2\right) \cdot \left(\textstyle\sum_{i=1}^r n_i^2 - n_{i_0}\right) \\
&= \textstyle\sum_{i=1}^r (m_i + 1)^2 \cdot \sum_{i=1}^r n_i^2 + (\alpha - 1) \cdot \Big(\sum_{i=1}^r (m_i + 1)^2 \cdot \big(\sum_{i=1}^r n_i^2 - \frac{\alpha}{\alpha - 1} \cdot n_{i_0}\big)\Big) \\
&\geq \left(\textstyle\sum_{i=1}^r (m_i + 1) \cdot n_i\right)^2 + (\alpha - 1) \cdot \Big(\sum_{i=1}^r (m_i + 1)^2 \cdot \big(\sum_{i=1}^r n_i^2 - \frac{\alpha}{\alpha - 1} \cdot n_{i_0}\big)\Big),
\end{aligned}$$

where $i_0 \in \{1, \ldots, r\}$ is such that $n_{i_0} = \min\{n_i \mid n_i \neq 0\}$. Since $D - K_\Sigma$ is nef, condition (b') is satisfied as soon as we have

$$\sum_{i=1}^r n_i^2 \geq \tfrac{\alpha}{\alpha - 1} \cdot n_{i_0}.$$

If this is not fulfilled, then $n_i < \frac{\alpha}{\alpha - 1}$ for all $i = 1, \ldots, r$, and thus

$$\sum_{i=1}^r (m_i + 1) \cdot n_i \leq k_\alpha \cdot (m_1 + 1).$$

Hence, for the remaining considerations (b') may be replaced by the worst case

$$(D - K_\Sigma).B \geq k_\alpha \cdot (m_1 + 1).$$

Note that since the $z_i$ are in very general position and $z_{i_0} \in B$ we have that $B^2 \geq 0$ and $\dim|B|_a > 0$ (cf. Corollary A.6). If $B^2 > 0$ then we are done by the Hodge Index Theorem E.4 and Hypothesis (1.4), since $D - K_\Sigma$ is nef:

$$(D - K_\Sigma).B \geq \sqrt{(D - K_\Sigma)^2} \geq \sqrt{(k_\alpha \cdot m_1 + k_\alpha)^2} \geq k_\alpha \cdot (m_1 + 1).$$

It remains to consider the case $B^2 = 0$ which is covered by Hypothesis (1.5).

For the "in particular" part we just note that $\mathcal{J}_{X(m_1;z_1)/\Sigma} \otimes \cdots \otimes \mathcal{J}_{X(m_r;z_r)/\Sigma} \otimes \mathcal{O}_\Sigma(D) \cong \mathcal{J}_{X(\underline{m};\underline{z})/\Sigma} \otimes \mathcal{O}_\Sigma(D)$ (see Lemma C.4) and that, using the Leray spectral sequence (compare **[Laz97]** Lemma 5.1)

$$H^\nu\left(\Sigma, \bigotimes_{i=1}^{r} \mathcal{J}_{X(m_i;z_i)/\Sigma} \otimes \mathcal{O}_\Sigma(D)\right) = H^\nu\left(\widetilde{\Sigma}, \pi^* D - \sum_{i=1}^{r} m_i E_i\right).$$

□

Choosing the constant $\alpha = 2$ in Theorem 1.1, then $\frac{\alpha}{\alpha-1} = 2$ and thus $k_\alpha = 1$. We therefore get the following corollary, which has the advantage that the conditions look simpler, and that the hypotheses on the "exceptional" curves are not too hard.

**1.2 Corollary**

*Let* $m_1, \ldots, m_r \in \mathbb{N}_0$*, and* $D \in \mathrm{Div}(\Sigma)$ *be a divisor satisfying the following three conditions*

(1.4) $\quad (D - K_\Sigma)^2 \geq 2 \cdot \sum_{i=1}^{r} (m_i + 1)^2,$

(1.5) $\quad (D - K_\Sigma).B > \max\{m_i \mid i = 1, \ldots, r\}$ *for any irreducible curve* $B$ *with* $B^2 = 0$ *and* $\dim|B|_a > 0$*, and*

(1.6) $\quad D - K_\Sigma$ *is nef.*

*Then for* $z_1, \ldots, z_r \in \Sigma$ *in very general position and* $\nu > 0$

$$H^\nu\left(\mathrm{Bl}_{\underline{z}}(\Sigma), \pi^* D - \sum_{i=1}^{r} m_i E_i\right) = 0.$$

*In particular,*

$$H^\nu\big(\Sigma, \mathcal{J}_{X(\underline{m};\underline{z})/\Sigma}(D)\big) = 0.$$

**1.3 Remark**

Condition (1.3) respectively Condition (1.6) are in several respects "expectable". First, Theorem 1.1 is a corollary of the Kawamata–Viehweg Vanishing Theorem, and if we take all $m_i$ to be zero, our assumptions should basically be the same, i. e. $D - K_\Sigma$ nef and big. The latter is more or less just (1.1) respectively (1.4). Secondly, we want to apply the theorem to an existence problem. A divisor being nef means it is somehow close to being effective, or better its linear system is close to being non-empty. If we want that some linear system $|D|_l$ contains a curve with certain properties, then it seems not to be so unreasonable to restrict to systems where already $|D - K_\Sigma|_l$, or even $|D - L - K_\Sigma|_l$ with $L$ some fixed divisor, is of positive dimension, thus nef.

In many interesting examples, such as $\mathbb{P}^2_{\mathbb{C}}$, Condition (1.2) respectively (1.5) turn out to be obsolete or easy to handle. So finally the most restrictive obstruction seems to be (1.1) respectively (1.4).

If we consider the situation where the largest multiplicity $m_1$ occurs in a large number, more precisely, if $m_1 = \ldots = m_{l_\alpha}$ with $l_\alpha = \min\{n \in \mathbb{N} \mid \alpha \cdot n \geq k_\alpha^2\}$, then Condition (1.1) comes down to

$$(1.1') \qquad (D - K_\Sigma)^2 \geq \alpha \cdot \sum_{i=1}^{r} (m_i + 1)^2.$$

**1.4 Remark**

Even though we said that condition (1.4) was the really restrictive condition we would like to understand better what condition (1.5) means. We therefore show in Appendix B that an algebraic system $|B|_a$ of dimension greater than zero with $B$ irreducible and $B^2 = 0$ gives rise to a fibration $f : \Sigma \to H$ of $\Sigma$ over a smooth projective curve $H$ whose fibres are just the elements of $|B|_a$.

## 2. Generalisation of a Lemma of Geng Xu

Throughout the proof of Theorem 1.1 we need the following generalisation of a lemma of Geng Xu.

**2.1 Lemma**

*Let $\underline{z} = (z_1, \ldots, z_r) \in \Sigma^r$ be in very general position, $\underline{n} \in \mathbb{N}_0^r$, and let $B \subset \Sigma$ be an irreducible curve with $\mathrm{mult}_{z_i}(B) \geq n_i$, then*

$$B^2 \geq \sum_{i=1}^{r} n_i^2 - \min\{n_i \mid n_i \neq 0\}.$$

**2.2 Remark**

(a) A proof for the above lemma in the case $\Sigma = \mathbb{P}^2_{\mathbb{C}}$ can be found in **[Xu94]** and in the case $r = 1$ in **[EiL93]**. Here we just extend the arguments given there to the slightly more general situation.

(b) For better estimates of the self intersection number of curves in the situation where one has some knowledge on equisingular deformations inside the algebraic system see **[GuS84]**.

(c) With the notation of Lemma A.2 respectively Corollary A.3 the assumption in Lemma 2.1 could be formulated more precisely as "let $B \subset \Sigma \subseteq \mathbb{P}^N_{\mathbb{C}}$ be an irreducible curve such that $V_{B,\underline{n}} = \Sigma^r$", or "let $\underline{z} \in \Sigma^r \setminus V$".[2]

(d) Note, that one cannot expect to get rid of the "$-\min\{n_i \mid n_i \neq 0\}$". E. g. $\Sigma = \mathrm{Bl}_z\left(\mathbb{P}^2_{\mathbb{C}}\right)$, the projective plane blown up in a point $z$, and $B \subset \Sigma$

[2] Since $B$ is irreducible, the general element in $|B|_a$ will be irreducible. Since $V_{B,\underline{n}} = \Sigma^r$ there will be some family of curves in $\mathrm{Hilb}^h_\Sigma$ satisfying the requirements of Lemma 2.3.

the strict transform of a line through $z$. Let now $r = 1$, $n_1 = 1$ and $z_1 \in \Sigma$ be any point. Then there is of course a (unique) curve $B_1 \in |B|_a$ through $z_1$, but $B^2 = 0 < 1 = n_1^2$.

**Idea of the proof:** Set $e_1 := n_1 - 1$ and $e_i := n_i$ for $i \neq 1$, where w. l. o. g. $n_1 = \min\{n_i \mid n_i \neq 0\}$. By assumption there is a family $\{C_t\}_{t\in\mathbb{C}}$ in $|B|_a$ satisfying the requirements of Lemma 2.3. Setting $C := C_0$ the proof is done in three steps:

**Step 1:** We show that $H^0\big(C, \mathcal{J}_{X(\underline{e};\underline{z})/\Sigma} \cdot \mathcal{O}_C(C)\big) \neq 0$. (Lemma 2.3)

**Step 2:** We deduce that $H^0\big(C, \pi_*\mathcal{O}_{\widetilde{C}}\big(-\sum_{i=1}^r e_iE_i\big) \otimes \mathcal{O}_C(C)\big) \neq 0$. (Lemma 2.4)

**Step 3:** It follows that deg $\big(\pi_*\mathcal{O}_{\widetilde{C}}\big(-\sum_{i=1}^r e_iE_i\big) \otimes \mathcal{O}_C(C)\big) \geq 0$, but this degree is just $C^2 - \sum_{i=1}^r e_in_i$.

□

**2.3 Lemma**

*Given* $e_1, \dots, e_r \in \mathbb{N}_0$, $r \geq 1$. *Let* $\{C_t\}_{t\in U}$, $U \subseteq \mathbb{C}$ *an open neighbourhood of* $0$, *be a non-trivial family of curves in* $\Sigma$ *together with a section* $U \to \Sigma : t \mapsto z_{1,t} \in C_t$ *such that*

$$\text{mult}_{z_{1,t}}(C_t) \geq e_1 + 1 \quad \textit{for all} \quad t \in U$$

*and fixed points* $z_2, \dots, z_r \in \Sigma$ *such that*

$$\text{mult}_{z_i}(C_t) \geq e_i \quad \textit{for all} \quad i = 2, \dots, r \;\; \textit{and} \;\; t \in U.$$

*Then with* $z_1 = z_{1,0}$

$$H^0\big(C, \mathcal{J}_{X(\underline{e};\underline{z})/\Sigma} \cdot \mathcal{O}_C(C)\big) \neq 0,$$

*i. e. there is a non-trivial section of the normal bundle of* $C$, *vanishing at* $z_i$ *to the order of at least* $e_i$ *for* $i = 1, \dots, r$.[3]

**Proof:** We stick to the convention $n_1 = e_1 + 1$ and $n_i = e_i$ for $i = 2, \dots, r$, and we set $z_{i,t} := z_i$ for $i = 2, \dots, r$ and $t \in U$. Let $\Delta \subset U$ be a small disc around $0$ with coordinate $t$, and choose coordinates $(x_i, y_i)$ on $\Sigma$ around $z_i$ such that

- $z_{i,t} = (a_i(t), b_i(t))$ for $t \in \Delta$ with $a_i, b_i \in \mathbb{C}\{t\}$,
- $z_i = (a_i(0), b_i(0)) = (0, 0)$, and
- $F_i(x_i, y_i, t) = f_{i,t}(x_i, y_i) \in \mathbb{C}\{x_i, y_i, t\}$, where $C_t = \{f_{i,t} = 0\}$ locally at $z_{i,t}$ (for $t \in \Delta$).

We view $\{C_t\}_{t\in\Delta}$ as a non-trivial deformation of $C$, which implies that the image of $\frac{\partial}{\partial t}_{|t=0} \in T_0(\Delta)$ under the Kodaira-Spencer map is a non-zero section $s$ of $H^0\big(C, \mathcal{O}_C(C)\big)$. $s$ is locally at $z_i$ given by $\frac{\partial F_i}{\partial t}_{|t=0}$.

**Idea:** Show that $\frac{\partial F_i}{\partial t}_{|t=0} \in (x_i, y_i)^{e_i}$, which are the stalks of $\mathcal{J}_{X(\underline{e};\underline{z})/\Sigma} \cdot \mathcal{O}_C(C)$ at the $z_i$, and hence $s$ is actually a global section of the subsheaf $\mathcal{J}_{X(\underline{e};\underline{z})/\Sigma} \cdot \mathcal{O}_C(C)$.

Set $\Phi_{i,t}(x_i, y_i) := F_{i,t}(x_i + a_i(t), y_i + b_i(t), t) = \sum_{k=0}^{\infty} \varphi_{i,k}(x_i, y_i) \cdot t^k \in \mathbb{C}\{x_i, y_i, t\}$. By assumption for any $t \in \Delta$ the multiplicity of $\Phi_{i,t}$ at $(0,0)$ is at least $n_i$,

[3]Note, that $\mathcal{J}_{X(\underline{e};\underline{z})/\Sigma} \cdot \mathcal{O}_C(C) = \mathfrak{m}_{\Sigma,z_1}^{e_1} \cdots \mathfrak{m}_{\Sigma,z_r}^{e_r} \cdot \mathcal{O}_C(C)$.

i. e. $\Phi_{i,t}(x_i, y_i) \in (x_i, y_i)^{n_i}$ for every fixed complex number $t \in \Delta$. Hence, $\varphi_{i,k}(x_i, y_i) \in (x_i, y_i)^{n_i}$ for every k.[4]
On the other hand we have

$$\begin{aligned}\varphi_{i,1}(x_i, y_i) &= \frac{\partial\Phi_{i,t}(x_i,y_i)}{\partial t}_{|t=0} \\ &= \Big\langle \big(\tfrac{\partial F_i}{\partial x_i}(x_i, y_i, 0), \tfrac{\partial F_i}{\partial y_i}(x_i, y_i, 0), \tfrac{\partial F_i}{\partial t}(x_i, y_i, 0)\big), \big(\dot{a}_i(0), \dot{b}_i(0), 1\big) \Big\rangle \\ &= \tfrac{\partial f_{i,0}}{\partial x_i}(x_i, y_i) \cdot \dot{a}_i(0) + \tfrac{\partial f_{i,0}}{\partial y_i}(x_i, y_i) \cdot \dot{b}_i(0) + \tfrac{\partial F_i}{\partial t}(x_i, y_i, 0).\end{aligned}$$

Since $f_{i,0} \in (x_i, y_i)^{n_i}$, we have $\frac{\partial f_{i,0}}{\partial x_i}(x_i, y_i), \frac{\partial f_{i,0}}{\partial y_i}(x_i, y_i) \in (x_i, y_i)^{n_i-1}$, and hence $\frac{\partial F_i}{\partial t}(x_i, y_i, 0) \in (x_i, y_i)^{e_i}$. For this note that $\dot{a}_i(0) = \dot{b}_i(0) = 0$, if $i \neq 1$. □

The main idea of the next lemma is the following commutative diagram:

$$\begin{array}{ccccccc} 0 \longrightarrow \mathrm{Ker}(\beta) \longrightarrow & H^0\big(\Sigma, \pi_*\mathcal{O}_{\widetilde{\Sigma}}\big(-\sum_{i=1}^r e_iE_i\big) \otimes_{\mathcal{O}_\Sigma} \mathcal{O}_C(C)\big) & \xrightarrow{\beta} & H^0\big(C, \mathcal{J}_{X(\underline{e};\underline{z})/\Sigma} \cdot \mathcal{O}_C(C)\big) \longrightarrow 0 \\ & \Big\downarrow \alpha & \nearrow \overline{\beta} & \\ & H^0\big(\Sigma, \pi_*\mathcal{O}_{\widetilde{C}}\big(-\sum_{i=1}^r e_iE_i\big) \otimes_{\mathcal{O}_C} \mathcal{O}_C(C)\big) & & \end{array}$$

i. e. the fact that $\mathrm{Ker}(\alpha) \subseteq \mathrm{Ker}(\beta)$, or in other words, that $\beta$ factorises over $\alpha$.

**2.4 Lemma**

*Given* $e_1, \ldots, e_r \in \mathbb{N}_0$ *and* $z_1, \ldots, z_r \in \Sigma$, $r \geq 1$.

*The canonical morphism*[5] $\mathcal{J}_{X(e_1;z_1)/\Sigma} \otimes \cdots \otimes \mathcal{J}_{X(e_r;z_r)/\Sigma} \otimes \mathcal{O}_C(C) \longrightarrow \mathcal{J}_{X(\underline{e};\underline{z})/\Sigma} \cdot \mathcal{O}_C(C)$ *induces a surjective morphism* $\beta$ *on the level of global sections.*[6]
*If* $s \in H^0\big(C, \mathcal{J}_{X(e_1;z_1)/\Sigma} \otimes_{\mathcal{O}_\Sigma} \cdots \otimes_{\mathcal{O}_\Sigma} \mathcal{J}_{X(e_r;z_r)/\Sigma} \otimes_{\mathcal{O}_\Sigma} \mathcal{O}_C(C)\big)$*, but not in* $\mathrm{Ker}(\beta)$*, then* $s$ *induces a non-zero section* $\tilde{s}$ *in* $H^0\big(C, \pi_*\mathcal{O}_{\widetilde{C}}\big(-\sum_{i=1}^r e_iE_i\big) \otimes_{\mathcal{O}_C} \mathcal{O}_C(C)\big)$.

**Proof:** Set $E := -\sum_{i=1}^r e_iE_i$.
We start with the structure sequence for $\widetilde{C}$:

$$0 \longrightarrow \mathcal{O}_{\widetilde{\Sigma}}\big(-\widetilde{C}\big) \longrightarrow \mathcal{O}_{\widetilde{\Sigma}} \longrightarrow \mathcal{O}_{\widetilde{C}} \longrightarrow 0.$$

Tensoring with the locally free sheaf $\mathcal{O}_{\widetilde{\Sigma}}(E)$ and then applying $\pi_*$ we get a morphism:

$$\pi_*\mathcal{O}_{\widetilde{\Sigma}}(E) \longrightarrow \pi_*\mathcal{O}_{\widetilde{C}}(E).$$

Now tensoring by $\mathcal{O}_C(C)$ over $\mathcal{O}_\Sigma$ we have an exact sequence:

$$0 \longrightarrow \mathrm{Ker}(\gamma) \longrightarrow \pi_*\mathcal{O}_{\widetilde{\Sigma}}(E) \otimes_{\mathcal{O}_\Sigma} \mathcal{O}_C(C) \xrightarrow{\gamma} \pi_*\mathcal{O}_{\widetilde{C}}(E) \otimes_{\mathcal{O}_\Sigma} \mathcal{O}_C(C).$$

And finally taking global sections, we end up with:

$$0 \longrightarrow H^0\big(\Sigma, \mathrm{Ker}(\gamma)\big) \longrightarrow H^0\big(\Sigma, \pi_*\mathcal{O}_{\widetilde{\Sigma}}(E) \otimes \mathcal{O}_C(C)\big) \xrightarrow{\alpha} H^0\big(\Sigma, \pi_*\mathcal{O}_{\widetilde{C}}(E) \otimes \mathcal{O}_C(C)\big).$$

Since the sheaves we look at are actually $\mathcal{O}_C$-sheaves and since $C$ is a closed subscheme of $\Sigma$, the global sections of the sheaves as sheaves on $\Sigma$ and

[4]See Lemma C.1.

[5]I. e. $\mathfrak{m}_{\Sigma,z_1}^{e_1} \otimes \cdots \otimes \mathfrak{m}_{\Sigma,z_r}^{e_r} \otimes \mathcal{O}_C(C) \longrightarrow \mathfrak{m}_{\Sigma,z_1}^{e_1} \cdots \mathfrak{m}_{\Sigma,z_r}^{e_r} \cdot \mathcal{O}_C(C)$.

[6]$\beta$ is surjective, since $\mathrm{supp}\big(\mathrm{Ker}(\beta)\big) \subseteq \{z_1, \ldots, z_r\}$ by Lemma C.4, and hence $H^1(\Sigma, \mathrm{Ker}(\beta)) = 0$.

as sheaves on $C$ coincide (cf. **[Har77]** III.2.10 - for more details, see Corollary C.3). Furthermore, $\pi_*\mathcal{O}_{\widetilde{\Sigma}}(E) = \bigotimes_{i=1}^r \mathcal{J}_{X(e_i;z_i)/\Sigma}$.

Thus it suffices to show that Ker $(\alpha) \subseteq$ Ker $(\beta)$.

Since $\pi_| : \widetilde{\Sigma} \setminus (\bigcup_{i=1}^r E_i) \to \Sigma \setminus \{z_1, \dots, z_r\}$ is an isomorphism, we have that supp $\big(\text{Ker}(\gamma)\big) \subseteq \{z_1, \dots, z_r\}$ is finite by Lemma C.5. Hence, by Lemma C.6 Ker$(\gamma)$ is a torsion sheaf, and thus

$$\text{Ker}(\alpha) \;=\; H^0\big(\Sigma, \text{Ker}(\gamma)\big) \subseteq H^0\Big(\Sigma, \text{Tor}\,\Big(\bigotimes_{i=1}^r \mathcal{J}_{X(e_i;z_i)/\Sigma} \otimes \mathcal{O}_C(C)\Big)\Big).$$

Let now $t \in \text{Ker}(\alpha)$ be given. We have to show that $\beta(t) = 0$, i. e. $\beta_z(t_z) = 0$ for every $z \in \Sigma$. If $z \notin \{z_1, \dots, z_r\}$, then $t_z = 0$. Thus we may assume $z = z_k$. As we have shown,

$$t_{z_k} \in \text{Tor}\,\big(\mathfrak{m}_{\Sigma,z_k}^{e_k} \otimes_{\mathcal{O}_{\Sigma,z_k}} \mathcal{O}_{C,z_k}\big) = \text{Tor}\,\big(\mathfrak{m}_{\Sigma,z_k}^{e_k}/f_{z_k}\mathfrak{m}_{\Sigma,z_k}^{e_k}\big) = (f_{z_k})/f_{z_k}\mathfrak{m}_{\Sigma,z_k}^{e_k},$$

where $f_{z_k}$ is a local equation of $C$ at $z_k$. Therefore, there exists a $0 \neq g_{z_k} \in \mathcal{O}_{\Sigma,z_k}$ such that $t_{z_k} = f_{z_k}g_{z_k} \pmod{f_{z_k}\mathfrak{m}_{\Sigma,z_k}^{e_k}} \equiv f_{z_k} \otimes g_{z_k}$ (note that $f_{z_k} \in \mathfrak{m}_{\Sigma,z_k}^{n_k} \subseteq \mathfrak{m}_{\Sigma,z_k}^{e_k}$!). But then $\beta_{z_k}(t_{z_k})$ is just the residue class of $f_{z_k}g_{z_k}$ in $\mathfrak{m}_{\Sigma,z_k}^{e_k}\mathcal{O}_{C,z_k} = \mathfrak{m}_{\Sigma,z_k}^{e_k}/(f_{z_k})$, and is thus zero. □

**Proof of Lemma 2.1:** Using the notation of the idea of the proof given on page 45, we have, by Lemma 2.3, a non-zero section $s \in H^0\big(C, \mathcal{J}_{X(\underline{e};\underline{z})/\Sigma} \cdot \mathcal{O}_C(C)\big)$. This lifts under the surjection $\beta$ to a section $s' \in H^0\big(C, \bigotimes_{i=1}^r \mathcal{J}_{X(e_i;z_i)/\Sigma} \otimes \mathcal{O}_C(C)\big)$ which is not in the kernel of $\beta$. Again setting $E := -\sum_{i=1}^r e_iE_i$, by Lemma 2.4, we have a non-zero section $\tilde{s} \in H^0\big(C, \pi_*\mathcal{O}_{\widetilde{C}}(E) \otimes_{\mathcal{O}_C} \mathcal{O}_C(C)\big)$, where by the projection formula the latter is just $H^0\big(C, \pi_*\big(\mathcal{O}_{\widetilde{C}}(E) \otimes_{\mathcal{O}_{\widetilde{C}}} \pi^*\mathcal{O}_C(C)\big)\big) =_{\text{def}} H^0\big(\widetilde{C}, \mathcal{O}_{\widetilde{C}}(E) \otimes_{\mathcal{O}_{\widetilde{C}}} \pi^*\mathcal{O}_C(C)\big)$.
Since $\mathcal{O}_{\widetilde{C}}(E) \otimes_{\mathcal{O}_{\widetilde{C}}} \pi^*\mathcal{O}_C(C)$ has a global section and since $\widetilde{C}$ is irreducible and reduced, we get by Lemma D.2:

$$0 \leq \deg\big(\mathcal{O}_{\widetilde{C}}(E) \otimes_{\mathcal{O}_{\widetilde{C}}} \pi^*\mathcal{O}_C(C)\big) = \deg\big(\mathcal{O}_{\widetilde{C}}(E)\big) + \deg\big(\pi^*\mathcal{O}_C(C)\big)$$

$$= E.\widetilde{C} + \deg\big(\mathcal{O}_C(C)\big) = \sum_{i=1}^r -e_in_i + C^2.$$

□

## 3. Examples

In this section we are going to examine the conditions in the vanishing theorem (Corollary 1.2). Unless otherwise stated, $r \geq 1$ is a positive integer, and $m_1, \dots, m_r \in \mathbb{N}_0$ are non-negative.

### 3.a. The Classical Case - $\Sigma = \mathbb{P}^2_{\mathbb{C}}$

Since in $\mathbb{P}^2_{\mathbb{C}}$ there are no irreducible curves of self-intersection number zero, condition (1.5) is redundant. Moreover, condition (1.6) takes in view of (1.4) the form $d + 3 \geq \sqrt{2}$. Corollary 1.2 thus takes the following form, where $L \in |\mathcal{O}_{\mathbb{P}^2_{\mathbb{C}}}(1)|_l$ is a generic line.

**1.2a Corollary**

*Let* $d$ *be any integer such that*

(1.4a) $\quad (d+3)^2 \geq 2\sum_{i=1}^{r}(m_i+1)^2,$

(1.6a) $\quad d \geq -1.$

*Then for* $z_1, \dots, z_r \in \mathbb{P}^2_{\mathbb{C}}$ *in very general position and* $\nu > 0$

$$H^{\nu}\left(\mathrm{Bl}_{\underline{z}}\left(\mathbb{P}^2_{\mathbb{C}}\right), d\,\pi^*L - \sum_{i=1}^{r} m_iE_i\right) = 0.$$

### 3.b. Geometrically Ruled Surfaces

Throughout this section we use the notation and the results from Section G.a, in particular Lemma G.2 for the irreducible curves with selfintersection zero on $\Sigma$.

**1.2b Corollary**

*Given two integers* $a, b \in \mathbb{Z}$ *satisfying*

(1.4b) $\quad a\left(b - \left(\frac{a}{2} - 1\right)e\right) \geq \sum_{i=1}^{r}(m_i+1)^2,$

(1.5b.i) $\quad a > \max\{m_i \mid i = 1, \dots, r\},$

(1.5b.ii) $\quad b > \max\{m_i \mid i = 1, \dots, r\},$ *if* $e = 0$,

(1.5b.iii) $\quad 2\left(b - \left(\frac{a}{2} - 1\right)e\right) > \max\{m_i \mid i = 1, \dots, r\},$ *if* $e < 0$, *and*

(1.6b) $\quad b \geq (a-1)e,$ *if* $e > 0$.

*For* $z_1, \dots, z_r \in \Sigma$ *in very general position and* $\nu > 0$

$$H^{\nu}\left(\mathrm{Bl}_{\underline{z}}(\Sigma), (a-2)\cdot\pi^*C_0 + (b-2+2g)\cdot\pi^*F - \sum_{i=1}^{r} m_iE_i\right) = 0.$$

**Proof:** Note that if the invariant $e$ is non-positive, then $\left(b - \left(\frac{a}{2} - 1\right)e\right) > 0$ implies

$$b \geq (a-1)e, \tag{3.1}$$

so that this inequality is fulfilled for any choice of $e$.

Setting $D = (a-2)C_0 + (b-2+2g)F$ we have

$$(D-K_\Sigma)^2 = \big(aC_0 + (b+e)F\big)^2 = 2a\left(b - \left(\frac{a}{2}-1\right)e\right) \geq 2\sum_{i=1}^{r}(m_i+1)^2,$$

which is just (1.4b.i/ii/iii). Similarly, by (1.5b) and Lemma G.2 condition (1.5) is satisfied.[7] Finally, in view of Lemma G.1, condition (1.6b) implies that $D - K_\Sigma$ is nef. □

## 3.c. Products of Curves

As we have seen in Proposition G.12, for a generic choice of smooth projective curves of genera $g_1 \geq 1$ and $g_2 \geq 1$ respectively the surface $\Sigma = C_1 \times C_2$ has Picard number two and according to Remark G.11 the only irreducible curves $B \subset \Sigma$ with selfintersection $B^2 = 0$ are the fibres $C_1$ and $C_2$, and for any irreducible curve $B \sim_a aC_1 + bC_2$ the coefficients $a$ and $b$ must be non-negative. Taking into account that $K_\Sigma \sim_a (2g_2-2)C_1 + (2g_1-2)C_2$ Corollary 1.2 comes down to the following.

**1.2c Corollary**

*Let $C_1$ and $C_2$ be two generic curves with $g(C_i) = g_i \geq 1$, $i = 1,2$, and let $a, b \in \mathbb{Z}$ be integers satisfying*

(1.4c) $\quad (a - 2g_2 + 2)\cdot(b - 2g_1 + 2) \geq \sum_{i=1}^{r}(m_i+1)^2$, *and*

(1.5c) $\quad (a - 2g_2 + 2), (b - 2g_1 + 2) > \max\{m_i \mid i = 1,\ldots,r\}$,

*then for $z_1,\ldots,z_r \in \Sigma = C_1 \times C_2$ in very general position and $\nu > 0$*

$$H^\nu\left(\mathrm{Bl}_{\underline{z}}(\Sigma),\ a\,\pi^*C_1 + b\,\pi^*C_2 - \sum_{i=1}^{r} m_iE_i\right) = 0.$$

## 3.d. Products of Elliptic Curves

That $C_1$ and $C_2$ be "generic" in the above sense means for elliptic curves just that they are non-isogenous.

In view of (1.5d) and Lemma G.19 (iv) the condition (1.6) becomes obsolete, and Corollary 1.2 has the following form, taking Lemma G.19 (iii) and $K_\Sigma = 0$ into account.

**1.2d Corollary**

*Let $C_1$ and $C_2$ be two non-isogenous elliptic curves, $a, b \in \mathbb{Z}$ be such that*

(1.4d) $\quad ab \geq \sum_{i=1}^{r}(m_i+1)^2$, *and*

[7] To see this, let $B \sim_a a'C_0 + b'F$ be an irreducible curve with $B^2 = 0$. Then by Lemma G.2 either $a' = 0$ and $b' = 1$, or $e = 0$, $a' \geq 1$ and $b' = 0$, or $e < 0$, $a' \geq 2$, and $b' = \frac{a'}{2}e < 0$. In the first case, $(D-K_\Sigma).B = a > \max\{m_i \mid i = 1,\ldots,r\}$ by (1.5b.i). In the second case, $(D-K_\Sigma).B = ba' \geq b > \max\{m_i \mid i = 1,\ldots,r\}$ by (1.5b.ii). And finally, in the third case, we have $(D-K_\Sigma).B = a'\cdot\big(b - \left(\frac{a}{2}-1\right)e\big) > \max\{m_i \mid i = 1,\ldots,r\}$ by (1.5b.iii).

(1.5d) $a, b > \max\{m_i \mid i = 1, \dots, r\}$,

*then for* $z_1, \dots, z_r \in \Sigma = C_1 \times C_2$ *in very general position and* $\nu > 0$

$$H^\nu\left(\mathrm{Bl}_{\underline{z}}(\Sigma),\ a\,\pi^*C_1 + b\,\pi^*C_2 - \sum_{i=1}^{r} m_i E_i\right) = 0.$$

## 3.e. Surfaces in $\mathbb{P}^3_{\mathbb{C}}$

Since we consider the case of rational surfaces separately the following considerations thus give a full answer for the "general case" of a surface in $\mathbb{P}^3_{\mathbb{C}}$.

**1.2e Corollary**

*Let* $\Sigma \subset \mathbb{P}^3_{\mathbb{C}}$ *be a surface in* $\mathbb{P}^3_{\mathbb{C}}$ *of degree* $n$, $H \in \mathrm{NS}(\Sigma)$ *be the algebraic class of a hyperplane section, and* $d$ *an integer satisfying*

(1.4e) $n \cdot (d - n + 4)^2 \geq 2 \sum_{i=1}^{r} (m_i + 1)^2$, *and*

(1.5e) $(d - n + 4) \cdot H.B > \max\{m_i \mid i = 1, \dots, r\}$ *for any irreducible curve* $B$ *with* $B^2 = 0$ *and* $\dim |B|_a \geq 1$, *and*

(1.6e) $d \geq n - 4$,

*then for* $z_1, \dots, z_r \in \Sigma$ *in very general position and* $\nu > 0$

$$H^\nu\left(\mathrm{Bl}_{\underline{z}}(\Sigma), d\,\pi^*H - \sum_{i=1}^{r} m_i E_i\right) = 0.$$

**3.1 Remark**

(a) If $\mathrm{NS}(\Sigma) = H\mathbb{Z}$, then (1.5e) is redundant, since there are no irreducible curves $B$ with $B^2 = 0$. Otherwise we would have $B \sim_a kH$ for some $k \in \mathbb{Z}$ and $k^2 n = B^2 = 0$ would imply $k = 0$, but then $H.B = 0$ in contradiction to $H$ being ample (see Lemma E.1).

(b) By a Theorem of Noether a generic surface in $\mathbb{P}^3_{\mathbb{C}}$ has Picard number one.However, a quadric in $\mathbb{P}^3_{\mathbb{C}}$ or the K3-surface given by $w^4+x^4+y^4+z^4 = 0$ contain irreducible curves of self-intersection zero.

(c) If $\sum_{i=1}^{r}(m_i+1)^2 > \frac{n}{2} m_i^2$ for all $i = 1, \dots, r$ then again (1.5e) becomes obsolete in view of (1.4e), since $H.B > 0$ anyway. The above inequality is, for instance, fulfilled if the highest multiplicity occurs at least $\frac{n}{2}$ times.

(d) In the existence theorems the condition depending on curves of self-intersection will vanish in any case, see Section III.3.e.

## 3.f. K3-Surfaces

3.f.i. *Generic K3-Surfaces.* Since a generic K3-surface does not possess an elliptic fibration the following version of Corollary 1.2 applies for generic K3-surfaces. (cf. **[FrM94]** I.1.3.7)

**1.2f.i Corollary**
*Let $\Sigma$ be a K3-surface which is not elliptic, and let $D$ a divisor on $\Sigma$ satisfying*

(1.4f) $\qquad D^2 \geq 2 \cdot \sum_{i=1}^{r} (m_i + 1)^2$, *and*

(1.6f) $\qquad D$ *nef,*

*then for $z_1, \ldots, z_r \in \Sigma$ in very general position and $\nu > 0$*

$$H^\nu \left( \mathrm{Bl}_{\underline{z}}(\Sigma), \pi^* D - \sum_{i=1}^{r} m_i E_i \right) = 0.$$

3.f.ii. *K3-Surfaces with an Elliptic Structure.* The hypersurface in $\mathbb{P}^3_{\mathbb{C}}$ given by the equation $x^4 + y^4 + z^4 + u^4 = 0$ is an example of a K3-surface which is endowed with an elliptic fibration. Among the elliptic K3-surfaces the general one will possess a unique elliptic fibration while there are examples with infinitely many different such fibrations. (cf. [**FrM94**] I.1.3.7)

**1.2f.ii Corollary**
*Let $\Sigma$ be a K3-surface which possesses an elliptic fibration, and let $D$ be a divisor on $\Sigma$ satisfying*

(1.4f) $\qquad D^2 \geq 2 \cdot \sum_{i=1}^{r} (m_i + 1)^2$,

(1.5f) $\qquad D.B > \max\{m_i \mid i = 1, \ldots, r\}$ *for any irreducible curve $B$ with $B^2 = 0$, and*

(1.6f) $\qquad D$ *nef,*

*then for $z_1, \ldots, z_r \in \Sigma$ in very general position and $\nu > 0$*

$$H^\nu \left( \mathrm{Bl}_{\underline{z}}(\Sigma), \pi^* D - \sum_{i=1}^{r} m_i E_i \right) = 0.$$

**3.2 Remark**
If $\Sigma$ is generic among the elliptic K3-surfaces, i. e. admits exactly one elliptic fibration, then condition (1.5f) means that a curve in $|D|_l$ meets a general fibre in at least $k = \max\{m_i \mid i = 1, \ldots, r\}$ distinct points.

## 4. Yet Another Vanishing Theorem

In [**GLS98c**] a different approach is used for the existence of curves with prescribed topological singularity types in the plane case. It relies once more on vanishing theorems for singularity schemes and may be applied to other surfaces as well, once we have the corresponding vanishing theorems. It is the aim of this section to generalise the first of these vanishing theorems (cf. [**GLS98c**] Lemma 3.1 or [**Los98**] Lemma 3.10), and we claim that the second one (cf. [**GLS98c**] Lemma 4.1 or [**Los98**] Lemma 3.11) can be treated analogously.

The proof uses an induction based on the reduction of a singularity scheme $X$ by a smooth curve $L$, that is, replacing $X$ by $X : L$. Unfortunately, the class of singularity schemes is not closed under reduction, it therefore becomes necessary to consider a wider class of zero-dimensional schemes. We only give the definitions and state the results, which are necessary for Theorem 4.5. For a more thorough investigation of these schemes, including many examples, we refer to [**GLS98c**] Section 2 or [**Los98**] Section 2.2.

**4.1 Definition**

Let $(C, z) \subset (\Sigma, z)$ be a reduced plane curve singularity with complete embedded resolution tree $T(C, z)$, and let $T^*(C, z)$ denote the essential subtree of $T(C, z)$.

(a) If $T^*$ is a finite, connected subtree of $T(C, z)$ containing the essential subtree $T^*(C, z)$, we call the zero-dimensional scheme $X(C, T^*)$ defined by the ideal sheaf with stalks

$$\mathcal{J}_{X(C,T^*)/\Sigma,z} = I(C, T^*) = \left\{ g \in \mathcal{O}_{\Sigma,z} \,\middle|\, g \text{ goes through the cluster } \mathcal{C}\ell\left(C, T^*\right) \right\}$$

and $\mathcal{J}_{X(C,T^*)/\Sigma,z'} = \mathcal{O}_{\Sigma,z'}$, whenever $z' \neq z$, a *generalised singularity scheme* (with centre $z$).
We denote by $\mathcal{GS}$ the class of zero-dimensional schemes in $\Sigma$ which may be obtained that way.

(b) The subclass of $\mathcal{GS}$ of zero-dimensional schemes of the form $X(C, T^*)$ with centre $z$, where the plane curve singularity $(C, z)$ has only *smooth branches*, is denoted by $\mathcal{GS}_1$.

(c) The subclass of $\mathcal{GS}$ of zero-dimensional schemes of the form $X(C, T^*)$ with centre $z$ where $T^* = T^*(C, z)$ is denote by $\mathcal{S}$, and its members are called *singularity schemes*.
We note that in this sense the singularity schemes $X^s(C)$ introduced in Definition I.2.7 are finite unions of singularity schemes. (See Remark 4.2.)

(d) By $\mathcal{OS}$ we denote the subclass of $\mathcal{S}$ of ordinary fat point schemes $X(m; z)$.

(e) Since the support of a scheme $X \in \mathcal{GS}$ consists of a unique point $z$, we may define $\text{mult}(X) = \text{mult}(X, z)$.

(f) Let $X = X(C, T^*) \in \mathcal{GS}$ and $L \subset \Sigma$ be a curve which is smooth at $z$. We then define $T^* \cap L = \{q \in T^* \mid \text{the strict transform of } L \text{ goes through } q\}$, that is, $T^* \cap L$ is the maximal subtree $T$ of $T^*$ such that $L$ goes through the cluster $\mathcal{C}\ell(C, T)$.

(g) Two generalised singularity schemes $X_0, X_1 \in \mathcal{GS}$ with centre $z$ are called *isomorphic*, if they are isomorphic as subschemes of $\Sigma$. We then write $X_0 \cong X_1$.

(h) An *equimultiple family* of plane curve singularities over a (reduced) algebraic scheme $T$ over $\mathbb{C}$ is a flat family

$$\begin{array}{ccc} \mathcal{C} & \hookrightarrow & S \times T \\ {\scriptstyle\sigma}\nwarrow & & \swarrow \\ & T, & \end{array}$$

of reduced plane curve singularities $\big(\mathcal{C}_t, \sigma(t)\big) \subset \Sigma \times \{t\} \cong \Sigma$ with section $\sigma$ , which admits a simultaneous embedded resolution, together with sections $\sigma_q$ through infinitely near points, defining a family $\mathcal{T}^*$ of finite connected trees $T^*_t$ with $T^*\big(\mathcal{C}_t, \sigma(t)\big) \subseteq T^*_t \subset T\big(\mathcal{C}_t, \sigma(t)\big)$, such that the total transform of $\mathcal{C}$ is equimultiple along $\sigma_q$, $\sigma_q(t) \in T^*_t$. (Cf. **[GLS98c]** Definition 2.9, and **[Los98]** Definition 2.14.)

(i) Two generalised singularity schemes $X_0, X_1 \in \mathcal{GS}$ with centre $z$ are called *equivalent*, if there exist germs $(C_0, z)$ and $(C_1, z)$ defining $X_0$ respectively $X_1$, and a $T^*$-equimultiple family of plane curve singularities over some (reduced) open connected subset $T$ of $\mathbb{A}^1_{\mathbb{C}}$ having $(C_0, T^*_0)$ and $(C_1, T^*_1)$ as fibres. We then write $X_0 \sim X_1$. (Cf. **[GLS98c]** Definition 2.9, and **[Los98]** Definition 2.28.)

**4.2 Remark**

The classes $\mathcal{GS}$, $\mathcal{GS}_1$ and $\mathcal{OS}$ are closed with respect to the equivalence relation $\sim$ and with respect to reduction by a smooth curve $L$. (Cf. **[GLS98c]** Proposition 2.11 and Lemma 2.13, or **[Los98]** Proposition 2.23 and Lemma 2.31.)

**4.3 Remark**

The concepts introduced so far immediately generalise to multigerms $(C, \underline{z})$, and also the remaining part of this section does not change in this situation (cf. **[GLS98c]** p. 545). Moreover, the singularity schemes introduced in Definition I.2.7 then become precisely the elements of $\mathcal{S}$. Just in order to save some notation we will avoid the multigerms.

**4.4 Definition**

Let $L \subset \Sigma$ be a smooth curve such that the corresponding divisor is very ample.

(a) Let $f_1, \ldots, f_r : \mathrm{Num}(\Sigma) \times \bigcup_{n \geq 1} \mathrm{Hilb}^n_\Sigma \to \mathbb{Q}$ be some functions possibly depending on the numerical class of $L$, and let $\mathcal{K} \subseteq \bigcup_{n \geq 1} \mathrm{Hilb}^n_\Sigma$ be some class of zero-dimensional schemes (not necessarily closed under reduction by $L$).
We then call $(f_1, \ldots, f_r)$ a tuple of $L$-*reduction resistant conditions on* $\mathcal{K}$ if, whenever $f_i(D, X) \geq 0$, $i = 1, \ldots, r$, for some zero-dimensional scheme $X \in \mathcal{K}$ and some divisor class $D \in \mathrm{Num}(\Sigma)$ with $D.L > L^2$, then also $f_i(D - L, X : L) \geq 0$, $i = 1, \ldots, r$.
We say, an $L$-reduction resistant condition $f$ is *satisfied* by some zero-dimensional scheme $X$ and some divisor class $D$, if $f(D, X) \geq 0$.

(b) We call a tuple $(f_1, \ldots, f_r)$ of L-reduction resistant conditions on $\mathcal{OS}$ *suitable* if for any ordinary fat point scheme $X = X(\underline{m}, \underline{z}) \subset \Sigma$ and any divisor $D$, with $D.L > L^2$, satisfying $f_i$, for $i = 1, \ldots, r$, there exists a scheme $X' \in \mathcal{OS}$ with
   (1) $X' \sim X$,
   (2) $\deg(X' \cap L) = \deg(X \cap L)$, and
   (3) $h^1\big(\Sigma, \mathcal{J}_{X'/\Sigma}(D)\big) = 0$.

Given a generalised singularity scheme $X$ and a divisor $D$ we would like to know under which circumstances $H^1\big(\Sigma, \mathcal{J}_{X/\Sigma}(D)\big)$ vanishes. However, it turns out that this is hard to answer, and indeed, for the applications in the existence theorems (cf. [**GLS98c**]) we can do with less. There, we may replace $X$ by an equivalent scheme $X'$ for which the cohomology group vanishes, and it is the aim of Theorem 4.5 to provide numerical conditions ensuring the existence of $X'$. Well, actually the theorem reduces the problem to finding conditions which guarantee the existence of $X'$ if $X$ is an ordinary fat point scheme (cf. (4.1)), which is much simpler to handle and where the geometry of $\Sigma$ will come into play.

**4.5 Theorem**
*Let $L \subset \Sigma$ be a smooth curve of genus $g = g(L)$ such that the corresponding divisor is very ample, and let $D \in \text{Div}(\Sigma)$ with $D.L \geq L^2$ and $X \in \mathcal{GS}_1$ such that:*

(4.1) *There is a suitable tuple of L-reduction resistent conditions on $\mathcal{OS}$ which are satisfied by $X$ and $D$.*

(4.2) $\deg(X) \leq \beta \cdot \left( \left( \frac{D.L}{L^2} - \text{mult}(X) \right)^2 - \frac{2\alpha g}{(\alpha - 1)} \cdot \frac{D.L}{L^2} \right)$, *and*

(4.3) $\deg(X \cap L) \leq \frac{D.L}{L^2} - \frac{\alpha \cdot L^2}{D.L} \cdot \deg(X) - 2g$,

*where $\alpha = 1 + \sqrt{2}$ and $\beta = 3 - 2 \cdot \sqrt{2}$.*

*Then there is some $X' \in \mathcal{GS}_1$ with $X' \sim X$ and $\deg(X' \cap L) = \deg(X \cap L)$ such that*

$$h^1\big(\Sigma, \mathcal{J}_{X'/\Sigma}(D)\big) = 0.$$

**Proof:** We do the proof by induction on $d = D.L$, starting with $d = L^2$. We note that we may suppose that $X$ is not an ordinary fat point scheme, since the theorem holds for these by Assumption (4.1). This gives in particular the induction basis, since for $d = L^2$ (4.2) can only be satisfied if $X = \emptyset$.

Setting $\mathcal{K}$ the subclass of $\mathcal{GS}_1$ which are not ordinary fat point schemes, Condition (4.2) and Condition (4.3) are L-reduction resistant on $\mathcal{K}$ by Lemma 4.7. We then may replace $X$ and $D$ by $X : L$ and $D - L$, satisfying now (4.1)–(4.3). Thus, by induction there is a scheme $Y' \sim X : L$ in $\mathcal{GS}_1$ such that $\deg\big(Y' \cap L\big) = \deg\big((X : L) \cap L\big)$ and $h^1\big(\Sigma, \mathcal{J}_{Y'/\Sigma}(D - L)\big) = 0$. Due to the Semicontinuity Theorem for $h^1$ (cf. [**Har77**] III.12.8) and Lemma 4.8 (a) we then find a scheme $X' \in \mathcal{GS}$ such that $X' \sim X$ and $\deg\big(X' \cap L\big) = \deg(X \cap L)$. Moreover, in view of Remark 4.2 we have $X' \in \mathcal{GS}_1$, and by Lemma III.1.1 and Condition (4.3) we find

$$h^1\big(\Sigma, \mathcal{J}_{X'/\Sigma}(D)\big) = 0,$$

which finishes the proof. □

**4.6 Remark**

(a) We note that for an ordinary fat point scheme $X(m;z)$ the cohomology group $H^1\big(\Sigma, \mathcal{J}_{X(m;z)/\Sigma}(D)\big)$ vanishes if and only if
   (1) $h^1\big(\Sigma, \mathcal{O}_\Sigma(D)\big) = 0$, and
   (2) $H^0\big(\Sigma, \mathcal{O}_\Sigma(D)\big) \to H^0\big(\Sigma, \mathcal{O}_{X(m;z)}(D)\big) = \mathcal{O}_{\Sigma,z}/\mathfrak{m}_{\Sigma,z}^m$ is surjective.

(b) We call a surface $\Sigma$ *regular with respect to* $L$ if for any $d \geq 1$ we have $h^1\big(\Sigma, \mathcal{O}_\Sigma(dL)\big) = 0$.
We then see that the examples which we are considering throughout this thesis are all regular with respect to the obvious choices of very ample divisors.
   (1) Complete intersections are regular with respect to hyperplane sections by Proposition G.20. In particular $\mathbb{P}^2_\mathbb{C}$ and surfaces in $\mathbb{P}^3_\mathbb{C}$ are so.
   (2) On a Hirzebruch surfaces $\mathbb{F}_e$, $e \geq 0$, a divisor $L = \alpha C_0 + \beta F$ is very ample if and only if $\alpha > 0$ and $\beta > \alpha e$ (cf. [**Har77**] V.2.18). Then, however, $\mathbb{F}_e$ is regular with respect to $L$ (cf. Remark G.4 and Lemma G.6).
   (3) For a geometrically ruled surface $\Sigma$ with invariant $e$ over a curve $C$ of genus $g = g(C) \geq 1$, we may choose a suitable integer $l \geq \{2g-1, e+2g-1\}$ such that there is a very ample divisor $L \sim_a C_0 + lF$. By Lemma G.5 $\Sigma$ is then regular with respect to $L$.
   (4) Let $\Sigma = C_1 \times C_2$ be a product of two smooth projective curves of genera $g_i = g(C_i)$ and denote by $\mathrm{pr}_i : \Sigma \to C_i$ the canonical projection, $i = 1, 2$. If $L = \mathrm{pr}_2^*\mathfrak{a} \otimes \mathrm{pr}_1^*\mathfrak{b}$ with $\mathfrak{a} \in \mathrm{Div}(C_2)$ of degree $a \geq 2g_2 + 1$ and $\mathfrak{b} \in \mathrm{Div}(C_1)$ of degree $b \geq 2g_1 + 1$, then $L$ is very ample by Lemma G.7, and since $\mathfrak{a}$ and $\mathfrak{b}$ are non-special by Riemann-Roch (cf. [**Har77**] IV.1.3.4) the Künneth formula in Lemma G.8 implies that $h^1(\Sigma, L) = 0$. But then $\Sigma$ is regular with respect to $L$.
   (5) If $\Sigma$ is a product of two elliptic curves, then $K_\Sigma = 0$, and thus by the Kodaira Vanishing Theorem $\Sigma$ is regular with respect to any very ample divisor.
   (6) If $\Sigma$ is a K3-surface, again the Kodaira Vanishing Theorem implies that $\Sigma$ is regular with respect to any very ample divisor $L$.

(c) If $L \in \mathrm{Div}(\Sigma)$ is very ample such that $\Sigma$ is regular with respect to $L$ and if $D = d \cdot L$ is a multiple of the very ample divisor $L$, then (4.1) just becomes obsolete.
In this situation Condition (4.2) and Condition (4.3) take the form:

$$\deg(X) \leq (3 - 2 \cdot \sqrt{2}) \cdot \Big( (d - \mathrm{mult}(X))^2 - (2 + \sqrt{2}) \cdot g \cdot d \Big), \quad (4.2^*)$$

$$\deg(X \cap L) \leq d - \tfrac{1+\sqrt{2}}{d} \cdot \deg(X) - 2g. \quad (4.3^*)$$

(d) In the case of the Hirzebruch surface $\Sigma = \mathbb{F}_0 = \mathbb{P}_{\mathbb{C}}1 \times \mathbb{P}^1_{\mathbb{C}}$ we would like to use the chance to replace (4.1) by some condition, which applies to arbitrary divisors $D = aC_0 + bF$ on $\Sigma$ rather than only to multiples of $L = C_0 + F$. We claim that (4.1) may be replaced by

$$D.F = a \geq \mathrm{mult}(X) - 1 \quad \text{and} \quad D.C_0 = b \geq \mathrm{mult}(X) - 1. \qquad (4.4)$$

**Proof:** (a) and (b) are obvious.

(c) $L$ is a hyperplane section of some embedding $\Sigma \hookrightarrow \mathbb{P}^N_{\mathbb{C}}$, and we thus may find two curves $L_x, L_y \in |L|_l$ such that the germs $L_{x,z}$ and $L_{y,z}$ at $z$ are local coordinates of $(\Sigma, z)$, that is $\mathfrak{m}_{\Sigma,z} = (L_{x,z}, L_{y,z})$.
If $X = X(m; z)$ is an ordinary fat point scheme such that (4.2) and (4.3) are satisfied by $D$ and $X$. We then deduce from (4.11) below, which is an immediate consequence of (4.2), that

$$m = \mathrm{mult}\, X \leq \tfrac{D.L}{2 \cdot L^2} < d.$$

But then the map in (a) is surjective.

(d) The global sections of $\mathcal{O}_\Sigma(D)$ may be identified with the polynomials in $\mathbb{C}[x_0, x_1, y_0, y_1]$ which are bihomogenous of bidegree $(a, b)$. Assuming that $z = \big((0:1), (0:1)\big)$ the map in (a) comes down to

$$H^0(\Sigma, aC_0 + bF) \to \mathcal{O}_{\Sigma,z}/\mathfrak{m}^m_{\Sigma,z} \cong C[x_0, y_0]/(x_0, y_0)^m : f \mapsto f(x_0, 1, y_0, 1),$$

which is surjective by Assumption (4.4).
It remains to show that the conditions are $L$-reduction resistant on $\mathcal{OS}$. However, if $X(m; z)$ is an ordinary fat point scheme and $L$ is smooth at $z$, then $\mathrm{mult}\,\big(X(m; z) : L\big) = \mathrm{mult}\,\big(X(m; z)\big) - 1$, while $D - L = (a - 1) \cdot C_0 + (b - 1) \cdot F$, which finishes the claim.

□

**4.7 Lemma**

*Let $\mathcal{K}$ denote the subclass of $\mathcal{GS}_1$ of schemes which are not ordinary fat point schemes, let $L \subset \Sigma$ be a smooth curve of genus $g = g(L)$ such that the corresponding divisor is very ample, and let $\alpha = 1 + \sqrt{2}, \beta = 3 - 2 \cdot \sqrt{2}$ and $\gamma = \frac{2\alpha}{(\alpha - 1)} \cdot g = (2 + \sqrt{2}) \cdot g$.*

*We define two conditions $f_A, f_B : \mathrm{Num}(\Sigma) \times \bigcup_{n \geq 1} \mathrm{Hilb}^n_\Sigma \to \mathbb{Q}$ by*

$$f_A(D, X) = A(D, X) - \deg(X) \quad \textit{and} \quad f_B(D, X) = B(D, X) - \deg(X \cap L),$$

*where*

$$A(D,X) = \beta \cdot \left(\left(\tfrac{D.L}{L^2} - \mathrm{mult}(X)\right)^2 - \gamma \cdot \tfrac{D.L}{L^2}\right)$$

*and*

$$B(D,X) = \tfrac{D.L}{L^2} - \tfrac{\alpha \cdot L^2}{D.L} \cdot \deg(X) - 2g.$$

*The tuple* $(f_A, f_B)$ *is* $L$*-reduction resistant on* $\mathcal{K}$*.*

**Proof:** For $X \in \mathcal{K}$ and $D \in \mathrm{Num}(\Sigma)$ with $D.L > L^2$, $\deg(X) \leq A(D,X)$ and $\deg(X \cap L) \leq B(D,X)$ we have to show

$$\deg(X : L) \leq A(D - L, X : L) \tag{4.5}$$

and

$$\deg\left((X : L) \cap L\right) \leq B(D - L, X : L). \tag{4.6}$$

For this we consider four different cases, where the first case shall illustrate that the constants $\alpha$ and $\beta$ are chosen optimal.

**Step 1:** Some useful considerations on $\alpha$, $\beta$ and $\gamma$.

We claim that $\alpha > 0$ is minimal and $\beta > 0$ maximal such that

$$\beta \cdot \left(\alpha^2 + \alpha\right) = \alpha - 1. \tag{4.7}$$

To see this we consider for a fixed $b > 0$ the equation

$$b \cdot a^2 + (b-1) \cdot a + 1 = 0. \tag{4.8}$$

The discriminant $(b-1)^2 - 4b$ vanishes if and only if $b = \beta$, and in this case $a = -\frac{\beta - 1}{2\beta} = \alpha$, which proves the claim.

Furthermore, by the definition of $\alpha$, $\beta$ and $\gamma$ we have

$$(\alpha + 1) \cdot \beta \cdot \gamma = 2g, \tag{4.9}$$

$$1 - \alpha\beta - 2\beta = 3\sqrt{2} - 4 > 0, \tag{4.10}$$

and $\mathrm{mult}(X)^2 \leq 2 \cdot \deg(X) \leq 2 \cdot A(D,X) \leq 2\beta \cdot \left(\frac{D.L}{L^2} - \mathrm{mult}(X)\right)^2$, which implies

$$\mathrm{mult}(X) < \tfrac{1+\sqrt{2\beta}}{\sqrt{2\beta}} \cdot \mathrm{mult}(X) \leq \tfrac{D.L}{L^2}. \tag{4.11}$$

**Step 2:** Suppose that $\deg(X \cap L) = B(D,X)$.

Since $\beta \cdot \left(\alpha^2 + \alpha\right) \leq \alpha - 1$ by (4.7), we have

$$\begin{aligned}\left(L^2\right)^2 \cdot \left(\alpha^2 + \alpha\right) \cdot \deg(X) &\leq \left(L^2\right)^2 \cdot \tfrac{(\alpha-1)}{\beta} \cdot A(D,X)\\ &\leq (\alpha - 1) \cdot (D.L)^2 - L^2 \cdot (\alpha - 1) \cdot \gamma \cdot D.L\\ &< (\alpha - 1) \cdot (D.L)^2 + \left(L^2 - 2\alpha g \cdot L^2\right) \cdot D.L.\end{aligned}$$

Taking the assumption of Step 2 into account, this implies

$$\tfrac{\alpha \cdot L^2}{D.L} \cdot \deg(X) \geq 1 + \tfrac{\alpha \cdot L^2}{D.L - L^2} \cdot \left(\deg(X) - B(D,X)\right) = \tfrac{L^2}{L^2} + \tfrac{\alpha \cdot L^2}{D.L - L^2} \cdot \deg(X : L).$$

But then (4.6) follows:

$$\deg\big((X:L)\cap L\big) \le \deg(X\cap L) \le B(D,X) \le B(D-L, X:L).$$

Due to (4.9)–(4.11) we have

$$C(D,X) := (D.L)^2\cdot(1-\alpha\beta-2\beta) + \alpha\beta\cdot \mathrm{mult}(X)\cdot L^2\cdot\big(2\cdot D.L - L^2\cdot\mathrm{mult}(X)\big) + D.L\cdot L^2\cdot\big(2\beta\cdot\mathrm{mult}(X)+\beta+(\alpha+1)\cdot\beta\gamma-2g\big)\ge 0.$$

In view of the assumption of Step 2 and (4.9) a tedious calculation shows

$$\begin{aligned}\deg(X:L) &= \deg(X)-\deg(X\cap L) = \deg(X) - B(D,X)\\ &= \big(1+\tfrac{\alpha\cdot L^2}{D.L}\big)\cdot\deg(X) - \tfrac{D.L}{L^2}+2g \le \big(1+\tfrac{\alpha\cdot L^2}{D.L}\big)\cdot A(D,X) - \tfrac{D.L}{L^2}+2g\\ &= \beta\cdot\Big(\big(\tfrac{D.L-L^2}{L^2}-\mathrm{mult}(X)\big)^2 - \gamma\cdot\tfrac{D.L-L^2}{L^2}\Big) - \tfrac{C(D,X)}{D.L\cdot L^2}\\ &\le \beta\cdot\Big(\big(\tfrac{D.L-L^2}{L^2}-\mathrm{mult}(X)\big)^2 - \gamma\cdot\tfrac{D.L-L^2}{L^2}\Big) \le A(D-L,X:L),\end{aligned}$$

which gives (4.5).

**Step 3:** We now suppose that $X = X(C,T^*)$ satisfies the following conditions:

(a) $\deg(X\cap L) < B(D,X)$.

(b) There exists an irreducible branch $Q$ of $(C,z)$ such that $T^*\cap L = T^*\cap Q$.

(c) There exists no irreducible branch $Q'$ of $(C,z)$ such that $T^*\cap L \subsetneqq T^*\cap Q'$.

Since $X$ is not an ordinary fat point scheme, the tree $T^*$ has at least three vertices, and due to (c) the tree $T^*\cap L$, therefore, has at least two. But then $\deg(X:L) < \deg(X)$ and $\deg\big((X:L)\cap L\big) \le \deg(X\cap L)-2$. Moreover, since $\alpha\beta<1$ and by (4.11) we have

$$\alpha\cdot\deg(X) < \big(\tfrac{D.L}{L^2}-\mathrm{mult}(X)\big)^2 \le \tfrac{D.L}{L^2}\cdot\big(D.L-L^2\big),$$

and thus, taking (a) into account, we get (4.6):

$$\begin{aligned}\deg\big((X:L)\cap L\big) &< B(D,X)-2\\ &= B(D-L,X:L) + \tfrac{\alpha\cdot L^2}{D.L\cdot(D.L-L^2)}\cdot\deg(X) - 1 \le B(D-L,X:L).\end{aligned}$$

By (b) and Lemma 4.8 (b) we know that $\mathrm{mult}(X:L)=\mathrm{mult}(X)-1$. Therefore,

$$\deg(X:L) < \deg(X) \le A(D,X) = A(D-L,X:L)-\gamma,$$

which implies (4.5).

**Step 4:** We now suppose that $X = X(C,T^*)$ satisfies the following conditions:

(a) $\deg(X\cap L) < B(D,X)$.

(b) There exists an irreducible branch $Q$ of $(C,z)$ such that
   (1) $T^*\cap Q = \{z=q_0<q_1<\ldots<q_l\}$,
   (2) $T^*\cap L = \{z=q_0<q_1<\ldots<q_s\}$, $s<l$, and
   (3) $\mathrm{mult}\big(C_{(q_{s+1})}, q_{s+1}\big) = m$ with $\deg(X\cap L)+m\ge B(D,X)$.

Due to (4.10) and (4.11) we have

$$C'(D,X) := (1-\alpha\beta-2\beta)\cdot\big(\tfrac{D.L}{L^2}-\mathrm{mult}(X)\big)+\beta+\tfrac{\alpha\beta\cdot\mathrm{mult}(X)\cdot L^2}{D.L}\cdot\big(\tfrac{D.L}{L^2}-\mathrm{mult}(X)\big)\geq 0.$$

But then, taking (3) and (4.9) into account, a simple calculation shows

$$\begin{aligned}\deg(X:L) &= \deg(X)-\deg(X\cap L)\leq \deg(X)-B(D,X)+\mathrm{mult}(X)\\ &= \big(1+\tfrac{\alpha\cdot L^2}{D.L}\big)\cdot\deg(X)-\big(\tfrac{D.L}{L^2}-\mathrm{mult}(X)\big)+2g\\ &\leq \big(1+\tfrac{\alpha\cdot L^2}{D.L}\big)\cdot A(D,X)-\big(\tfrac{D.L}{L^2}-\mathrm{mult}(X)\big)+2g\\ &= \beta\cdot\Big(\big(\tfrac{D.L-L^2}{L^2}-\mathrm{mult}(X)\big)^2-\gamma\cdot\tfrac{D.L-L^2}{L^2}\Big)-C'(D,X)\\ &\leq \beta\cdot\Big(\big(\tfrac{D.L-L^2}{L^2}-\mathrm{mult}(X)\big)^2-\gamma\cdot\tfrac{D.L-L^2}{L^2}\Big)\leq A(D-L,X:L),\end{aligned}$$

which gives (4.5).

Consider next the concave function $\varphi:\mathbb{R}\to\mathbb{R}:x\mapsto\beta\cdot(1+\alpha)\cdot\big(2x-\frac{L^2}{D.L}\cdot x^2\big)$, which takes its maximum at $x=\frac{D.L}{L^2}$ and is thus concavely increasing on the interval $\Big[0,\frac{\sqrt{2\beta}}{1+\sqrt{2\beta}}\cdot\frac{D.L}{L^2}\Big]$. In particular, since the line $\{y=x\}$ intersects the graph of $\varphi$ in $x=0$ and $x=\frac{\sqrt{\beta}}{1+\sqrt{\beta}}\cdot\frac{D.L}{L^2}$, $\varphi(x)\geq x$ for all $0\leq x\leq\frac{\sqrt{\beta}}{1+\sqrt{\beta}}\cdot\frac{D.L}{L^2}$.

Now consider a second function $\psi:\Big[0,\frac{\sqrt{2\beta}}{1+\sqrt{2\beta}}\cdot\frac{D.L}{L^2}\Big]\to\mathbb{R}$ given by

$$\psi(x):=\begin{cases} x, & \text{for } 0\leq x\leq\frac{\sqrt{\beta}}{1+\sqrt{\beta}}\cdot\frac{D.L}{L^2},\\ \sqrt{2\beta\cdot\big(\frac{D.L}{L^2}-x\big)^2-x^2}, & \text{for } \frac{\sqrt{\beta}}{1+\sqrt{\beta}}\cdot\frac{D.L}{L^2}\leq x\leq\frac{\sqrt{2\beta}}{1+\sqrt{2\beta}}\cdot\frac{D.L}{L^2}.\end{cases}$$

This function is concave as well and takes its maximum at $x=\frac{\sqrt{\beta}}{1+\sqrt{\beta}}\cdot\frac{D.L}{L^2}$, where $\psi$ and $\varphi$ coincide. In particular, $\psi$ is bounded from above by $\varphi$.

Note that by Lemma 4.8 (c) we have $\frac{\mathrm{mult}(X)^2+m^2}{2}<\deg(X)\leq\beta\cdot\big(\frac{D.L}{L^2}-\mathrm{mult}(X)\big)^2$; and $m\leq\mathrm{mult}(X)$ anyway. We thus find

$$m\leq\psi\big(\mathrm{mult}(X)\big)\leq\varphi\big(\mathrm{mult}(X)\big),$$

since by (4.11) mult(X) takes its values in the domain of definition of $\psi$.

Taking (4.8) and (4.9) into account, we deduce from this

$$\begin{aligned}C''(D,X) &:= \tfrac{D.L}{L^2}\cdot(\alpha-1)+(1-\alpha m-2\alpha g)-\tfrac{L^2}{D.L}\cdot\big(\alpha+\alpha^2\big)\cdot\deg(X)\\ &\geq \tfrac{D.L}{L^2}\cdot(\alpha-1)+(1-\alpha m-2\alpha g)-\tfrac{L^2}{D.L}\cdot\big(\alpha+\alpha^2\big)\cdot A(D,X)\\ &\geq \alpha\cdot\Big(\varphi\big(\mathrm{mult}(X)\big)-m\Big)\geq 0.\end{aligned}$$

Moreover, we note that due to (3) we have

$$\deg(X:L)=\deg(X)-\deg(X\cap L)\leq\deg(X)-B(D,X)+m.$$

But then

$$\begin{aligned}&\deg\big((X:L)\cap L\big)\leq\deg(X\cap L)\leq B(D,X)\\ &=\tfrac{D.L-L^2}{L^2}-\alpha\cdot\tfrac{L^2}{D.L-L^2}\cdot\big(\deg(X)-B(D,X)+m\big)-2g-\tfrac{L^2}{D.L-L^2}\cdot C''(D,X)\\ &\qquad\leq\tfrac{D.L-L^2}{L^2}-\alpha\cdot\tfrac{L^2}{D.L-L^2}\cdot\deg(X:L)-2g=B(D-L,X:L),\end{aligned}$$

which gives (4.6).

**Step 5:** We finally suppose that $X = X(C, T^*)$ satisfies the following conditions:

(a) $\deg(X \cap L) < B(D, X)$.

(b) There exists an irreducible branch $Q$ of $(C, z)$ such that
  (1) $T^* \cap Q = \{z = q_0 < q_1 < \ldots < q_l\}$,
  (2) $T^* \cap L = \{z = q_0 < q_1 < \ldots < q_s\}$, $s < l$, and
  (3) $\operatorname{mult}\big(C_{(q_{s+1})}, q_{s+1}\big) = m$ with $\deg(X \cap L) + m < B(D, X)$.

Then by Lemma 4.8 (d) we may specialise the point $q_{s+1}$ onto the curve $L$ and consider the new scheme $X_1 \sim X$ with $\deg(X_1 \cap L) = \deg(X \cap L) + m$. (Note that $X_1 \sim X$ implies that $X_1 \in \mathcal{K}$.) By the Semicontinuity Theorem for $h^1$ (cf. [**Har77**] III.12.8) the vanishing of $H^1\big(\Sigma, \mathcal{J}_{X''/\Sigma}(D)\big)$ for some $X'' \sim X_1$ with $\deg\big(X'' \cap L\big) = \deg(X_1 \cap L)$ implies the vanishing of $H^1\big(\Sigma, \mathcal{J}_{X/\Sigma}(D)\big)$ itself. Thus, specialising points to $L$ we come down to the cases which we have already studied. □

**4.8 Lemma**

*Let $X = X(C, T^*) \in \mathcal{GS}$, let $Q$ be a smooth branch of $(C, z)$ and let $L \subset \Sigma$ be a curve which is smooth at $z$. Then:*

(a) *For almost all $Y' \sim X : L$ in $\mathcal{GS}$ with $\deg\big(Y' \cap L\big) = \deg\big((X : L) \cap L\big)$ there exists a generalised singularity scheme $X' \sim X$ in $\mathcal{GS}$ such that $Y' = X' : L$ and $\deg\big(X' \cap L\big) = \deg(X \cap L)$.*

(b) *If $T^* \cap Q \subseteq T^* \cap L$, then $\operatorname{mult}(X : L) = \operatorname{mult}(X) - 1$.*

(c) $\deg(X) = \sum_{q \in T^*} \frac{m_q \cdot (m_q + 1)}{2}$.

(d) *If $T^* \cap L \subseteq T \subseteq T^* \cap Q$ is a connected subtree, then there exists an analytical isomorphism $\varphi : (\Sigma, z) \to (\Sigma, z)$ mapping $(C, z)$ to a plane curve singularity $(C_1, z)$ such that the subtree $T_1^*$ corresponding to $T^*$ satisfies:*
  (1) $X_1 = X(C_1, T_1^*) \cong X$,
  (2) *$T_1^* \cap L$ is the subtree of $T_1^*$ corresponding to $T$, and*
  (3) *$T_1^* \cap L = T_1^* \cap Q_1$, where $Q_1 = \varphi(Q)$ is a smooth branch of $(C_1, z)$.*

  *In particular, $X_1 \sim X$ and $\deg(X_1 \cap L) = \deg(X \cap L) + \sum_{q \in T \setminus (T^* \cap L)} m_q$.*

**Proof:** (a) Cf. [**GLS98c**] Proposition 2.18 or [**Los98**] Proposition 2.32.

(b) Cf. [**GLS98c**] Lemma 2.15 or [**Los98**] Lemma 2.26.

(c) Cf. [**GLS98c**] Lemma 2.6 or [**Los98**] Lemma 2.8.

(d) Cf. [**GLS98c**] Lemma 2.14 or [**Los98**] Lemma 2.25.

□

## 5. Examples

We are now stating the results which we derive from Theorem 4.5 in view of Remark 4.6. We leave it to the reader to exploit the possibilities, which (4.1) offers, in a better way.

### 5.a. The Classical Case - $\Sigma = \mathbb{P}^2_{\mathbb{C}}$

Choosing $L$ as a line in $\mathbb{P}^2_{\mathbb{C}}$, which has genus $g = g(L) = 0$, we get precisely the same result as in [**GLS98c**].

**4.5a Theorem**
*Let* $d > 1$ *be an integer and let* $X \in \mathcal{GS}_1$ *such that:*

(4.2a) $\qquad \deg(X) \leq \left(3 - 2 \cdot \sqrt{2}\right) \cdot \left(d - \operatorname{mult}(X)\right)^2,$

(4.3a) $\qquad \deg(X \cap L) \leq d - \frac{1+\sqrt{2}}{d} \cdot \deg(X).$

*Then there is some* $X' \in \mathcal{GS}_1$ *with* $X' \sim X$ *and* $\deg(X' \cap L) = \deg(X \cap L)$ *such that*

$$h^1\left(\mathbb{P}^2_{\mathbb{C}}, \mathcal{J}_{X'/\Sigma}(dL)\right) = 0.$$

### 5.b. Hirzebruch Surfaces

Working now with the very ample divisor $L = C_0 + (e+1) \cdot L$, which has genus $g = g(L) = 0$, the theorem takes for divisors $dL$ precisely the same form as in the case of $\mathbb{P}^2_{\mathbb{C}}$. We might as well work with any other very ample divisor $L$, of possibly larger genus $g = g(L)$. The Theorem then takes the same form as for K3-surfaces.

For the Hirzebruch surface $\mathbb{F}_0 = \mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$ we also get another, somewhat more general version.

**4.5b Theorem**
*Let* $a, b > 1$ *be two integers and let* $X \in \mathcal{GS}_1$ *such that:*

(4.1b) $\qquad a, b \geq \operatorname{mult}(X) - 1,$

(4.2b) $\qquad \deg(X) \leq \left(3 - 2 \cdot \sqrt{2}\right) \cdot \left(d - \operatorname{mult}(X)\right)^2,$

(4.3b) $\qquad \deg(X \cap L) \leq d - \frac{1+\sqrt{2}}{d} \cdot \deg(X).$

*Then there is some* $X' \in \mathcal{GS}_1$ *with* $X' \sim X$ *and* $\deg(X' \cap L) = \deg(X \cap L)$ *such that*

$$h^1\left(\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}, \mathcal{J}_{X'/\Sigma}(aC_0 + bF)\right) = 0.$$

### 5.c. Geometrically Ruled Surfaces

Using the notation of Section G.a $\pi : \Sigma \to C$ is a geometrically ruled surface with invariant $e$. As in Remark 4.6 we may choose some integer $l \geq \max\{2g - 1, e + 2g - 1\}$ such that $L = C_0 + lF$ is very ample. Its genus is then just $g(L) = g(C)$.

**4.5c Theorem**
*Let* $d > 1$ *be an integer and let* $X \in \mathcal{GS}_1$ *such that*

$$\text{(4.2c)} \qquad \deg(X) \leq \left(3 - 2\cdot\sqrt{2}\right)\cdot\Big(\big(d - \operatorname{mult}(X)\big)^2 - \left(2+\sqrt{2}\right)\cdot g(C)\cdot d\Big),$$

$$\text{(4.3c)} \qquad \deg(X \cap L) \leq d - \tfrac{1+\sqrt{2}}{d}\cdot\deg(X) - 2g(C).$$

*Then there is some* $X' \in \mathcal{GS}_1$ *with* $X' \sim X$ *and* $\deg(X' \cap L) = \deg(X \cap L)$ *such that*

$$h^1\big(\Sigma, \mathcal{J}_{X'/\Sigma}(dL)\big) = 0.$$

## 5.d. Products of Curves

Let $\Sigma = C_1 \times C_2$ be a product of two smooth projective curves of genera $g_1$ and $g_2$ respectively. Then we may choose some very ample divisor $L \sim_a (2g_2 + 1)\cdot C_1 + (2g_1 + 1)\cdot C_2$ of genus $g = 8g_1g_2 + g_1 + g_2$.

**4.5d Theorem**
*Let* $d > 1$ *be an integer and let* $X \in \mathcal{GS}_1$ *such that*

$$\text{(4.2d)} \qquad \deg(X) \leq \left(3-2\cdot\sqrt{2}\right)\cdot\Big(\big(d-\operatorname{mult}(X)\big)^2-\left(2+\sqrt{2}\right)\cdot(8g_1g_2+g_1+g_2)\cdot d\Big),$$

$$\text{(4.3d)} \qquad \deg(X \cap L) \leq d - \tfrac{1+\sqrt{2}}{d}\cdot\deg(X) - 2\cdot(8g_1g_2 + g_1 + g_2).$$

*Then there is some* $X' \in \mathcal{GS}_1$ *with* $X' \sim X$ *and* $\deg(X' \cap L) = \deg(X \cap L)$ *such that*

$$h^1\Big(C_1 \times C_2, \mathcal{J}_{X'/\Sigma}\big(d\cdot(2g_2+1)\cdot C_1 + d\cdot(2g_1+1)\cdot C_2\big)\Big) = 0.$$

## 5.e. Surfaces in $\mathbb{P}^3_{\mathbb{C}}$

For a surface in $\mathbb{P}^3_{\mathbb{C}}$ of degree $n$ with $L$ a hyperplane section, the formula for the genus $g = g(L)$ comes down to $g = \frac{(n-1)\cdot(n-2)}{2}$.

**4.5e Theorem**
*Let* $d > 1$ *be an integer and let* $X \in \mathcal{GS}_1$ *such that*

$$\text{(4.2e)} \qquad \deg(X) \leq \left(3 - 2\cdot\sqrt{2}\right)\cdot\Big(\big(d - \operatorname{mult}(X)\big)^2 - \tfrac{2+\sqrt{2}}{2}\cdot(n-1)\cdot(n-2)\cdot d\Big),$$

$$\text{(4.3e)} \qquad \deg(X \cap L) \leq d - \tfrac{1+\sqrt{2}}{d}\cdot\deg(X) - (n-1)\cdot(n-2).$$

*Then there is some* $X' \in \mathcal{GS}_1$ *with* $X' \sim X$ *and* $\deg(X' \cap L) = \deg(X \cap L)$ *such that*

$$h^1\big(\Sigma, \mathcal{J}_{X'/\Sigma}(dL)\big) = 0.$$

If $\Sigma$ was a complete intersection of type $(d_1, \ldots, d_{N-2})$ and $L$ is a hyperplane section, then we just would have to replace the genus $g = g(L) = \frac{d_1\cdots d_{N-2}\cdot\left(\sum_{i=1}^{N-2} d_i - N\right) + 2}{2}$.

## 5.f. K3-Surfaces

If $\Sigma$ is a K3-surface and $L$ any very ample divisor of genus $g(L) = g$, Theorem 4.5 comes down to.

**4.5f Theorem**
*Let* $d > 1$ *be an integer and let* $X \in \mathcal{GS}_1$ *such that*

$$\deg(X) \leq \left(3 - 2 \cdot \sqrt{2}\right) \cdot \Big(\big(d - \operatorname{mult}(X)\big)^2 - \big(2 + \sqrt{2}\big) \cdot g \cdot d\Big), \tag{4.2f}$$

$$\deg(X \cap L) \leq d - \tfrac{1+\sqrt{2}}{d} \cdot \deg(X) - 2g. \tag{4.3f}$$

*Then there is some* $X' \in \mathcal{GS}_1$ *with* $X' \sim X$ *and* $\deg(X' \cap L) = \deg(X \cap L)$ *such that*

$$h^1\big(\Sigma, \mathcal{J}_{X'/\Sigma}(dL)\big) = 0.$$

CHAPTER III

# Existence

The question whether there exist plane curves of given degree having singular points of given types was already studied by the Italian geometers during the last century. A very nice and inspiring result in this direction is that of Severi (cf. [**Sev21**]) which says that there is an irreducible plane curve of degree $d$ with $r$ nodes as only singularities if and only if $0 \leq r \leq \frac{(d-1)\cdot(d-2)}{2}$. In our terminology this means

$$V^{irr}_{|d\cdot H|}(rA_1) \neq \emptyset \qquad \Longleftrightarrow \qquad 0 \leq r \leq \frac{(d-1)\cdot(d-2)}{2},$$

where $H$ is a line in $\mathbb{P}^2_{\mathbb{C}}$. Severi deduced this by showing that nodes on plane curves can be "smoothed" independently.

Looking for generalisations of this result one might concentrate on different aspects such as

- looking at ordinary multiple points of higher multiplicities on plane curves, or
- looking at arbitrary singularities on plane curves, or
- looking at nodal curves on arbitrary surfaces, or, finally,
- looking at arbitrary singularities on arbitrary surfaces.

All these generalisations have one problem in common – we may hardly expect criteria which are necessary *and* sufficient for the existence at the same time. Already when replacing "nodes" by "cusps" on plane curves there is no such complete answer known – except for small degrees $d$, $d \leq 10$ –, and the existence of irreducible plane curves with more cusps than the number of conditions imposed by them should allow shows that one may hardly expect such a result (cf. [**Hir92**]). Similar problems arise in the other situations, and the reason is mainly that in general the conditions which the singular points impose are not independent.

The direction of looking at plane curves with ordinary multiple points of arbitrary multiplicities was very much inspired by a conjecture of Nagata who proposed in [**Nag59**] that $\sqrt{r} \cdot \sum_{i=1}^{r} m_i$ should be a lower bound for the degree of a plane curve passing through $r$ general points with multiplicities $m_1, \ldots, m_r$, whenever $r \geq 9$. He himself gave a complete answer for the case $r \leq 8$ and when $r$ is a square. Many efforts have been taken to fully prove the conjecture and much progress has been made – cf. [**Nag59, Seg62, ArC81, Hir85, Har86, Gim87, Hir89, Ran89, CiM98,**

**Bru99, Eva99, Mig00, AlH00, Roé01, Mig01, Sei01**]. However, a full proof is still missing, though several special cases have been cracked and the generally known bounds have been improved largely. We refer to [**Cil01**] and [**Har01**] for an overview on the state of the art for the Nagata Conjecture and for the related conjectures of Segre [**Seg62**], Gimigliano [**Gim87**] and of Harbourne–Hirschowitz [**Har85a, Hir89**].

Also for the existence of plane curves with one or with several singularities of given topological singularity types there is a large number of results, giving necessary or sufficient conditions for the existence or just giving strange examples – cf. [**GTU66, ArC81, Bru81, ArC83, GuN83, Var83, Ura83, Tan84, Ura84, GrM85, Ivi85, Koe86, Hir86, Ura86, Deg90, Shu87a, GrM88, Hir92, Bar93b, Sak93, Shu93, Deg00, Wal95, Wal96, GLS98c, Los98, Shu98, Shu99, Mig01, Shu03**]. For an overview on the results known we refer to [**Los98**], [**GrS99**] or [**GLS05**]. Just recently the asymptotically best known conditions for the existence of curves in the plane with arbitrary topological singularities have been considerably improved using Castelnuovo function arguments, and at the same time the first general conditions for analytical singularities have been found (cf. [**Shu03**]). The existence of an irreducible plane curve of degree $d$ with topological singularity types $\mathcal{S}_1, \ldots, \mathcal{S}_r$, among them $k$ nodes, $m$ cusps, $t$ ordinary triple points and $u$ singularities of type $A_{2l}$ with $l \geq 2$, as only singular points is ensured by

$$6k + 10m + \tfrac{169}{6} \cdot t + \tfrac{25}{3} \cdot u + \tfrac{27}{2} \cdot \sum_{\mathcal{S}_i \neq A_1, A_2, D_4} \delta(\mathcal{S}_i) \leq d^2 - 2d + 3,$$

where $\delta(\mathcal{S}_i)$ is the delta-invariant of $\mathcal{S}_i$, and similarly for analytical singularity types

$$6k + 10m + 9 \cdot \sum_{\mathcal{S}_i \neq A_1, A_2} \mu(\mathcal{S}_i) \leq d^2 - 2d + 3,$$

suffices, where $\mu(\mathcal{S}_i)$ is the Milnor number of $\mathcal{S}_i$. (Cf. [**Shu03**].) We will compare this particular result to the corresponding special case of our general results in Section 3.a and see that our results are weaker – which was to be expected, since the techniques applied in [**Shu03**] are optimised for the plane case.

In the past the question of the existence of curves with prescribed analytical singularity types has attained much less attention, due to the lack of suitable methods to tackle the problem. However, the Viro method which Shustin uses in [**Shu99**] Section 6 on hypersurfaces in $\mathbb{P}^n_{\mathbb{C}}$ and which we also use here allows to glue topological singularities as well as analytical ones, so that our main results are valid for both kinds of singularity types. (See also [**Shu03**].)

The basic idea may be described as follows: suppose we have a "suitable" irreducible curve $C \in |D|_l$ with ordinary multiple points $z_1, \ldots, z_r$ of multiplicities $m_1, \ldots, m_r$ as singularities, and suppose we have "good" affine representatives $g_1, \ldots, g_r$ for the singularity types $\mathcal{S}_1, \ldots, \mathcal{S}_r$, then we may glue locally at the $z_i$ the equations $g_i$ into the curve $C$. In Definition 2.1 we define what it precisely means that a representative is "good"; that the curve $C$ then is "suitable", just means that the $m_i$-jet of $C$ at $z_i$ coincides with the $m_i$-jet of $g_i$, where $m_i = \deg(g_i)$, and that $V_{|D|}(m_1, \ldots, m_r)$ is smooth at $C$ of the expected dimension. Finding general conditions for the existence thus comes down to finding conditions for the existence of curves with ordinary multiple points satisfying the T-smoothness property and to finding upper bounds for the degree of a good representative. Section 1 is devoted to the first problem, and we get satisfactory results in Theorem 1.2 and Corollary 1.3, while for the second problem we are bound to known results (cf. [**Shu03**]).

The most restrictive among the *sufficient* conditions which we find in Corollary 2.5 is (2.7) respectively (2.11) and could be characterised as a condition of the type

$$\sum_{i=1}^{r} \delta(\mathcal{S}_i) \leq \alpha D^2 + \beta D.K + \gamma,$$

for topological singularity types, respectively for analytical ones as

$$\sum_{i=1}^{r} \mu(\mathcal{S}_i) \leq \alpha D^2 + \beta D.K + \gamma,$$

where $K$ is some fixed divisor class and $\alpha$, $\beta$ and $\gamma$ are some absolute constants. (See also Remark II.1.3.) However, it seems that this condition is of the right type and with the right "exponents" for the invariants of the $\mathcal{S}_i$ as well as for the divisor $D$, since there are also *necessary* conditions of this type, e. g.

$$\sum_{i=1}^{r} \mu(\mathcal{S}_i) \leq 2 \cdot \sum_{i=1}^{r} \delta(\mathcal{S}_i) \leq D^2 + D.K_\Sigma + 2,$$

which follows from the genus formula: If $D$ is an irreducible curve with precisely $r$ singular points of topological or analytical types $\mathcal{S}_1, \ldots, \mathcal{S}_r$ and $\nu : \widetilde{D} \to D$ its normalisation, then $p_a(D) = g\big(\widetilde{D}\big) + \delta(D) \geq \delta(D)$, where $\delta(D) = \dim_{\mathbb{C}} \big(\nu_* \mathcal{O}_{\widetilde{D}} / \mathcal{O}_D\big)$ is the delta invariant of $D$ (cf. [**BPV84**] II.11). Moreover, by definition $\delta(D) = \sum_{z \in \mathrm{Sing}(D)} \delta(D, z)$, and it is well known that $2\delta(D, z) = \mu(D, z) + r(D, z) - 1 \geq \mu(D, z)$ (see Remark I.2.2). Using now the genus formula we get: $D^2 + D.K_\Sigma + 2 = 2p_a(D) \geq 2 \cdot \sum_{i=1}^{r} \delta(\mathcal{S}_i) \geq \sum_{i=1}^{r} \mu(\mathcal{S}_i)$.

## 1. Existence Theorem for Ordinary Fat Point Schemes

In order to be able to apply the existence theorem for ordinary fat point schemes to the general case it is important that the existing curve has the T-smoothness property, that is that some $H^1$ vanishes. This vanishing is ensured by reducing the problem to a "lower degree" and applying the following lemma as a kind of induction step.

**1.1 Lemma**

*Let* $L \subset \Sigma$ *be a smooth curve and* $X \subset \Sigma$ *a zero-dimensional scheme. If* $D \in \mathrm{Div}(\Sigma)$ *such that*

(1.1) $h^1\big(\Sigma, \mathcal{J}_{X:L/\Sigma}(D-L)\big) = 0$, *and*

(1.2) $\deg(X\cap L) \leq D.L + 1 - 2g(L)$,

*then*

$$h^1\big(\Sigma, \mathcal{J}_{X/\Sigma}(D)\big) = 0.$$

**Proof:** Condition (1.2) implies

$$2g(L) - 2 < D.L - \deg(X\cap L) = \deg\big(\mathcal{O}_L(D)\big) + \deg\big(\mathcal{J}_{X\cap L/L}\big) = \deg\big(\mathcal{J}_{X\cap L/L}(D)\big),$$

and thus by Riemann-Roch (cf. [**Har77**] IV.1.3.4)

$$h^1\big(\mathcal{J}_{X\cap L/L}(D)\big) = 0.$$

Consider now the exact sequence

$$0 \longrightarrow \mathcal{J}_{X:L/\Sigma}(D-L) \xrightarrow{\cdot L} \mathcal{J}_{X/\Sigma}(D) \longrightarrow \mathcal{J}_{X\cap L/L}(D) \longrightarrow 0.$$

The result then follows from the corresponding long exact cohomology sequence

$$0 = H^1\big(\mathcal{J}_{X:L/\Sigma}(D-L)\big) \longrightarrow H^1\big(\mathcal{J}_{X/\Sigma}(D)\big) \longrightarrow H^1\big(\mathcal{J}_{X\cap L/L}(D)\big) = 0.$$

□

**1.2 Theorem**
*Given* $m_1, \ldots, m_r \in \mathbb{N}$*, not all zero, and* $z_1, \ldots, z_r \in \Sigma$*, in very general position. Let* $L \in \mathrm{Div}(\Sigma)$ *be very ample over* $\mathbb{C}$*, and let* $D \in \mathrm{Div}(\Sigma)$ *be such that*

(1.3) $h^1\big(\Sigma, \mathcal{J}_{X(\underline{m};\underline{z})/\Sigma}(D-L)\big) = 0$, *and*

(1.4) $D.L - 2g(L) \geq m_i + 1$ *for all* $i = 1, \ldots, r$.

*Then there exists an irreducible curve* $C \in |D|_l$ *with ordinary singular points of multiplicity* $m_i$ *at* $z_i$ *for* $i = 1, \ldots, r$ *and no other singular points. Furthermore,*

$$h^1\big(\Sigma, \mathcal{J}_{X(\underline{m};\underline{z})/\Sigma}(D)\big) = 0,$$

*and in particular,* $V_{|D|}(\underline{m})$ *is T-smooth at* $C$.
**Addendum:** *Given local coordinates* $x_j, y_j$ *in* $z_j$ *and open dense subsets* $U_j \subseteq \mathbb{C}[x_j, y_j]_{m_j}$*,*[1] $j = 1, \ldots, r$*, the curve* $C$ *may be chosen such that the* $m_j$*-jet at* $z_j$ *of a section defining* $C$ *belongs to* $U_j$.
**Idea of the proof:** For each $z_j$ find a curve $C_j \in \big|\mathcal{J}_{X(\underline{m},\underline{z})/\Sigma}(D)\big|_l$ with an ordinary singular point of multiplicity $m_j$ and show that this linear system has no other base points than $z_1, \ldots, z_r$. Then the generic element is smooth outside $z_1, \ldots, z_r$ and has an ordinary singularity of multiplicity $m_j$ in $z_j$ for $j = 1, \ldots, r$.

[1]For the definition of $\mathbb{C}[x, y]_d$ see Page 72.

**Proof:** We may assume that $\Sigma \subseteq \mathbb{P}^n_{\mathbb{C}}$ is embedded via $L$, in particular the elements of $|L|_l$ are the hyperplane sections. For $i \neq j$ we denote by $\overline{z_iz_j}$ the secant line joining $z_i$ and $z_j$.

Let us choose $z_0 \in \Sigma \setminus \bigcup_{i\neq j} \overline{z_iz_j}$. Then by Corollary E.8 for every point $z^*$ in the open dense subset $\Sigma \setminus B_{z_0}$ of $\Sigma$ there is a smooth irreducible curve, say $L_{z^*}$, in $|L|_l$ through $z_0$ and $z^*$ containing at most one of the $z_i$, $i = 1, \ldots, r$. If $\Sigma$ is the projective plane $B_{z_0}$ is empty, otherwise it consists of the finite union of secant lines through $z_0$ completely contained in $\Sigma$.

If $Z \subseteq \Sigma$ is any zero-dimensional subscheme and $z^* \in \Sigma$ is any point, we define the zero-dimensional scheme $Z \cup \{z^*\}$ by the ideal sheaf

$$\mathcal{J}_{Z\cup\{z^*\}/\Sigma,z} = \begin{cases} \mathcal{J}_{Z/\Sigma,z}, & \text{if } z \neq z^*, \\ \mathfrak{m}_{\Sigma,z^*} \cdot \mathcal{J}_{Z/\Sigma,z^*}, & \text{if } z = z^*. \end{cases}$$

Writing $X$ for $X(\underline{m};\underline{z})$ we introduce in particular the zero-dimensional schemes $X_j = X \cup \{z_j\}$ for $j = 0, \ldots, r$.

**Step 1:** $h^1\big(\mathcal{J}_{X_0\cup\{z^*\}/\Sigma}(D)\big) = h^1\big(\mathcal{J}_{X_j/\Sigma}(D)\big) = 0$ for $z^* \in \Sigma \setminus B_{z_0}$ and $j = 0, \ldots, r$.

Since $L_{z^*}$ passes through $z_0$, $z^*$ and at most one $z_j$, $j \in \{1, \ldots, r\}$, where $z^*$ might be this $z_j$, Condition (1.4) implies

$$\deg\big(X_0 \cap L_{z^*}\big) \leq m_j + 2 \leq D.L + 1 - 2g(L). \tag{1.5}$$

Moreover, if $z^* = z_j$ for some $j \in \{1, \ldots, r\}$ or if $L_{z^*}$ does not pass through any $z_j$, $j \in \{1, \ldots, r\}$, then $X_0 \cup \{z^*\} : L_{z^*} = X$, otherwise we get the exact sequence

$$0 \longrightarrow \mathcal{J}_{X/\Sigma} \longrightarrow \mathcal{J}_{X_0\cup\{z^*\}:L_{z^*}/\Sigma} \longrightarrow \mathfrak{m}^{m_j-1}_{\Sigma,z_j}/\mathfrak{m}^{m_j}_{\Sigma,z_j} \longrightarrow 0,$$

but in any case we have by Condition (1.3) that

$$h^1\big(\mathcal{J}_{X_0\cup\{z^*\}:L_{z^*}/\Sigma}(D - L)\big) = 0. \tag{1.6}$$

(1.5) and (1.6) allow us to apply Lemma 1.1 in order to obtain

$$h^1\big(\mathcal{J}_{X_0\cup\{z^*\}/\Sigma}(D)\big) = 0.$$

The inclusion $\mathcal{J}_{X_0\cup\{z_j\}/\Sigma} \hookrightarrow \mathcal{J}_{X_j/\Sigma}$, for $j = 1, \ldots, r$, respectively the inclusion $\mathcal{J}_{X_0\cup\{z^*\}/\Sigma} \hookrightarrow \mathcal{J}_{X_0/\Sigma}$, for some $z^* \neq z_0$, then imply that for any $j = 0, \ldots, r$ also

$$h^1\big(\mathcal{J}_{X_j/\Sigma}(D)\big) = 0.$$

**Step 2:** For each $j = 1, \ldots, r$ there exists a curve $C_j \in |D|_l$ with an ordinary singular point of multiplicity $m_j$ at $z_j$ and with $\text{mult}_{z_i}(C_j) \geq m_i$ for $1 \leq i \neq j$.

Consider the exact sequence

$$0 \longrightarrow \mathcal{J}_{X_j/\Sigma} \longrightarrow \mathcal{J}_{X/\Sigma} \longrightarrow \mathfrak{m}^{m_j}_{\Sigma,z_j}/\mathfrak{m}^{m_j+1}_{\Sigma,z_j} \longrightarrow 0$$

twisted by $D$ and the corresponding long exact cohomology sequence

$$H^0\big(\mathcal{J}_{X/\Sigma}(D)\big) \to \mathfrak{m}_{\Sigma,z_j}^{m_j}/\mathfrak{m}_{\Sigma,z_j}^{m_j+1} \to \underset{\underset{0}{\|\,\text{Step 1}}}{H^1\big(\mathcal{J}_{X_j/\Sigma}(D)\big)} \to H^1\big(\mathcal{J}_{X/\Sigma}(D)\big) \to 0. \quad (1.7)$$

Thus we may choose the $C_j$ to be given by a section in $H^0\big(\mathcal{J}_{X/\Sigma}(D)\big)$ where the $m_j$ tangent directions at $z_j$ are all different – *indeed, we may choose the* $m_j$*-jet at* $z_j$ *of the section defining* $C_j$ *arbitrary!*

**Step 3:** The base locus of $\big|\mathcal{J}_{X_0/\Sigma}(D)\big|_l$ is contained in $\{z_0,\ldots,z_r\} \cup B_{z_0}$.

Suppose $z^* \in \Sigma$ was an additional base point of $\big|\mathcal{J}_{X_0/\Sigma}(D)\big|_l$. By Step 1

$$h^1\big(\mathcal{J}_{X_0\cup\{z^*\}/\Sigma}(D)\big) = 0,$$

and thus as in Step 2

$$h^0\big(\mathcal{J}_{X_0/\Sigma}(D)\big) = h^0\big(\mathcal{J}_{X_0\cup\{z^*\}/\Sigma}(D)\big) + 1.$$

But by assumption $z^*$ is a base point, and thus

$$h^0\big(\mathcal{J}_{X_0/\Sigma}(D)\big) = h^0\big(\mathcal{J}_{X_0\cup\{z^*\}/\Sigma}(D)\big),$$

which gives us the desired contradiction.

**Step 4:** The base locus of $\big|\mathcal{J}_{X/\Sigma}(D)\big|_l$ is $\{z_1,\ldots,z_r\}$.

By Step 3 and since $\big|\mathcal{J}_{X_0/\Sigma}(D)\big|_l \subset \big|\mathcal{J}_{X/\Sigma}(D)\big|_l$ we know that the base locus is contained in $\{z_0,\ldots,z_r\} \cup B_{z_0}$. However, the intersection over the $B_{z_0}$ for all admissible $z_0$ is empty, so that the base locus of $\big|\mathcal{J}_{X/\Sigma}(D)\big|_l$ is just $\{z_1,\ldots,z_r\}$.

**Step 5:** There is an irreducible curve $C \in \big|\mathcal{J}_{X/\Sigma}(D)\big|_l$, which has an ordinary singular point of multiplicity $m_i$ at $z_i$ for $i = 1,\ldots,r$ and no other singular points – indeed, there is an open and dense subset $U$ of curves in $\big|\mathcal{J}_{X/\Sigma}(D)\big|_l$ satisfying these requirements.

Because of Step 2 the general element in $\big|\mathcal{J}_{X/\Sigma}(D)\big|_l$ has an ordinary singular point of multiplicity $m_i$ at $z_i$ and is by Bertini's Theorem (cf. **[Har77]** III.10.9.2) smooth outside its base locus. It remains to show that $\big|\mathcal{J}_{X/\Sigma}(D)\big|_l$ contains an irreducible curve, then the generic curve is irreducible as well.

Suppose therefore that all curves in $\big|\mathcal{J}_{X/\Sigma}(D)\big|_l$ are reducible. Then by a Theorem of Bertini (cf. Theorem E.5) there is an irreducible one-dimensional family $\mathcal{F}$ of curves such that the irreducible components of any curve in $\big|\mathcal{J}_{X/\Sigma}(D)\big|_l$ belong to $\mathcal{F}$, and the base locus of $\mathcal{F}$ is $\{z_1,\ldots,z_r\}$. Moreover, through a very general point $z_0 \in \Sigma$ there is a unique, irreducible curve, say $C_{z_0}$ in $\mathcal{F}$. The elements of $\big|\mathcal{J}_{X_0/\Sigma}(D)\big|_l$ belong to $\big|\mathcal{J}_{X/\Sigma}(D)\big|_l$ and pass through $z_0$. Thus $C_{z_0}$ is a fixed component of $\big|\mathcal{J}_{X_0/\Sigma}(D)\big|_l$, which implies that $\Sigma$ is not the projective plane and $C_{z_0}$ is one of the lines in $B_{z_0}$. But, $z_0$ being general, then there are infinitely many lines in $\mathcal{F}$, thus passing through $z_1$, in contradiction to Lemma E.7.

**Step 6:** $h^1\big(\mathcal{J}_{X/\Sigma}(D)\big) = 0$, which follows immediately from equation (1.7).

**Step 7:** $V_{|D|}(\underline{m})$ is T-smooth at $C$.

By [**GLS98c**], Lemma 2.7, we have

$$\mathcal{J}_{X/\Sigma} \subseteq \mathcal{J}_{X^{es}(C)/\Sigma},$$

and thus by Step 6

$$h^1\big(\mathcal{J}_{X^{es}(C)/\Sigma}(D)\big) = 0,$$

which proves the claim.

**Step 8:** Prove the addendum.

Fixing local coordinates $x_j, y_j$ at $z_j$ we may consider the *linear* maps

$$\varphi_j : H^0\big(\Sigma, \mathcal{J}_{X/\Sigma}(D)\big) \to \mathbb{C}[x_j, y_j]_{m_j} : s \mapsto \text{jet}_{m_j}(s_{z_j}),$$

where $\text{jet}_{m_j}(s_{z_j})$ denotes the $m_j$-jet of the section $s$ in the local coordinates at $z_j$. According to the considerations in Step 2 we know that the $\varphi_j$ are indeed surjective. Thus the sets $\varphi_j^{-1}(U_j)$ are open dense subsets of $H^0\big(\Sigma, \mathcal{J}_{X/\Sigma}(D)\big)$, and if $V \subset H^0\big(\Sigma, \mathcal{J}_{X/\Sigma}(D)\big)$ denotes the open dense subset of sections corresponding to the curves in the set $U$ from Step 5, then $W = V \cap \bigcap_{j=1}^r \varphi_j^{-1}(U_j)$ is non-empty and a curve defined by a section in $W$ satisfies the requirements of the addendum. □

**1.3 Corollary**

*Let* $m_1, \ldots, m_r \in \mathbb{N}_0$*, not all zero, and let* $L \in \text{Div}(\Sigma)$ *be very ample over* $\mathbb{C}$*. Suppose* $D \in \text{Div}(\Sigma)$ *such that*

(1.8) $\quad (D - L - K_\Sigma)^2 \geq 2 \cdot \sum_{i=1}^r (m_i + 1)^2,$

(1.9) $\quad (D-L-K_\Sigma).B > \max\{m_1, \ldots, m_r\}$ *for any irreducible curve* $B \subset \Sigma$ *with* $B^2 = 0$ *and* $\dim |B|_a \geq 1$*,*

(1.10) $\quad D.L - 2g(L) > \max\{m_1, \ldots, m_r\}$*, and*

(1.11) $\quad D - L - K_\Sigma$ *is nef.*

*Then for* $z_1, \ldots, z_r \in \Sigma$ *in very general position there exists an irreducible curve* $C \in |D|_l$ *with ordinary singular points of multiplicity* $m_i$ *at* $z_i$ *for* $i = 1, \ldots, r$ *and no other singular points. Furthermore,*

$$h^1\big(\Sigma, \mathcal{J}_{X(\underline{m};\underline{z})/\Sigma}(D)\big) = 0,$$

*and in particular,* $V_{|D|}(\underline{m})$ *is T-smooth in* $C$.

**Addendum:** *Given local coordinates* $x_j, y_j$ *in* $z_j$ *and open dense subsets* $U_j \subseteq \mathbb{C}[x_j, y_j]_{m_j}$*,* $j = 1, \ldots, r$*, the curve* $C$ *may be chosen such that the* $m_j$*-jet at* $z_j$ *of a section defining* $C$ *belongs to* $U_j$*.*

**Proof:** Follows from Theorem 1.2 and Corollary II.1.2. □

## 2. Existence Theorem for General Singularity Schemes

**Notation**

Througout this section we will be concerned with equisingular respectively equianalytical families of curves. We are treating these two cases simultaneousely, using the suffixes "es" in the equisingular case, that is whenever topological singularity types are considered, "ea" in the equanalytical case, that is when analytical singularity types are considered, and finally, using "*" when both cases are considered at once.

In the following we will denote by $\mathbb{C}[x,y]_d$, respectively by $\mathbb{C}[x,y]_{\leq d}$ the $\mathbb{C}$-vector spaces of polynomials of degree $d$, respectively of degree at most $d$. If $f \in \mathbb{C}[x,y]_{\leq d}$ we denote by $f_k \in \mathbb{C}[x,y]_k$ for $k = 0, \ldots, d$ the homogeneous part of degree $k$ of $f$, and thus $f = \sum_{k=0}^{d} f_k$. By $\underline{a} = (a_{k,l} | 0 \leq k+l \leq d)$ we will denote the coordinates of $\mathbb{C}[x,y]_{\leq d}$ with respect to the basis $\{x^k y^l \mid 0 \leq k+l \leq d\}$.

For any $f \in \mathbb{C}[x,y]_{\leq d}$ the tautological family

$$\mathbb{C}[x,y]_{\leq d} \times \mathbb{C}^2 \supset \bigcup_{g \in \mathbb{C}[x,y]_{\leq d}} \{g\} \times g^{-1}(0) \longrightarrow \mathbb{C}[x,y]_{\leq d}$$

induces a deformation of the plane curve singularity $\big(f^{-1}(0), 0\big)$ whose base space is the germ $\big(\mathbb{C}[x,y]_{\leq d}, f\big)$ of $\mathbb{C}[x,y]_{\leq d}$ at $f$. Given any deformation $(X,x) \hookrightarrow (\mathcal{X},x) \to (S,s)$ of a plane curve singularity $(X,x)$, we will denote by $S^* = (S^*, s)$ the germ of the equisingular respectively equianalytical stratum of $(S,s)$.[2] Thus, fixed an $f \in \mathbb{C}[x,y]_{\leq d}$, $\mathbb{C}[x,y]^*_{\leq d} = \big(\mathbb{C}[x,y]^*_{\leq d}, f\big)$ is the (local) equisingular respectively equianalytical stratum of $\mathbb{C}[x,y]_{\leq d}$ at $f$.

**2.1 Definition**

(a) We say the family $\mathbb{C}[x,y]_{\leq d}$ is topologically respectively analytically *T-smooth* at $f \in \mathbb{C}[x,y]_{\leq d}$ if for any $e \geq d$ there exists a $\Lambda \subset \big\{(k,l) \in \mathbb{N}_0^2 \;\big|\; 0 \leq k+l \leq d\big\}$ with $\#\Lambda = \tau^* = \tau^*\big(f^{-1}(0), 0\big)$ such that $\mathbb{C}[x,y]^*_{\leq e}$ is given by equations

$$a_{k,l} = \phi_{k,l}\big(\underline{a}_{(1)}, \underline{a}_{(2)}\big), \quad (k,l) \in \Lambda,$$

with $\phi_{k,l} \in \mathbb{C}\big\{\underline{a}_{(1)}, \underline{a}_{(2)}\big\}$ where $\underline{a}_{(0)} = (a_{k,l} \mid (k,l) \in \Lambda)$, $\underline{a}_{(1)} = (a_{k,l} \mid 0 \leq k+l \leq d, (k,l) \notin \Lambda)$, and $\underline{a}_{(2)} = (a_{k,l} \mid d+1 \leq k+l \leq e)$.

(b) A polynomial $f \in \mathbb{C}[x,y]_{\leq d}$ is said to be a *good representative* of the topological respectively analytical singularity type $\mathcal{S}$ in $\mathbb{C}[x,y]_{\leq d}$ if it meets the following conditions:

(a) $\operatorname{Sing}\big(f^{-1}(0)\big) = \Big\{p \in \mathbb{C}^2 \;\Big|\; f(p) = 0, \frac{\partial f}{\partial x}(p) = 0, \frac{\partial f}{\partial y}(p) = 0\Big\} = \{0\}$,

(b) $\big(f^{-1}(0), 0\big) \sim_t \mathcal{S}$ respectively $\big(f^{-1}(0), 0\big) \sim_c \mathcal{S}$,

(c) $\mathbb{C}[x,y]_{\leq d}$ is T-smooth at $f$, and

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[2] That is, $S^*$ is the analytical space germ parametrising the subfamily of $(\mathcal{X},x) \to (S,s)$ of singularities which are topologically respectively analytically equivalent to $(X,x)$.

(d) $f_d$ is a generic reduced d-form, that is, there is an open subset $U \subseteq \mathbb{C}[x,y]_d$ of reduced d-forms such that for each $g_d \in U$ there is an $f \in \mathbb{C}[x,y]_{\leq d}$ satisfying (a)-(c) and $f_d = g_d$.

(c) Given a topological respectively analytical singularity type $\mathcal{S}$ we define $e^s(\mathcal{S})$ respectively $e^a(\mathcal{S})$ to be the minimal number d such that $\mathcal{S}$ has a good representative of degree d.
Whenever we consider topological and analytical singularity types at the same time, we will use $e^*(\mathcal{S})$ to denote $e^s(\mathcal{S})$ and $e^a(\mathcal{S})$ respectively.

## 2.2 Remark

(a) The condition for T-smoothness just means that for any $e \geq d$ the equisingular respectively equianalytical stratum $\mathbb{C}[x,y]^*_{\leq e}$ is smooth at the point f of the expected codimension in $\big(\mathbb{C}[x,y]_{\leq e}, f\big)$.

(b) Note that for a polynomial of degree d the highest homogeneous part $f_d$ defines the normal cone, i. e. the intersection of the curve $\{\hat{f} = 0\}$ with the line at infinity in $\mathbb{P}^2_{\mathbb{C}}$, where $\hat{f}$ is the homogenisation of f. Thus the condition "$f_d$ reduced" just means that the line at infinity intersects the curve transversally in d different points.

(c) If $f \in \mathbb{C}[x,y,z]_d$ is an irreducible polynomial such that $(0:0:1)$ is the only singular point of the plane curve $\{f = 0\} \subset \mathbb{P}^2_{\mathbb{C}}$, then a linear change of coordinates of the type $(x,y,z) \mapsto (x,y,z+ax+by)$ will ensure that the dehomogenisation $\check{f}$ of f satisfies "$\check{f}_d$ reduced". Note for this that the coordinate change corresponds to choosing a line in $\mathbb{P}^2_{\mathbb{C}}$, not passing through $(0:0:1)$ and meeting the curve in d distinct points. Therefore, the bounds for $e^s(\mathcal{S})$ and $e^a(\mathcal{S})$ given in **[Shu03]** do apply here.

(d) In order to obtain good numerical conditions for the existence it is vital to find good upper bounds for $e^s(\mathcal{S})$ and $e^a(\mathcal{S})$. We gather here the best known results in this direction obtained by Eugenii Shustin in **[Shu03]** (see also **[GLS05]** Chapter V.4).
If $\mathcal{S}$ is a simple singularity, then of course $e^s(\mathcal{S}) = e^a(\mathcal{S})$.

| $\mathcal{S}$ | $e^s(\mathcal{S}) = e^a(\mathcal{S})$ | $\mathcal{S}$ | $e^s(\mathcal{S}) = e^a(\mathcal{S})$ |
|---|---|---|---|
| $A_1$ | 2 | $D_4$ | 3 |
| $A_2$ | 3 | $D_5$ | 4 |
| $A_\mu, \mu = 3,\ldots,7$ | 4 | $D_\mu, \mu \leq 6,\ldots,10$ | 5 |
| $A_\mu, \mu = 8,\ldots,10$ | 5 | $D_\mu, \mu \leq 11,\ldots,13$ | 6 |
| $A_\mu, \mu \geq 1$ | $\leq 2 \cdot \lfloor\sqrt{\mu+5}\rfloor$ | $D_\mu, \mu \geq 1$ | $\leq 2 \cdot \lfloor\sqrt{\mu+7}\rfloor + 1$ |

| $\mathcal{S}$ | $e^s(\mathcal{S}) = e^a(\mathcal{S})$ | $\mathcal{S}$ | $e^s(\mathcal{S}) = e^a(\mathcal{S})$ | $\mathcal{S}$ | $e^s(\mathcal{S}) = e^a(\mathcal{S})$ |
|---|---|---|---|---|---|
| $E_6$ | 4 | $E_7$ | 4 | $E_8$ | 5 |

For non-simple singularities the invariants for $e^s(\mathcal{S})$ and $e^a(\mathcal{S})$ differ in general. If $\mathcal{S} \notin \{A_1, A_2\}$ is a topological singularity type, then

$$e^s(\mathcal{S}) \le \tfrac{9}{\sqrt{6}} \cdot \sqrt{\delta(\mathcal{S})} - 1,$$

and if $\mathcal{S} \notin \{A_1, A_2\}$ is an analytical singularity type, then

$$e^a(\mathcal{S}) \le 3 \cdot \sqrt{\mu(\mathcal{S})} - 1.$$

Finally, in Lemma V.3.10 we are going to show that for an arbitrary topological or analytical singularity type we have the inequality

$$e^*(\mathcal{S}) \le \tau^*(\mathcal{S}) + 1.$$

(e) For refined results using the techniques of the following proof we refer to [**Shu99**].

**2.3 Theorem (Existence)**
*Let $\mathcal{S}_1, \ldots, \mathcal{S}_r$ be topological or analytical singularity types, and suppose there exists an irreducible curve $C \in |D|_l$ with $\mathrm{Sing}(C) = \{z_1, \ldots, z_r\}$ such that in local coordinates at $z_i$ the $m_i$-jet of a section defining $C$ coincides with the $m_i$-jet of a good representative of $\mathcal{S}_i$,[3] where $m_i = e^s(\mathcal{S}_i)$ respectively $m_i = e^a(\mathcal{S}_i)$, $i = 1, \ldots, r$, and*

$$h^1\big(\Sigma, \mathcal{J}_{X(\underline{m};\underline{z})/\Sigma}(D)\big) = 0.$$

*Then there exists an irreducible curve $\widetilde{C} \in |D|_l$ with precisely $r$ singular points of types $\mathcal{S}_1, \ldots, \mathcal{S}_r$ as its only singularities.*

**Idea of the proof:** The basic idea is to glue locally at the $z_i$ equations of good representatives for the $\mathcal{S}_i$ into the curve $C$. Let us now explain more detailed what we mean by this.

If $g_i = \sum_{k+l=0}^{m_i} a_{k,l}^{i,fix} x_i^k y_i^l$, $i = 1, \ldots, r$, are good representatives of the $\mathcal{S}_i$, then we are looking for a family $F_t$, $t \in (\mathbb{C}, 0)$, in $H^0\big(\Sigma, \mathcal{O}_\Sigma(D)\big)$ which in local coordinates $x_i, y_i$ at $z_i$ looks like

$$F_t^i = \sum_{k+l=0}^{m_i-1} t^{m_i-k-l} \tilde{a}_{k,l}^i(t) \cdot x_i^k y_i^l + \sum_{k+l=m_i} \big(a_{k,l}^{i,fix} + t \cdot \text{h.o.t.}\big) \cdot x_i^k y_i^l + \text{h.o.t.},$$

where the $\tilde{a}_{k,l}^i(t)$ should be convergent power series in $t$ with $\tilde{a}_{k,l}^i(0) = a_{k,l}^{i,fix}$. The curve defined by $F_0$ will just be the curve $C$, while $F_t^i$ for $t \neq 0$ can be transformed, by $(x_i, y_i) \mapsto (tx_i, ty_i)$, into a member of some family

$$\tilde{F}_t^i = \sum_{k+l=0}^{m_i-1} \tilde{a}_{k,l}^i(t) \cdot x_i^k y_i^l + \sum_{k+l=m_i} \big(a_{k,l}^{i,fix} + t \cdot \text{h.o.t.}\big) \cdot x_i^k y_i^l + \text{h.o.t.}, \quad t \in \mathbb{C},$$

[3]This implies in particular that $C$ has an ordinary $m_i$-fold point.

with

$$\tilde{F}_0^i = g_i.$$

Using now the T-smoothness property of $g_i$, $i = 1, \ldots, r$, we can choose the $\tilde{a}^i_{k,l}(t)$ such that this family is equisingular respectively equianalytical. Hence, for small $t \neq 0$, the curve given by $F_t$ will have the right singularities at the $z_i$. Finally, the knowledge on the singularities of $C$ and the Conservation of Milnor numbers will ensure that the curve given by $F_t$ has no further singularities, for $t \neq 0$ sufficiently small.

The proof will be done in several steps. First of all we are going to fix some notation by choosing good representatives for the $\mathcal{S}_i$ (Step 1.1) and by choosing a basis of $H^0\big(\Sigma, \mathcal{O}_\Sigma(D)\big)$ which reflects the "independence" of the coordinates at the different $z_i$ ensured by $h^1\big(\Sigma, \mathcal{J}_{X(\underline{m};\underline{z})/\Sigma}(D)\big) = 0$ (Step 1.1). In a second step we are making an "Ansatz" for the family $F_t$, and, for the local investigation of the singularity type, we are switching to some other families $\tilde{F}^i_t$, $i = 1, \ldots, r$ (Step 2.1). We, then, reduce the problem of $F_t$, for $t \neq 0$ small, having the right singularities to a question about the equisingular respectively equianalytical strata of some families of polynomials (Step 2.2), which in Step 2.3 will be solved. The final step serves to show that the curves $F_t$ have only the singularities which we controlled in the previous steps.

**Proof:**

**Step 1.1:** By the definition of $m_i = e^s(\mathcal{S}_i)$ respectively $m_i = e^a(\mathcal{S}_i)$, we may choose good representatives

$$g_i = \sum_{k+l=0}^{m_i} a^{i,fix}_{k,l} x_i^k y_i^l \in \mathbb{C}[x_i, y_i]_{\leq m_i}$$

for the $\mathcal{S}_i$, $i = 1, \ldots, r$. Let $\underline{a}^{i,fix} = \big(a^{i,fix}_{k,l} \;\big|\; 0 \leq k+l \leq m_i\big)$ and $\underline{a}^{fix} = \big(\underline{a}^{1,fix}, \ldots, \underline{a}^{r,fix}\big)$.

**Step 1.2:** Parametrise $|D|_l = \mathbb{P}\big(H^0\big(\mathcal{O}_\Sigma(D)\big)\big)$.

Consider the following exact sequence:

$$0 \longrightarrow \mathcal{J}_{X(\underline{m};\underline{z})/\Sigma}(D) \longrightarrow \mathcal{O}_\Sigma(D) \longrightarrow \bigoplus_{i=1}^r \mathcal{O}_{\Sigma,z_i}/\mathfrak{m}^{m_i}_{\Sigma,z_i} \longrightarrow 0.$$

Since $h^1\big(\mathcal{J}_{X(\underline{m};\underline{z})/\Sigma}(D)\big) = 0$, the long exact cohomology sequence gives

$$H^0\big(\mathcal{O}_\Sigma(D)\big) = \bigoplus_{i=1}^r \mathbb{C}\{x_i, y_i\}/(x_i, y_i)^{m_i} \oplus H^0\big(\mathcal{J}_{X(\underline{m};\underline{z})/\Sigma}(D)\big),$$

where $x_i, y_i$ are local coordinates of $(\Sigma, z_i)$.

We, therefore, can find a basis $\big\{s^i_{k,l}, s_j \;\big|\; j = 1, \ldots, e,\; i = 1, \ldots, r,\; 0 \leq k+l \leq m_i - 1\big\}$ of $H^0\big(\mathcal{O}_\Sigma(D)\big)$, with $e = h^0\big(\mathcal{J}_{X(\underline{m};\underline{z})/\Sigma}(D)\big)$, such that[4]

[4]Throughout this proof we will use the multi index notation $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}^2$ and $|\alpha| = \alpha_1 + \alpha_2$.

(1) $C$ is the curve defined by $s_1$,

(2) $(s_1)_{z_i} = \sum_{|\alpha|\geq m_i} B^i_\alpha x_i^{\alpha_1} y_i^{\alpha_2}$ for $i = 1, \ldots, r$, with $B^i_\alpha = a^{i,fix}_{\alpha_1,\alpha_2}$ when $|\alpha| = m_i$,

(3) $$\left(s^j_{k,l}\right)_{z_i} = \begin{cases} x_i^k y_i^l + \sum\limits_{|\alpha|\geq m_i} A^{i,i}_{\alpha,k,l} x_i^{\alpha_1} y_i^{\alpha_2}, & \text{if } i = j, \\ \sum\limits_{|\alpha|\geq m_i} A^{j,i}_{\alpha,k,l} x_i^{\alpha_1} y_i^{\alpha_2}, & \text{if } i \neq j. \end{cases}$$

Let us now denote the coordinates of $H^0\big(\mathcal{O}_\Sigma(D)\big)$ w. r. t. this basis by $(\underline{a}, \underline{b}) = \big(\underline{a}^1, \ldots, \underline{a}^r, \underline{b}\big)$ with $\underline{a}^i = \big(a^i_{k,l} \mid 0 \leq k+l \leq m_i - 1\big)$ and $\underline{b} = (b_j \mid j = 1, \ldots, e)$.

Thus the family

$$F_{(\underline{a},\underline{b})} = \sum_{i=1}^{r} \sum_{k+l=0}^{m_i-1} a^i_{k,l} s^i_{k,l} + \sum_{j=1}^{e} b_j s_j, \quad (\underline{a}, \underline{b}) \in \mathbb{C}^N \text{ with } N = e + \sum_{i=1}^{r} \binom{m_i+1}{2},$$

parametrises $H^0\big(\mathcal{O}_\Sigma(D)\big)$.

**Step 2:** We are going to glue the good representatives for the $\mathcal{S}_i$ into the curve $C$. More precisely, we are constructing a subfamily $F_t$, $t \in (\mathbb{C}, 0)$, in $H^0\big(\mathcal{O}_\Sigma(D)\big)$ such that $F_0 = s_1$ and locally in $z_i$, $i = 1, \ldots, r$, the $F_t$, for small $t \neq 0$, can be transformed into members of a fixed $\mathcal{S}_i$-equisingular respectively $\mathcal{S}_i$-equianalytical family.

**Step 2.1:** "Ansatz" and first reduction for a local investigation.

Let us make the following "Ansatz":

$$b_1 = 1,\ b_2 = \ldots = b_e = 0,$$

$$a^i_{k,l} = t^{m_i-k-l} \cdot \tilde{a}^i_{k,l}, \ \text{ for } i = 1, \ldots, r,\ 0 \leq k+l \leq m_i - 1.$$

This gives rise to a family

$$F_{(t,\underline{\tilde{a}})} = s_1 + \sum_{i=1}^{r} \sum_{k+l=0}^{m_i-1} t^{m_i-k-l} \tilde{a}^i_{k,l} s^i_{k,l} \in H^0\big(\mathcal{O}_\Sigma(D)\big)$$

with $t \in \mathbb{C}$ and $\underline{\tilde{a}} = \big(\underline{\tilde{a}}^1, \ldots, \underline{\tilde{a}}^r\big)$ where $\underline{\tilde{a}}^i = \big(\tilde{a}^i_{k,l} \mid 0 \leq k+l \leq m_i - 1\big) \in \mathbb{C}^{N_i}$ with $N_i = \binom{m_i+1}{2}$. In particular, $F_{(0,\underline{\tilde{a}})} = s_1$.

Fixing $i \in \{1, \ldots, r\}$, in local coordinates at $z_i$ the family looks like

$$F^i_{(t,\underline{\tilde{a}})} := \big(F_{(t,\underline{\tilde{a}})}\big)_{z_i} = \sum_{k+l=0}^{m_i-1} t^{m_i-k-l} \tilde{a}^i_{k,l} x_i^k y_i^l + \sum_{|\alpha|\geq m_i} \varphi^i_\alpha(t, \underline{\tilde{a}})\, x_i^{\alpha_1} y_i^{\alpha_2},$$

with

$$\varphi^i_\alpha(t, \underline{\tilde{a}}) = B^i_\alpha + \sum_{j=1}^{r} \sum_{k+l=0}^{m_j-1} t^{m_j-k-l} \tilde{a}^j_{k,l} A^{j,i}_{\alpha,k,l},$$

and thus

$$\varphi^i_\alpha(0, \underline{\tilde{a}}) = B^i_\alpha = a^{i,fix}_{\alpha_1,\alpha_2}, \quad \text{for } |\alpha| = m_i.$$

For $t \neq 0$ the transformation $\psi_t^i : (x_i, y_i) \mapsto (tx_i, ty_i)$ is indeed a coordinate transformation, and thus $F^i_{(t,\underline{\tilde{a}})}$ is contact equivalent to

$$\tilde{F}^i_{(t,\underline{\tilde{a}})} := t^{-m_i} \cdot F^i_{(t,\underline{\tilde{a}})}(tx_i, ty_i) = \sum_{k+l=0}^{m_i-1} \tilde{a}^i_{k,l} x_i^k y_i^l + \sum_{|\alpha| \geq m_i} t^{|\alpha|-m_i} \varphi^i_\alpha(t, \underline{\tilde{a}})\, x_i^{\alpha_1} y_i^{\alpha_2}.$$

Note that for this new family in $\mathbb{C}\{x_i, y_i\}$ we have

$$\tilde{F}^i_{(0,\underline{a}^{fix})} = \sum_{k+l=0}^{m_i} a^{i,fix}_{k,l} x_i^k y_i^l = g_i,$$

and hence it gives rise to a deformation of $\big(g_i^{-1}(0), 0\big)$.

**Step 2.2:** Reduction to the investigation of the equisingular respectively equianalytical strata of certain families of polynomials.

It is basically our aim to verify the $\underline{\tilde{a}}$ as convergent power series in $t$ such that the corresponding family is equisingular respectively equianalytical. However, since the $\tilde{F}^i_{(t,\underline{\tilde{a}})}$ are power series in $x_i$ and $y_i$, we cannot right away apply the T-smoothness property of $g_i$, but we rather have to reduce to polynomials. For this let $e_i$ be the determinacy bound[5] of $\mathcal{S}_i$ and define

$$\hat{F}^i_{(t,\underline{\tilde{a}})} := \sum_{k+l=0}^{m_i-1} \tilde{a}^i_{k,l} x_i^k y_i^l + \sum_{|\alpha|=m_i}^{e_i} t^{|\alpha|-m_i} \varphi^i_\alpha(t, \underline{\tilde{a}})\, x_i^{\alpha_1} y_i^{\alpha_2} \equiv \tilde{F}^i_{(t,\underline{\tilde{a}})} \big(\mathrm{mod}\ (x_i, y_i)^{e_i+1}\big).$$

Thus $\hat{F}^i_{(t,\underline{\tilde{a}})}$ is a family in $\mathbb{C}[x_i, y_i]_{\leq e_i}$, and still

$$\hat{F}^i_{(0,\underline{a}^{fix})} = \tilde{F}^i_{(0,\underline{a}^{fix})} = g_i.$$

We claim that it suffices to find $\underline{\tilde{a}}(t) \in \mathbb{C}\{t\}$ with $\underline{\tilde{a}}(0) = \big(a^{i,fix}_{k,l} \mid i = 1, \ldots, r,\ 0 \leq k + l \leq m_i - 1\big)$, such that the families $\hat{F}^i_t := \hat{F}^i_{(t,\underline{\tilde{a}}(t))}$, $t \in (\mathbb{C}, 0)$, are in the equisingular respectively equianalytical strata $\mathbb{C}[x_i, y_i]^*_{\leq e_i}$, for $i = 1, \ldots, r$.

Since then we have, for small[6] $t \neq 0$,

$$g_i = \hat{F}^i_0 \sim_c \hat{F}^i_t \sim \tilde{F}^i_{(t,\underline{\tilde{a}}(t))} \sim_c F^i_{(t,\underline{\tilde{a}}(t))} = \big(F_{(t,\underline{\tilde{a}}(t))}\big)_{z_i(t)},$$

by the $e_i$-determinacy and since $\psi_t^i$ is a coordinate change for $t \neq 0$, which proves condition (2). Note that the singular points $z_i$ will move with $t$.

**Step 2.3:** Find $\underline{\tilde{a}}(t) \in \mathbb{C}\{t\}^n$ with $\underline{\tilde{a}}(0) = \big(a^{i,fix}_{k,l}, i = 1, \ldots, r,\ 0 \leq k+l \leq m_i - 1\big)$, $n = \sum_{i=1}^r \binom{m_i+1}{2}$, such that the families $\hat{F}^i_t = \hat{F}^i_{(t,\underline{\tilde{a}}(t))}$, $t \in (\mathbb{C}, 0)$, are in the equisingular respectively equianalytical strata $\mathbb{C}[x_i, y_i]^*_{\leq e_i}$, for $i = 1, \ldots r$.

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[5]A power series $f \in \mathcal{O}_2 = \mathbb{C}\{x, y\}$ (respectively the singularity $\big(V(f), 0\big)$ defined by $f$) is said to be *finitely determined* with respect to some equivalence relation $\sim$ if there exists some positive integer $e$ such that $f \sim g$ whenever $f$ and $g$ have the same $e$-jet. If $f$ is finitely determined, the smallest possible $e$ is called the *determinacy bound*. Isolated singularities are finitely determined with respect to analytical equivalence and hence as well with respect to topological equivalence. C. f. [**DeP00**] Theorem 9.1.3 and Footnote 6.

[6] Here $f \sim g$ means $f \sim_t g$ respectively $f \sim_c g$ in the sense of Definition I.2.1. Note that if $f$ and $g$ are contact equivalent, then there exists even an analytic coordinate change $\Phi$, that is, $f \sim_c g$ implies $f \sim_t g$.

In the sequel we adopt the notation of Definition 2.1 adding indices $i$ in the obvious way.

Since $\mathbb{C}[x_i, y_i]_{\le m_i}$ is T-smooth at $g_i$, for $i = 1, \ldots, r$, there exist $\Lambda_i \subseteq \{(k,l) \mid 0 \le k + l \le m_i\}$ and power series $\phi^i_{k,l} \in \mathbb{C}\{\underline{\tilde{a}}^i_{(1)}, \underline{\tilde{a}}^i_{(2)}\}$, for $(k,l) \in \Lambda_i$, such that the equisingular respectively equianalytical stratum $\mathbb{C}[x_i, y_i]^*_{\le e_i}$ is given by the $\tau^*_i = \tau^*(\mathcal{S}_i) = \#\Lambda_i$ equations

$$\tilde{a}^i_{k,l} = \phi^i_{k,l}\big(\underline{\tilde{a}}^i_{(1)}, \underline{\tilde{a}}^i_{(2)}\big), \quad \text{for } (k,l) \in \Lambda_i.$$

Setting $\Lambda = \bigcup_{j=1}^r \{j\} \times \Lambda_j$ we use the notation $\underline{\tilde{a}}_{(0)} = \big(\underline{\tilde{a}}^1_{(0)}, \ldots, \underline{\tilde{a}}^r_{(0)}\big) = \big(\tilde{a}^i_{k,l} \;\big|\; (i,k,l) \in \Lambda\big)$ and, similarly $\underline{\tilde{a}}_{(1)}$, $\underline{\tilde{a}}_{(2)}$, $\underline{a}^{i,fix}_{(1)}$, $\underline{a}^{fix}_{(0)}$, and $\underline{a}^{fix}_{(1)}$. Moreover, setting $\tilde{\varphi}^i\big(t, \underline{\tilde{a}}_{(0)}\big) = \big(t^{|\alpha| - m_i} \varphi^i_\alpha\big(t, \underline{\tilde{a}}_{(0)}, \underline{a}^{fix}_{(1)}\big) \;\big|\; m_i \le |\alpha| \le e_i\big)$, we define an analytic map germ

$$\Phi : \Big(\mathbb{C} \times \mathbb{C}^{\tau^*_1} \times \cdots \times \mathbb{C}^{\tau^*_r}, \big(0, \underline{a}^{fix}_{(0)}\big)\Big) \to \Big(\mathbb{C}^{\tau^*_1} \times \cdots \times \mathbb{C}^{\tau^*_r}, 0\Big)$$

by

$$\Phi^i_{k,l}\big(t, \underline{\tilde{a}}_{(0)}\big) = \tilde{a}^i_{k,l} - \phi^i_{k,l}\Big(\underline{a}^{i,fix}_{(1)}, \tilde{\varphi}^i\big(t, \underline{\tilde{a}}_{(0)}\big)\Big), \quad \text{for } (i,k,l) \in \Lambda,$$

and we consider the system of equations

$$\Phi^i_{k,l}\big(t, \underline{\tilde{a}}_{(0)}\big) = 0, \quad \text{for } (i,k,l) \in \Lambda.$$

We show in Step 2.4 that

$$\left(\frac{\partial \Phi^i_{k,l}}{\partial \tilde{a}^j_{\kappa,\lambda}}\big(0, \underline{a}^{fix}_{(0)}\big)\right)_{(i,k,l),(j,\kappa,\lambda) \in \Lambda} = \mathrm{id}_{\mathbb{C}^n}\,.$$

Thus by the Inverse Function Theorem there exist $\tilde{a}^i_{k,l}(t) \in \mathbb{C}\{t\}$ with $\tilde{a}^i_{k,l}(0) = a^{i,fix}_{k,l}$ such that

$$\tilde{a}^i_{k,l}(t) = \phi^i_{k,l}\Big(\underline{a}^{i,fix}_{(1)}, \tilde{\varphi}^i\big(t, \underline{\tilde{a}}_{(0)}(t)\big)\Big), \quad (i,k,l) \in \Lambda.$$

Now, setting $\underline{\tilde{a}}_{(1)}(t) \equiv \underline{a}^{fix}_{(1)}$, the families $\hat{F}^i_t = \hat{F}^i_{(t,\underline{\tilde{a}}(t))}$ are in the equisingular respectively equianalytical strata $\mathbb{C}[x_i, y_i]^*_{\le e_i}$, for $i = 1, \ldots, r$.

**Step 2.4:** Show that $\left(\frac{\partial \Phi^i_{k,l}}{\partial \tilde{a}^j_{\kappa,\lambda}}\big(0, \underline{a}^{fix}_{(0)}\big)\right)_{(i,k,l),(j,\kappa,\lambda) \in \Lambda} = \mathrm{id}_n$.

For this it suffices to show that

$$\left.\frac{\partial \phi^i_{k,l}\big(\underline{a}^{i,fix}_{(1)}, \tilde{\varphi}^i\big(t, \underline{\tilde{a}}_{(0)}\big)\big)}{\partial \tilde{a}^j_{\kappa,\lambda}}\right|_{t=0, \underline{\tilde{a}}_{(0)} = \underline{a}^{fix}_{(0)}} = 0,$$

for any $(i,k,l), (j,\kappa,\lambda) \in \Lambda$, which is fulfilled since the map

$$\theta : \big(t, \underline{\tilde{a}}_{(0)}\big) \mapsto \Big(\underline{a}^{fix}_{(1)}, \tilde{\varphi}\big(t, \underline{\tilde{a}}_{(0)}\big)\Big)$$

satisfies

$$\left.\frac{\partial \theta}{\partial \tilde{a}^j_{\kappa,\lambda}}\right|_{t=0, \underline{\tilde{a}}_{(0)} = \underline{a}^{fix}_{(0)}} = \left(\underline{0}, \left.\frac{\partial \tilde{\varphi}^i\big(t, \underline{\tilde{a}}_{(0)}\big)}{\partial \tilde{a}^j_{\kappa,\lambda}}\right|_{t=0, \underline{\tilde{a}}_{(0)} = \underline{a}^{fix}_{(0)}}\right) = 0.$$

For the latter notice that

$$\frac{\partial\tilde{\varphi}^i\big(t,\underline{\tilde{a}}_{(0)}\big)}{\partial\tilde{a}^j_{\kappa,\lambda}}\Bigg|_{t=0,\underline{\tilde{a}}_{(0)}=\underline{a}^{fix}_{(0)}} = \left( t^{|\alpha|-m_i}\cdot\frac{\partial\varphi^i_\alpha\big(t,\underline{\tilde{a}}_{(0)},\underline{a}^{fix}_{(1)}\big)}{\partial\tilde{a}^j_{\kappa,\lambda}}\Bigg|_{t=0,\underline{\tilde{a}}_{(0)}=\underline{a}^{fix}_{(0)}} \;\Bigg|\; m_i\le|\alpha|\le e_i\right)$$

and for $|\alpha|=m_i$

$$\varphi^i_\alpha(t,\underline{\tilde{a}}) = a^{i,fix}_{\alpha_1,\alpha_2} + t\cdot\sum_{j=1}^{r}\sum_{k+l=0}^{m_j-1} t^{m_j-1-k-l}\tilde{a}^j_{k,l}A^{j,i}_{\alpha,k,l}.$$

**Step 3:** It finally remains to show that $F_t$, for small $t\neq 0$, has no other singular points than $z_1(t),\ldots,z_r(t)$.

$C_t\in|D|_l$ shall denote the curve defined by $F_t$, $t\in(\mathbb{C},0)$, in particular $C_0=C$.

Since for any $i=1,\ldots,r$ the family $F_t$, $t\in(\mathbb{C},0)$, induces a deformation of the singularity $(C_0,z_i)$ there are, by the Conservation of Milnor Numbers (cf. [**DeP00**], Chapter 6), (Euclidean) open neighbourhoods $U(z_i)\subset\Sigma$ and $V(0)\subset\mathbb{C}$ such that for any $t\in V(0)$

(2.1) $\operatorname{Sing}(C_t)\subset\bigcup_{i=1}^r U(z_i)$, i. e. singular points of $C_t$ come from singular points of $C_0$,

(2.2) $\mu(C_0,z_i)=\sum_{z\in\operatorname{Sing}(F^i_t)\cap U(z_i)}\mu\big(F^i_t,z\big),\quad i=1,\ldots,r.$

Let $i\in\{1,\ldots,r\}$. For $t\neq 0$ fixed we consider the transformation $\psi'^i_t$ defined by the coordinate change $\psi^i_t$,

$$\begin{array}{ccccccc}\mathbb{C}^2 & \supset & U(z_i) & \longrightarrow & U_t(z_i) & \subset & \mathbb{C}^2\\ & & \cup\!\!\!\!\!\;\shortmid & & \cup\!\!\!\!\!\;\shortmid & & \\ & & (x_i,y_i) & \mapsto & \big(\frac{1}{t}x_i,\frac{1}{t}y_i\big), & & \end{array}$$

and the transformed equations

$$\tilde{F}^i_t(x_i,y_i)=t^{-m_i}F^i_t(tx_i,ty_i)=0.$$

Then obviously,

$$F^i_t(z)=0\quad\Longleftrightarrow\quad\big(\tilde{F}^i_t\circ\psi'^i_t\big)(z)=0,$$

and

$$\nabla\psi'^i_t\equiv\frac{1}{t}\,\mathrm{id}_{\mathbb{C}^2}\,.$$

Thus we have

$$(m_i-1)^2=\mu(C_0,z_i)=\sum_{z\in\operatorname{Sing}(F^i_t)\cap U(z_i)}\mu\big(F^i_t,z\big)=\sum_{z\in\operatorname{Sing}(\tilde{F}^i_t)\cap U_t(z_i)}\mu(\tilde{F}^i_t,z).$$

For $t\neq 0$ very small $U_t(z_i)$ becomes very large, so that, by shrinking $V(0)$ we may suppose that for any $0\neq t\in V(0)$

$$\operatorname{Sing}(g_i)\subset U_t(z_i),$$

and that for any $z \in \mathrm{Sing}(g_i)$ there is an open neighbourhood $U(z) \subset U_t(z_i)$ such that

$$\mu(g_i, z) = \sum_{z' \in \mathrm{Sing}(\tilde{F}_t^i) \cap U(z)} \mu\big(\tilde{F}_t^i, z'\big).$$

If we now take into account that $g_i$ has precisely one critical point, $z_i$, on its zero level, and that the critical points on the zero level of $\tilde{F}_t^i$ all contribute to the Milnor number $\mu(g_i, z_i)$, then we get the following sequence of inequalities:[7]

$$(m_i - 1)^2 - \mu(\mathcal{S}_i) = \sum_{z \in \mathrm{Sing}(g_i)} \mu(g_i, z) - \sum_{z \in \mathrm{Sing}(g_i^{-1}(0))} \mu(g_i, z)$$

$$\leq \sum_{z \in \mathrm{Sing}(\tilde{F}_t^i) \cap U_t(z_i)} \mu\big(\tilde{F}_t^i, z\big) - \sum_{z \in \mathrm{Sing}((\tilde{F}_t^i)^{-1}(0)) \cap U_t(z_i)} \mu\big(\tilde{F}_t^i, z\big)$$

$$= \sum_{z \in \mathrm{Sing}(F_t^i) \cap U(z_i)} \mu(F_t^i, z) - \sum_{z \in \mathrm{Sing}((F_t^i)^{-1}(0)) \cap U(z_i)} \mu(F_t^i, z)$$

$$\leq \mu(C_0, z_i) - \mu\big(F_t^i, z_i\big) = (m_i - 1)^2 - \mu(\mathcal{S}_i).$$

Hence all inequalities must have been equalities, and, in particular,

$$\mathrm{Sing}(C_t) \cap U(z_i) = \mathrm{Sing}\left((F_t^i)^{-1}(0)\right) \cap U(z_i) = \{z_i\},$$

which in view of condition (2.1) finishes the proof.

Note that $C_t$, being a small deformation of the irreducible reduced curve $C = C_0$, will again be irreducible and reduced. □

Now applying the existence theorem of Section 1 for ordinary fat point schemes we deduce the following corollary, giving explicit numerical criteria for the existence of curves with prescribed topological respectively analytical singularities on an arbitrary smooth projective surface.

**2.4 Corollary**

*Let* $L \in \mathrm{Div}(\Sigma)$ *be very ample over* $\mathbb{C}$. *Suppose that* $D \in \mathrm{Div}(\Sigma)$ *and* $\mathcal{S}_1, \ldots, \mathcal{S}_r$ *are topological respectively analytical singularity types with* $e^*(\mathcal{S}_1) \geq \ldots \geq e^*(\mathcal{S}_r)$ *satisfying*

(2.3) $\qquad (D - L - K_\Sigma)^2 \geq 2 \sum_{i=1}^r \left(e^*(\mathcal{S}_i) + 1\right)^2,$

(2.4) $\qquad (D - L - K_\Sigma).B > e^*(\mathcal{S}_1)$

*for any irreducible curve* $B$ *with* $B^2 = 0$ *and* $\dim |B|_a > 0$,

[7] Note, an ordinary plane curve singularity $(X, x)$ of multiplicity $m$ has Milnor number $\mu(X, x) = \dim_{\mathbb{C}} \left(\mathbb{C}\{x, y\}/(x^{m-1}, y^{m-1})\right) = (m-1)^2$. And, for an affine plane curve given by an equation $g$ such that the equation has no critical point at infinity we have by Bézout's Theorem:

$$(m-1)^2 = \sum_{p \in \mathbb{A}^2} i\left(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}; p\right),$$

where by definition $i\big(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}; p\big)$ and $\mu(g, p)$ are defined as the dimension of the same vector spaces.

$$D.L - 2g(L) > e^*(\mathcal{S}_1), \text{ and} \tag{2.5}$$

$$D - L - K_\Sigma \text{ is nef,} \tag{2.6}$$

*then there is an irreducible reduced curve* $C$ *in* $|D|_l$ *with* $r$ *singular points of topological respectively analytical types* $\mathcal{S}_1, \dots, \mathcal{S}_r$ *as its only singularities.*

**Proof:** Due to the definition of a good representative there is an open dense subset $U_i$ of $m_i$-forms for which there exists a good representative of $\mathcal{S}_i$, where $m_i = e^*(\mathcal{S}_i)$. The above conditions then imply with the aid of Corollary 1.3 the existence of an irreducible curve $C \in |D|_l$ with $\mathrm{Sing}(C) = \{z_1, \dots, z_r\}$ whose $m_i$-jet in local coordinates at $z_i$ coincides with the $m_i$-jet of a good representative. Moreover, $h^1\big(\Sigma, \mathcal{J}_{X(\underline{m};\underline{z})/\Sigma}(D)\big) = 0$ and thus Theorem 2.3 applies. □

In view of the bounds for $m_i$ in Remark 2.2 we get the following corollary.

**2.5 Corollary**

*Let* $L \in \mathrm{Div}(\Sigma)$ *be very ample over* $\mathbb{C}$ *and* $D \in \mathrm{Div}(\Sigma)$.

(a) *Suppose that* $\mathcal{S}_1, \dots, \mathcal{S}_r$ *are topological singularity types with* $\delta(\mathcal{S}_1) \geq \dots \geq \delta(\mathcal{S}_r)$*, among them* $k$ *nodes and* $m$ *cusps, satisfying*

$$(D - L - K_\Sigma)^2 \geq 18k + 32m + 27 \cdot \sum_{i=1}^{r-k-m} \delta(\mathcal{S}_i), \tag{2.7}$$

$$(D - L - K_\Sigma).B > \frac{9}{\sqrt{6}} \cdot \sqrt{\delta(\mathcal{S}_1)} - 1 \tag{2.8}$$

*for any irreducible curve* $B$ *with* $B^2 = 0$ *and* $\dim |B|_a > 0$,

$$D.L - 2g(L) > \frac{9}{\sqrt{6}} \cdot \sqrt{\delta(\mathcal{S}_1)} - 1, \text{ and} \tag{2.9}$$

$$D - L - K_\Sigma \text{ is nef,} \tag{2.10}$$

*then there is an irreducible reduced curve* $C$ *in* $|D|_l$ *with* $r$ *singular points of topological types* $\mathcal{S}_1, \dots, \mathcal{S}_r$ *as its only singularities.*

(b) *Suppose that* $\mathcal{S}_1, \dots, \mathcal{S}_r$ *are analytical singularity types with* $\mu(\mathcal{S}_1) \geq \dots \geq \mu(\mathcal{S}_r)$*, among them* $k$ *nodes and* $m$ *cusps, satisfying*

$$(D - L - K_\Sigma)^2 \geq 18k + 32m + 18 \cdot \sum_{i=1}^{r-k-m} \mu(\mathcal{S}_i), \tag{2.11}$$

$$(D - L - K_\Sigma).B > 3 \cdot \sqrt{\mu(\mathcal{S}_1)} - 1 \tag{2.12}$$

*for any irreducible curve* $B$ *with* $B^2 = 0$ *and* $\dim |B|_a > 0$,

$$D.L - 2g(L) > 3 \cdot \sqrt{\mu(\mathcal{S}_1)} - 1, \text{ and} \tag{2.13}$$

$$D - L - K_\Sigma \text{ is nef,} \tag{2.14}$$

*then there is an irreducible reduced curve* $C$ *in* $|D|_l$ *with* $r$ *singular points of analytical types* $\mathcal{S}_1, \dots, \mathcal{S}_r$ *as its only singularities.*

**2.6 Remark**

Knowing something more about the singularity type one can achieve much better results, applying the corresponding bounds for the $e^s(\mathcal{S}_i)$ respectively $e^a(\mathcal{S}_i)$.

## 3. Examples

In this section we are going to examine the conditions in the existence results for various types of surfaces. $m_1, \ldots, m_r \in \mathbb{N}$ are positive integers.

### 3.a. The Classical Case - $\Sigma = \mathbb{P}^2_{\mathbb{C}}$

We note that in $\mathbb{P}^2_{\mathbb{C}}$ there are no irreducible curves of self-intersection number zero, and we take $L \in |\mathcal{O}_{\mathbb{P}^2_{\mathbb{C}}}(1)|_l$ to be a generic line.

In the existence theorem Corollary 1.3 for ordinary fat point schemes, we, of course, find that condition (1.9) is obsolete, and so is (1.11), taking into account that (1.10) implies $d > 0$. But then conditions (1.10) becomes also redundant in view of condition (1.8) and the assumption $d \geq 3$.

Thus Corollary 1.3 and Corollary 2.5 reduce to the following versions.

**1.3a Corollary**
*Let $H$ be a line in $\mathbb{P}^2_{\mathbb{C}}$, and $d \geq 3$ an integer satisfying*

(1.8a) $$(d+2)^2 \geq 2 \cdot \sum_{i=1}^{r} (m_i + 1)^2,$$

*then for $z_1, \ldots, z_r \in \Sigma$ in very general position there is an irreducible reduced curve $C \in |dH|_l$ with ordinary singularities of multiplicities $m_i$ at the $z_i$ as only singularities. Moreover, $V_{|dH|}(\underline{m})$ is T-smooth at $C$.*

**2.5a Corollary**
*Let $H$ be a line in $\mathbb{P}^2_{\mathbb{C}}$, $\mathcal{S}_1, \ldots, \mathcal{S}_r$ topological respectively analytical singularity types, which are neither nodes nor cusps, and $d \geq 3$ an integer such that*

(2.7a) $$(d+2)^2 \geq 18k + 32m + 27 \cdot \sum_{i=1}^{r} \delta(\mathcal{S}_i)$$

*respectively*

(2.11a) $$(d+2)^2 \geq 18k + 32m + 18 \cdot \sum_{i=1}^{r} \mu(\mathcal{S}_i).$$

*Then there is an irreducible reduced curve $C$ in $|D|_l$ with $r$ singular points of topological respectively analytical types $\mathcal{S}_1, \ldots, \mathcal{S}_r$, $k$ nodes and $m$ cusps as its only singularities.*

Our results for ordinary multiple points are weaker than those in [**GLS98c**] (see also [**Los98**] Proposition 4.11), where the factor 2 is replaced by $\frac{10}{9}$) which use the Vanishing Theorem of Geng Xu (cf. [**Xu95**] Theorem 3), particularly designed for $\mathbb{P}^2_{\mathbb{C}}$. Similarly, our general conditions are weaker than the conditions which recently have been found by Shustin applying the Castelnuovo function (cf. [**Shu03**]). – Using $L \in |\mathcal{O}_\Sigma(l)|_l$ with $l > 1$ instead of $\mathcal{O}_\Sigma(1)$ in Corollary 1.3 does not improve the conditions.

## 3.b. Geometrically Ruled Surfaces

Throughout this section we use the notation of Section G.a.

With the choice of $L = C_0 + lF$ as indicated in Remark G.4 we have $g(L) = g$, and hence the generic curve in $|L|_l$ is a smooth curve whose genus equals the genus of the base curve.

In order to obtain nice formulae we considered $D = (a-2)C_0 + (b-2+2g)F$ in the formulation of the vanishing theorem (Corollary II.1.2b). For the existence theorems it turns out that the formulae look best if we work with the divisor $D = (a-1)C_0 + (b+l+2g-2-e)F$ instead. In the case of Hirzebruch surfaces this is just $D = (a-1)C_0 + (b-1)F$.

**1.3b Corollary**

*Given integers* $a, b \in \mathbb{Z}$ *satisfying*

(1.8b) $\quad a\left(b - \frac{a}{2}e\right) \geq \sum_{i=1}^{r} (m_i + 1)^2,$

(1.9b.i) $\quad a > \max\{m_i \mid i = 1, \ldots, r\},$

(1.9b.ii) $\quad b > \max\{m_i \mid i = 1, \ldots, r\},$ *if* $e = 0$,

(1.9b.iii) $\quad 2 \cdot \left(b - \frac{a}{2}e\right) > \max\{m_i \mid i = 1, \ldots, r\},$ *if* $e < 0$, *and*

(1.11b) $\quad b \geq ae$, *if* $e > 0$,

*then for* $z_1, \ldots, z_r \in \Sigma$ *in very general position there is an irreducible reduced curve* $C \in |(a-1)C_0 + (b+l+2g-2-e)F|_a$ *with ordinary singularities of multiplicities* $m_i$ *at the* $z_i$ *as only singularities. Moreover,* $V_{|C|}(\underline{m})$ *is T-smooth at* $C$.

**Proof:** Note that by (1.8b) and (1.9b.i) $b > \frac{a}{2}e \geq ae$, if $e \leq 0$, and thus the inequality

$$b \geq ae, \tag{3.1}$$

is fulfilled no matter what $e$ is.

Noting that $D - L - K_\Sigma \sim_a aC_0 + bF$, it is in view of Lemma G.2 clear, that the conditions (1.8) and (1.9) take the form (1.8b) respectively (1.9b). It, therefore, remains to show that (1.10) is obsolete, and that (1.11) takes the form (1.11b), which in particular means that it is obsolete in the case $\Sigma \cong C \times \mathbb{P}^1_{\mathbb{C}}$.

**Step 1:** (1.10) is obsolete.

If $\Sigma \not\cong \mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$, then $l \geq 2$. Since, moreover, $g(L) = g$ and $D.L = a(l-e) + b + 2g - 2$, condition (1.9b.i) and (3.1) imply (1.10), i. e. for all $i, j$

$$D.L - 2g(L) = a(l-e) + b - 2 \geq \begin{cases} a + b - 2 > m_i, & \text{if } \Sigma \cong \mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}, \\ 2a + (b - ae) - 2 > m_i, & \text{else.} \end{cases}$$

**Step 2:** (1.11) takes the form (1.11b).

If $e \leq 0$, then by **[Har77]** V.2.20 and V.2.21 we find in view of (1.9b.i)-(1.9b.iii) that $D - L - K_\Sigma$ is even ample, while, if $e > 0$, the result follows from Lemma G.1. □

**2.5b Corollary**
*Given integers* $a$ *and* $b$.

(a) *Let* $\mathcal{S}_1, \ldots, \mathcal{S}_r$ *topological singularity types with* $\delta(\mathcal{S}_1) \geq \ldots \geq \delta(\mathcal{S}_r)$, *among them* $k$ *nodes and* $m$ *cusps, such that*

(2.7b) $a(b - \frac{a}{2}e) \geq 9k + 16m + \frac{27}{2} \cdot \sum_{i=1}^{r-k-m} \delta(\mathcal{S}_i)$,

(2.8b.i) $a > \frac{9}{\sqrt{6}} \cdot \sqrt{\delta(\mathcal{S}_1)} - 1$

(2.8b.ii) $b > \frac{9}{\sqrt{6}} \cdot \sqrt{\delta(\mathcal{S}_1)} - 1$ *if* $e = 0$,

(2.8b.iii) $2\big(b - \frac{a}{2}e\big) > \frac{9}{\sqrt{6}} \cdot \sqrt{\delta(\mathcal{S}_1)} - 1$ *if* $e < 0$, *and*

(2.10b) $b \geq ae$, *if* $e > 0$.

*Then there is an irreducible reduced curve* $C$ *in* $|(a-1)C_0 + (b+l+2g-2-e)F|_a$ *with* $r$ *singular points of topological types* $\mathcal{S}_1, \ldots, \mathcal{S}_r$ *as its only singularities.*

(b) *Let* $\mathcal{S}_1, \ldots, \mathcal{S}_r$ *analytical singularity types with* $\mu(\mathcal{S}_1) \geq \ldots \geq \mu(\mathcal{S}_r)$, *among them* $k$ *nodes and* $m$ *cusps, such that*

(2.11b) $a(b - \frac{a}{2}e) \geq 9k + 16m + 9 \cdot \sum_{i=1}^{r-k-m} \mu(\mathcal{S}_i)$,

(2.12b.i) $a > 3 \cdot \sqrt{\mu(\mathcal{S}_1)} - 1$

(2.12b.ii) $b > 3 \cdot \sqrt{\mu(\mathcal{S}_1)} - 1$ *if* $e = 0$,

(2.12b.iii) $2\big(b - \frac{a}{2}e\big) > 3 \cdot \sqrt{\mu(\mathcal{S}_1)} - 1$ *if* $e < 0$, *and*

(2.14b) $b \geq ae$, *if* $e > 0$.

*Then there is an irreducible reduced curve* $C$ *in* $|(a-1)C_0 + (b+l+2g-2-e)F|_a$ *with* $r$ *singular points of analytical types* $\mathcal{S}_1, \ldots, \mathcal{S}_r$ *as its only singularities.*

## 3.c. Products of Curves

As we have seen in Proposition G.12, for a generic choice of smooth projective curves of genera $g_1 \geq 1$ and $g_2 \geq 1$ respectively the surface $\Sigma = C_1 \times C_2$ has Picard number two. Furthermore, according to Remark G.11 the only irreducible curves $B \subset \Sigma$ with selfintersection $B^2 = 0$ are the fibres $C_1$ and $C_2$, and for any irreducible curve $B \sim_a aC_1 + bC_2$ the coefficients $a$ and $b$ must be non-negative.

Then choosing for the existence theorem Corollary 1.3 $l \geq 3$ minimal such that $L = lC_1 + lC_2$ is very ample, we claim that the conditions (1.10) and (1.11) become obsolete and the corollary takes the following form.

**1.3c Corollary**

*Let $\Sigma = C_1 \times C_2$ for two smooth projective curves $C_1$ and $C_2$ of genera $g_1, g_2 \geq 1$ such that the Picard number is two, and let $l > 0$ be minimal such that $lC_1+lC_2$ is very ample. Suppose that $a, b \in \mathbb{Z}$ are two integers satisfying*

(1.8c) $$(a - l - 2g_2 + 2) \cdot (b - l - 2g_1 + 2) \geq \sum_{i=1}^{r} (m_i + 1)^2, \text{ and}$$

(1.9c) $$(a - l - 2g_2 + 2), (b - l - 2g_1 + 2) > \max\{m_i \mid i = 1, \ldots, r\}.$$

*Then for $z_1, \ldots, z_r \in \Sigma$ in very general position there is an irreducible reduced curve $C \in |aC_1 + bC_2|_a$ with ordinary singularities of multiplicities $m_i$ at the $z_i$ as only singularities. Moreover, $V_{|C|}(\underline{m})$ is T-smooth at $C$.*

**Proof:** (1.11) becomes redundant in view of (1.9c) and since an irreducible curve $B \sim_a a'C_1 + b'C_2$ has non-negative coefficients $a'$ and $b'$.

It remains to show that $D.L-2g(L) > m_i$ for all $i, j$. However, by the adjunction formula $g(L) = 1+\frac{1}{2}(L^2+L.K_\Sigma) = 1+l\cdot(l+g_1+g_2-2)$, and by (1.9c) $D.L-2g(L) > l\cdot\big((a-l-2g_2+2)+(b-l-2g_1+2)\big) > 6m_i > m_i$. Thus the claim is proved. □

From these considerations we at once deduce the conditions for the existence of an irreducible curve in $|D|_l$, $D \sim_a aC_1 + bC_2$, with prescribed singularities of arbitrary type, i. e. the conditions in Corollary 2.5.

**2.5c Corollary**

*Let $\Sigma = C_1 \times C_2$ for two smooth projective curves $C_1$ and $C_2$ of genera $g_1, g_2 \geq 1$ such that the Picard number is two, and let $l > 0$ be minimal such that $lC_1+lC_2$ is very ample.*

(a) *Suppose that $a, b \in \mathbb{Z}$ are two integers and $\mathcal{S}_1, \ldots, \mathcal{S}_r$ topological singularity types with $\delta(\mathcal{S}_1) \geq \ldots \geq \delta(\mathcal{S}_r)$, among them $k$ nodes and $m$ cusps, satisfying*

(2.7c) $$(a-l-2g_2+2)\cdot(b-l-2g_1+2) \geq 9k+16m+\tfrac{27}{2}\cdot\sum_{i=1}^{r-k-m} \delta(\mathcal{S}_i),$$

(2.8c) $$(a - l - 2g_2 + 2), (b - l - 2g_1 + 2) > \tfrac{9}{\sqrt{6}} \cdot \sqrt{\delta(\mathcal{S}_1)} - 1.$$

*Then there is an irreducible reduced curve $C$ in $|aC_1 + bC_2|_a$ with $r$ singular points of topological types $\mathcal{S}_1, \ldots, \mathcal{S}_r$ as its only singularities.*

(b) *Let $\mathcal{S}_1, \ldots, \mathcal{S}_r$ analytical singularity types with $\mu(\mathcal{S}_1) \geq \ldots \geq \mu(\mathcal{S}_r)$, among them $k$ nodes and $m$ cusps, such that*

(2.11c) $$(a-l-2g_2+2)\cdot(b-l-2g_1+2) \geq 9k+16m+9\cdot\sum_{i=1}^{r-k-m} \mu(\mathcal{S}_i),$$

(2.12c) $$(a - l - 2g_2 + 2), (b - l - 2g_1 + 2) > 3 \cdot \sqrt{\mu(\mathcal{S}_1)} - 1.$$

*Then there is an irreducible reduced curve $C$ in $|aC_1 + bC_2|_a$ with $r$ singular points of analytical types $\mathcal{S}_1, \ldots, \mathcal{S}_r$ as its only singularities.*

### 3.d. Products of Elliptic Curves

That $C_1$ and $C_2$ be "generic" in the above sense means elliptic curves for just that they are non-isogenous. Working with the very ample divisor class $L = 3C_1 + 3C_2$ the theorems in Section 3.c look as follows.

**1.3d Corollary**

*Let $\Sigma = C_1 \times C_2$ for two non-isogenous elliptic curves $C_1$ and $C_2$. Suppose that $a, b \in \mathbb{Z}$ are two integers satisfying*

$$(1.8d) \qquad (a-3)\cdot(b-3) \geq \sum_{i=1}^{r}(m_i+1)^2, \text{ and}$$

$$(1.9d) \qquad (a-3),(b-3) > \max\{m_i \mid i = 1,\ldots,r\}.$$

*Then for $z_1,\ldots,z_r \in \Sigma$ in very general position there is an irreducible reduced curve $C \in |aC_1 + bC_2|_a$ with ordinary singularities of multiplicities $m_i$ at the $z_i$ as only singularities. Moreover, $V_{|C|}(\underline{m})$ is T-smooth at $C$.*

**2.5d Corollary**

*Let $\Sigma = C_1 \times C_2$ for two non-isogenous elliptic curves $C_1$.*

(a) *Suppose that $a, b \in \mathbb{Z}$ are two integers and $\mathcal{S}_1,\ldots,\mathcal{S}_r$ topological singularity types with $\delta(\mathcal{S}_1) \geq \ldots \geq \delta(\mathcal{S}_r)$, among them $k$ nodes and $m$ cusps, satisfying*

$$(2.7d) \qquad (a-3)\cdot(b-3) \geq 9k + 16m + \tfrac{27}{2}\cdot\textstyle\sum_{i=1}^{r-k-m}\delta(\mathcal{S}_i) \text{ and}$$

$$(2.8d) \qquad a-3, b-3 > \tfrac{9}{\sqrt{6}}\cdot\sqrt{\delta(\mathcal{S}_1)} - 1.$$

*Then there is an irreducible reduced curve $C$ in $|aC_1 + bC_2|_a$ with $r$ singular points of topological types $\mathcal{S}_1,\ldots,\mathcal{S}_r$ as its only singularities.*

(b) *Let $\mathcal{S}_1,\ldots,\mathcal{S}_r$ analytical singularity types with $\mu(\mathcal{S}_1) \geq \ldots \geq \mu(\mathcal{S}_r)$, among them $k$ nodes and $m$ cusps, such that*

$$(2.11d) \qquad (a-3)\cdot(b-3) \geq 9k + 16m + 9\cdot\textstyle\sum_{i=1}^{r-k-m}\mu(\mathcal{S}_i) \text{ and}$$

$$(2.12d) \qquad a-3, b-3 > 3\cdot\sqrt{\mu(\mathcal{S}_1)} - 1.$$

*Then there is an irreducible reduced curve $C$ in $|aC_1 + bC_2|_a$ with $r$ singular points of analytical types $\mathcal{S}_1,\ldots,\mathcal{S}_r$ as its only singularities.*

### 3.e. Surfaces in $\mathbb{P}^3_{\mathbb{C}}$

Since we consider the case of rational surfaces separately the following considerations thus give a full answer for the "general case" of a surface in $\mathbb{P}^3_{\mathbb{C}}$.

**1.3e Corollary**

*Let $\Sigma$ be a smooth projective surface of degree $n$ in $\mathbb{P}^3_{\mathbb{C}}$ whose Picard group is generated by a hyperplane section $H$, and let $d \geq 3$ be an integer such that*

$$(1.8e) \qquad n\cdot(d-n+3)^2 \geq 2\cdot\sum_{i=1}^{r}(m_i+1)^2.$$

*Then for* $z_1, \ldots, z_r \in \Sigma$ *in very general position there is an irreducible reduced curve* $C \in |dH|_a$ *with ordinary singularities of multiplicities* $m_i$ *at the* $z_i$ *as only singularities. Moreover,* $V_{|dH|}(\underline{m})$ *is T-smooth at* $C$.

**Proof:** The role of the very ample divisor $L$ is filled by a hyperplane section, and thus $g(L) = 1 + \frac{L^2+L.K_\Sigma}{2} = \binom{n-1}{2}$. Therefore, (1.8e) obviously implies (1.8), and (1.10) takes the form

$$n \cdot (d-n+3) > m_i + 2 \text{ for all } i = 1, \ldots, r. \tag{3.2}$$

However, from (1.8e) we deduce for any $i \in \{1, \ldots, r\}$

$$n \cdot (d-n+3) \geq \sqrt{n} \cdot \sqrt{2} \cdot (m_i + 1) \geq m_i + 2,$$

unless $n = r = m_1 = 1$, in which case we are done by the assumption $d \geq 3$. Thus (1.10) is redundant.

Moreover, there are no curves of self-intersection zero on $\Sigma$, and it thus remains to verify (1.11), which in this situation takes the form

$$d \geq n - 3,$$

and follows at once from (3.2). □

**2.5e Corollary**
*Let* $\Sigma$ *be a smooth projective surface of degree* $n$ *in* $\mathbb{P}^3_{\mathbb{C}}$ *whose Picard group is generated by a hyperplane section* $H$.

(a) *Let* $d$ *be an integer and* $\mathcal{S}_1, \ldots, \mathcal{S}_r$ *topological singularity types with* $\delta(\mathcal{S}_1) \geq \ldots \geq \delta(\mathcal{S}_r)$*, among them* $k$ *nodes and* $m$ *cusps, satisfying*

(2.7e) $\quad n(d-n+3)^2 \geq 18k + 32m + 27 \cdot \sum_{i=1}^{r-k-m} \delta(\mathcal{S}_i)$.

*Then there is an irreducible reduced curve* $C$ *in* $|dH|_a$ *with* $r$ *singular points of topological types* $\mathcal{S}_1, \ldots, \mathcal{S}_r$ *as its only singularities.*

(b) *Let* $\mathcal{S}_1, \ldots, \mathcal{S}_r$ *analytical singularity types with* $\mu(\mathcal{S}_1) \geq \ldots \geq \mu(\mathcal{S}_r)$*, among them* $k$ *nodes and* $m$ *cusps, such that*

(2.11e) $\quad n \cdot (d-n+3)^2 \geq 18k + 32m + 18 \cdot \sum_{i=1}^{r-k-m} \mu(\mathcal{S}_i)$.

*Then there is an irreducible reduced curve* $C$ *in* $|dH|_a$ *with* $r$ *singular points of analytical types* $\mathcal{S}_1, \ldots, \mathcal{S}_r$ *as its only singularities.*

### 3.f. K3-Surfaces

3.f.i. *Generic K3-Surfaces.* A generic K3-surface does not possess an elliptic fibration, and hence it does not possess any irreducible curve of self-intersection zero. (cf. **[FrM94]** I.1.3.7)

Therefore, the conditions in Corollary 1.3 reduce to

(1.8f.i) $\quad (D-L)^2 \geq 2 \cdot \sum_{i=1}^{r} (m_i + 1)^2,$

(1.10f.i) $D.L - 2g(L) > m_i$ for all $i, j$, and

(1.11f.i) $D - L$ nef,

and, analogously, the conditions in Corollary 2.5 in the topological case reduce to (2.9) and

(2.7f.i) $(D-L)^2 \geq 18k + 32m + 27 \cdot \sum_{i=1}^{r-k-m} \delta(\mathcal{S}_i)$, and

(2.10f.i) $D - L$ nef,

and in the analytical case to (2.13) and

(2.11f.i) $(D-L)^2 \geq 18k + 32m + 18 \cdot \sum_{i=1}^{r-k-m} \mu(\mathcal{S}_i)$, and

(2.14f.i) $D - L$ nef.

3.f.ii. *K3-Surfaces with an Elliptic Structure.* The conditions in Corollary 1.3 then reduce to (1.8f.i), (1.11f.i), (1.10f.i), and

(1.9f.ii) $(D-L).B > \max\{m_i \mid i = 1, \ldots, r\}$

for any irreducible curve $B$ with $B^2 = 0$.

Similarly, the conditions in Corollary 2.5 reduce to (2.7f.i) / (2.11f.i), (2.10f.i) / (2.14f.i), (2.5), and

(2.12f.ii) $(D-L).B > \frac{9}{\sqrt{6}} \cdot \sqrt{\delta(\mathcal{S}_1)} - 1$

for any irreducible curve $B$ with $B^2 = 0$,

respectively

(2.12f.ii) $(D-L).B > 3 \cdot \sqrt{\mu(\mathcal{S}_1)} - 1$

for any irreducible curve $B$ with $B^2 = 0$.

CHAPTER IV

# T-Smoothness

The varieties $V_{|D|}(rA_1)$ (respectively the open subvarieties $V^{irr}_{|D|}(rA_1)$) of reduced (respectively reduced and irreducible) nodal curves in a fixed linear system $|D|_l$ on a smooth projective surface $\Sigma$ are also called *Severi varieties*. When $\Sigma = \mathbb{P}^2_{\mathbb{C}}$ Severi showed that these varieties are smooth of the expected dimension, whenever they are non-empty – that is, nodes always impose independent conditions. It seems natural to study this question on other surfaces, but it is not surprising that the situation becomes harder.

Tannenbaum showed in **[Tan82]** that also on K3-surfaces $V_{|D|}(rA_1)$ is always smooth, that, however, the dimension is larger than the expected one and thus $V_{|D|}(rA_1)$ is not T-smooth in this situation. If we restrict our attention to the subvariety $V^{irr}_{|D|}(rA_1)$ of *irreducible* curves with $r$ nodes, then we gain T-smoothness again whenever the variety is non-empty. That is, while on a K3-surface the conditions which nodes impose on irreducible curves are always independent, they impose dependent conditions on reducible curves.

On more complicated surfaces the situation becomes even worse. Chiantini and Sernesi study in **[ChS97]** Severi varieties on surfaces in $\mathbb{P}^3_{\mathbb{C}}$. They show that on a generic quintic $\Sigma$ in $\mathbb{P}^3_{\mathbb{C}}$ with hyperplane section $H$ the variety $V^{irr}_{|dH|}\big(\frac{5d(d-2)}{4}\cdot A_1\big)$ has a non-smooth reduced component of the expected dimension, if $d$ is even. They construct their examples by intersecting a general cone over $\Sigma$ in $\mathbb{P}^4_{\mathbb{C}}$ with a general complete intersection surface of type $\big(2, \frac{d}{2}\big)$ in $\mathbb{P}^4_{\mathbb{C}}$ and projecting the resulting curve to $\Sigma$ in $\mathbb{P}^3_{\mathbb{C}}$. Moreover, Chiantini and Ciliberto give in **[ChC99]** examples showing that the Severi varieties $V^{irr}_{|dH|}(rA_1)$ on a surface in $\mathbb{P}^3_{\mathbb{C}}$ also may have components of dimension larger than the expected one.

Hence, one can only ask for numerical conditions ensuring that $V^{irr}_{|dH|}(rA_1)$ is T-smooth, and Chiantini and Sernesi answer this question by showing that on a surface of degree $n \geq 5$ the condition

$$r < \frac{d(d-2n+8)n}{4} \tag{0.1}$$

implies that $V^{irr}_{|dH|}(rA_1)$ is T-smooth for $d > 2n-8$. Note that the above example shows that this bound is even sharp. Actually Chiantini and Sernesi prove a somewhat more general result for surfaces with ample canonical divisor $K_\Sigma$ and curves which are in $|pK_\Sigma|_l$ for some $p \in \mathbb{Q}$. For their proof they suppose that for some curve $C \in V^{irr}_{|dH|}(rA_1)$ the cohomology group $H^1\big(\Sigma, \mathcal{J}_{X^{es}(C)/\Sigma}(D)\big)$ does not vanish and derive from this the existence of a Bogomolov unstable rank-two bundle $E$. This bundle in turn provides them with a divisor $\Delta_0$ such that they are able to find a lower and an upper bound for $\Delta_0.K_\Sigma$ contradicting each other.

This is basically the same approach which we use as well. However, we allow arbitrary singularities rather than only nodes, we weaken the assumption of $K_\Sigma$ being ample to $K_\Sigma^2 \geq -3$ and we study a wider range of divisors. Yet we still get the same optimal condition (0.1) for the T-smoothness of nodal curves on surfaces in $\mathbb{P}_\mathbb{C}^3$ – see Corollary 2.6. We should like to point out that our general result for the T-smoothness of $V_{|D|}^{irr}(\mathcal{S}_1, \ldots, \mathcal{S}_r)$ with the main condition

$$\left(\sum_{i=1}^{r} \big(\tau^*(\mathcal{S}_i)+1\big) + \tfrac{D.K_\Sigma}{2}\right)^2 < \left(r + \tfrac{K_\Sigma^2}{4}\right)^2 \cdot D^2,$$

does apply to a wider range of surfaces than most of the previously known results. E. g. we deduce in Section 4.d that on a product $\Sigma = C_1 \times C_2$ of elliptic curves the Severi variety $V_{|aC_1+bC_2|}^{irr}(rA_1)$ is T-smooth as soon as

$$2r \leq ab.$$

For the plane case there are of course better results like the condition

$$\sum_{i=1}^{r} \big(\tau^*(\mathcal{S}_i)+1\big)^2 \leq (d+3)^2,$$

for T-smoothness (cf. [**GLS00**] or [**Los98**] Theorem 5.4), where $d$ is the degree of $D$. Note that the a series of irreducible plane curves of degree $d$ with $r$ singularities of type $A_\mu$, $\mu$ arbitrarily large, satisfying

$$r\mu^2 = \sum_{i=1}^{r} \tau^*(A_\mu)^2 = 9d^2 + \text{ terms of lower order}$$

constructed by Shustin (cf. [**Shu97**]) shows that asymptotically we cannot expect to do essentially better in general. For a survey on other known results on $\Sigma = \mathbb{P}_\mathbb{C}^2$ we refer to [**Los98**] and [**GLS00**].

In [**GLS98a**] nodal curves on the projective plane blown up in $m \geq 10$ points are considered. This is a case to which our theorem does not apply in general since $K_\Sigma^2 = 9 - m < -3$ for $m \geq 13$. Nodal curves on arbitrary rational surfaces were also studied by Tannenbaum in [**Tan80**]. There he gives for the T-smoothness a condition which basically coincides with the only previously known general condition for the T-smoothness of $V_{|D|}^{irr}(\mathcal{S}_1, \ldots, \mathcal{S}_r)$

$$\sum_{i=1}^{r} \tau^*(\mathcal{S}_i) - \epsilon^*(\mathcal{S}_i) < -K_\Sigma.D,$$

where $\epsilon^*(\mathcal{S}_i) \geq 1$ is the so-called isomorphism defect of $\mathcal{S}_i$ (cf. [**GrK89, GrL96**]). In [**Fla01a**] Flamini studies the case of Severi varieties on surfaces of general type and he receives conditions for the T-smoothness of $V^{irr}_{|D|}(rA_1)$ of the type

$$r < \frac{D.(D-4H) + \sqrt{D^2 \cdot (D-4H)^2}}{8},$$

where $H$ is some hyperplane section (cf. **[Fla01a]** Theorem 5.1 and 5.3). For a generalisation of the examples of Chiantini and Sernesi of families of nodal curves which are not T-smooth to general non-degenerate complete intersection surfaces with ample canonical bundle we refer to **[FlM01]**. These show that the results of **[Fla01a]** are sharp, see **[Fla01a]** Remark 5.6.

The theorem which we here present in the following section and its proof are a slight modification of Theorem 1 in **[GLS97]**. We sacrifice the sophisticated examination of the zero-dimensional schemes involved in the proof for the sake of simplified conditions. A somewhat stronger version may be found in the book **[GLS05]**.

## 1. T-Smoothness

**1.1 Theorem**
*Let $\mathcal{S}_1, \ldots, \mathcal{S}_r$ be topological or analytical singularity types with $\tau^*(\mathcal{S}_1) \geq \ldots \geq \tau^*(\mathcal{S}_r)$ and let $D \in \text{Div}(\Sigma)$ be a divisor such that*[1]

(1.1) $D + K_\Sigma$ *nef,*

(1.2) $D - K_\Sigma$ *is big and nef,*

(1.3) $D^2 - 2 \cdot D.K_\Sigma \geq \frac{(D.K_\Sigma)^2 - D^2 \cdot K_\Sigma^2}{4} + 4$, *if* $D.K_\Sigma < -4$,

(1.4) $\sum_{i=1}^{r} \tau^*(\mathcal{S}_i) < \frac{1}{4} \cdot (D - K_\Sigma)^2$, *and*

(1.5) $\left( \sum_{i=1}^{s} \big(\tau^*(\mathcal{S}_i) + 1\big) + \frac{D.K_\Sigma}{2} \right)^2 < \left( s + \frac{K_\Sigma^2}{4} \right) \cdot D^2$ *for all* $s = 1, \ldots, r$.

*Then either $V^{irr}_{|D|}(\mathcal{S}_1, \ldots, \mathcal{S}_r)$ is empty or T-smooth.*

**Idea of the Proof:** T-smoothness at a point $C \in V^{irr}_{|D|}(\mathcal{S}_1, \ldots, \mathcal{S}_r)$ follows once we know $h^1\big(\Sigma, \mathcal{J}_{X^*(C)/\Sigma}(D)\big) = 0$. Supposing this is not the case we find a Bogomolov unstable rank-two bundle $E$ which provides us with some "nice" divisor $\Delta_0$. This means, we get an upper and a lower bound for $D^2 \cdot (D - K_\Sigma - \Delta_0)^2$ which contradict each other due to Condition (1.5).

**Proof:** Let $C \in V^{irr}_{|D|}(\mathcal{S}_1, \ldots, \mathcal{S}_r)$ be arbitrary. We set $X^*(C) = X^{es}(C)$ respectively $X^*(C) = X^{ea}(C)$, so that $\deg\big(X^*(C)\big) = \sum_{i=1}^{r} \tau^*(\mathcal{S}_i)$.

We are going to show that

$$h^1\big(\Sigma, \mathcal{J}_{X^*(C)/\Sigma}(D)\big) = 0,$$

[1]Here $\tau^*(\mathcal{S}_i) = \tau^{es}(\mathcal{S}_i)$, the codimension of the $\mu$-constant stratum in the base of the semi-universal deformation of $\mathcal{S}_i$, respectively $\tau^*(\mathcal{S}_i) = \tau(\mathcal{S}_i)$, the Tjurina number of $\mathcal{S}_i$.

which by Remark I.2.15 implies the T-smoothness of $V^{\text{irr}}_{|D|}(\mathcal{S}_1,\ldots,\mathcal{S}_r)$ at $C$.

Suppose the contrary, that is $h^1\big(\Sigma,\mathcal{J}_{X^*(C)/\Sigma}(D)\big)>0$.

**Step 1:** Find a *non-empty* subscheme $X_0\subseteq X^*(C)$ and a rank-two vector bundle $E$ satisfying the relations (1.6) and (1.7).

Choose $X_0\subseteq X^*(C)$ minimal such that still $h^1\big(\Sigma,\mathcal{J}_{X_0/\Sigma}(D)\big)>0$. By (1.2) the divisor $D-K_\Sigma$ is big and nef, and thus $h^1\big(\Sigma,\mathcal{O}_\Sigma(D)\big)=0$ by the Kawamata–Viehweg Vanishing Theorem. Hence $X_0$ cannot be empty.

Due to the Grothendieck-Serre duality (cf. **[Har77]** III.7.6) we have $0\neq H^1\big(\Sigma,\mathcal{J}_{X_0/\Sigma}(D)\big)\cong \mathrm{Ext}^1\big(\mathcal{J}_{X_0/\Sigma}(D),\mathcal{O}_\Sigma(K_\Sigma)\big)$, and thus, since $\mathcal{O}_\Sigma(K_\Sigma)$ is locally free, (cf. **[Har77]** III.6.7)

$$\mathrm{Ext}^1\big(\mathcal{J}_{X_0/\Sigma}(D-K_\Sigma),\mathcal{O}_\Sigma\big)\neq 0.$$

That is, there is an extension (cf. **[Har77]** Ex. III.6.1)

$$0\to\mathcal{O}_\Sigma\to E\to\mathcal{J}_{X_0/\Sigma}(D-K_\Sigma)\to 0. \tag{1.6}$$

The minimality of $X_0$ implies that $E$ is locally free (cf. **[Laz97]** Proposition 3.9). Moreover, we have (cf. **[Laz97]** Exercise 4.3)

$$c_1(E)=D-K_\Sigma \quad\text{and}\quad c_2(E)=\deg(X_0). \tag{1.7}$$

**Step 2:** Find a divisor $\Delta_0$ such that

(a) $(2\Delta_0-D+K_\Sigma)^2\geq c_1(E)^2-4\cdot c_2(E)>0$, and

(b) $(2\Delta_0-D+K_\Sigma).H>0$ for all $H\in\mathrm{Div}(\Sigma)$ ample.

By (1.4) and (1.7) we have

$$c_1(E)^2-4\cdot c_2(E)=(D-K_\Sigma)^2-4\cdot\deg(X_0)>0,$$

and thus $E$ is Bogomolov unstable (cf. **[Laz97]** Theorem 4.2). This, however, implies that there exists a divisor $\Delta_0\in\mathrm{Div}(\Sigma)$ and a zero-dimensional scheme $Z\subset\Sigma$ such that

$$0\to\mathcal{O}_\Sigma(\Delta_0)\to E\to\mathcal{J}_{Z/\Sigma}(D-K_\Sigma-\Delta_0)\to 0 \tag{1.8}$$

is exact (cf. **[Laz97]** Theorem 4.2) and such that (a) and (b) are fulfilled.

**Step 3:** Find a curve $\Delta\subset\Sigma$ such that

(c) $\Delta+\Delta_0\sim_l D-K_\Sigma$,

(d) $D.\Delta\geq\deg(X_0)+\#X_0$, and

(e) $\deg(X_0)\geq\Delta_0.\Delta$.

Tensoring (1.8) with $\mathcal{O}_\Sigma(-\Delta_0)$ leads to the following exact sequence

$$0\to\mathcal{O}_\Sigma\to E(-\Delta_0)\to\mathcal{J}_{Z/\Sigma}(D-K_\Sigma-2\Delta_0)\to 0, \tag{1.9}$$

and we deduce $h^0\big(\Sigma,E(-\Delta_0)\big)\neq 0$.

Now tensoring (1.6) with $\mathcal{O}_\Sigma(-\Delta_0)$ leads to

$$0 \to \mathcal{O}_\Sigma(-\Delta_0) \to E(-\Delta_0) \to \mathcal{J}_{X_0/\Sigma}(D-K_\Sigma-\Delta_0) \to 0. \tag{1.10}$$

Let $H$ be some ample divisor. By (b) and since $D-K_\Sigma$ is nef by (1.2):

$$-\Delta_0.H < -\tfrac{1}{2}\cdot(D-K_\Sigma).H \le 0. \tag{1.11}$$

Hence $-\Delta_0$ cannot be effective, that is $H^0(\Sigma,-\Delta_0) = 0$. But the long exact cohomology sequence of (1.10) then implies

$$0 \ne H^0\big(\Sigma, E(-\Delta_0)\big) \hookrightarrow H^0\left(\Sigma, \mathcal{J}_{X_0/\Sigma}(D-K_\Sigma-\Delta_0)\right).$$

In particular we may choose a curve

$$\Delta \in \left|\mathcal{J}_{X_0/\Sigma}(D-K_\Sigma-\Delta_0)\right|_l.$$

Thus (c) is obviously fulfilled and it remains to show (d) and (e).

We note that $C \in |D|_l$ is irreducible and that $\Delta$ cannot contain $C$ as an irreducible component: otherwise applying (b) with some ample divisor $H$ we would get the following contradiction, since $D+K_\Sigma$ is nef by (1.1),

$$0 \le (\Delta - C).H < -\tfrac{1}{2}\cdot(D+K_\Sigma).H \le 0.$$

Since $X_0 \subseteq X^*(C)$, Lemma 1.3 applies[2] to the local ideals of $X_0$, that is for the points $z \in \text{supp}(X_0)$ we have $i(C,\Delta;z) \ge \deg(X_0,z)+1$. Thus, since $X_0 \subset C\cap\Delta$ the Theorem of Bézout implies

$$D.\Delta = C.\Delta = \sum_{z\in C\cap\Delta} i(C,\Delta;z) \ge \sum_{z\in\text{supp}(X_0)} \big(\deg(X_0,z)+1\big) = \deg(X_0) + \#X_0.$$

Finally, by (a), (c) and (1.7) we get

$$(\Delta_0-\Delta)^2 \ge c_1(E)^2 - 4\cdot c_2(E) = (\Delta_0+\Delta)^2 - 4\cdot\deg(X_0),$$

and thus $\deg(X_0) \ge \Delta_0.\Delta$.

**Step 4:** Find a lower and an upper bound for $D^2\cdot(\Delta_0-\Delta)^2$, contradicting each other.

From (a) we derive

$$(\Delta_0-\Delta)^2 \ge (D-K_\Sigma)^2 - 4\deg(X_0). \tag{1.12}$$

(1.1) and (1.2) imply that $D$ is nef, and since the strict inequality for ample divisors in (b) comes down to "$\ge$" for nef divisors, we get in view of (c)

$$\begin{aligned} 0 \le D.(\Delta_0-\Delta) &= D.(\Delta_0+\Delta) - 2\cdot D.\Delta \\ &\le D.(\Delta_0+\Delta) - 2\deg(X_0) - 2\cdot\#X_0 \\ &= D.(D-K_\Sigma) - 2\deg(X_0) - 2\cdot\#X_0. \end{aligned} \tag{1.13}$$

[2]We note that $X^{es}(C) \subseteq X^{ea}(C)$, i. e. if $X_0 \subseteq X^{es}(C)$ then $X_0 \subseteq X^{ea}(C)$ as well.

But then applying the Hodge Index Theorem E.4 to the divisors $D$ and $\Delta_0 - \Delta$, taking (a) into account, we get

$$D^2 \cdot (\Delta_0 - \Delta)^2 \leq \big(D.(\Delta_0 - \Delta)\big)^2 \leq \big(D.(D - K_\Sigma) - 2\deg(X_0) - 2 \cdot \#X_0\big)^2. \quad (1.14)$$

From (1.12) and (1.14) we deduce that

$$\big(D.(D - K_\Sigma) - 2\deg(X_0) - 2 \cdot \#X_0\big)^2 \geq D^2 \cdot \big((D - K_\Sigma)^2 - 4\deg(X_0)\big),$$

and hence

$$\big(2\deg(X_0) + 2 \cdot \#X_0 + D.K_\Sigma\big)^2 \geq \big(4 \cdot \#X_0 + K_\Sigma^2\big) \cdot D^2 \geq 0. \quad (1.15)$$

Suppose first that $-D.K_\Sigma > 2\deg(X_0) + 2 \cdot \#X_0 \geq 4$, then by (1.5) and (1.3)

$$\big(2\deg(X_0) + 2 \cdot \#X_0 + D.K_\Sigma\big)^2 < (4 + D.K_\Sigma)^2 \leq \big(4 + K_\Sigma^2\big) \cdot D^2 \leq \big(4 \cdot \#X_0 + K_\Sigma^2\big) \cdot D^2 \quad (1.16)$$

in contradiction to (1.15).

Thus we must have $2\deg(X_0) + 2 \cdot \#X_0 + D.K_\Sigma \geq 0$, and since

$$\deg(X_0) \leq \sum_{z \in \mathrm{supp}(X_0)} \tau^*(C, z) \leq \sum_{i=1}^{\#X_0} \tau^*(\mathcal{S}_i)$$

we find once more a contradiction to (1.15) with the aid of (1.5):

$$\big(2\deg(X_0) + 2 \cdot \#X_0 + D.K_\Sigma\big)^2 \leq \left(2\sum_{i=1}^{\#X_0} \big(\tau^*(\mathcal{S}_i) + 1\big) + D.K_\Sigma\right)^2 < \big(4 \cdot \#X_0 + K_\Sigma^2\big) \cdot D^2.$$

□

In the following remark we replace (1.5) by a number of other conditions, which partially may even be better in certain situations.

**1.2 Remark**

(a) For a surface with $K_\Sigma^2 \leq -3$ condition (1.5) will never be satisfied.

(b) If $K_\Sigma$ is nef, then (1.1) and (1.3) are superfluous.

(c) If $D.K_\Sigma \leq 0$, we can replace (1.5) by

$$\sum_{i=1}^{r} \big(\tau^*(\mathcal{S}_i) + 1\big)^2 < D^2 - 2 \cdot D.K_\Sigma - \tfrac{(D.K_\Sigma)^2 - D^2 \cdot K_\Sigma^2}{4}. \quad (1.17)$$

Moreover, (1.3) is always fulfilled.

(d) If $D.K_\Sigma \geq 0$, we may replace (1.5) in Theorem 1.1 by

$$\sum_{i=1}^{r} \big(\tau^*(\mathcal{S}_i) + 1\big)^2 < D^2 - D.K_\Sigma \cdot \big(\tau^*(\mathcal{S}_1) + 1\big) - \tfrac{(D.K_\Sigma)^2 - D^2 \cdot K_\Sigma^2}{4}. \quad (1.18)$$

(e) For divisors $D$ of the form $D \sim_n pK_\Sigma$ for some $0 \neq p \in \mathbb{Q}$ the inequalities (1.17) and (1.18) take the simpler forms

$$\sum_{i=1}^{r} \big(\tau^*(\mathcal{S}_i)+1\big)^2 < D^2 - 2\cdot D.K_\Sigma \tag{1.19}$$

respectively

$$\sum_{i=1}^{r} \big(\tau^*(\mathcal{S}_i)+1\big)^2 < D^2 - D.K_\Sigma \cdot \big(\tau^*(\mathcal{S}_1)+1\big). \tag{1.20}$$

Moreover, (1.3) is always fulfilled.

(f) If $D$ is of the form $D \sim_n pK_\Sigma$ for some $0 \neq p \in \mathbb{Q}$, Condition (1.5) can only be fulfilled if $\tau^*(\mathcal{S}_i)+1 < \frac{D^2}{D.K_\Sigma}$ for all $i = 1,\dots,r$. Assuming this, the inequality (1.5) may be replaced by

$$\sum_{i=1}^{r} \frac{\big(\tau^*(\mathcal{S}_i)+1\big)^2}{1-\frac{D.K_\Sigma}{D^2}\cdot\big(\tau^*(\mathcal{S}_i)+1\big)} < D^2. \tag{1.21}$$

(g) We could replace (1.5) by

$$\sum_{i=1}^{r} \tau^*(\mathcal{S}_i) < \sqrt{1+\tfrac{K_\Sigma^2}{4}}\cdot\sqrt{D^2} - \big|\tfrac{K_\Sigma.D}{2}+1\big|. \tag{1.22}$$

(h) If $-K_\Sigma$ is ample and $D \sim_n pK_\Sigma$ for some $p \in \mathbb{Q}$ with $p < 1-\sqrt{1+\frac{4}{K_\Sigma^2}} < 0$, we could replace (1.5) in Theorem 1.1 by

$$\sum_{i=1}^{r} \tau^*(\mathcal{S}_i) < \sqrt{1+\tfrac{K_\Sigma^2}{4}}\cdot\sqrt{D^2} - \tfrac{1}{2}\cdot K_\Sigma.D - 1. \tag{1.23}$$

(i) If $D^2 \geq 8 + K_\Sigma^2 + 4\cdot\sqrt{K_\Sigma^2+4}$, then (1.22) respectively (1.23) replaces both (1.4) and (1.5).

(j) We note that always

$$4\cdot\sum_{i=1}^{r} \tau^*(\mathcal{S}_i) \leq \sum_{i=1}^{r} \big(\tau^*(\mathcal{S}_i)+1\big)^2. \tag{1.24}$$

**Proof:** (a) & (b) These are obvious.

(c) We note that

$$\left(\sum_{i=1}^{s} \big(\tau^*(\mathcal{S}_i)+1\big)\right)^2 \leq s\cdot\sum_{i=1}^{s} \big(\tau^*(\mathcal{S}_i)+1\big)^2.$$

Thus the inequalities

$$\sum_{i=1}^{s} \big(\tau^*(\mathcal{S}_i)+1\big)^2 < D^2 - \frac{D.K_\Sigma}{s}\cdot\sum_{i=1}^{s}\big(\tau^*(\mathcal{S}_i)+1\big) - \tfrac{(D.K_\Sigma)^2-D^2\cdot K_\Sigma^2}{4\cdot s} \tag{1.25}$$

for $s = 1,\dots,r$ will imply (1.5). However, by (1.1) and (1.2) $D$ is nef and thus $D^2 \geq 0$ and by the Hodge Index Theorem E.4 $\frac{(D.K_\Sigma)^2-D^2\cdot K_\Sigma^2}{4} \geq 0$. Moreover, $D.K_\Sigma \leq 0$ and we therefore get the desired result noting that 2 is a lower bound for the coefficient $\frac{1}{s}\cdot\sum_{i=1}^{s}\big(\tau^*(\mathcal{S}_i)+1\big)$ of $D.K_\Sigma$.
Since $\tau^*(\mathcal{S}_1) \geq 1$, from (1.17) we have $4 < D^2 - 2\cdot D.K_\Sigma - \frac{(D.K_\Sigma)^2-D^2\cdot K_\Sigma^2}{4}$, which implies (1.3).

(d) It again suffices to verify (1.25) for $s = 1, \ldots, r$. However, since this time $D^2$, $\frac{(D.K_\Sigma)^2 - D^2 \cdot K_\Sigma^2}{4}$ and $D.K_\Sigma$ are all non-negative, (1.25) follows from (1.18), once we note that $\tau^*(\mathcal{S}_1) + 1 \geq \frac{1}{s} \cdot \sum_{i=1}^{s} \big(\tau^*(\mathcal{S}_i) + 1\big)$.

(e) If $D \sim_n p \cdot K_\Sigma$, then $(D.K_\Sigma)^2 - D^2 \cdot K_\Sigma^2 = 0$.

(f) Once (1.5) is satisfied, we get from $s = 1$ that $D.K_\Sigma \cdot \big(\tau^*(\mathcal{S}_1) + 1\big) < D^2$. Let us therefore suppose that $\tau^*(\mathcal{S}_i) + 1 < \frac{D^2}{D.K_\Sigma}$ for each $i = 1, \ldots, r$. We note that then for any $s = 1, \ldots, r$

$$\sum_{i=1}^{s} \frac{\big(\tau^*(\mathcal{S}_i) + 1\big)^2}{1 - \frac{D.K_\Sigma}{D^2} \cdot \big(\tau^*(\mathcal{S}_i) + 1\big)} \leq \sum_{i=1}^{r} \frac{\big(\tau^*(\mathcal{S}_i) + 1\big)^2}{1 - \frac{D.K_\Sigma}{D^2} \cdot \big(\tau^*(\mathcal{S}_i) + 1\big)}.$$

The Cauchy-Schwartz Inequality thus shows

$$\left(\sum_{i=1}^{s} \big(\tau^*(\mathcal{S}_i) + 1\big)\right)^2 = \left(\sum_{i=1}^{s} \frac{\tau^*(\mathcal{S}_i)+1}{\sqrt{1 - \frac{D.K_\Sigma}{D^2} \cdot (\tau^*(\mathcal{S}_i)+1)}} \cdot \sqrt{1 - \frac{D.K_\Sigma}{D^2} \cdot \big(\tau^*(\mathcal{S}_i) + 1\big)}\right)^2$$

$$\leq \sum_{i=1}^{s} \frac{(\tau^*(\mathcal{S}_i)+1)^2}{1 - \frac{D.K_\Sigma}{D^2} \cdot (\tau^*(\mathcal{S}_i)+1)} \cdot \sum_{i=1}^{s} \left(1 - \frac{D.K_\Sigma}{D^2} \cdot \big(\tau^*(\mathcal{S}_i) + 1\big)\right)$$

$$< s \cdot D^2 - D.K_\Sigma \cdot \sum_{i=1}^{s} \big(\tau^*(\mathcal{S}_i) + 1\big),$$

which is just (1.5), since $(D.K_\Sigma)^2 = D^2 \cdot K_\Sigma^2$.

(g) In the proof of Theorem 1.1 we could replace $D.\Delta$ in (1.13) by $\deg(X_0) + 1$ instead of $\deg(X_0) + \#X_0$. Then (1.15) would be replaced by

$$\big(2\deg(X_0) + 2 + D.K_\Sigma\big)^2 \geq \big(4 + K_\Sigma^2\big) \cdot D^2 \geq 0, \tag{1.26}$$

and hence

$$\Big(2\deg(X_0) + |2 + D.K_\Sigma|\Big)^2 \geq \big(4 + K_\Sigma^2\big) \cdot D^2 \geq 0$$

which contradicts (1.22).

(h) We have to derive from (1.26) a contradiction, replacing (1.22) by (1.23). For this we consider the convex function

$$\varphi : \Big[1, \deg\big(X^*(C)\big)\Big] \to \mathbb{R} : t \mapsto (2t + 2 + D.K_\Sigma)^2.$$

Since $\varphi$ is convex, it takes its maximum either in $t^* = 1$ or in $t^* = \deg\big(X^*(C)\big)$.
If $t^* = 1$, then we get from (1.26)

$$\big(4 + p \cdot K_\Sigma^2\big)^2 \geq \big(2 \cdot \deg(X_0) + 2 + D.K_\Sigma\big)^2 \geq \big(4 + K_\Sigma^2\big) \cdot p^2 \cdot K_\Sigma^2,$$

that is $4p^2K_\Sigma^2 - 8pK_\Sigma^2 - 16 \leq 0$, which contradicts the assumptions on $p$.
If $t^* = \deg\big(X^*(C)\big)$, then we get from (1.26)

$$\Big(2 \cdot \deg\big(X^*(C)\big) + 2 + D.K_\Sigma\Big)^2 \geq \big(4 + K_\Sigma^2\big) \cdot D^2,$$

and, moreover, in this situation we must have $2\cdot\deg\big(X^*(C)\big)+2+D.K_\Sigma \geq 0$. But then we may take square roots on both sides and get

$$2\cdot\deg\big(X^*(C)\big) \geq \sqrt{4+K_\Sigma^2}\cdot\sqrt{D^2}-D.K_\Sigma-2,$$

which is in contradiction to (1.22).

(i) By the above condition we know that

$$\Big(D^2-\big(8+K_\Sigma^2\big)\Big)^2 \geq 16\cdot K_\Sigma^2+64.$$

This, however, implies

$$\big(1+\tfrac{K_\Sigma^2}{4}\big)\cdot D^2 \leq \tfrac{1}{16}\cdot\big(D^2+K_\Sigma^2\big)^2,$$

and thus finally

$$\sqrt{1+\tfrac{K_\Sigma^2}{4}}\cdot\sqrt{D^2}-\big|\tfrac{K_\Sigma.D}{2}+1\big| \leq \sqrt{1+\tfrac{K_\Sigma^2}{4}}\cdot\sqrt{D^2}-\tfrac{D.K_\Sigma}{2}-1 \leq \tfrac{1}{4}\cdot(D-K_\Sigma)^2,$$

which finishes the proof.

(j) $0 \leq \sum_{i=1}^r\big(\tau^*(\mathcal{S}_i)-1\big)^2 = \sum_{i=1}^r\big(\tau^*(\mathcal{S}_i)+1\big)^2-4\cdot\sum_{i=1}^r\tau^*(\mathcal{S}_i).$

□

Throughout the proof of Theorem 1.1 we used the following lemma.

**1.3 Lemma**

*Let* $(C,z)$ *be a reduced plane curve singularity given by* $f\in\mathcal{O}_{\Sigma,z}$ *and let* $I\subseteq\mathfrak{m}_{\Sigma,z}\subset\mathcal{O}_{\Sigma,z}$ *be an ideal containing the Tjurina ideal* $I^{ea}(C,z)$*. Then for any* $g\in I$ *we have*

$$\dim_{\mathbb{C}}(\mathcal{O}_{\Sigma,z}/I) < \dim_{\mathbb{C}}\big(\mathcal{O}_{\Sigma,z}/(f,g)\big) = i(f,g;z).$$

**Proof:** Cf. [**Shu97**] Lemma 4.1. □

## 2. Examples

Throughout this section for a topological respectively analytical singularity type $\mathcal{S}$ we will denote by $\tau^*(\mathcal{S})=\tau^{es}(\mathcal{S})$, the codimension of the $\mu$-constant stratum in the base of the semi-universal deformation of $\mathcal{S}$, respectively $\tau^*(\mathcal{S})=\tau(\mathcal{S})$, the Tjurina number of $\mathcal{S}$.

### 2.a. The Classical Case - $\Sigma=\mathbb{P}^2_{\mathbb{C}}$

In view of Remark 1.2 we get the following version of Theorem 1.1.

**1.1a Theorem**

*Let* $\mathcal{S}_1,\ldots,\mathcal{S}_r$ *be topological or analytical singularity types, let* $d\geq 3$ *and suppose that*

$$\sum_{i=1}^r\big(\tau^*(\mathcal{S}_i)+1\big)^2 < d^2+6d. \tag{1.5a}$$

*Then either* $V^{irr}_{|dH|}(\mathcal{S}_1,\ldots,\mathcal{S}_r)$ *is empty or T-smooth.*

**Proof:** (1.1) and (1.2) are fulfilled since $d \geq 3$, while by Remark 1.2 (e) (1.3) is redundant and (1.5) takes the form (1.5a). It remains to show that (1.5a) implies (1.4), which follows at once from (1.24) in Remark 1.2 (j). □

**2.1 Remark**

(a) The sufficient condition

$$\sum_{i=1}^{r} \big(\tau^*(\mathcal{S}_i) + 1\big)^2 \leq (d+3)^2 \tag{2.1}$$

in [**Los98**] Corollary 5.5 (see also [**GLS00**]) is always slightly better than (1.5a).

(b) Going back to the original equations in (1.5) we could replace them by

$$\sum_{i=1}^{r} \big(\tau^*(\mathcal{S}_i) + 1\big)^2 < \Big(1 + \tfrac{3\sqrt{4r+9}+9}{2r}\Big) \cdot d^2. \tag{2.2}$$

For a fixed $r$ this condition is asymptotically better even than the one in (2.1) – more precisely, $d \geq 3\sqrt{r}$ suffices. However, in order to get rid of (1.4) we then need some assumptions on the singularities – e. g. $r \geq 18$ and $\tau^*(\mathcal{S}_i) \geq 6$ for all $i$.

(c) Working with condition (1.23) from Remark 1.2 (h) we could replace (1.5a) by

$$\sum_{i=1}^{r} \tau^*(\mathcal{S}_i)^2 < \tfrac{10}{r} \cdot d^2, \tag{2.3}$$

if we require $d \geq 8$. This result is better than (2.1) only for $r \leq 9$.

**Proof:** (a) This is obvious.

(b) Since the derivative

$$\frac{\partial f}{\partial r}(r) = \frac{-3 \cdot \big(2r + 3\sqrt{4r+9} + 9\big)}{2 \cdot \sqrt{4r+9} \cdot r^2}$$

of $f : \mathbb{R} \to \mathbb{R} : r \mapsto \frac{3\sqrt{4r+9}+9}{2r}$ is negative for all $r > 0$, the function is monotonously decreasing, that is

$$\frac{3\sqrt{4r+9}+9}{2r} \leq \frac{3\sqrt{4s+9}+9}{2s}$$

for all $s = 1, \ldots, r$, and therefore

$$\sum_{i=1}^{s} \big(\tau^*(\mathcal{S}_i) + 1\big)^2 \leq \sum_{i=1}^{r} \big(\tau^*(\mathcal{S}_i) + 1\big)^2$$
$$< \left(1 + \frac{3\sqrt{4r+9}+9}{2r}\right) \cdot d^2 \leq \left(1 + \frac{3\sqrt{4s+9}+9}{2s}\right) \cdot d^2.$$

But this implies

$$\left(\sum_{i=1}^{s}\big(\tau^*(\mathcal{S}_i)+1\big)\right)^2 \le s\cdot\sum_{i=1}^{s}\big(\tau^*(\mathcal{S}_i)+1\big)^2$$
$$< \left(s+\tfrac{3}{2}\cdot\sqrt{4s+9}+\tfrac{9}{2}\right)\cdot d^2 = \left(\sqrt{s+\tfrac{9}{4}}+\tfrac{3}{2}\right)^2\cdot d^2,$$

and then

$$\sum_{i=1}^{s}\big(\tau^*(\mathcal{S}_i)+1\big)-\tfrac{3}{2}\cdot d < \sqrt{s+\tfrac{9}{4}}\cdot d,$$

which in turn gives (1.5), supposed that the left hand side is non-negative. However, if it is negative, then

$$\left(\sum_{i=1}^{s}\big(\tau^*(\mathcal{S}_i)+1\big)-\tfrac{3}{2}\cdot d\right)^2 < \frac{9d^2}{4} < \left(s+\tfrac{9}{4}\right)\cdot d^2,$$

that is, (1.5) is still fulfilled.
If $d \ge 3\sqrt{r}$, then $\frac{3\cdot\sqrt{4r}}{2r}\cdot d^2 \ge 9d \ge 6d+9$. Thus (2.1) implies (2.2). Furthermore, if $r \ge 18$, then $1+\frac{3\cdot\sqrt{4r+9}+9}{2r} \le 2$, and if moreover $\tau^*(\mathcal{S}_i) \ge 6$ for all $i$, then

$$\sum_{i=1}^{r}\tau^*(\mathcal{S}_i) \le \tfrac{1}{8}\cdot\sum_{i=1}^{r}\big(\tau^*(\mathcal{S}_i)+1\big)^2 < \tfrac{1}{4}\cdot d^2,$$

which implies (1.4).

(c) We note that (2.3) implies

$$\left(\sum_{i=1}^{r}\tau^*(\mathcal{S}_i)\right)^2 \le r\cdot\sum_{i=1}^{r}\tau^*(\mathcal{S}_i)^2 < 10\cdot d^2 < \left(\tfrac{\sqrt{13}+3}{2}\cdot d-1\right)^2,$$

once $d \ge 8$. This however is sufficient for the T-smoothness according to Remark 1.2 (h) and (i).

□

For further results in the plane case see [**Wah74a, GrK89, Lue87a, Lue87b, Shu87b, Vas90, Shu91b, Shu94, GrL96, Shu96b, Shu97, GLS98a, Los98, GLS00**].

### 2.b. Geometrically Ruled Surfaces

Throughout this section we use the notation of Section G.a, thus in particular $\pi : \Sigma \to C$ is a geometrically ruled surface with $g = g(C)$ and $K_\Sigma^2 = 8-8g$. Theorem 1.1 therefore only applies for Hirzebruch surfaces and if $C$ is elliptic. Below we state the theorem in its general version for the invariant $e \ge 0$, even though the conditions look far too nasty to be of much use. We afterwards consider the cases separately, so that the conditions take a nicer form, in particular we consider the unique geometrically ruled surface with negative invariant $e$ and $g = 1$.

Throughout the section it is convenient to set $\kappa = 2 + e - 2g$, so that $K_\Sigma \sim_l -2C_0 - \kappa F$. We note that for $g \leq 1$ and $e \geq 0$ we have $\kappa \geq 0$. Thus $-K_\Sigma$ is nef in this situation and we may replace (1.5) by (1.17) in Remark 1.2.

**1.1b Theorem**

*Let $\pi : \Sigma \to C$ be a geometrically ruled surface with invariant $e \geq 0$ and $g \in \{0, 1\}$, let $D = aC_0 + bF \in \text{Div}(\Sigma)$, and let $\mathcal{S}_1, \ldots, \mathcal{S}_r$ be topological or analytical singularity types.*

*Suppose that*

(1.1b) $\quad a \geq 2,\ b \geq e \cdot (a-2) + \kappa,$

(1.2b) $\quad b > e \cdot (a+2) - \kappa,$

(1.5b) $\quad \sum_{i=1}^{r} \big(\tau^*(\mathcal{S}_i) + 1\big)^2 < (2+\kappa) \cdot ab + 4b + (2\kappa - 4e) \cdot a - \big(\frac{\kappa^2}{4} + e\big) \cdot a^2 - b^2.$

*Then either $V^{irr}_{|D|}(\mathcal{S}_1, \ldots, \mathcal{S}_r)$ is empty or T-smooth.*

**Proof:** The conditions (1.1) and (1.2) just come down to (1.1b) and (1.2b), and (1.17) takes the form (1.5b). Moreover, by Remark 1.2 (c) Condition (1.3) is obsolete.

It thus remains to show that (1.5b) implies (1.4), which in this situation looks like

$$\sum_{i=1}^{r} \tau^*(\mathcal{S}_i) < \tfrac{1}{4} \cdot \big(2 \cdot (a+2) \cdot (b+\kappa) - e \cdot (a+2)^2\big).$$

In view of (1.24) in Remark 1.2 (j) and by (1.5b) we know already that

$$\sum_{i=1}^{r} \tau^*(\mathcal{S}_i) < \tfrac{1}{4} \cdot \Big((2+\kappa) \cdot ab + 4b + (2\kappa - 4e) \cdot a - \big(\tfrac{\kappa^2}{4} + e\big) \cdot a^2 - b^2\Big).$$

Since $\kappa - e = 2 - 2g \geq 0$, the claim follows from

$$\begin{aligned} &(2+\kappa) \cdot ab + 4b + (2\kappa - 4e) \cdot a - \big(\tfrac{\kappa^2}{4} + e\big) \cdot a^2 - b^2 \\ &\quad = 2 \cdot (a+2) \cdot (b+\kappa) - e \cdot (a+2)^2 - \big(\tfrac{\kappa}{2} \cdot a - b\big)^2 - 4 \cdot (\kappa - e) \\ &\quad \leq 2 \cdot (a+2) \cdot (b+\kappa) - e \cdot (a+2)^2. \end{aligned}$$

□

We note that (1.5b) gives an obstruction on $a$ and $b$, namely

$$(2+\kappa) \cdot ab + 4b + (2\kappa - 4e) \cdot a \ \big(\tfrac{\kappa^2}{4} + e\big) \cdot a^2 + b^2 + 4. \qquad (2.4)$$

Let us first consider Hirzebruch surfaces.

**2.2 Corollary**

*Let $\Sigma \cong \mathbb{F}_0 = \mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$, let $D \in \text{Div}(\Sigma)$ of type $(a, b)$ with $a, b \geq 2$, and let $\mathcal{S}_1, \ldots, \mathcal{S}_r$ be topological or analytical singularity types.*

*Suppose that*

$$\sum_{i=1}^{r} \left(\tau^*(\mathcal{S}_i)+1\right)^2 < 4ab+4b+4a-a^2-b^2. \tag{2.5}$$

*Then either* $V_{|D|}^{irr}(\mathcal{S}_1,\ldots,\mathcal{S}_r)$ *is empty or T-smooth.*

We note that the obstruction (2.4) in this case reads

$$4ab+4b+4a \geq a^2+b^2+4, \tag{2.6}$$

which could be replaced by

$$a \leq 2b+3 \quad \text{and} \quad b \leq 2a+3. \tag{2.7}$$

If we consider the case $a=b$, that is, if $D$ is a rational multiple of the canonical divisor, then (2.5) looks like

$$\sum_{i=1}^{r} \left(\tau^*(\mathcal{S}_i)+1\right)^2 < 2a^2+8a. \tag{2.8}$$

In this situation we could even use (1.21) in Remark 1.2 instead and replace (2.8) by the better inequality

$$\sum_{i=1}^{r} \frac{(\tau^*(\mathcal{S}_i)+1)^2}{1+\frac{2}{a}\cdot(\tau^*(\mathcal{S}_i)+1)} < 2a^2. \tag{2.9}$$

Comparing these results with the conditions for irreducibility in Section V.4.b the right hand side there differs by a factor of about $\frac{1}{200}$.

The only Hirzebruch surface, which is not minimal, is $\mathbb{F}_1$. Since the obstructions for $b$ there differ from the remaining cases we give the result separately.

**2.3 Corollary**

*Let* $\Sigma=\mathbb{F}_1$, *let* $D=aC_0+bF\in \mathrm{Div}(\Sigma)$ *with* $a\geq 2$ *and* $b\geq a+1$, *and let* $\mathcal{S}_1,\ldots,\mathcal{S}_r$ *be topological or analytical singularity types.*

*Suppose that*

$$\sum_{i=1}^{r} \left(\tau^*(\mathcal{S}_i)+1\right)^2 < 5ab+4b+2a-\tfrac{13}{4}\cdot a^2-b^2. \tag{2.10}$$

*Then either* $V_{|D|}^{irr}(\mathcal{S}_1,\ldots,\mathcal{S}_r)$ *is empty or T-smooth.*

In contrast to the Hirzebruch surfaces with $e\geq 2$, the corollary here applies to divisors, which are multiples of the canonical divisor, that is when $b=\frac{3}{2}\cdot a$. Then, however, we may in view of Remark 1.2 (f) replace (2.10) by the inequality

$$\sum_{i=1}^{r} \frac{(\tau^*(\mathcal{S}_i)+1)^2}{1+\frac{3}{a}\cdot(\tau^*(\mathcal{S}_i)+1)} < 2a^2. \tag{2.11}$$

For the Hirzebruch surfaces with $e\geq 2$ we have to study the conditions (1.2b) and (2.4) a bit closer. We then see that they may be replaced by

$$b \geq ea+e-1 \tag{2.12}$$

and

$$\varphi_{a,e}(b) := -\left(b - \tfrac{e+2}{2} \cdot a\right)^2 + \left(2ab - ea^2\right) + 4b + (4-2e) \cdot a - 4 > 0. \qquad (2.13)$$

For $a \geq 2$ fixed we therefore may consider $\varphi_{a,e} : [ea+e-1, \infty[ \to \mathbb{R}$ as a concave function, and we find that

$$\varphi_{a,e}(b) > 0 \iff b < \left(2 + \tfrac{e}{2}\right) \cdot a + \sqrt{3a^2 + 12a} + 2.$$

In particular, (2.12) and (2.13) cannot be satisfied both at the same time unless $e \leq 7$! And if $e \leq 7$, then we may replace condition (1.1b) and (1.2b) by

$$a \geq 2 \quad \text{and} \quad b \in \left[ea + e - 1, \left(2 + \tfrac{e}{2}\right) \cdot a + \sqrt{3a^2 + 12a} + 2\right[. \qquad (2.14)$$

**2.4 Corollary**

*Let $\Sigma = \mathbb{F}_e$ with $2 \leq e \leq 7$, let $D = aC_0 + bF \in \mathrm{Div}(\Sigma)$ with $a \geq 2$ and $b \in \left[ea + e - 1, \left(2 + \frac{e}{2}\right) \cdot a + \sqrt{3a^2 + 12a} + 2\right[$, and let $\mathcal{S}_1, \ldots, \mathcal{S}_r$ be topological or analytical singularity types.*

*Suppose that*

$$\sum_{i=1}^{r} \left(\tau^*(\mathcal{S}_i) + 1\right)^2 < \left(2ab - ea^2\right) + 4b + (4-2e) \cdot a - \left(b - \tfrac{e+2}{2} \cdot a\right)^2. \qquad (2.15)$$

*Then either $V^{\mathrm{irr}}_{|D|}(\mathcal{S}_1, \ldots, \mathcal{S}_r)$ is empty or T-smooth.*

Next we consider the case of an elliptic base curve. As in the case of Hirzebruch surfaces, we study Condition (1.2b) and Obstruction (2.4) a bit closer. We then see that they may be replaced by

$$b \geq ea + e + 1 \qquad (2.16)$$

and

$$\psi_{a,e}(b) := -\left(b - \tfrac{e}{2} \cdot a\right)^2 + \left(2ab - ea^2\right) + 4b + -2ea - 4 > 0. \qquad (2.17)$$

For $a \geq 2$ fixed we consider $\psi_{a,e} : [ea + e + 1, \infty[ \to \mathbb{R}$ as a concave function, and we find that

$$\psi_{a,e}(b) > 0 \iff b < a + \tfrac{e}{2} \cdot a + \sqrt{a^2 + 4a} + 2.$$

Thus (2.16) and (2.17) will never be satisfied at the same time if $e \geq 4$, while if $0 \leq e \leq 3$ we may replace (1.1b) and (1.2b) by

$$a \geq 2 \quad \text{and} \quad b \in \left[ea + e + 1, a + \tfrac{e}{2} \cdot a + \sqrt{a^2 + 4a} + 2\right[. \qquad (2.18)$$

**2.5 Corollary**

*Let $\pi : \Sigma \to C$ be a geometrically ruled surface with $g(C) = 1$ and $0 \leq e \leq 3$, let $D = aC_0 + bF \in \mathrm{Div}(\Sigma)$ with $a \geq 2$ and $b \in \left[ea + e + 1, a + \frac{e}{2} \cdot a + \sqrt{a^2 + 4a} + 2\right[$, and let $\mathcal{S}_1, \ldots, \mathcal{S}_r$ be topological or analytical singularity types.*

*Suppose that*

$$\sum_{i=1}^{r} \left(\tau^*(\mathcal{S}_i) + 1\right)^2 < \left(2ab - ea^2\right) + 4b - 2ea - \left(b - \tfrac{e}{2} \cdot a\right)^2. \qquad (2.19)$$

*Then either* $V^{irr}_{|D|}(\mathcal{S}_1,\ldots,\mathcal{S}_r)$ *is empty or T-smooth.*

We note that in the most interesting case, namely when $e = 0$, which includes the product $C \times \mathbb{P}^1_{\mathbb{C}}$, we just need

$$0 < b \le a + \sqrt{a^2+4a} + 2 \tag{2.20}$$

and

$$\sum_{i=1}^{r} \left(\tau^*(\mathcal{S}_i)+1\right)^2 < 2ab + 4b - b^2. \tag{2.21}$$

Finally we have to consider the unique case of a geometrically ruled surface over an elliptic curve with negative invariant $e$. (1.1) and (1.2) come down to

$$a \ge 2 \quad \text{and} \quad b > -\frac{a}{2}. \tag{2.22}$$

Moreover, then $(aC_0 + bF).K_\Sigma = -2b - a < 0$, so that Remark 1.2 (c) still applies, that is, (1.3) is redundant and (1.5) may be replaced by (1.17), which takes the form

$$\sum_{i=1}^{r} \left(\tau^*(\mathcal{S}_i)+1\right)^2 < \tfrac{3}{4}\cdot a^2 + ab + 2a + 4b - b^2.$$

This, however, leads to the obstruction

$$\psi_{a,-1}(b) = \tfrac{3}{4}\cdot a^2 + ab + 2a + 4b - b^2 - 4 > 0.$$

Thus, considering $\psi_{a,-1}$ now as a concave function $\psi_{a,-1} : \left]-\frac{a}{2},\infty\right[ \to \mathbb{R}$ we find that

$$\psi_{a,-1}(b) > 0 \quad\Longleftrightarrow\quad -\frac{a}{2} < \tfrac{a}{2} + 2 - \sqrt{a^2+4a} < b < \tfrac{a}{2} + 2 + \sqrt{a^2+4a}.$$

And we may therefore replace (1.1b)-(1.2b) by

$$a \ge 2 \quad \text{and} \quad b \in \left]\frac{a}{2} + 2 - \sqrt{a^2+4a}, \frac{a}{2} + 2 + \sqrt{a^2+4a}\right[. \tag{2.23}$$

**2.6 Corollary**

*Let* $\pi : \Sigma \to C$ *be a geometrically ruled surface with* $g(C) = 1$ *and* $e = -1$, *let* $D = aC_0 + bF \in \mathrm{Div}(\Sigma)$ *with* $a \ge 2$ *and* $b \in \left]\frac{a}{2}+2-\sqrt{a^2+4a}, \frac{a}{2}+2+\sqrt{a^2+4a}\right[$, *and let* $\mathcal{S}_1,\ldots,\mathcal{S}_r$ *be topological or analytical singularity types.*

*Suppose that*

$$\sum_{i=1}^{r} \left(\tau^*(\mathcal{S}_i)+1\right)^2 < \tfrac{3}{4}\cdot a^2 + ab + 2a + 4b - b^2. \tag{2.24}$$

*Then either* $V^{irr}_{|D|}(\mathcal{S}_1,\ldots,\mathcal{S}_r)$ *is empty or T-smooth.*

In the above corollary we could for instance let $b = \frac{a}{2} + 2$, in which case (2.19) reads

$$\sum_{i=1}^{r} \left(\tau^*(\mathcal{S}_i)+1\right)^2 < (a+2)^2.$$

## 2.c. Products of Curves

Throughout this section we use the notation of Section G.b. In particular, $\Sigma = C_1 \times C_2$ where $C_1$ and $C_2$ are smooth projective curves over $\mathbb{C}$ of genera $g_1$ and $g_2$ respectively. Since $\Sigma$ is geometrically ruled if some $g_i = 0$, we may restrict our attention to the case $g_1, g_2 \geq 1$.

For a generic choice of $C_1$ and $C_2$ the Néron–Severi group $\mathrm{NS}(\Sigma)$ is two-dimensional by Proposition G.12. Thus the following theorem answers the general case completely.

**1.1c Theorem**
*Let $C_1$ and $C_2$ be two smooth projective curves of genera $g_1$ and $g_2$ respectively with $g_1 \geq g_2 \geq 1$.*

*Let $D = aC_1 + bC_2 \in \mathrm{Div}(\Sigma)$ with $a > 2g_2 - 2$ and $b > 2g_1 - 2$, and let $\mathcal{S}_1, \ldots, \mathcal{S}_r$ be topological or analytical singularity types with $\tau^*(\mathcal{S}_1) \geq \tau^*(\mathcal{S}_i)$ for all $i = 1, \ldots, r$.*

*Suppose that*

$$\text{(1.5c)} \qquad \sum_{i=1}^{r} \big(\tau^*(\mathcal{S}_i) + 1\big)^2 < 2ab - 2\cdot\big(\tau^*(\mathcal{S}_1)+1\big)\cdot\big((g_1-1)a + (g_2-1)b\big) - \big((g_1-1)a - (g_2-1)b\big)^2.$$

*Then either $V^{irr}_{|D|}(\mathcal{S}_1, \ldots, \mathcal{S}_r)$ is empty or T-smooth.*

**Proof:** We note that $K_\Sigma = (2g_2-2)\cdot C_1 + (2g_1-2)\cdot C_2$ is nef. Thus by Remark 1.2 (b) and (d) and due to the assumptions on $a$ and $b$ (1.1)-(1.3) are fulfilled, while (1.5) may be replaced by (1.18), which in this case is just (1.5c). It thus remains to show that (1.4) is satisfied, which in this case takes the form

$$\sum_{i=1}^{r} \tau^*(\mathcal{S}_i) < \tfrac{1}{2} \cdot (a + 2 - 2g_2) \cdot (b + 2 - 2g_1).$$

This, however, follows from (1.24) in Remark 1.2 since

$$\begin{aligned} 2ab - 2 \cdot \big(\tau^*(\mathcal{S}_1) + 1\big) \cdot \big((g_1 - 1)a + (g_2 - 1)b\big) - \big((g_1 - 1)a - (g_2 - 1)b\big)^2 \\ \leq 2ab - 4 \cdot \big((g_1 - 1)a + (g_2 - 1)b\big) \leq 2 \cdot (a + 2 - 2g_2) \cdot (b + 2 - 2g_1). \end{aligned}$$

□

Of course, if $(g_1 - 1) \cdot a$ and $(g_2 - 1) \cdot b$ differ too much from each other, then Condition (1.5c) can never be satisfied, since then $\big((g_1 - 1) \cdot a - (g_2 - 1) \cdot b\big)^2$ becomes larger than $2ab$ – e. g. if $g_1, g_2 > 1$ and $a > 4 \cdot \frac{g_2-1}{g_1-1} \cdot b$. Moreover, we see that the largest $\tau^*(\mathcal{S}_i)$ should be considerably smaller than $a$ or $b$.

If $g_1, g_2 > 1$ and if $a = p \cdot (g_2 - 1)$ and $b = p \cdot (g_1 - 1)$ for some $p \in \mathbb{Q}$, that is if $D$ is a multiple of the ample divisor $K_\Sigma$ then (1.5c) takes the simpler form

$$\sum_{i=1}^{r} \big(\tau^*(\mathcal{S}_i) + 1\big)^2 < 2 \cdot \Big(p^2 - 2p \cdot \big(\tau^*(\mathcal{S}_1) + 1\big)\Big) \cdot (g_1 - 1) \cdot (g_2 - 1). \qquad (2.25)$$

In this case, however, we could replace (2.25) in view of Remark 1.2 (f) also by

$$\sum_{i=1}^{r} \frac{(\tau^*(\mathcal{S}_i)+1)^2}{1-\frac{4}{p}\cdot(\tau^*(\mathcal{S}_i)+1)} < 2p^2\cdot(g_1-1)\cdot(g_2-1). \tag{2.26}$$

## 2.d. Products of Elliptic Curves

If in Section V.4.c the curves $C_1$ and $C_2$ are chosen to be both elliptic curves, Theorem 1.1c looks much nicer, since $K_\Sigma = 0$.

**1.1d Theorem**

*Let* $C_1$ *and* $C_2$ *be two smooth elliptic curves, let* $D \in \mathrm{Div}(\Sigma)$ *big and nef, and let* $\mathcal{S}_1, \dots, \mathcal{S}_r$ *be topological or analytical singularity types.*

*Suppose that*

(1.5d) $$\sum_{i=1}^{r} \big(\tau^*(\mathcal{S}_i)+1\big)^2 < D^2.$$

*Then either* $V^{irr}_{|D|}(\mathcal{S}_1, \dots, \mathcal{S}_r)$ *is empty or T-smooth.*

If $C_1$ and $C_2$ are non-isogenous, then any divisor of $\Sigma$ is of the form $D = aC_1 + bC_2$, and for such a divisor the condition "$D$ big and nef" means that $a, b > 0$.

The Severi variety $V^{irr}_{|aC_1+bC_2|}(rA_1)$ is therefore T-smooth as soon as

$$r \le \frac{ab}{2},$$

and in the case of cusps and nodes the condition for the T-smoothness of $V^{irr}_{|aC_1+bC_2|}(kA_1 + mA_2)$ thus just reads

$$4k + 9m < 2ab.$$

## 2.e. Surfaces in $\mathbb{P}^3_{\mathbb{C}}$

For a generic surface of degree $n \ge 4$ in $\mathbb{P}^3_{\mathbb{C}}$ any curve is a hypersurface section by a Theorem of Noether (see Section G.d). Therefore the following result on hypersurface sections answers the problem completely on a generic surface in $\mathbb{P}^3_{\mathbb{C}}$. We note that a hypersurface section is always a rational multiple of the canonical bundle and thus we may use Remark 1.2 (b) and (1.20).

**1.1e Theorem**

*Let* $\Sigma \subset \mathbb{P}^3_{\mathbb{C}}$ *be a smooth hypersurface of degree* $n \ge 4$, $H \subset \Sigma$ *be a hyperplane section,* $d > n-4$ *and let* $\mathcal{S}_1, \dots, \mathcal{S}_r$ *be topological or analytical singularity types with* $\tau^*(\mathcal{S}_1) \ge \tau^*(\mathcal{S}_i)$ *for all* $i = 1, \dots, r$. *Suppose that*

(1.4e) $$\sum_{i=1}^{r} \tau^*(\mathcal{S}_i) < \tfrac{(d+4-n)^2\cdot n}{4}, \text{ and}$$

(1.5e) $$\sum_{i=1}^{r} \big(\tau^*(\mathcal{S}_i)+1\big)^2 < d^2\cdot n - d\cdot(n-4)\cdot n\cdot\big(\tau^*(\mathcal{S}_1)+1\big).$$

*Then either* $V^{irr}_{|dH|}(\mathcal{S}_1, \dots, \mathcal{S}_r)$ *is empty or T-smooth.*

If we suppose $n > 4$ and the $\tau^*(\mathcal{S}_i) + 1 < \frac{d}{n-4}$, which is necessary anyway in order to have (1.5e), then in view of Remark 1.2 (f) we may replace (1.5e) by the better inequality

$$\sum_{i=1}^{r} \frac{\big(\tau^*(\mathcal{S}_i)+1\big)^2}{1-\frac{n-4}{d}\cdot\big(\tau^*(\mathcal{S}_i)+1\big)} < d^2\cdot n. \tag{2.27}$$

Let us consider the two cases of nodal curves and of curves with nodes and cusps more closely.

**2.6 Corollary**

*Let $\Sigma \subset \mathbb{P}^3_{\mathbb{C}}$ be a smooth hypersurface of degree $n \geq 4$, $H \subset \Sigma$ be a hyperplane section and $d > 2n - 8$. Suppose that*

$$r < \frac{d\cdot(d-2n+8)\cdot n}{4}. \tag{2.28}$$

*Then either $V^{irr}_{|dH|}(rA_1)$ is empty or T-smooth.*

**Proof:** In this situation (1.5e) is just (2.28), and (1.4e) is fulfilled as well in view of

$$\frac{d\cdot(d-2n+8)\cdot n}{4} \leq \frac{d\cdot(d-2n+8)\cdot n+(4-n)^2 n}{4} = \frac{(d+4-n)^2\cdot n}{4}.$$

□

**2.7 Corollary**

*Let $\Sigma \subset \mathbb{P}^3_{\mathbb{C}}$ be a smooth hypersurface of degree $n \geq 4$, $H \subset \Sigma$ be a hyperplane section and $d > 3n - 12$. Suppose that*

$$4k + 9m < d\cdot(d-3n+12)\cdot n. \tag{2.29}$$

*Then either $V^{irr}_{|dH|}(kA_1, mA_2)$ is empty or T-smooth.*

**Proof:** In this situation (1.5e) is just (2.29), and (1.4e) is fulfilled in view of

$$4\cdot(k+2m) \leq 4k+9m < d\cdot(d-3n+12)\cdot n \leq (d+4-n)^2\cdot n.$$

□

## 2.f. K3-Surfaces

By Remark 1.2 (b), (c) and (j) Theorem 1.1 for K3-surfaces takes the following form.

**1.1f Theorem**

*Let $\Sigma$ be a smooth K3-surface, $D$ a big and nef divisor on $\Sigma$ and $\mathcal{S}_1, \ldots, \mathcal{S}_r$ be topological or analytical singularity types.*

*Suppose that*

(1.5f) $\quad \sum_{i=1}^{r} \big(\tau^*(\mathcal{S}_i)+1\big)^2 < D^2.$

*Then either $V^{irr}_{|D|}(\mathcal{S}_1, \ldots, \mathcal{S}_r)$ is empty or T-smooth.*

## 3. Another Criterion for T-Smoothness

In the proof of Theorem 1.1 we used the result of Lemma 1.3 in a crude way, i. e. by replacing the difference $i(f,g;z) - \dim_{\mathbb{C}}(\mathcal{O}_{\Sigma,z}/I)$ by its minimum possible value one. In the following we are more careful. Using the Cauchy-Schwartz Inequality we derive conditions which involve new invariants for which the above difference plays a crucial role. The proofs are an adaptation of [**GLS00**], replacing the Castelnuovo function arguments by Bogomolov instability methods. In the case of the projective plane we reproduce their result (see Theorem 3.1 and Remark 3.2), but the results apply to much larger classes of surfaces.

Compared with the results in Section 1 the new results have mainly two advantages - firstly they apply to surfaces to which the previous results did not apply, and secondly the involved invariants are in many cases much better, e. g. for ordinary multiple points. However, for families of nodal or cuspidal curves the results of Section 1 better wherever they apply - apart from the plane case.

**3.1 Theorem**
*Let $\Sigma$ be a surface such that*[3] $\mathrm{NS}(\Sigma) = L \cdot \mathbb{Z}$*, let* $D = d \cdot L \in \mathrm{Div}(\Sigma)$*, let* $\mathcal{S}_1, \ldots, \mathcal{S}_r$ *be topological or analytical singularity types, and let* $K_\Sigma = \kappa \cdot L$.

*Suppose that* $d \geq \max\{\kappa+1, -\kappa\}$ *and*

$$\sum_{i=1}^{r} \gamma_\alpha^*(\mathcal{S}_i) < \alpha \cdot (D - K_\Sigma)^2 = \alpha \cdot (d-\kappa)^2 \cdot L^2 \quad \text{with } \alpha = \tfrac{1}{\max\{1, 1+\kappa\}}. \tag{3.1}$$

*Then either* $V_{|D|}^{irr}(\mathcal{S}_1, \ldots, \mathcal{S}_r)$ *is empty or it is T-smooth.*

**Proof:** Let $C \in V_{|D|}^{irr}(\mathcal{S}_1, \ldots, \mathcal{S}_r)$. It suffices to show that the cohomology group $h^1\big(\Sigma, \mathcal{J}_{X^*(C)/\Sigma}(D)\big)$ vanishes.

Suppose this is not the case. Since for $X_0 \subseteq X^*(C)$ any local complete intersection scheme and $z \in \mathrm{supp}(X_0)$ we have

$$4 \cdot \deg(X_z) \leq \frac{4}{(1+\alpha)^2} \cdot \gamma_\alpha^*(C,z) \leq \frac{1}{\alpha} \cdot \gamma_\alpha^*(C,z) \tag{3.2}$$

Lemma 3.5 applies and there is curve $\Delta \in |\delta \cdot L|_l$ and a local complete intersection scheme $X_0 \subseteq X^*(C)$ satisfying the assumptions (a)-(d) there and Equation (3.9). That is, fixing the notation $l = \sqrt{L^2}$, $\mathrm{supp}(X_0) = \{z_1, \ldots, z_s\}$, $X_i = X_{0,z_i}$ and $\varepsilon_i = \min\{\deg(X_i), i(C,\Delta;z_i) - \deg(X_i)\} \geq 1$, we have

(a) $d \cdot \delta \cdot l^2 \geq \deg(X_0) + \sum_{i=1}^{s} \varepsilon_i$,

(b) $\deg(X_0) \geq (d - \kappa - \delta) \cdot \delta \cdot l^2$,

[3] By Lemma E.1 we may assume w. l. o. g. that $L$ is ample.

and

$$\delta \cdot l \leq \frac{(d-\kappa)\cdot l}{2} - \sqrt{\frac{(d-\kappa)^2\cdot l^2}{4} - \deg(X_0)} = \frac{2\cdot \deg(X_0)}{(d-\kappa)\cdot l + \sqrt{(d-\kappa)^2\cdot l^2 - 4\cdot \deg(X_0)}}.$$

But then together with (a) and (b) we deduce

$$\sum_{i=1}^{s} \varepsilon_i \leq \delta\cdot(\delta+\kappa)\cdot l^2 \leq \frac{1}{\alpha}\cdot\left(\frac{2\cdot\deg(X_0)}{(d-\kappa)\cdot l + \sqrt{(d-\kappa)^2\cdot l^2 - 4\cdot\deg(X_0)}}\right)^2. \quad (3.3)$$

Applying the Cauchy inequality this leads to

$$\sum_{i=1}^{s} \frac{\deg(X_i)^2}{\varepsilon_i} \geq \frac{\deg(X_0)^2}{\sum_{i=1}^{s}\varepsilon_i} \geq \frac{\alpha\cdot(d-\kappa)^2\cdot l^2}{4}\cdot\left(1+\sqrt{1-\frac{4\cdot\deg(X_0)}{(d-\kappa)^2\cdot l^2}}\right)^2.$$

Setting

$$\beta = \frac{\sum_{i=1}^{s}\frac{\deg(X_i)^2}{\varepsilon_i}}{\alpha\cdot(d-\kappa)^2\cdot l^2}, \quad \gamma = \frac{\sum_{i=1}^{s}\frac{\deg(X_i)^2}{\varepsilon_i}}{\alpha\cdot\deg(X_0)},$$

we thus have

$$\beta \geq \frac{1}{4}\cdot\left(1+\sqrt{1-\frac{4\beta}{\gamma}}\right)^2,$$

and hence, $\beta \geq \left(\frac{\gamma}{\gamma+1}\right)^2$. But then, applying the Cauchy inequality once more, we find

$$\alpha\cdot(d-\kappa)^2\cdot l^2 = \frac{\alpha\cdot\gamma}{\beta}\cdot\deg(X_0) \leq \alpha\cdot\left(\gamma+2+\frac{1}{\gamma}\right)\cdot\deg(X_0)$$

$$\leq \sum_{i=1}^{s}\left(\frac{\deg(X_i)^2}{\varepsilon_i} + 2\alpha\deg(X_i) + \alpha^2\varepsilon_i\right) \leq \sum_{i=1}^{r}\gamma_\alpha^*(\mathcal{S}_i),$$

in contradiction to Equation (3.1). □

### 3.2 Remark

If in Theorem 3.1 we have $\kappa > 0$, i. e. $\alpha < 1$, then the strict inequality in Condition (3.1) may be replaced by "$\leq$", since in (3.2) the second inequality is strict, as is the second inequality in (3.3). This applies e. g. in the case of the general surfaces in $\mathbb{P}^3_{\mathbb{C}}$.

Similarly, if in Theorem 3.1 $\kappa < 0$ and for some for some $\mathcal{S}_i$ we have $\gamma_\alpha^*(\mathcal{S}_i) > (1+\alpha)^2\tau_{ci}^*(\mathcal{S}_i)$, then the inequalities in (3.2) and (3.3) are strict, so that the strict inequality in (3.1) may be replaced by "$\leq$". This implies e. g. in the case of $\mathbb{P}^2_{\mathbb{C}}$, considering simple singularities or ordinary multiple points not all of which are $A_1$. We then get the same result as **[GLS00]** Theorem 1.

### 3.3 Theorem

*Let* $C_1$ *and* $C_2$ *be two smooth projective curves of genera* $g_1$ *and* $g_2$*, such that for* $\Sigma = C_1\times C_2$ *the Néron–Severi group is* $\mathrm{NS}(\Sigma) = C_1\mathbb{Z}\oplus C_2\mathbb{Z}$*.*

*Let* $D \in \mathrm{Div}(\Sigma)$ *such that* $D \sim_a aC_1 + bC_2$ *with* $a \geq \max\{2-2g_2, 2g_2-1\}$ *and* $b \geq \max\{2-2g_1, 2g_1-1\}$*, let* $\mathcal{S}_1,\ldots,\mathcal{S}_r$ *be topological or analytical singularity types.*

*Suppose that*

$$\sum_{i=1}^{r} \gamma_0^*(\mathcal{S}_i) \;<\; \gamma \cdot (D - K_\Sigma)^2, \tag{3.4}$$

*where the constant* $\gamma$ *may be read off the following table with* $A = \frac{a-2g2+2}{b-2g1+2}$

| $g_1$ | $g_2$ | $\gamma$ |
|---|---|---|
| $0,1$ | $0,1$ | $\frac{1}{4}$ |
| $\geq 2$ | $0,1$ | $\min\left\{\frac{1}{4g_1}, \frac{1}{4\cdot(g_1-1)\cdot A}\right\}$ |
| $\geq 2$ | $\geq 2$ | $\min\left\{\frac{1}{4g_1+4g_2-4}, \frac{A}{4\cdot(g_2-1)}, \frac{1}{4\cdot(g_1-1)\cdot A}\right\}$ |

*Then either* $V_{|D|}^{irr}(\mathcal{S}_1,\ldots,\mathcal{S}_r)$ *is empty or it is T-smooth.*

**Proof:** Let $C \in V_{|D|}^{irr}(\mathcal{S}_1,\ldots,\mathcal{S}_r)$. It suffices to show that the cohomology group $h^1\big(\Sigma, \mathcal{J}_{X^*(C)/\Sigma}(D)\big)$ vanishes.

Suppose this is not the case. Since for $X_0 \subseteq X^*(C)$ any local complete intersection scheme and $z \in \text{supp}(X)$ we have

$$\deg(X_z) \leq \gamma_0^*(C, z),$$

and since $\gamma \leq \frac{1}{4}$, Lemma 3.5 applies and there is curve $\Delta \sim_n \alpha \cdot C_1 + \beta \cdot C_2$ and a local complete intersection scheme $X_0 \subseteq X^*(C)$ satisfying the assumptions (a)-(d) there and Equation (3.9). That is, fixing the notation $\text{supp}(X_0) = \{z_1,\ldots,z_s\}$, $X_i = X_{0,z_i}$ and $\varepsilon_i = \min\{\deg(X_i), i(C,\Delta;z_i) - \deg(X_i)\} \geq 1$, we have

(a) $a\beta + b\alpha \geq \deg(X_0) + \sum_{i=1}^{s} \varepsilon_i$,

(b) $\deg(X_0) \geq (a - 2g_2 + 2 - \alpha)\cdot\beta + (b - 2g_1 + 2 - \beta)\cdot\alpha$, and

(c) $0 \leq \alpha \leq \frac{a-2g_2+2}{2}$ and $0 \leq \beta \leq \frac{b-2g_1+2}{2}$.

The last inequalities follow from (d) in Lemma 3.5 replacing the ample divisor $H$ by the nef divisors $C_2$ respectively $C_1$.

From (b) and (c) we deduce

$$\deg(X_0) \geq \frac{a - 2g_2 + 2}{2}\cdot\beta + \frac{b - 2g_1 + 2}{2}\cdot\alpha,$$

and thus

$$\deg(X_0)^2 \geq 4 \cdot \frac{a - 2g_2 + 2}{2} \cdot \frac{b - 2g_1 + 2}{2} \cdot \alpha \cdot \beta = \frac{(D - K_\Sigma)^2}{2} \cdot \alpha \cdot \beta. \tag{3.5}$$

Considering now (a) and (b) we get

$$0 < \sum_{i=1}^{s} \varepsilon_i \leq \Delta.(\Delta + K_\Sigma) = 2\alpha\beta + (2g_1 - 2)\cdot\alpha + (2g_2 - 2)\cdot\beta \leq \frac{\alpha\beta}{2\gamma},$$

where the last inequality holds only if $\alpha \neq 0 \neq \beta$. In particular, we see $\alpha \neq 0$ if $g_2 \leq 1$ and $\beta \neq 0$ if $g_1 \leq 1$. But this together with (3.5) gives

$$\sum_{i=1}^{s} \varepsilon_i \leq \frac{\deg(X_0)^2}{\gamma \cdot (D-K_\Sigma)^2}.$$

If $\alpha = 0$, then from (a) and (b) we deduce again

$$0 < \sum_{i=1}^{s} \varepsilon_i \leq (2g_2-2)\cdot\beta \leq \frac{4\cdot(g_1-1)}{A}\cdot\frac{\deg(X_0)^2}{(D-K_\Sigma)^2} \leq \frac{\deg(X_0)^2}{\gamma\cdot(D-K_\Sigma)^2},$$

and similarly, if $\beta = 0$,

$$0 < \sum_{i=1}^{s} \varepsilon_i \leq (2g_1-2)\cdot\alpha \leq 4\cdot(g_1-1)\cdot A\cdot\frac{\deg(X_0)^2}{(D-K_\Sigma)^2} \leq \frac{\deg(X_0)^2}{\gamma\cdot(D-K_\Sigma)^2}.$$

Applying the Cauchy inequality, we finally get

$$\gamma\cdot(D-K_\Sigma)^2 \leq \frac{\deg(X_0)^2}{\sum_{i=1}^{s}\varepsilon_i} \leq \sum_{i=1}^{s}\frac{\deg(X_i)^2}{\varepsilon_i} \leq \sum_{i=1}^{r}\gamma_0^*(\mathcal{S}_i),$$

in contradiction to Assumption (3.4). □

In the following theorem we use the notation of Section G.a.

**3.4 Theorem**

*Let $\pi : \Sigma \to C$ be a geometrically ruled surface with $e \leq 0$ and $g = g(C)$.*

*Let $D \in \mathrm{Div}(\Sigma)$ such that $D \sim_a aC_0 + bF$ with $b > \max\{2g-2, 2-2g\} + \frac{ae}{2}$ and $a > 2$, and let $\mathcal{S}_1, \ldots, \mathcal{S}_r$ be topological or analytical singularity types.*

*Suppose that*

$$\sum_{i=1}^{r}\gamma_0^*(\mathcal{S}_i) \ < \ \gamma\cdot(D-K_\Sigma)^2, \tag{3.6}$$

*where with $A = \frac{a+2}{b+2-2g-\frac{ae}{2}}$ the constant $\gamma$ satisfies*

$$\gamma = \begin{cases} \frac{1}{4}, & \text{if } g \in \{0,1\},\\ \min\left\{\frac{1}{4g}, \frac{1}{4\cdot(g-1)\cdot A}\right\}, & \text{if } g \geq 2.\end{cases}$$

*Then either $V_{|D|}^{\mathrm{irr}}(\mathcal{S}_1, \ldots, \mathcal{S}_r)$ is empty or it is T-smooth.*

**Proof:** Let $C \in V_{|D|}^{\mathrm{irr}}(\mathcal{S}_1, \ldots, \mathcal{S}_r)$. It suffices to show that the cohomology group $h^1\big(\Sigma, \mathcal{J}_{X^*(C)/\Sigma}(D)\big)$ vanishes.

Suppose this is not the case. Since for $X_0 \subseteq X^*(C)$ any local complete intersection scheme and $z \in \mathrm{supp}(X)$ we have

$$\deg(X_z) \leq \gamma_0^*(C, z),$$

and since $\gamma \leq \frac{1}{4}$, Lemma 3.5 applies and there is curve $\Delta \sim_a \alpha\cdot C_0 + \beta\cdot F$ and a local complete intersection scheme $X_0 \subseteq X^*(C)$ satisfying the assumptions (a)-(d) there and Equation (3.9).

Remember that the Néron–Severi group of $\Sigma$ is generated by a section $C_0$ of $\pi$ and a fibre $F$ with intersection pairing given by $\left(\begin{smallmatrix} -e & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Then $K_\Sigma \sim_a -2C_0 + (2g-2-e)\cdot F$. Note that by Lemma G.1 we have

$$\alpha \geq 0 \quad \text{ and } \quad \beta' := \beta - \tfrac{e}{2}\alpha \geq 0.$$

If we set $\kappa_1 = a+2$ and $\kappa_2 = b+2-2g-\frac{ae}{2}$, we get

$$(D-K_\Sigma)^2 = -e\cdot(a+2)^2 + 2\cdot(a+2)\cdot(b+2+e-2g) = 2\cdot\kappa_1\cdot\kappa_2. \tag{3.7}$$

Fixing the notation $\operatorname{supp}(X_0) = \{z_1,\ldots,z_s\}$, $X_i = X_{0,z_i}$, $b' = b - \frac{ea}{2}$, $\beta' = \beta - \frac{ea}{2}$, and $\varepsilon_i = \min\{\deg(X_i), i(C,\Delta;z_i) - \deg(X_i)\} \geq 1$, the conditions on $\Delta$ and $\deg(X_0)$ take the form

(a) $a\beta' + b'\alpha \geq \deg(X_0) + \sum_{i=1}^s \varepsilon_i$,

(b) $\deg(X_0) \geq \kappa_1\cdot\beta' + \kappa_2\cdot\alpha - 2\alpha\beta'$, and

(c) $0 \leq \alpha \leq \frac{\kappa_1}{2}$ and $0 \leq \beta' \leq \frac{\kappa_2}{2}$.

The last inequalities follow from (d) in Lemma 3.5 replacing the ample divisor $H$ by the nef divisors $F$ respectively $C_0 + \frac{e}{2}\cdot F$.

From (b) and (c) we deduce

$$\deg(X_0) \geq \frac{\kappa_1}{2}\cdot\beta' + \frac{\kappa_2}{2}\cdot\alpha,$$

and thus, taking (3.7) into account,

$$\deg(X_0)^2 \geq 4\cdot\frac{\kappa_1}{2}\cdot\frac{\kappa_2}{2}\cdot\alpha\cdot\beta' = \frac{(D-K_\Sigma)^2}{2}\cdot\alpha\cdot\beta'. \tag{3.8}$$

Considering now (a) and (b) we get

$$0 < \sum_{i=1}^s \varepsilon_i \leq \Delta.(\Delta+K_\Sigma) = 2\alpha\beta' + (2g-2)\cdot\alpha - 2\beta' \leq \frac{\alpha\beta'}{2\gamma},$$

where the last inequality holds if $\beta' \neq 0$. We see, in particular, that $\beta' \neq 0$ if $g \leq 1$. But this together with (3.8) gives for $\beta' \neq 0$

$$\sum_{i=1}^s \varepsilon_i \leq \frac{\deg(X_0)^2}{\gamma\cdot(D-K_\Sigma)^2}.$$

If $\beta' = 0$, then we deduce from (a) and (b)

$$0 < \sum_{i=1}^s \varepsilon_i \leq (2g-2)\cdot\alpha \leq 4\cdot(g-1)\cdot A\cdot\frac{\deg(X_0)^2}{(D-K_\Sigma)^2} \leq \frac{\deg(X_0)^2}{\gamma\cdot(D-K_\Sigma)^2}.$$

Applying the Cauchy inequality, we finally get

$$\gamma\cdot(D-K_\Sigma)^2 \leq \frac{\deg(X_0)^2}{\sum_{i=1}^s \varepsilon_i} \leq \sum_{i=1}^s \frac{\deg(X_i)^2}{\varepsilon_i} \leq \sum_{i=1}^r \gamma_0^*(\mathcal{S}_i),$$

in contradiction to Assumption (3.6). $\square$

The following Lemma is the technical key to the above results.

**3.5 Lemma**

*Let $\Sigma$ a smooth projective surface, and let $D \in \mathrm{Div}(\Sigma)$ and $X \subset \Sigma$ be a zero-dimensional scheme satisfying*

(0) $D - K_\Sigma$ *is big and nef, and* $D + K_\Sigma$ *is nef,*

(1) $\exists\, C \in |D|_l$ *irreducible* : $X \subseteq X^*(C)$,

(2) $h^1\big(\Sigma, \mathcal{J}_{X/\Sigma}(D)\big) > 0$, *and*

(3) $4 \cdot \deg(X_0) < (D - K_\Sigma)^2$ *for all local complete intersection schemes* $X_0 \subseteq X$.

*Then there exists a curve $\Delta \subset \Sigma$ and a zero-dimensional local complete intersection scheme $X_0 \subseteq X \cap \Delta$ such that with the notation* $\mathrm{supp}(X_0) = \{z_1, \ldots, z_s\}$, $X_i = X_{0,z_i}$ *and*[4] $\varepsilon_i = \min\{\deg(X_i), i(C, \Delta; z_i) - \deg(X_i)\} \geq 1$ *we have*

(a) $D.\Delta \geq \deg(X_0) + \sum_{i=1}^{s} \varepsilon_i$,

(b) $\deg\big(X_0\big) \geq \big(D - K_\Sigma - \Delta\big).\Delta$,

(c) $\big(D - K_\Sigma - 2 \cdot \Delta\big)^2 > 0$, *and*

(d) $\big(D - K_\Sigma - 2 \cdot \Delta\big).H > 0$ *for all* $H \in \mathrm{Div}(\Sigma)$ *ample.*

*Moreover, it follows*

$$0 \leq \tfrac{1}{4} \cdot (D - K_\Sigma)^2 - \deg\big(X_0\big) \leq \big(\tfrac{1}{2} \cdot (D - K_\Sigma) - \Delta\big)^2. \tag{3.9}$$

**Proof:** Choose $X_0 \subseteq X$ minimal such that still $h^1\big(\Sigma, \mathcal{J}_{X_0/\Sigma}(D)\big) > 0$. By Assumption (0) the divisor $D - K_\Sigma$ is big and nef, and thus $h^1\big(\Sigma, \mathcal{O}_\Sigma(D)\big) = 0$ by the Kawamata–Viehweg Vanishing Theorem. Hence $X_0$ cannot be empty.

Due to the Grothendieck-Serre duality (cf. **[Har77]** III.7.6) we have $0 \neq H^1\big(\Sigma, \mathcal{J}_{X_0/\Sigma}(D)\big) \cong \mathrm{Ext}^1\big(\mathcal{J}_{X_0/\Sigma}(D), \mathcal{O}_\Sigma(K_\Sigma)\big)$, and thus, since $\mathcal{O}_\Sigma(K_\Sigma)$ is locally free, (cf. **[Har77]** III.6.7)

$$\mathrm{Ext}^1\big(\mathcal{J}_{X_0/\Sigma}(D - K_\Sigma), \mathcal{O}_\Sigma\big) \neq 0.$$

That is, there is an extension (cf. **[Har77]** Ex. III.6.1)

$$0 \to \mathcal{O}_\Sigma \to E \to \mathcal{J}_{X_0/\Sigma}(D - K_\Sigma) \to 0. \tag{3.10}$$

The minimality of $X_0$ implies that $E$ is locally free and $X_0$ is a local complete intersection scheme (cf. **[Laz97]** Proposition 3.9). Moreover, we have (cf. **[Laz97]** Exercise 4.3)

$$c_1(E) = D - K_\Sigma \quad \text{and} \quad c_2(E) = \deg(X_0). \tag{3.11}$$

By Assumption (3) and (3.11) we have

$$c_1(E)^2 - 4 \cdot c_2(E) = (D - K_\Sigma)^2 - 4 \cdot \deg(X_0) > 0,$$

[4] Since $X_0 \subseteq X^*(C) \subseteq X^{ea}(C)$, Lemma 1.3 applies to the local ideals of $X_0$, that is for the points $z \in \mathrm{supp}(X_0)$ we have $i(C, \Delta; z) \geq \deg(X_0, z) + 1$.

and thus $E$ is Bogomolov unstable (cf. [**Laz97**] Theorem 4.2). This, however, implies that there exists a divisor $\Delta_0 \in \text{Div}(\Sigma)$ and a zero-dimensional scheme $Z \subset \Sigma$ such that

$$0 \to \mathcal{O}_\Sigma(\Delta_0) \to E \to \mathcal{J}_{Z/\Sigma}(D - K_\Sigma - \Delta_0) \to 0 \tag{3.12}$$

is exact (cf. [**Laz97**] Theorem 4.2), and such that

$$(2\Delta_0 - D + K_\Sigma)^2 \geq c_1(E)^2 - 4 \cdot c_2(E) > 0 \tag{3.13}$$

and

$$(2\Delta_0 - D + K_\Sigma).H > 0 \quad \text{for all ample} \quad H \in \text{Div}(\Sigma). \tag{3.14}$$

Tensoring (3.12) with $\mathcal{O}_\Sigma(-\Delta_0)$ leads to the following exact sequence

$$0 \to \mathcal{O}_\Sigma \to E(-\Delta_0) \to \mathcal{J}_{Z/\Sigma}(D - K_\Sigma - 2\Delta_0) \to 0, \tag{3.15}$$

and we deduce $h^0\big(\Sigma, E(-\Delta_0)\big) \neq 0$.

Now tensoring (3.10) with $\mathcal{O}_\Sigma(-\Delta_0)$ leads to

$$0 \to \mathcal{O}_\Sigma(-\Delta_0) \to E(-\Delta_0) \to \mathcal{J}_{X_0/\Sigma}(D - K_\Sigma - \Delta_0) \to 0. \tag{3.16}$$

Let $H$ be some ample divisor. By (3.14) and since $D - K_\Sigma$ is nef by (0):

$$-\Delta_0.H < -\tfrac{1}{2} \cdot (D - K_\Sigma).H \leq 0.$$

Hence $-\Delta_0$ cannot be effective, that is $H^0\big(\Sigma, \mathcal{O}_\Sigma(-\Delta_0)\big) = 0$. But the long exact cohomology sequence of (3.16) then implies

$$0 \neq H^0\big(\Sigma, E(-\Delta_0)\big) \hookrightarrow H^0\left(\Sigma, \mathcal{J}_{X_0/\Sigma}(D - K_\Sigma - \Delta_0)\right).$$

In particular we may choose a curve

$$\Delta \in \left|\mathcal{J}_{X_0/\Sigma}(D - K_\Sigma - \Delta_0)\right|_l.$$

Thus (c) and (d) follow from (3.13) and (3.14). It remains to show (a) and (b).

We note that $C \in |D|_l$ is irreducible and that $\Delta$ cannot contain $C$ as an irreducible component: otherwise applying (3.14) with some ample divisor $H$ we would get the following contradiction, since $D + K_\Sigma$ is nef by (0),

$$0 \leq (\Delta - C).H < -\tfrac{1}{2} \cdot (D + K_\Sigma).H \leq 0.$$

Since $X_0 \subset C \cap \Delta$ the Theorem of Bézout implies (a):

$$D.\Delta = C.\Delta = \sum_{z \in C \cap \Delta} i(C, \Delta; z) \geq \sum_{i=1}^{s} \big(\deg(X_i) + \varepsilon_i\big) = \deg(X_0) + \sum_{i=1}^{s} \varepsilon_i.$$

Finally, by (3.11) and (3.12) we get (b):

$$\deg(X_0) = c_2(E) = \Delta_0.(D - K_\Sigma - \Delta_0) + \deg(Z) \geq (D - K_\Sigma - \Delta).\Delta.$$

Equation (3.9) is just a reformulation of (b). □

## 4. Examples

Throughout this section for a topological respectively analytical singularity type $\mathcal{S}$ we will denote by $\gamma_\alpha^*(\mathcal{S}) = \gamma_\alpha^{es}(\mathcal{S})$, respectively $\gamma_\alpha^*(\mathcal{S}) = \gamma_\alpha^{ea}(\mathcal{S})$.

### 4.a. The Classical Case - $\Sigma = \mathbb{P}_{\mathbb{C}}^2$

In view of Remark 3.2 we get the following version of Theorem 3.1.

**3.1a Theorem**

*Let* $\mathcal{S}_1, \ldots, \mathcal{S}_r$ *be topological or analytical singularity types, let* $d \geq 3$ *and suppose that*

$$\sum_{i=1}^{r} \gamma_1^*(\mathcal{S}_i) < (d+3)^2. \tag{3.1a}$$

*Then either* $V_{|dH|}^{irr}(\mathcal{S}_1, \ldots, \mathcal{S}_r)$ *is empty or T-smooth.*

As soon as for one of the singularities we have $\gamma_1^*(\mathcal{S}_i) > 4 \cdot \tau_{ci}^*(\mathcal{S}_i)$, e. g. simple singularities or ordinary multiple points which are not simple double points, then the strict inequality in (3.1a) can be replaced by "$\leq$", which then is the same sufficient condition as in in [**Los98**] Corollary 5.5 (see also [**GLS00**]), and since $\gamma_1^*(\mathcal{S}) \leq \big(\tau^*(\mathcal{S})+1\big)^2$ it is better than the condition in Theorem 1.1a.

In particular, $V_{|dH|}^{irr}(kA_1, mA_2, M_{m_1}, \ldots, M_{m_r})$, $m_i \geq 3$, is therefore T-smooth as soon as

$$4k + 9m + \sum_{i=1}^{r} 2 \cdot m_i^2 \leq (d+3)^2.$$

### 4.b. Geometrically Ruled Surfaces

Throughout this section we use the notation of Section G.a.

**3.4b Theorem**

*Let* $\pi : \Sigma \to C$ *be a geometrically ruled surface with* $g = g(C)$.

*Let* $D \in \mathrm{Div}(\Sigma)$ *such that* $D \sim_a aC_0 + bF$ *with* $b > \max\{2g-2, 2-2g\} + \frac{ae}{2}$ *and* $a > 2$*, and let* $\mathcal{S}_1, \ldots, \mathcal{S}_r$ *be topological or analytical singularity types.*

*Suppose that*

$$\sum_{i=1}^{r} \gamma_0^*(\mathcal{S}_i) \;<\; \gamma \cdot (D - K_\Sigma)^2, \tag{3.6b}$$

*where with* $A = \frac{a+2}{b+2-2g-\frac{ae}{2}}$ *the constant* $\gamma$ *satisfies*

$$\gamma = \begin{cases} \frac{1}{4}, & \textit{if } g \in \{0,1\}, \\ \min\left\{\frac{1}{4g}, \frac{1}{4\cdot(g-1)\cdot A}\right\}, & \textit{if } g \geq 2. \end{cases}$$

*Then either* $V_{|D|}^{irr}(\mathcal{S}_1, \ldots, \mathcal{S}_r)$ *is empty or it is T-smooth.*

In particular, $V^{irr}_{|D|}(M_{m_1},\dots,M_{m_r})$, $m_i \geq 3$, is therefore T-smooth as soon as

$$\sum_{i=1}^{r} 2\cdot(m_i-1)^2 < \gamma\cdot(D-K_\Sigma)^2.$$

Note that, as for products of curves, the constant $\gamma$ in Theorem 3.4 depends on the ratio of $a$ and $b$ unless $g$ is at most one. This means that in general an asymptotical behaviour can only be examined if the ratio remains unchanged.

Let us write down the result explicitly for the Hirzebruch surfaces $\mathbb{F}_e$.

**4.1 Corollary**
*Let $\Sigma = \mathbb{F}_e$, let $D$ be a divisor of type $(a,b)$ with $a \geq 3$ and $b > 2 + \frac{ae}{2}$, and let $\mathcal{S}_1,\dots,\mathcal{S}_r$ be topological or analytical singularity types.*

*Suppose that*

$$\sum_{i=1}^{r} \gamma_0^*(\mathcal{S}_i) < \frac{(a+2)\cdot(2b+4-ae)}{4}. \tag{4.1}$$

*Then either $V^{irr}_{|D|}(\mathcal{S}_1,\dots,\mathcal{S}_r)$ is empty or T-smooth.*

The results in Subsection 2.b only applied to eight Hirzebruch surfaces and a few classes of fibrations over elliptic curves, while our results apply to all geometrically ruled surfaces. Moreover, the results are in general better, e. g. for the Hirzebruch surface $\mathbb{P}^1_{\mathbb{C}}\times\mathbb{P}^1_{\mathbb{C}}$ already the previous sufficient condition for T-smoothness of families of curves with $r$ cusps and $b = 3a$ the condition

$$9r < 2a^2 + 8a$$

has been replaced by the slightly better condition

$$8r < 3a^2 + 8a + 4.$$

For ordinary multiple points the difference will become more significant. Even for families of nodal curves the new conditions would always be slightly better, but for those families T-smoothness is guaranteed anyway by [**Tan80**].

## 4.c. Products of Curves

Throughout this section we use the notation of Section G.b. In particular, $\Sigma = C_1 \times C_2$ where $C_1$ and $C_2$ are smooth projective curves over $\mathbb{C}$ of genera $g_1$ and $g_2$ respectively. Since $\Sigma$ is geometrically ruled if some $g_i = 0$, we may restrict our attention to the case $g_1, g_2 \geq 1$.

For a generic choice of $C_1$ and $C_2$ the Néron–Severi group $\mathrm{NS}(\Sigma)$ is two-dimensional by Proposition G.12. Thus the following theorem answers the general case completely.

**3.3c Theorem**
*Let $C_1$ and $C_2$ be two smooth projective curves of genera $g_1$ and $g_2$, such that for $\Sigma = C_1\times C_2$ the Néron–Severi group is $\mathrm{NS}(\Sigma) = C_1\mathbb{Z}\oplus C_2\mathbb{Z}$.*

*Let* $D \in \mathrm{Div}(\Sigma)$ *such that* $D \sim_a aC_1 + bC_2$ *with* $a \geq \max\{2-2g_2, 2g_2-1\}$ *and* $b \geq \max\{2-2g_1, 2g_1-1\}$, *let* $\mathcal{S}_1, \ldots, \mathcal{S}_r$ *be topological or analytical singularity types.*

*Suppose that*

$$\sum_{i=1}^{r} \gamma_0^*(\mathcal{S}_i) \ < \ \gamma \cdot (D - K_\Sigma)^2, \tag{3.4c}$$

*where the constant* $\gamma$ *may be read off the following table with* $A = \frac{a-2g2+2}{b-2g1+2}$

| $g_1$ | $g_2$ | $\gamma$ |
|---|---|---|
| $0, 1$ | $0, 1$ | $\frac{1}{4}$ |
| $\geq 2$ | $0, 1$ | $\min\left\{\frac{1}{4g_1}, \frac{1}{4\cdot(g_1-1)\cdot A}\right\}$ |
| $\geq 2$ | $\geq 2$ | $\min\left\{\frac{1}{4g_1+4g_2-4}, \frac{A}{4\cdot(g_2-1)}, \frac{1}{4\cdot(g_1-1)\cdot A}\right\}$ |

*Then either* $V_{|D|}^{irr}(\mathcal{S}_1, \ldots, \mathcal{S}_r)$ *is empty or it is T-smooth.*

In particular, $V_{|D|}^{irr}(M_{m_1}, \ldots, M_{m_r})$, $m_i \geq 3$, is therefore T-smooth as soon as

$$\sum_{i=1}^{r} 2 \cdot (m_i - 1)^2 < \gamma \cdot (D - K_\Sigma)^2.$$

Note that the constant $\gamma$ in Theorem 3.3 depends on the ratio of $a$ and $b$ unless both $g_1$ and $g_2$ are at most one. This means that in general an asymptotical behaviour can only be examined if the ratio remains unchanged.

## 4.d. Products of Elliptic Curves

If in Section V.4.c the curves $C_1$ and $C_2$ are chosen to be both elliptic curves, Theorem 3.3 looks much nicer, since $K_\Sigma = 0$.

**3.3d Theorem**

*Let* $C_1$ *and* $C_2$ *be two smooth non-isogenous elliptic curves,* $a, b \geq 1$, *and* $\mathcal{S}_1, \ldots, \mathcal{S}_r$ *be topological or analytical singularity types.*

*Suppose that*

$$\sum_{i=1}^{r} \gamma_0^*(\mathcal{S}_i) < \frac{ab}{2}. \tag{3.4d}$$

*Then either* $V_{|aC_1+bC_2|}^{irr}(\mathcal{S}_1, \ldots, \mathcal{S}_r)$ *is empty or T-smooth.*

In particular, on a product of non-isogenous elliptic curves for nodal curves we reproduce the previous sufficient condition

$$r < ab,$$

for T-smoothness of $V_{|aC_1+bC_2|}^{irr}(rA_1)$, while the previous condition

$$\sum_{i=1}^{r} \frac{\left(m_i^2 + 2m_i + 5\right)^2}{32} < ab$$

for T-smoothness of $V^{irr}_{|aC_1+bC_2|}(M_{m_1},\ldots,M_{m_r})$, $m_i \geq 3$, has been replaced by the condition

$$\sum_{i=1}^{r} 8\cdot(m_i-1)^2 < ab,$$

which from $m_i \geq 7$ is better.

## 4.e. Surfaces in $\mathbb{P}^3_{\mathbb{C}}$

For a generic surface of degree $n \geq 4$ in $\mathbb{P}^3_{\mathbb{C}}$ any curve is a hypersurface section by a Theorem of Noether (see Section G.d). Therefore the following result on hypersurface sections answers the problem completely on a generic surface in $\mathbb{P}^3_{\mathbb{C}}$.

**3.1e Theorem**

*Let $\Sigma \subset \mathbb{P}^3_{\mathbb{C}}$ be a smooth hypersurface of degree $n \geq 4$ with Picard number one, let $H \subset \Sigma$ be a hyperplane section, $d \geq n-3$ and let $\mathcal{S}_1,\ldots,\mathcal{S}_r$ be topological or analytical singularity types.*

*Suppose that*

$$\sum_{i=1}^{r} \gamma^*_{\frac{1}{n-3}}(\mathcal{S}_i) < \frac{n}{n-3}\cdot(d-n+4)^2. \tag{3.1e}$$

*Then either $V^{irr}_{|D|}(\mathcal{S}_1,\ldots,\mathcal{S}_r)$ is empty or it is T-smooth.*

In particular, $V^{irr}_{|dH|}(M_{m_1},\ldots,M_{m_r})$, $m_i \geq 3$, is therefore T-smooth as soon as

$$\sum_{i=1}^{r} 2\cdot\left(m_i-\frac{n-2}{n-3}\right)^2 < \frac{n}{n-3}\cdot(d-n+4)^2.$$

The condition

$$r \leq \frac{n\cdot(n-3)}{(n-2)^2}\cdot(d-n+4)^2,$$

which gives the T-smoothness of $V_{|dH|}(rA_1)$ is weaker than the condition provided by Corollary 2.6, but for $n=5$ it reads $r \leq \frac{10}{9}\cdot(d-1)^2$ and it comes still close to the sharp bound $\frac{5}{4}\cdot(d-1)^2$ provided by [**ChS97**].

## 4.f. K3-Surfaces

**3.1f Theorem**

*Let $\Sigma$ be a smooth K3-surface with $\mathrm{NS}(\Sigma)=L\cdot\mathbb{Z}$, $L$ ample, and set $n=L^2$. Let $d\geq 1$, and let $\mathcal{S}_1,\ldots,\mathcal{S}_r$ be topological or analytical singularity types.*

*Suppose that*

$$\sum_{i=1}^{r} \gamma^*_1(\mathcal{S}_i) < d^2\cdot n. \tag{3.1f}$$

*Then either $V^{irr}_{|dL|}(\mathcal{S}_1,\ldots,\mathcal{S}_r)$ is empty or it is T-smooth.*

The condition in Theorem 1.1f for T-smoothness on K3-surfaces is thus completely replaced.

CHAPTER V

# Irreducibility

The question whether the variety $V^{\text{irr}} = V^{\text{irr}}_{|D|}(\mathcal{S}_1, \ldots, \mathcal{S}_r)$ of irreducible curves in $|D|_l$ having precisely $r$ singular points of the given types $\mathcal{S}_1, \ldots, \mathcal{S}_r$ is irreducible seems to be much harder to answer suitably than that of existence or smoothness. Even in the simplest case of nodal curves in $\mathbb{P}^2_{\mathbb{C}}$ the conjecture of Severi that $V^{\text{irr}}$ should be irreducible whenever it is non-empty resisted a full proof until 1985 (cf. [**Har85b**]), while already Severi himself (cf. [**Sev21**]) gave a complete characterisation for the existence and showed that the variety is always T-smooth. Moreover, while for the question of existence and smoothness in the plane curve case there exist sufficient criteria showing the same asymptotical behaviour as known necessary criteria, respectively as known (series of) non-smooth families, the asymptotics for the sufficient criteria for irreducibility seem to be worse (cf. [**Los98**] Chapter 6). To be more precise there are conditions linear in certain invariants and quadratic in the degree which guarantee the existence, while for the irreducibility (as for the T-smoothness) the conditions are quadratic in the degree and the invariants as well. Applying similar techniques our results carry the same stigma. However, apart from nodal curves on the blown up plane (cf. [**Ran89, GLS98a**]) we do not know at all of any criteria for the irreducibility–problem on surfaces other than the plane. (See also Section 4.f).

The main condition for irreducibility which we get in the different cases looks like

$$\sum_{i=1}^{r} \big(\tau^*(\mathcal{S}_i) + 2\big)^2 < \gamma \cdot (D - K_\Sigma)^2, \tag{0.1}$$

where $\gamma$ is some constant.

At this point we should like to point out that on more complicated surfaces than $\mathbb{P}^2_{\mathbb{C}}$ we cannot expect the results to be as nice as in the plane case. Already Harris' result for the variety of nodal curves does not extend to arbitrary surfaces $\Sigma$ – more precisely, to our knowledge there is no surface known, except the plane, where the result holds. In [**ChC99**] it is shown that on a generic surface in $\mathbb{P}^3_{\mathbb{C}}$ of degree $n \geq 8$ the variety $V^{\text{irr}}_{|dH|}(rA_1)$ is reducible for all even $d > 0$ and $r = \frac{\left(d^2+(4-n)d+2\right)\cdot n}{2}$ as soon as

$$\frac{n^3 - 12n^2 + 11n - 6}{6} < \frac{dn - d - 2}{2},$$

where the left hand side is the expected dimension of $V^{irr}_{|dH|}(rA_1)$ and the right hand side is a lower bound for the actual dimension. This lower bound is found by intersecting $\Sigma$ with a family of cones with fixed vertex over plane nodal curves of degree $d$ with $\frac{r}{n}$ nodes. The right hand side is the dimension of the family of plane nodal curves, and we thus only have to note that cones over different curves are different. The arguments given so far show that $V^{irr}_{|dH|}(rA_1)$ has a component which is not T-smooth and works for any surface of degree $n \geq 8$ in $\mathbb{P}^3_{\mathbb{C}}$, however, in **[ChC99]** Theorem 3.1 Chiantini and Ciliberto show that on a generic surface in $\mathbb{P}^3_{\mathbb{C}}$ the variety $V^{irr}_{|dH|}(rA_1)$ with $d \geq n$ also contains a T-smooth component whenever it is non-empty.

For an overview on the different approaches to the question of irreducibility in the case of plane curves we refer to **[GrS99]** or **[Los98]** Chapter 6 – see also **[BrG81, Ura83, GD84, Ura84, Har85b, Dav86, Ura86, Deg90, GrK89, Kan89a, Ran89, Shu91a, Bar93a, Shu94, Shu96b, Wal96, Bar98, Mig01]**. The case of $\mathbb{P}^2_{\mathbb{C}}$ blown up in $k$ generic points is treated in **[Ran89]**, if $k = 1$, and in **[GLS98a]** in the general case. Our proof proceeds along the lines of an unpublished result of Greuel, Lossen and Shustin (cf. **[GLS98b]**). The basic ideas are in some respect similar to the approach utilised in **[GLS00]**, replacing the "Castelnuovo-function" arguments by "Bogomolov unstability". We tackle the problem in three steps:

**Step 1:** We first show that the open subvariety $V^{irr,reg}$ of curves in $V^{irr}$ with $h^1\big(\Sigma, \mathcal{J}_{X(C)/\Sigma}(D)\big) = 0$ is always irreducible, and hence so is its closure in $V^{irr}$. (Cf. Corollary 1.2.)

**Step 2:** Then we find conditions which ensure that the open subvariety $V^{irr,fix}$ of curves in $V^{irr}$ with $h^1\big(\Sigma, \mathcal{J}_{X^*_{fix}(C)/\Sigma}(D)\big) = 0$ is dense in $V^{irr}$. (Cf. Section 2.)

**Step 3:** And finally, we combine these conditions with conditions which guarantee that $V^{irr,reg}$ is dense in $V^{irr,fix}$ by showing that they share some open dense subset $V^{gen}_U$ of curves with singularities in very general position (cf. Lemma 3.9). But then $V^{irr,reg}$ is dense in $V^{irr}$ and $V^{irr}$ is irreducible by Step 1.

The most difficult part is Step 2. For this one we consider the restriction of the morphism (cf. Definition I.2.16)

$$\Phi : V \to \mathrm{Sym}^r(\Sigma) =: \mathcal{B}$$

to an irreducible component $V^*$ of $V^{irr}$ not contained in the closure $\overline{V^{irr,fix}}$ in $V^{irr}$. Knowing, that the dimension of $V^*$ is at least the expected dimension $\dim\big(V^{irr,fix}\big)$ we deduce that the codimension of $\mathcal{B}^* = \Phi\big(V^*\big)$ in $\mathcal{B}$ is at most $h^1\big(\Sigma, \mathcal{J}_{X^*_{fix}(C)/\Sigma}(D)\big)$, where $C \in V^*$ (cf. Lemma 3.7). It thus suffices to find conditions which contradict this inequality, that is, we have to get our hands on $\mathrm{codim}_{\mathcal{B}}(\mathcal{B}^*)$. However, on the surfaces, which we consider, the non-vanishing of $h^1\big(\Sigma, \mathcal{J}_{X^*_{fix}(C)/\Sigma}(D)\big)$ means in some sense that the zero-dimensional scheme $X^*_{fix}(C)$ is in special position. We may thus hope to realise large parts $X^0_i$ of

$X^*_{fix}(C)$ on curves $\Delta_i$ of "small degree" ($i = 1, \ldots, m$), which would impose at least $\#X^0_i - \dim|\Delta_i|_l$ conditions on $X^*_{fix}(C)$, giving rise to a lower bound $\sum_{i=1}^m \#X^0_i - \dim|\Delta_i|_l$ for $\operatorname{codim}_{\mathcal{B}}(\mathcal{B}^*)$.[1] The $X^0_i$ and the $\Delta_i$ are found in Lemma 3.1 with the aid of certain Bogomolov unstable rank-two bundles. It thus finally remains (cf. Lemma 3.3, 3.4 and 3.6) to give conditions which imply

$$\sum_{i=1}^{m} \#X^0_i - \dim|\Delta_i|_l > h^1\big(\Sigma, \mathcal{J}_{X^*_{fix}(C)/\Sigma}(D)\big).$$

What are the obstructions by which our approach is restricted?

First, the Bogomolov unstability does not give us much information about the curves $\Delta_i$ apart from their existence and the fact that they are in some sense "small" compared with the divisor $D$. It thus is obvious that we are bound to the study of surfaces where we have a good knowledge of the dimension of arbitrary complete linear systems. Second, in order to derive the above inequality many nasty calculations are necessary which depend very much on the particular structure of the Néron–Severi group of the surface, that is, we are restricted to surfaces where the Néron–Severi group is not too large and the intersection pairing is not too hard (cf. Lemma 3.3, 3.4 and 3.6). Finally, in order to ensure the Bogomolov unstability of the vector bundle considered throughout the proof of Lemma 3.1 we heavily use the fact that the surface $\Sigma$ does not contain any curves of negative self-intersection, which excludes e. g. general Hirzebruch surfaces.

In Section 1 we do not only prove that $V^{irr,reg}$ is irreducible, but that this indeed remains true if we drop the requirement that the curves should be irreducible, i. e. we show that $V^{reg}$ is irreducible. However, unfortunately our approach does not give conditions implying that $V^{reg}$ is dense, and thus we cannot say anything about the irreducibility of the variety of possibly reducible curves in $|D|_l$ with prescribed singularities.[2] The reason for this is that in the proof of Lemma 3.1 we use the Theorem of Bézout to estimate $D.\Delta_i$. Since $\Delta_i$ may be about "half" of $D$, we need an irreducible curve in $|D|_l$ to be sure that at least for some curve in $|D|_l$ the curve $\Delta_i$ is not a component.

## 1. $V^{irr,reg}$ is irreducible

We are now showing that $V^{reg}$ and $V^{irr,reg}$ are always irreducible. We do this by showing that under $\Psi : V \to B$ (cf. Definition I.2.16) every irreducible

[1] Let $k_i = \dim|\Delta_i|_l$ and let $H^0(\Sigma, \Delta_i)$ be spanned by the linearly independent sections $s_{i,0}, \ldots, s_{i,k_i}$. Then for $k_i$ general points $p_1, \ldots, p_{k_i}$ in $\Sigma$ the linear system of equations $a_0 s_{i,0}(p_j) + \ldots + a_{k_i} s_{i,k_i}(p_j) = 0$, $j = 1, \ldots, k_i$, has a one-dimensional solution set, hence there is a unique curve $C$ in $|\Delta_i|_l$ through $p_1, \ldots, p_{k_i}$. But then for the remaining $k_i' = \#X^0_i - k_i$ points which must lie on $C$ as well there is only one degree of freedom left instead of two. Hence the dimension of $\mathcal{B}^*$ is at most the dimension of $\mathcal{B}$, which is $2r$, lowered by $\sum_{i=1}^m k_i'$.

[2] Note that e. g. the variety $V^{irr}_{|2H|}(A_1)$ of irreducible plane conics with one node is empty, while $V_{|2H|}(A_1)$ is four-dimensional – even though the latter is of course irreducible.

component of these is smooth and maps dominant to the irreducible variety $B$ with irreducible fibres.

**1.1 Theorem**

*Let* $D \in \mathrm{Div}(\Sigma)$*,* $\mathcal{S}_1, \ldots, \mathcal{S}_r$ *be pairwise distinct topological or analytical singularity types and* $k_1, \ldots, k_r \in \mathbb{N} \setminus \{0\}$*.*

*If* $V_{|D|}^{reg}(k_1\mathcal{S}_1, \ldots, k_r\mathcal{S}_r)$ *is non-empty, then it is a T-smooth, irreducible, open subset of* $V_{|D|}(k_1\mathcal{S}_1, \ldots, k_r\mathcal{S}_r)$ *of dimension* $\dim |D|_l - \sum_{i=1}^r k_i \cdot \tau^*(\mathcal{S}_i)$*.*

**Proof:** Let us consider the following maps from Definition I.2.16

$$\Psi = \Psi_D(k_1\mathcal{S}_1, \ldots, k_r\mathcal{S}_r) : V = V_{|D|}(k_1\mathcal{S}_1, \ldots, k_r\mathcal{S}_r) \longrightarrow B(k_1\mathcal{S}_1, \ldots, k_r\mathcal{S}_r)$$

and

$$\psi = \psi(k_1\mathcal{S}_1, \ldots, k_r\mathcal{S}_r) : B(k_1\mathcal{S}_1, \ldots, k_r\mathcal{S}_r) \longrightarrow \mathrm{Hilb}_\Sigma^n.$$

**Step 1:** Every irreducible component $V^*$ of $V^{reg}$ is T-smooth of dimension $\dim |D|_l - \sum_{i=1}^r k_i \cdot \tau^*(\mathcal{S}_i)$.

By **[Los98]** Proposition 2.1 (c2) $V^*$ is T-smooth at any $C \in V^*$ of dimension $\dim |D|_l - \deg\big(X^*(C)\big) = \dim |D|_l - \sum_{i=1}^r k_i \cdot \tau(\mathcal{S}_i)$, since $h^1\big(\Sigma, \mathcal{J}_{X^*/\Sigma}(D)\big) = 0$ according to Remark I.2.9. (See also Remark I.2.11.)

**Step 2:** $V^{reg}$ is open in $V$.

Let $C \in V^{reg}$. By assumption $h^1\big(\Sigma, \mathcal{J}_{X(C)/\Sigma}(D)\big) = 0$, and thus by Lemma F.1 there exists an open, dense neighbourhood $U$ of $X(C)$ in $\mathrm{Hilb}_\Sigma^n$ such that $h^1\big(\Sigma, \mathcal{J}_{Y/\Sigma}(D)\big) = 0$ for all $Y \in U$. But then $\Psi^{-1}\big(\psi^{-1}(U)\big) \subseteq V^{reg}$ is an open neighbourhood of $X(C)$ in $V$, and hence $V^{reg}$ is open in $V$.

**Step 3:** $\Psi = \Psi_D(k_1\mathcal{S}_1, \ldots, k_r\mathcal{S}_r)$ restricted to any irreducible component $V^*$ of $V^{reg}$ is dominant.

Let $V^*$ be an irreducible component of $V^{reg}$ and let $C \in V^*$. Since $\Psi^{-1}\big(\Psi(C)\big)$ is an open and dense subset of $\big|\mathcal{J}_{X(C)/\Sigma}(D)\big|_l$ and since $h^1\big(\Sigma, \mathcal{J}_{X(C)/\Sigma}(D)\big) = 0$, we have

$$\dim \Psi^{-1}\big(\Psi(C)\big) = h^0\big(\Sigma, \mathcal{J}_{X(C)/\Sigma}(D)\big) - 1 = \dim |D|_l - \deg\big(X(C)\big).$$

By Step 1 we know the dimension of $V^*$ and by a remark in Definition I.2.16 we also know the dimension of $B$. Thus we conclude

$$\begin{aligned} \dim \Psi\big(V^*\big) &= \dim V^* - \dim \Psi^{-1}\big(\Psi(C)\big) \\ &= \big(\dim |D|_l - \deg X^*(C)\big) - \big(\dim |D|_l - \deg X(C)\big) \\ &= \deg\big(X(C)\big) - \deg\big(X^*(C)\big) \;=\; \dim B. \end{aligned}$$

Since $B$ is irreducible, $\Psi\big(V^*\big)$ must be dense in $B$.

**Step 4:** $V^{reg}$ is irreducible.

Let $V^*$ and $V^{**}$ be two irreducible components of $V^{reg}$. Then $\Psi\big(V^*\big) \cap \Psi\big(V^{**}\big) \neq \emptyset$, and thus some fibre $F$ of $\Psi$ intersects both, $V^*$ and $V^{**}$. However, the fibre is irreducible and by Step 1 both $V^*$ and $V^{**}$ are smooth. Thus $F$ must be completely contained in $V^*$ and $V^{**}$, which, since both are smooth of the same dimension, implies that $V^* = V^{**}$. That is $V^{reg}$ is irreducible. □

**1.2 Corollary**

*Let* $D \in \mathrm{Div}(\Sigma)$*,* $\mathcal{S}_1, \ldots, \mathcal{S}_r$ *be pairwise distinct topological or analytical singularity types and* $k_1, \ldots, k_r \in \mathbb{N} \setminus \{0\}$*.*

*If* $V^{irr,reg}_{|D|}(k_1\mathcal{S}_1, \ldots, k_r\mathcal{S}_r)$ *is non-empty, then it is a T-smooth, irreducible, open subset of* $V_{|D|}(k_1\mathcal{S}_1, \ldots, k_r\mathcal{S}_r)$ *of dimension* $\dim |D|_l - \sum_{i=1}^r k_i \cdot \tau^*(\mathcal{S}_i)$*.*

**Proof:** This follows from Theorem 1.1 since $V^{irr,reg}_{|D|}(k_1\mathcal{S}_1, \ldots, k_r\mathcal{S}_r)$ is an open subset of the irreducible set $V^{reg}_{|D|}(k_1\mathcal{S}_1, \ldots, k_r\mathcal{S}_r)$. □

## 2. Conditions for the irreducibility of $V^{irr}$

Knowing that $V^{irr,reg}$, and hence its closure $\overline{V^{irr,reg}}$ in $V^{irr}$, is always irreducible, the search for sufficient conditions for the irreducibility of $V^{irr}$ may be reduced to the search for conditions ensuring that $V^{irr,reg}$ is dense in $V^{irr}$. As indicated in the introduction (cf. p. 120) we achieve this aim by combining conditions, which ensure that $V^{irr,reg}$ and $V^{irr,fix}$ share some dense subset $V^{gen}_U$ (cf. Step 3 on page 120), with conditions which ensure that $V^{irr,fix}$ is dense in $V^{irr}$ (cf. Step 2 on page 120). For the latter problem we apply a reduction technique involving the Bogomolov unstability of certain rank-two vector bundles on $\Sigma$. The main part of the work is carried out in Section 3, where we prove several technical lemmata which partly are useful in their own respect.

Let us now reformulate Step 2 and Step 3 in a more precise way.

**Step 2:** We derive from Lemma 3.1-3.7 conditions which ensure that $V^{irr,fix}$ is dense in $V^{irr}$.

**Step 3:** Taking Lemma 3.8 into account, we deduce from Lemma 3.9 conditions which ensure that there exists a very general subset $U \subset \Sigma^r$ such that the family $V^{gen}_U = V^{gen}_{|D|,U}(\mathcal{S}_1, \ldots, \mathcal{S}_r)$, as defined there, satisfies

(a) $V^{gen}_U$ is dense in $V^{irr,fix}$, and

(b) $V^{gen}_U \subseteq V^{irr,reg}$.

Thus in particular, $V^{irr,reg}$ is dense in $V^{irr,fix}$.

Throughout the proof of the following theorem we will heavily rely on the study of the zero-dimensional schemes $X^*_{fix}(C)$ for some curves $C$, and we therefore would like to remind the reader that in view of Definition I.2.11 for a topological respectively analytical singularity type $\mathcal{S}$ we have

$$\deg\left(X^*_{fix}(\mathcal{S}_i)\right) = \tau^*(\mathcal{S}_i) + 2.$$

**2.1 Theorem**

*Let* $\Sigma$ *be a surface such that*[3]

[3]By Lemma E.1 we may assume w. l. o. g. that $L$ is ample.

(i) $NS(\Sigma) = L \cdot \mathbb{Z}$,

(ii) $h^1(\Sigma, C) = 0$, *whenever* $C$ *is effective.*

*Let* $D \in \mathrm{Div}(\Sigma)$, *let* $\mathcal{S}_1, \ldots, \mathcal{S}_r$ *be topological or analytical singularity types. Suppose that*

(2.1) $D - K_\Sigma$ *is big and nef,*

(2.2) $D + K_\Sigma$ *is nef,*

(2.3) $\sum_{i=1}^{r} \big(\tau^*(\mathcal{S}_i) + 2\big) < \beta \cdot (D - K_\Sigma)^2$ *for some* $0 < \beta \leq \frac{1}{4}$, *and*

(2.4) $\sum_{i=1}^{r} \big(\tau^*(\mathcal{S}_i) + 2\big)^2 < \gamma \cdot (D - K_\Sigma)^2$, *where* $\gamma = \frac{\left(1+\sqrt{1-4\beta}\right)^2 \cdot L^2}{4 \cdot \chi(\mathcal{O}_\Sigma) + \max\{0, 2 \cdot K_\Sigma.L\} + 6 \cdot L^2}$.

*Then either* $V^{irr}_{|D|}(\mathcal{S}_1, \ldots, \mathcal{S}_r)$ *is empty or it is irreducible of the expected dimension.*

**Proof:** We may assume that $V^{irr} = V^{irr}_{|D|}(\mathcal{S}_1, \ldots, \mathcal{S}_r)$ is non-empty. As indicated above it suffices to show that:

**Step 2:** $V^{irr} = \overline{V^{irr,fix}}$, where $V^{irr,fix} = V^{irr,fix}_{|D|}(\mathcal{S}_1, \ldots, \mathcal{S}_r)$, and

**Step 3:** the conditions of Lemma 3.9 are fulfilled.

For Step 3 we note that $\nu^*(\mathcal{S}_i) \leq \tau^*(\mathcal{S}_i)$. Thus (2.4) implies that

$$\sum_{i=1}^{r} \big(\nu^*(\mathcal{S}_i) + 2\big)^2 \leq \sum_{i=1}^{r} \big(\tau^*(\mathcal{S}_i) + 2\big)^2 \leq \gamma \cdot (D - K_\Sigma)^2 \leq \tfrac{1}{2} \cdot (D - K_\Sigma)^2,$$

which gives the first condition in Lemma 3.9. Since a surface with Picard number one has no curves of selfintersection zero, the second condition in Lemma 3.9 is void, while the last condition is satisfied by (2.1).

It remains to show Step 2, i. e. $V^{irr} = \overline{V^{irr,fix}}$. Suppose the contrary, that is, there is an irreducible curve $C_0 \in V^{irr} \setminus \overline{V^{irr,fix}}$, in particular $h^1\big(\Sigma, \mathcal{J}_{X_0/\Sigma}(D)\big) > 0$ for $X_0 = X^*_{fix}(C_0)$. Since $\deg(X_0) = \sum_{i=1}^{r} \big(\tau^*(\mathcal{S}_i) + 2\big)$ and $\sum_{z \in \Sigma} \big(\deg(X_{0,z})\big)^2 = \sum_{i=1}^{r} \big(\tau^*(\mathcal{S}_i) + 2\big)^2$ the assumptions (0)-(3) of Lemma 3.1 and (4) of Lemma 3.3 are fulfilled. Thus Lemma 3.3 implies that $C_0$ satisfies Condition (3.28) in Lemma 3.7, which it cannot satisfy by the same Lemma. Thus we have derived a contradiction. □

**2.2 Remark**

If we set

$$\gamma = \frac{36\alpha}{(3\alpha + 4)^2} \quad \text{with} \quad \alpha = \frac{4 \cdot \chi(\mathcal{O}_\Sigma) + \max\{0, 2 \cdot K_\Sigma.L\} + 6 \cdot L^2}{L^2},$$

then a simple calculation shows that (2.3) becomes redundant. For this we have to take into account that $\tau^*(\mathcal{S}) \geq 1$ for any singularity type $\mathcal{S}$. The claim then follows with $\beta = \frac{1}{3} \cdot \gamma \leq \frac{1}{4}$.

**2.3 Theorem**

*Let* $C_1$ *and* $C_2$ *be two smooth projective curves of genera* $g_1$ *and* $g_2$ *respectively with* $g_1 \geq g_2 \geq 0$*, such that for* $\Sigma = C_1 \times C_2$ *the Néron–Severi group is* $\mathrm{NS}(\Sigma) = C_1\mathbb{Z} \oplus C_2\mathbb{Z}$*.*

*Let* $D \in \mathrm{Div}(\Sigma)$ *such that* $D \sim_a aC_1 + bC_2$ *with* $a \geq \max\{3-2g_2, 2g_2+\nu^*(\mathcal{S}_i) \mid i = 1,\ldots,r\}$ *and* $b \geq \max\{3-2g_1, 2g_1+\nu^*(\mathcal{S}_i) \mid i = 1,\ldots,r\}$*, let* $\mathcal{S}_1,\ldots,\mathcal{S}_r$ *be topological or analytical singularity types.*

*Suppose that*

$$\sum_{i=1}^{r} \big(\tau^*(\mathcal{S}_i)+2\big)^2 \;<\; \gamma\cdot(D-K_\Sigma)^2, \tag{2.5}$$

*where* $\gamma$ *may be taken from the following table with* $\alpha = \frac{a-2g_2+2}{b-2g_1+2} > 0$*.*

| $g_1$ | $g_2$ | $\gamma$ | $\gamma$, *if* $\alpha = 1$ |
|---|---|---|---|
| $0$ | $0$ | $\frac{1}{24}$ | $\frac{1}{24}$ |
| $1$ | $0$ | $\frac{1}{\max\{32,2\alpha\}}$ | $\frac{1}{32}$ |
| $\geq 2$ | $0$ | $\frac{1}{\max\{24+16g_1,4g_1\alpha\}}$ | $\frac{1}{24+16g_1}$ |
| $1$ | $1$ | $\frac{1}{\max\left\{32,2\alpha,\frac{2}{\alpha}\right\}}$ | $\frac{1}{32}$ |
| $\geq 2$ | $\geq 1$ | $\frac{1}{\max\left\{24+16g_1+16g_2,4g_1\alpha,\frac{4g_2}{\alpha}\right\}}$ | $\frac{1}{24+16g_1+16g_2}$ |

*Then* $V^{irr}_{|D|}(\mathcal{S}_1,\ldots,\mathcal{S}_r)$ *is empty or it is irreducible of the expected dimension.*

**Proof:** The assumptions on $a$ and $b$ ensure that $D - K_\Sigma$ is big and nef and that $D + K_\Sigma$ is nef. Thus, once we know that (2.5) implies Condition (3) in Lemma 3.1, Step 2 in the proof of Theorem 2.1 follows in the same way, just replacing Lemma 3.3 by Lemma 3.4.

For Condition (3) we note that

$$\sum_{i=1}^{r} \big(\tau^*(\mathcal{S}_i)+2\big) \leq \sum_{i=1}^{r} \big(\tau^*(\mathcal{S}_i)+2\big)^2 \leq \tfrac{1}{24}\cdot(D-K_\Sigma)^2 < \tfrac{1}{4}\cdot(D-K_\Sigma)^2.$$

Also Step 3 in the proof of Theorem 2.1 follows in the same way, except that Condition (b) in Lemma 3.9 is not void this time, but is fullfilled by the assumptions on $a$ and $b$, since the only irreducible curves of selfintersection zero are linearly equivalent either to $C_1$ or to $C_2$. □

In the following theorem we use the notation of Section G.a.

**2.4 Theorem**

*Let* $\pi : \Sigma \to C$ *be a geometrically ruled surface with* $e \leq 0$ *and* $g = g(C)$*.*

*Let* $D \in \mathrm{Div}(\Sigma)$ *such that* $D \sim_a aC_0 + bF$ *with* $a \geq \max\{2, \nu^*(\mathcal{S}_i) \mid i = 1,\ldots,r\}$*,* $b > 2g - 1 + \frac{ae}{2} + \max\{\nu^*(\mathcal{S}_i) \mid i = 1,\ldots,r\}$*, and if* $g = 0$ *then* $b \geq 2$*. Let* $\mathcal{S}_1,\ldots,\mathcal{S}_r$ *be topological or analytical singularity types.*

*Suppose that*

$$\sum_{i=1}^{r}\big(\tau^*(\mathcal{S}_i)+2\big)^2 \;<\; \gamma\cdot(D-K_\Sigma)^2, \tag{2.6}$$

*where* $\gamma$ *may be taken from the following table with* $\alpha=\frac{a+2}{b+2-2g-\frac{ae}{2}}>0$.

| $g$ | $e$ | $\gamma$ | $\gamma$, *if* $\alpha=1$ |
|---|---|---|---|
| $0$ | $0$ | $\frac{1}{24}$ | $\frac{1}{24}$ |
| $1$ | $0$ | $\frac{1}{\max\{24,2\alpha\}}$ | $\frac{1}{24}$ |
| $1$ | $-1$ | $\frac{1}{\max\left\{\min\left\{30+\frac{16}{\alpha}+4\alpha,40+9\alpha\right\},\frac{13}{2}\alpha\right\}}$ | $\frac{1}{49}$ |
| $\geq 2$ | $0$ | $\frac{1}{\max\{24+16g,4g\alpha\}}$ | $\frac{1}{24+16g}$ |
| $\geq 2$ | $<0$ | $\frac{1}{\max\left\{\min\left\{24+16g-9e\alpha,18+16g-9e\alpha-\frac{16}{e\alpha}\right\},4g\alpha-9e\alpha\right\}}$ | |

*Then* $V_{|D|}^{irr}(\mathcal{S}_1,\dots,\mathcal{S}_r)$ *is empty or it is irreducible of the expected dimension.*

**Proof:** The proof is identical to that of Theorem 2.3, just replacing Lemma 3.4 by Lemma 3.6 and applying Lemma G.2 for the irreducible curves of selfintersection zero. □

## 3. The Main Technical Lemmata

The following lemma is the heart of the proof. Given a curve $C\in|D|_l$ such that the scheme $X_0=X^{es}_{fix}(C)$ respectively $X_0=X^{ea}_{fix}(C)$ is *special with respect to* $D$ in the sense that $h^1\big(\Sigma,\mathcal{J}_{X_0/\Sigma}(D)\big)>0$, it provides a "small" curve $\Delta_1$ through a subscheme $X_1^0$ of $X_0$, so that we can reduce the problem by replacing $X_0$ and $D$ by $X_0:\Delta_1$ and $D-\Delta_1$ respectively. We can of course proceed inductively as long as the new zero-dimensional scheme is again special with respect to the new divisor.

In order to find $\Delta_1$ we choose a subscheme $X_1^0\subseteq X_0$ which is minimal among those subschemes special with respect to $D$. By Grothendieck-Serre duality

$$H^1\big(\Sigma,\mathcal{J}_{X_1^0/\Sigma}(D)\big)\cong \mathbf{Ext}^1\left(\mathcal{J}_{X_1^0/\Sigma}(D-K_\Sigma),\mathcal{O}_\Sigma\right)$$

and a non-trivial element of the latter group gives rise to an extension

$$0\to\mathcal{O}_\Sigma\to E_1\to\mathcal{J}_{X_1^0/\Sigma}(D-K_\Sigma)\to 0.$$

We then show that the rank-two bundle $E_1$ is Bogomolov unstable and deduce the existence of a divisor $\Delta_1^0$ such that

$$H^0\Big(\Sigma,\mathcal{J}_{X_1^0/\Sigma}\big(D-K_\Sigma-\Delta_1^0\big)\Big)\neq 0,$$

that is, we find a curve $\Delta_1\in\big|\mathcal{J}_{X_1^0/\Sigma}\big(D-K_\Sigma-\Delta_1^0\big)\big|_l$.

**3.1 Lemma**
*Let $\Sigma$ be a surface such that*

(*) *any curve* $C \subset \Sigma$ *is nef.*

*Let* $D \in \text{Div}(\Sigma)$ *and* $X_0 \subset \Sigma$ *a zero-dimensional scheme satisfying*

(0) $D - K_\Sigma$ *is big and nef, and* $D + K_\Sigma$ *is nef,*

(1) $\exists\, C_0 \in |D|_l$ *irreducible* : $X_0 \subset C_0$,

(2) $h^1\big(\Sigma, \mathcal{J}_{X_0/\Sigma}(D)\big) > 0$, *and*

(3) $\deg(X_0) < \beta \cdot (D - K_\Sigma)^2$ *for some* $0 < \beta \leq \frac{1}{4}$.

*Then there exist curves* $\Delta_1, \ldots, \Delta_m \subset \Sigma$ *and zero-dimensional locally complete intersections* $X_i^0 \subseteq X_{i-1} \cap \Delta_i$ *for* $i = 1, \ldots, m$, *where* $X_i = X_{i-1} : \Delta_i$ *for* $i = 1, \ldots, m$, *such that*

(a) $h^1\Big(\Sigma, \mathcal{J}_{X_m/\Sigma}\big(D - \sum_{i=1}^m \Delta_i\big)\Big) = 0$,

*and for* $i = 1, \ldots, m$

(b) $h^1\Big(\Sigma, \mathcal{J}_{X_i^0/\Sigma}\big(D - \sum_{k=1}^{i-1} \Delta_k\big)\Big) = 1$

(c) $D.\Delta_i \geq \deg(X_{i-1} \cap \Delta_i) \geq \deg\big(X_i^0\big) \geq \big(D - K_\Sigma - \sum_{k=1}^i \Delta_k\big).\Delta_i \geq \Delta_i^2 \geq 0$

(d) $\big(D - K_\Sigma - \sum_{k=1}^i \Delta_k - \Delta_i\big)^2 > 0$,

(e) $\big(D - K_\Sigma - \sum_{k=1}^i \Delta_k - \Delta_i\big).H > 0$ *for all* $H \in \text{Div}(\Sigma)$ *ample, and*

(f) $D - K_\Sigma - \sum_{k=1}^i \Delta_k$ *is big and nef.*

*Moreover, it follows*

$$0 \leq \tfrac{1}{4}(D - K_\Sigma)^2 - \sum_{i=1}^m \deg\big(X_i^0\big) \leq \left(\tfrac{1}{2}(D - K_\Sigma) - \sum_{i=1}^m \Delta_i\right)^2. \tag{3.1}$$

**Proof:** We are going to find the $\Delta_i$ and $X_i^0$ recursively. Let us therefore suppose that we have already found $\Delta_1, \ldots, \Delta_{i-1}$ and $X_1^0, \ldots, X_{i-1}^0$ satisfying (b)-(f), and suppose that still $h^1\Big(\Sigma, \mathcal{J}_{X_{i-1}/\Sigma}\big(D - \sum_{k=1}^{i-1} \Delta_k\big)\Big) > 0$.

We choose $X_i^0 \subseteq X_{i-1}$ minimal such that $h^1\Big(\Sigma, \mathcal{J}_{X_i^0/\Sigma}\big(D - \sum_{k=1}^{i-1} \Delta_k\big)\Big) > 0$.
**Step 1:** $h^1\Big(\Sigma, \mathcal{J}_{X_i^0/\Sigma}\big(D - \sum_{k=1}^{i-1} \Delta_k\big)\Big) = 1$, i. e. (b) is fulfilled.

Suppose it was strictly larger than one. By (0) respectively (f), and the Kawamata–Viehweg Vanishing Theorem we have $h^1\Big(\Sigma, \mathcal{O}_\Sigma\big(D - \sum_{k=1}^{i-1} \Delta_k\big)\Big) = 0$.

Thus $X_i^0$ cannot be empty, that is $\deg\big(X_i^0\big) \geq 1$ and we may choose a subscheme $Y \subset X_i^0$ of degree $\deg(Y) = \deg\big(X_i^0\big) - 1$. The inclusion $\mathcal{J}_{X_i^0} \hookrightarrow \mathcal{J}_Y$

implies $h^0\Big(\Sigma,\mathcal{J}_{X_i^0/\Sigma}\big(D-\sum_{k=1}^{i-1}\Delta_k\big)\Big)\leq h^0\Big(\Sigma,\mathcal{J}_{Y/\Sigma}\big(D-\sum_{k=1}^{i-1}\Delta_k\big)\Big)$ and the structure sequences of $Y$ and $X_i^0$ thus lead to

$$\begin{aligned}
&h^1\Big(\Sigma,\mathcal{J}_{Y/\Sigma}\big(D-\textstyle\sum_{k=1}^{i-1}\Delta_k\big)\Big)\\
=\ &h^0\Big(\Sigma,\mathcal{J}_{Y/\Sigma}\big(D-\textstyle\sum_{k=1}^{i-1}\Delta_k\big)\Big)-h^0\Big(\Sigma,\mathcal{O}_\Sigma\big(D-\textstyle\sum_{k=1}^{i-1}\Delta_k\big)\Big)+\deg(Y)\\
\geq\ &h^0\Big(\Sigma,\mathcal{J}_{X_i^0/\Sigma}\big(D-\textstyle\sum_{k=1}^{i-1}\Delta_k\big)\Big)-h^0\Big(\Sigma,\mathcal{O}_\Sigma\big(D-\textstyle\sum_{k=1}^{i-1}\Delta_k\big)\Big)+\deg\big(X_i^0\big)-1\\
=\ &h^1\Big(\Sigma,\mathcal{J}_{X_i^0/\Sigma}\big(D-\textstyle\sum_{k=1}^{i-1}\Delta_k\big)\Big)-1>0
\end{aligned}$$

in contradiction to the minimality of $X_i^0$.

**Step 2:** $\deg\big(X_i^0\big)\leq\deg(X_0)-\sum_{k=1}^{i-1}\deg(X_{k-1}\cap\Delta_k)$.

The case $i=1$ follows from the fact that $X_1^0\subseteq X_0$, and for $i>1$ the inclusion $X_i^0\subseteq X_{i-1}=X_{i-2}:\Delta_{i-1}$ implies

$$\deg\big(X_i^0\big)\leq\deg(X_{i-2}:\Delta_{i-1})=\deg(X_{i-2})-\deg(X_{i-2}\cap\Delta_{i-1}).$$

It thus suffices to show, that

$$\deg(X_{i-2})-\deg(X_{i-2}\cap\Delta_{i-1})=\deg(X_0)-\sum_{k=1}^{i-1}\deg(X_{k-1}\cap\Delta_k).$$

If $i=2$, there is nothing to show. Otherwise $X_{i-2}=X_{i-3}:\Delta_{i-2}$ implies

$$\begin{aligned}
&\deg(X_{i-2})-\deg(X_{i-2}\cap\Delta_{i-1})\\
=\ &\deg(X_{i-3}:\Delta_{i-2})-\deg(X_{i-2}\cap\Delta_{i-1})\\
=\ &\deg(X_{i-3})-\deg(X_{i-3}\cap\Delta_{i-2})-\deg(X_{i-2}\cap\Delta_{i-1})
\end{aligned}$$

and we are done by induction.

**Step 3:** There exists a “suitable” locally free rank-two vector bundle $E_i$.

By the Grothendieck-Serre duality (cf. [**Har77**] III.7.6) we have $0\neq H^1\Big(\Sigma,\mathcal{J}_{X_i^0/\Sigma}\big(D-\sum_{k=1}^{i-1}\Delta_k\big)\Big)\cong\operatorname{Ext}^1\Big(\mathcal{J}_{X_i^0/\Sigma}\big(D-\sum_{k=1}^{i-1}\Delta_k\big),\mathcal{O}_\Sigma(K_\Sigma)\Big)$, and thus, since $\mathcal{O}_\Sigma(K_\Sigma)$ is locally free, (cf. [**Har77**] III.6.7)

$$\operatorname{Ext}^1\Big(\mathcal{J}_{X_i^0/\Sigma}\big(D-K_\Sigma-\textstyle\sum_{k=1}^{i-1}\Delta_k\big),\mathcal{O}_\Sigma\Big)\neq0.$$

That is, there is an extension (cf. [**Har77**] Ex. III.6.1)

$$0\to\mathcal{O}_\Sigma\to E_i\to\mathcal{J}_{X_i^0/\Sigma}\Big(D-K_\Sigma-\textstyle\sum_{k=1}^{i-1}\Delta_k\Big)\to0. \tag{3.2}$$

The minimality of $X_i^0$ implies that $E_i$ is locally free (cf. [**Laz97**] Proposition 3.9) and hence that $X_i^0$ is a locally complete intersection (cf. [**Laz97**] p. 175). Moreover, we have (cf. [**Laz97**] Exercise 4.3)

$$c_1(E_i)=D-K_\Sigma-\sum_{k=1}^{i-1}\Delta_k\quad\text{and}\quad c_2(E_i)=\deg\big(X_i^0\big). \tag{3.3}$$

**Step 4:** $E_i$ is Bogomolov unstable.

According to the Theorem of Bogomolov we only have to show $c_1(E_i)^2 > 4c_2(E_i)$ (cf. [**Bog79**] or [**Laz97**] Theorem 4.2). In order to show this we note that

$$\begin{aligned}
&-2\sum_{k=1}^{i-1}\Delta_k.\big(D-K_\Sigma-\sum_{j=1}^{k}\Delta_j\big)\\
=&\ -2\left(\sum_{k=1}^{i-1}\Delta_k\right).(D-K_\Sigma)+2\sum_{k=1}^{i-1}\sum_{j=1}^{k}\Delta_k.\Delta_j\\
=&\ -2\left(\sum_{k=1}^{i-1}\Delta_k\right).(D-K_\Sigma)+\sum_{k=1}^{i-1}\Delta_k^2+\left(\sum_{k=1}^{i-1}\Delta_k\right)^2.
\end{aligned}\tag{3.4}$$

Since $(4\beta-1)\cdot(D-K_\Sigma)^2\le 0$ by (3) and since $\Delta_k^2\ge 0$ by (*) we deduce:

$$\begin{aligned}
4c_2(E_i) &= 4\deg\left(X_i^0\right) \le_{\text{Step 2}} 4\deg(X_0)-4\sum_{k=1}^{i-1}\deg(X_{k-1}\cap\Delta_k)\\
&<_{(3)/(c)} 4\beta(D-K_\Sigma)^2-2\sum_{k=1}^{i-1}\Delta_k.\big(D-K_\Sigma-\sum_{j=1}^{k}\Delta_j\big)-2\sum_{k=1}^{i-1}\Delta_k^2\\
&=_{(3.4)} 4\beta(D-K_\Sigma)^2-2\left(\sum_{k=1}^{i-1}\Delta_k\right).(D-K_\Sigma)+\left(\sum_{k=1}^{i-1}\Delta_k\right)^2-\sum_{k=1}^{i-1}\Delta_k^2\\
&= \left(D-K_\Sigma-\sum_{k=1}^{i-1}\Delta_k\right)^2+(4\beta-1)\cdot(D-K_\Sigma)^2-\sum_{k=1}^{i-1}\Delta_k^2\\
&\le \left(D-K_\Sigma-\sum_{k=1}^{i-1}\Delta_k\right)^2 = c_1(E_i)^2.
\end{aligned}$$

**Step 5:** Find $\Delta_i$.

Since $E_i$ is Bogomolov unstable there exists a zero-dimensional scheme $Z_i\subset\Sigma$ and a divisor $\Delta_i^0\in\text{Div}(\Sigma)$ such that

$$0\to\mathcal{O}_\Sigma(\Delta_i^0)\to E_i\to\mathcal{J}_{Z_i/\Sigma}\left(D-K_\Sigma-\sum_{k=1}^{i-1}\Delta_k-\Delta_i^0\right)\to 0\tag{3.5}$$

is exact (cf. [**Laz97**] Theorem 4.2) and that

(d') $\left(2\Delta_i^0-D+K_\Sigma+\sum_{k=1}^{i-1}\Delta_k\right)^2\ge c_1(E_i)^2-4\cdot c_2(E_i)>0$, and

(e') $\left(2\Delta_i^0-D+K_\Sigma+\sum_{k=1}^{i-1}\Delta_k\right).H>0$ for all $H\in\text{Div}(\Sigma)$ ample.

We note that (e') implies $h^0\Big(\Sigma,\mathcal{O}_\Sigma\big(D-K_\Sigma-\sum_{k=1}^{i-1}\Delta_k-2\Delta_i^0\big)\Big)=0$ and the inclusion $\mathcal{J}_{Z_i/\Sigma}\hookrightarrow\mathcal{O}_\Sigma$ thus gives

$$h^0\Big(\Sigma,\mathcal{J}_{Z_i/\Sigma}\big(D-K_\Sigma-\sum_{k=1}^{i-1}\Delta_k-2\Delta_i^0\big)\Big)=0.\tag{3.6}$$

Tensoring (3.5) with $\mathcal{O}_\Sigma(-\Delta_i^0)$ leads to the following exact sequence

$$0\to\mathcal{O}_\Sigma\to E_i(-\Delta_i^0)\to\mathcal{J}_{Z_i/\Sigma}\left(D-K_\Sigma-\sum_{k=1}^{i-1}\Delta_k-2\Delta_i^0\right)\to 0,\tag{3.7}$$

and we deduce with (3.6) $h^0\Big(\Sigma,E_i(-\Delta_i^0)\Big)=h^0(\Sigma,\mathcal{O}_\Sigma)=1$.

Now tensoring (3.2) with $\mathcal{O}_\Sigma(-\Delta_i^0)$ leads to

$$0\to\mathcal{O}_\Sigma(-\Delta_i^0)\to E_i(-\Delta_i^0)\to\mathcal{J}_{X_i^0/\Sigma}\left(D-K_\Sigma-\sum_{k=1}^{i-1}\Delta_k-\Delta_i^0\right)\to 0.\tag{3.8}$$

By (e'), and (0) respectively (f)

$$-\Delta_i^0.H<-\tfrac{1}{2}\big(D-K_\Sigma-\sum_{k=1}^{i-1}\Delta_k\big).H\le 0$$

for an ample divisor $H$, hence $-\Delta_i^0$ cannot be effective, that is $H^0(\Sigma, -\Delta_i^0) = 0$. But the long exact cohomology sequence of (3.8) then implies

$$0 \neq H^0\Big(\Sigma, E_i\big(-\Delta_i^0\big)\Big) \hookrightarrow H^0\left(\Sigma, \mathcal{J}_{X_i^0/\Sigma}\left(D - K_\Sigma - \sum_{k=1}^{i-1}\Delta_k - \Delta_i^0\right)\right).$$

In particular the linear system $\left|\mathcal{J}_{X_i^0/\Sigma}\big(D - K_\Sigma - \sum_{k=1}^{i-1}\Delta_k - \Delta_i^0\big)\right|_l$ is non-empty, and we may choose

$$\Delta_i \in \left|\mathcal{J}_{X_i^0/\Sigma}\big(D - K_\Sigma - \sum_{k=1}^{i-1}\Delta_k - \Delta_i^0\big)\right|_l.$$

**Step 6:** $\Delta_i$ satisfies (d)-(f).

We note that by the choice of $\Delta_i$ we have the following linear equivalences

$$\Delta_i^0 \sim_l D - K_\Sigma - \sum_{k=1}^{i}\Delta_k \tag{3.9}$$

$$\Delta_i^0 - \Delta_i \sim_l 2\Delta_i^0 - D + K_\Sigma + \sum_{k=1}^{i-1}\Delta_k \sim_l D - K_\Sigma - \sum_{k=1}^{i}\Delta_k - \Delta_i. \tag{3.10}$$

Thus (d) and (e) is a reformulation of (d') and (e').

Moreover, $\big(\Delta_i^0 - \Delta_i\big).H > 0$ for any ample $H$, implies $\big(\Delta_i^0 - \Delta_i\big).H \geq 0$ for any $H$ in the closure of the ample cone, in particular

$$\big(\Delta_i^0 - \Delta_i\big).H \geq 0 \quad \text{for all } H \text{ nef.} \tag{3.11}$$

But then

$$\Delta_i^0.H \geq \Delta_i.H \geq 0 \quad \text{for all } H \text{ nef,} \tag{3.12}$$

since $\Delta_i$ is effective. And finally, since by assumption (*) any effective divisor is nef, we deduce that $\Delta_i^0.C \geq 0$ for any curve $C$, that is, $\Delta_i^0$ is nef. In view of (3.9) for (f) it remains to show that $\big(\Delta_i^0\big)^2 > 0$. Taking once more into account that $\Delta_i$ is nef by (*) we have by (d'), (3.10), (3.11) and (3.12)

$$\big(\Delta_i^0\big)^2 = \big(\Delta_i^0 - \Delta_i\big)^2 + \big(\Delta_i^0 - \Delta_i\big).\Delta_i + \Delta_i^0.\Delta_i > 0.$$

**Step 7:** $\Delta_i$ satisfies (c).

We would like to apply the Theorem of Bézout to $C_0$ and $\Delta_i$. Thus suppose that the irreducible curve $C_0$ is a component of $\Delta_i$ and let $H$ be any ample divisor. Applying (d) and the fact that $D + K_\Sigma$ is nef by (0), we derive the contradiction

$$0 \leq (\Delta_i - C_0).H < -\frac{1}{2} \cdot \left(D + K_\Sigma + \sum_{k=1}^{i-1}\Delta_k\right).H \leq -\frac{1}{2} \cdot (D + K_\Sigma).H \leq 0.$$

Since $X_{i-1} \subseteq X_0 \subset C_0$ the Theorem of Bézout therefore implies

$$D.\Delta_i = C_0.\Delta_i \geq \deg(X_{i-1} \cap \Delta_i).$$

By definition $X_i^0 \subseteq X_{i-1}$ and $X_i^0 \subset \Delta_i$, thus

$$\deg(X_{i-1} \cap \Delta_i) \geq \deg\big(X_i^0\big).$$

By assumption (*) the curve $\Delta_i$ is nef and thus (3.12) gives

$$\big(D - K_\Sigma - \textstyle\sum_{k=1}^{i} \Delta_k\big).\Delta_i = \Delta_i^0.\Delta_i \geq \Delta_i^2 \geq 0.$$

Finally from (d') and by (3.3) it follows that

$$\big(\Delta_i^0 - \Delta_i\big)^2 \geq c_1(E_i)^2 - 4 \cdot c_2(E_i) = \big(\Delta_i^0 + \Delta_i\big)^2 - 4 \cdot \deg\big(X_i^0\big),$$

and thus

$$\deg\big(X_i^0\big) \geq \Delta_i^0.\Delta_i.$$

**Step 8:** After a finite number $m$ of steps $h^1\Big(\Sigma, \mathcal{J}_{X_m/\Sigma}\big(D - \sum_{i=1}^{m} \Delta_i\big)\Big) = 0$, i. e. (a) is fulfilled.

As we have mentioned in Step 1 $\deg\big(X_i^0\big) > 0$. This ensures that

$$\deg(X_i) = \deg(X_{i-1}) - \deg(X_{i-1} \cap \Delta_i) \leq \deg(X_{i-1}) - \deg\big(X_i^0\big) < \deg(X_{i-1}),$$

i. e. the degree of $X_i$ is strictly diminished each time. Thus the procedure must stop after a finite number $m$ of steps, which is equivalent to the fact that $h^1\Big(\Sigma, \mathcal{J}_{X_m/\Sigma}\big(D - \sum_{i=1}^{m} \Delta_i\big)\Big) = 0$.

**Step 9:** It remains to show (3.1).

By assumption (*) the curves $\Delta_i$ are nef, in particular $\Delta_i.\Delta_j \geq 0$ for all $i, j$. Thus (c) implies

$$\begin{aligned} \textstyle\sum_{i=1}^{m} \deg\big(X_i^0\big) &\geq \textstyle\sum_{i=1}^{m} \big(D - K_\Sigma - \sum_{k=1}^{i} \Delta_k\big).\Delta_i \\ &= (D - K_\Sigma). \textstyle\sum_{i=1}^{m} \Delta_i - \frac{1}{2}\Big(\big(\sum_{i=1}^{m} \Delta_i\big)^2 + \sum_{i=1}^{m} \Delta_i^2\Big) \\ &\geq (D - K_\Sigma). \textstyle\sum_{i=1}^{m} \Delta_i - \big(\sum_{i=1}^{m} \Delta_i\big)^2. \end{aligned}$$

But then, taking condition (3) into account,

$$\begin{aligned} 0 \leq \tfrac{1}{4}(D - K_\Sigma)^2 - \deg(X_0) &\leq \tfrac{1}{4}(D - K_\Sigma)^2 - \textstyle\sum_{i=1}^{m} \deg\big(X_i^0\big) \\ &\leq \tfrac{1}{4}(D - K_\Sigma)^2 - (D - K_\Sigma). \textstyle\sum_{i=1}^{m} \Delta_i + \big(\sum_{i=1}^{m} \Delta_i\big)^2 \\ &= \big(\tfrac{1}{2}(D - K_\Sigma) - \textstyle\sum_{i=1}^{m} \Delta_i\big)^2. \end{aligned}$$

□

It is our overall aim to compare the dimension of a cohomology group of the form $H^1\big(\Sigma, \mathcal{J}_{X_0/\Sigma}(D)\big)$ with some invariants of the $X_i^0$ and $\Delta_i$. The following lemma will be vital for the necessary estimations.

**3.2 Lemma**

*Let $D \in \text{Div}(\Sigma)$ and let $X_0 \subset \Sigma$ be a zero-dimensional scheme such that there exist curves $\Delta_1, \ldots, \Delta_m \subset \Sigma$ and zero-dimensional schemes $X_i^0 \subseteq X_{i-1}$ for $i = 1, \ldots, m$, where $X_i = X_{i-1} : \Delta_i$ for $i = 1, \ldots, m$, such that (a)-(f) in Lemma 3.1 are fulfilled.*

*Then:*

$$\begin{aligned} h^1\big(\Sigma,\mathcal{J}_{X_0/\Sigma}(D)\big) &\le \sum_{i=1}^m h^1\Big(\Delta_i,\mathcal{J}_{X_{i-1}\cap\Delta_i/\Delta_i}\big(D-\textstyle\sum_{k=1}^{i-1}\Delta_k\big)\Big)\\ &\le \sum_{i=1}^m \Big(1+\deg(X_{i-1}\cap\Delta_i)-\deg\big(X_i^0\big)\Big)\\ &\le \sum_{i=1}^m \Big(\Delta_i\cdot\big(K_\Sigma+\textstyle\sum_{k=1}^i\Delta_k\big)+1\Big). \end{aligned}$$

**Proof:** Throughout the proof we use the following notation

$$\mathcal{G}_i=\mathcal{J}_{X_{i-1}\cap\Delta_i/\Delta_i}\Big(D-\textstyle\sum_{k=1}^{i-1}\Delta_k\Big)\quad\text{and}\quad \mathcal{G}_i^0=\mathcal{J}_{X_i^0/\Delta_i}\big(D-\textstyle\sum_{k=1}^{i-1}\Delta_k\big)$$

for $i=1,\dots,m$, and

$$\mathcal{F}_i=\mathcal{J}_{X_i/\Sigma}\Big(D-\textstyle\sum_{k=1}^{i}\Delta_k\Big),$$

for $i=0,\dots,m$.

Since $X_{i+1}=X_i:\Delta_{i+1}$ we have the following short exact sequence

$$0\longrightarrow\mathcal{F}_{i+1}\xrightarrow{\cdot\Delta_{i+1}}\mathcal{F}_i\longrightarrow\mathcal{G}_{i+1}\longrightarrow 0 \tag{3.13}$$

for $i=0,\dots,m-1$ and the corresponding long exact cohomology sequence

$$\begin{array}{c} 0\longrightarrow H^0(\Sigma,\mathcal{F}_{i+1})\longrightarrow H^0(\Sigma,\mathcal{F}_i)\longrightarrow H^0(\Sigma,\mathcal{G}_{i+1})\to H^1(\Sigma,\mathcal{F}_{i+1})\\ \hfill\downarrow\\ 0=H^2(\Sigma,\mathcal{G}_{i+1})\longleftarrow H^2(\Sigma,\mathcal{F}_i)\longleftarrow H^2(\Sigma,\mathcal{F}_{i+1})\leftarrow H^1(\Sigma,\mathcal{G}_{i+1})\longleftarrow H^1(\Sigma,\mathcal{F}_i)\end{array} \tag{3.14}$$

**Step 1:** $h^1(\Sigma,\mathcal{F}_i)\le\sum_{j=i+1}^m h^1(\Sigma,\mathcal{G}_j)$ for $i=0,\dots,m-1$.

We prove the claim by descending induction on $i$. From (3.14) we deduce

$$0=H^1(\Sigma,\mathcal{F}_m)\longrightarrow H^1(\Sigma,\mathcal{F}_{m-1})\longrightarrow H^1(\Sigma,\mathcal{G}_m),$$

which implies $h^1(\Sigma,\mathcal{F}_{m-1})\le h^1(\Sigma,\mathcal{G}_m)$ and thus proves the case $i=m-1$.

We may therefore assume that $1\le i\le m-2$. Once more from (3.14) we deduce

$$a=h^0(\Sigma,\mathcal{F}_{i+1})-h^0(\Sigma,\mathcal{F}_i)+h^0(\Sigma,\mathcal{G}_{i+1})\ge 0,$$

and

$$b=h^2(\Sigma,\mathcal{F}_{i+1})-h^2(\Sigma,\mathcal{F}_i)\ge 0,$$

and finally

$$\begin{aligned} h^1(\Sigma,\mathcal{F}_i) &= h^1(\Sigma,\mathcal{G}_{i+1})+h^1(\Sigma,\mathcal{F}_{i+1})-a-b\\ &\le h^1(\Sigma,\mathcal{G}_{i+1})+h^1(\Sigma,\mathcal{F}_{i+1})\\ &\le_{\text{Ind.}} h^1(\Sigma,\mathcal{G}_{i+1})+\textstyle\sum_{j=i+2}^m h^1(\Sigma,\mathcal{G}_j)\\ &= \textstyle\sum_{j=i+1}^m h^1(\Sigma,\mathcal{G}_j). \end{aligned}$$

**Step 2:** $h^1(\Delta_i,\mathcal{G}_i)=h^0(\Delta_i,\mathcal{G}_i)-\chi\Big(\mathcal{O}_{\Delta_i}\big(D-\sum_{k=1}^{i-1}\Delta_k\big)\Big)+\deg(X_{i-1}\cap\Delta_i)$.

We consider the exact sequence

$$0 \longrightarrow \mathcal{G}_i \longrightarrow \mathcal{O}_{\Delta_i}\left(D - \sum_{k=1}^{i-1} \Delta_k\right) \longrightarrow \mathcal{O}_{X_{i-1}\cap\Delta_i/\Delta_i}\left(D - \sum_{k=1}^{i-1} \Delta_k\right) \longrightarrow 0.$$

The result then follows from the long exact cohomology sequence.

**Step 3:** $h^0(\Delta_i, \mathcal{G}_i^0) - \chi\Big(\mathcal{O}_{\Delta_i}(D - \sum_{k=1}^{i-1}\Delta_k)\Big) = h^1(\Delta_i, \mathcal{G}_i^0) - \deg(X_i^0)$.

This follows analogously, replacing $X_{i-1}$ by $X_i^0$, since $X_i^0 = X_i^0 \cap \Delta_i$.

**Step 4:** $h^1(\Delta_i, \mathcal{G}_i^0) \leq h^1\left(\Sigma, \mathcal{J}_{X_i^0/\Sigma}(D - \sum_{k=1}^{i-1}\Delta_k)\right) = 1$.

Note that $X_i^0 : \Delta_i = \emptyset$, and hence $\mathcal{J}_{X_i^0:\Delta_i/\Sigma} = \mathcal{O}_\Sigma$. We thus have the following short exact sequence

$$0 \longrightarrow \mathcal{O}_\Sigma\left(D - \sum_{k=1}^{i} \Delta_k\right) \xrightarrow{\cdot\Delta_i} \mathcal{J}_{X_i^0/\Sigma}\left(D - \sum_{k=1}^{i-1} \Delta_k\right) \longrightarrow \mathcal{G}_i^0 \longrightarrow 0. \quad (3.15)$$

By assumption (f) the divisor $D - K_\Sigma - \sum_{k=1}^{i}\Delta_k$ is big and nef and hence

$$0 = h^0\Big(\Sigma, \mathcal{O}_\Sigma(-D + K_\Sigma + \sum_{k=1}^{i}\Delta_k)\Big) = h^2\Big(\Sigma, \mathcal{O}_\Sigma(D - \sum_{k=1}^{i}\Delta_k)\Big).$$

Thus the long exact cohomology sequence of (3.15) gives

$$H^1\Big(\Sigma, \mathcal{J}_{X_i^0/\Sigma}(D - \sum_{k=1}^{i-1}\Delta_k)\Big) \longrightarrow H^1(\Delta_i, \mathcal{G}_i^0) \longrightarrow 0,$$

and

$$h^1(\Delta_i, \mathcal{G}_i^0) \leq h^1\left(\Sigma, \mathcal{J}_{X_i^0/\Sigma}(D - \sum_{k=1}^{i-1}\Delta_k)\right).$$

However, by assumption (b) the latter is just one.

**Step 5:** $h^1(\Delta_i, \mathcal{G}_i) \leq 1 + \deg(X_{i-1}\cap\Delta_i) - \deg\left(X_i^0\right)$.

We note that $\mathcal{G}_i \hookrightarrow \mathcal{G}_i^0$, and thus

$$h^0(\Delta_i, \mathcal{G}_i) \leq h^0(\Delta_i, \mathcal{G}_i^0).$$

But then

$$\begin{aligned} h^1(\Delta_i, \mathcal{G}_i) &\leq_{\text{Step 2/3}} h^1(\Delta_i, \mathcal{G}_i^0) - \deg\left(X_i^0\right) + \deg(X_{i-1}\cap\Delta_i) \\ &\leq_{\text{Step 4}} 1 - \deg\left(X_i^0\right) + \deg(X_{i-1}\cap\Delta_i). \end{aligned}$$

**Step 6:** Finish the proof.

Taking into account, that $h^1(\Sigma, \mathcal{G}_i) = h^1(\Delta_i, \mathcal{G}_i)$, since $\mathcal{G}_i$ is concentrated on $\Delta_i$ (cf. **[Har77]** III.2.10), the first inequality follows from Step 1, while the second inequality is a consequence of Step 5 and the last inequality follows from assumption (c). ☐

In the Lemmata 3.3, 3.4 and 3.6 we consider special classes of surfaces which allow us to do the necessary estimations in order to finally derive

$$\sum_{i=1}^{m}\left(\#X_i^0 - \dim|\Delta_i|_l\right) > h^1\big(\Sigma, \mathcal{J}_{X_0/\Sigma}(D)\big).$$

We first consider surfaces with Picard number one.

**3.3 Lemma**

*Let $\Sigma$ be a surface such that*[4]

(i) $\mathrm{NS}(\Sigma) = L \cdot \mathbb{Z}$,

(ii) $h^1(\Sigma, C) = 0$, *whenever* $C$ *is effective.*

*Let* $D \in \mathrm{Div}(\Sigma)$ *and* $X_0 \subset \Sigma$ *a zero-dimensional scheme satisfying (0)–(3) from Lemma 3.1 and*

(4) $\sum_{z\in\Sigma} \big(\deg(X_{0,z})\big)^2 < \gamma \cdot (D - K_\Sigma)^2$, *where* $\gamma = \frac{\left(1+\sqrt{1-4\beta}\right)^2 \cdot L^2}{4\cdot\chi(\mathcal{O}_\Sigma)+\max\{0,2\cdot K_\Sigma.L\}+6\cdot L^2}$.

*Then, using the notation of Lemma 3.1 and setting* $X_S = \bigcup_{i=1}^m X_i^0$,

$$h^1\big(\Sigma, \mathcal{J}_{X_0/\Sigma}(D)\big) + \sum_{i=1}^m \Big(h^0\big(\Sigma, \mathcal{O}_\Sigma(\Delta_i)\big) - 1\Big) < \#X_S.$$

**Proof:** We note that by Lemma E.1 we may assume that $L$ is ample. We fix the following notation:

$$D \sim_a d \cdot L,\quad K_\Sigma \sim_a \kappa \cdot L,\quad \Delta_i \sim_a \delta_i \cdot L,\quad \text{and } l = \sqrt{L^2} > 0.$$

Furthermore, we set

$$\alpha = \frac{4\cdot\chi(\mathcal{O}_\Sigma)+\max\{0,2\cdot K_\Sigma.L\}+6\cdot L^2}{4\cdot L^2} = \begin{cases} \frac{\chi(\mathcal{O}_\Sigma)}{l^2} + \frac{\kappa+3}{2}, & \text{if } \kappa \geq 0,\\ \frac{\chi(\mathcal{O}_\Sigma)}{l^2} + \frac{3}{2}, & \text{if } \kappa < 0,\end{cases}$$

and thus $\gamma = \frac{\left(1+\sqrt{1-4\beta}\right)^2}{4\alpha}$.

**Step 1:** $\Sigma$ satisfies the assumption (*) of Lemma 3.1.

If $c \cdot L \sim_a C \subset \Sigma$ is effective, then in particular $c = \frac{1}{l^2} \cdot C.L > 0$, and thus $C$ is ample, in particular nef. Hence (*) in Lemma 3.1 is fulfilled.

**Step 2:** $\sum_{i=1}^m \delta_i \cdot l \leq \frac{(d-\kappa)\cdot l}{2} - \sqrt{\frac{(d-\kappa)^2\cdot l^2}{4} - \deg(X_S)}$, by (3.1).

**Step 3:** $h^1\big(\Sigma, \mathcal{J}_{X_0}(D)\big) \leq \big(\kappa \cdot \sum_{i=1}^m \delta_i\big) \cdot l^2 + \frac{1}{2}\Big(\big(\sum_{i=1}^m \delta_i\big)^2 + \sum_{i=1}^m \delta_i^2\Big) \cdot l^2 + m.$

By Lemma 3.2 we know:

$$\begin{aligned} h^1\big(\Sigma, \mathcal{J}_{X_0}(D)\big) &\leq \sum_{i=1}^m \Big(\Delta_i \cdot \big(K_\Sigma + \sum_{k=1}^i \Delta_k\big) + 1\Big)\\ &= \sum_{i=1}^m \delta_i \cdot \big(\kappa + \sum_{k=1}^i \delta_k\big) \cdot l^2 + m\\ &= \big(\kappa \cdot \sum_{i=1}^m \delta_i\big) \cdot l^2 + \big(\sum_{i=1}^m \sum_{k=1}^i \delta_i \cdot \delta_k\big) \cdot l^2 + m\\ &= \big(\kappa \cdot \sum_{i=1}^m \delta_i\big) \cdot l^2 + \frac{1}{2}\Big(\big(\sum_{i=1}^m \delta_i\big)^2 + \sum_{i=1}^m \delta_i^2\Big) \cdot l^2 + m. \end{aligned}$$

**Step 4:** $\sum_{i=1}^m \Big(h^0\big(\Sigma, \mathcal{O}_\Sigma(\Delta_i)\big) - 1\Big) \leq m \cdot \big(\chi(\mathcal{O}_\Sigma) - 1\big) + \frac{l^2}{2} \cdot \sum_{i=1}^m \delta_i^2 - \frac{\kappa\cdot l^2}{2} \cdot \sum_{i=1}^m \delta_i.$

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[4] By Lemma E.1 we may assume w. l. o. g. that $L$ is ample. – Remember that $\#X_S$ is the number of points in the support of $X_S$.

Since $\Delta_i$ is effective by (ii), $h^1(\Sigma,\Delta_i)=0$. Hence by Riemann-Roch

$$h^0(\Sigma,\Delta_i)\leq\chi\big(\mathcal{O}_\Sigma(\Delta_i)\big)=\frac{\Delta_i^2-K_\Sigma.\Delta_i}{2}+\chi(\mathcal{O}_\Sigma).$$

This implies

$$\begin{aligned}\textstyle\sum_{i=1}^m\Big(h^0\big(\Sigma,\mathcal{O}_\Sigma(\Delta_i)\big)-1\Big) &\leq -m+m\cdot\chi(\mathcal{O}_\Sigma)+\tfrac{1}{2}\textstyle\sum_{i=1}^m\big(\Delta_i^2-K_\Sigma.\Delta_i\big)\\ &= m\cdot\big(\chi(\mathcal{O}_\Sigma)-1\big)+\tfrac{l^2}{2}\cdot\textstyle\sum_{i=1}^m\delta_i^2-\tfrac{\kappa\cdot l^2}{2}\cdot\textstyle\sum_{i=1}^m\delta_i.\end{aligned}$$

**Step 5:** Finish the proof.

In the following consideration we use the facts that

$$m\leq\left(\sum_{i=1}^m\delta_i\right)^2,\quad\sum_{i=1}^m\delta_i\leq\left(\sum_{i=1}^m\delta_i\right)^2,\quad\sum_{i=1}^m\delta_i^2\leq\left(\sum_{i=1}^m\delta_i\right)^2,\tag{3.16}$$

and that

$$\deg(X_S)\leq\deg(X_0)\leq\beta\cdot(d-\kappa)^2\cdot l^2.\tag{3.17}$$

We thus get:

$$\begin{aligned}&h^1\big(\Sigma,\mathcal{J}_{X_0}(D)\big)+\textstyle\sum_{i=1}^m\Big(h^0\big(\Sigma,\mathcal{O}_\Sigma(\Delta_i)\big)-1\Big)\\
&\leq_{\text{Step 3 / 4}}\ m\cdot\chi(\mathcal{O}_\Sigma)+l^2\cdot\textstyle\sum_{i=1}^m\delta_i^2+\frac{\kappa\cdot l^2}{2}\cdot\sum_{i=1}^m\delta_i+\frac{l^2}{2}\cdot\big(\sum_{i=1}^m\delta_i\big)^2\\
&\leq_{(3.16)}\ \alpha\cdot\big(l\cdot\textstyle\sum_{i=1}^m\delta_i\big)^2\ \leq_{\text{Step 2}}\ \alpha\cdot\left(\frac{(d-\kappa)\cdot l}{2}-\sqrt{\frac{(d-\kappa)^2\cdot l^2}{4}-\deg(X_S)}\right)^2\\
&\leq\ \alpha\cdot\left(\frac{\frac{(d-\kappa)^2\cdot l^2}{4}-\Big(\frac{(d-\kappa)^2\cdot l^2}{4}-\deg(X_S)\Big)}{\frac{(d-\kappa)\cdot l}{2}+\sqrt{\frac{(d-\kappa)^2\cdot l^2}{4}-\deg(X_S)}}\right)^2\ =\ \alpha\cdot\left(\frac{2\cdot\deg(X_S)\big)}{(d-\kappa)\cdot l+\sqrt{(d-\kappa)^2\cdot l^2-4\cdot\deg(X_S)}}\right)^2\\
&\leq_{(3.17)}\ \frac{4\alpha}{\big(1+\sqrt{1-4\beta}\big)^2\cdot(d-\kappa)^2\cdot l^2}\cdot\big(\deg(X_S)\big)^2\ =\ \frac{1}{\gamma\cdot(D-K_\Sigma)^2}\cdot\big(\textstyle\sum_{z\in\Sigma}\deg(X_{S,z})\big)^2\\
&\leq\ \frac{\#X_S}{\gamma\cdot(D-K_\Sigma)^2}\cdot\textstyle\sum_{z\in\Sigma}\deg(X_{S,z})^2\ \leq\ \frac{\#X_S}{\gamma\cdot(D-K_\Sigma)^2}\cdot\sum_{z\in\Sigma}\deg(X_{0,z})^2\ <_{(4)}\ \#X_S.\end{aligned}$$

□

The second class of surfaces which we consider, are products of curves. We use the notation of Section G.b.

**3.4 Lemma**

*Let $C_1$ and $C_2$ be two smooth projective curves of genera $g_1$ and $g_2$ respectively with $g_1\geq g_2\geq 0$, such that for $\Sigma=C_1\times C_2$ the Néron–Severi group is $\mathrm{NS}(\Sigma)=C_1\mathbb{Z}\oplus C_2\mathbb{Z}$, and let $D\in\mathrm{Div}(\Sigma)$ such that $D\sim_a aC_1+bC_2$ with $a>\max\{2g_2-2,2-2g_2\}$ and $b>\max\{2g_1-2,2-2g_1\}$. Suppose moreover that $X_0\subset\Sigma$ is a zero-dimensional scheme satisfying (1)–(3) from Lemma 3.1 and*

(4) $\sum\limits_{z\in\Sigma}\big(\deg(X_{0,z})\big)^2\ <\ \gamma\cdot(D-K_\Sigma)^2,$

*where* $\gamma$ *may be taken from the following table with* $\alpha = \frac{a-2g_2+2}{b-2g_1+2} > 0$.

| $g_1$ | $g_2$ | $\gamma$ |
|---|---|---|
| 0 | 0 | $\frac{1}{24}$ |
| 1 | 0 | $\frac{1}{\max\{32,2\alpha\}}$ |
| $\geq 2$ | 0 | $\frac{1}{\max\{24+16g_1,4g_1\alpha\}}$ |
| 1 | 1 | $\frac{1}{\max\left\{32,2\alpha,\frac{2}{\alpha}\right\}}$ |
| $\geq 2$ | $\geq 1$ | $\frac{1}{\max\left\{24+16g_1+16g_2,4g_1\alpha,\frac{4g_2}{\alpha}\right\}}$ |

*Then, using the notation of Lemma 3.1 and setting*[5] $X_S = \bigcup_{i=1}^m X_i^0$,

$$h^1\big(\Sigma, \mathcal{J}_{X_0}(D)\big) + \sum_{i=1}^m \Big(h^0\big(\Sigma, \mathcal{O}_\Sigma(\Delta_i)\big) - 1\Big) < \#X_S.$$

**Proof:** Then $K_\Sigma \sim_a (2g_2 - 2) \cdot C_1 + (2g_1 - 2) \cdot C_2$ and we fix the notation:

$$\Delta_i \sim_a a_i C_1 + b_i C_2.$$

**Step 1:** $\Sigma$ satisfies the assumption (*) of Lemma 3.1 by Lemma G.7. Moreover, due to the assumptions on $a$ and $b$ we know that $D - K_\Sigma$ is ample and $D + K_\Sigma$ is nef, i. e. (0) in Lemma 3.1 is fulfilled as well.

**Step 2a:** $\left(\frac{a-2g_2+2}{4}\right) \cdot \sum_{i=1}^m b_i + \left(\frac{b-2g_1+2}{4}\right) \cdot \sum_{i=1}^m a_i \leq \deg(X_S)$.

Let us notice first that the strict inequality "$<$" in Lemma 3.1 (e) for ample divisors $H$ comes down to "$\leq$" for nef divisors $H$. We may apply this for $H = C_1$ and $H = C_2$ and deduce the following inequalities:

$$0 \leq \left(D - K_\Sigma - \sum_{k=1}^i \Delta_k - \Delta_i\right).C_1 = b - 2g_1 + 2 - \sum_{k=1}^i b_k - b_i, \tag{3.18}$$

and

$$0 \leq \left(D - K_\Sigma - \sum_{k=1}^i \Delta_k - \Delta_i\right).C_2 = a - 2g_2 + 2 - \sum_{k=1}^i a_k - a_i. \tag{3.19}$$

For the following consideration we choose $i_0, j_0 \in \{1, \ldots, m\}$ such that $a_{i_0} \geq a_i$ for all $i = 1, \ldots, m$ and $b_{j_0} \geq b_j$ for all $j = 1, \ldots, m$. Then

$$a - 2g_2 + 2 \geq \sum_{k=1}^{i_0} a_k + a_{i_0} \geq 2a_{i_0} \geq 2a_i \tag{3.20}$$

and

$$b - 2g_1 + 2 \geq \sum_{k=1}^{j_0} b_k + b_{j_0} \geq 2b_{j_0} \geq 2b_j \tag{3.21}$$

[5]Remember that $\#X_S$ is the number of points in the support of $X_S$.

for all $i, j = 1, \ldots, m$, and we finally get

$$\begin{aligned}
\deg(X_S) \;&=\; \sum_{i=1}^m \deg\big(X_i^0\big) \;\geq_{\text{Lemma 3.1 (c)}}\; \sum_{i=1}^m \big(D - K_\Sigma - \sum_{k=1}^i \Delta_k\big).\Delta_i \\
&= (D - K_\Sigma).\sum_{i=1}^m \Delta_i - \sum_{1\leq k\leq i\leq 1} \Delta_k.\Delta_i \\
&= (D - K_\Sigma).\sum_{i=1}^m \Delta_i - \tfrac{1}{2}\sum_{i=1}^m \Delta_i^2 - \tfrac{1}{2}\big(\sum_{i=1}^m \Delta_i\big)^2 \\
&= (a - 2g_2 + 2)\sum_{i=1}^m b_i + (b - 2g_1 + 2)\sum_{i=1}^m a_i - \sum_{i=1}^m a_ib_i - \sum_{i=1}^m a_i \sum_{i=1}^m b_i \\
&\geq_{(3.18)\,/\,(3.19)} \tfrac{a-2g_2+2}{2}\sum_{i=1}^m b_i + \tfrac{b-2g_1+2}{2}\sum_{i=1}^m a_i + \tfrac{1}{2}\big(\sum_{i=1}^m a_i + a_m\big)\cdot\sum_{i=1}^m b_i \\
&\qquad + \tfrac{1}{2}\big(\sum_{i=1}^m b_i + b_m\big)\cdot\sum_{i=1}^m a_i - \sum_{i=1}^m a_ib_i - \sum_{i=1}^m a_i\sum_{i=1}^m b_i \\
&= \tfrac{a-2g_2+2}{2}\sum_{i=1}^m b_i + \tfrac{b-2g_1+2}{2}\sum_{i=1}^m a_i + \tfrac{a_m}{2}\sum_{i=1}^m b_i + \tfrac{b_m}{2}\sum_{i=1}^m a_i - \sum_{i=1}^m a_ib_i \\
&\geq_{(3.20)\,/\,(3.21)} \tfrac{a-2g_2+2}{4}\sum_{i=1}^m b_i + \tfrac{1}{4}\sum_{i=1}^m 2a_ib_i + \tfrac{b-2g_1+2}{4}\sum_{i=1}^m a_i + \tfrac{1}{4}\sum_{i=1}^m 2a_ib_i \\
&\qquad + \tfrac{a_m}{2}\sum_{i=1}^m b_i + \tfrac{b_m}{2}\sum_{i=1}^m a_i - \sum_{i=1}^m a_ib_i \\
&\geq \tfrac{a-2g_2+2}{4}\sum_{i=1}^m b_i + \tfrac{b-2g_1+2}{4}\sum_{i=1}^m a_i.
\end{aligned}$$

**Step 2b:** $\sum_{i=1}^m a_i \cdot \sum_{i=1}^m b_i \leq \frac{8}{(D-K_\Sigma)^2}\cdot\big(\deg(X_S)\big)^2$.

Using Step 2a we deduce

$$\begin{aligned}
\big(\deg(X_S)\big)^2 \;&>\; \Big(\tfrac{b-2g_1+2}{4}\cdot\sum_{i=1}^m a_i + \tfrac{a-2g_2+2}{4}\cdot\sum_{i=1}^m b_i\Big)^2 \\
&\geq \tfrac{4\cdot(a-2g_2+2)\cdot(b-2g_1+2)}{16}\cdot\sum_{i=1}^m a_i\cdot\sum_{i=1}^m b_i \\
&= \tfrac{(D-K_\Sigma)^2}{8}\cdot\sum_{i=1}^m a_i\cdot\sum_{i=1}^m b_i.
\end{aligned}$$

**Step 2c:** $\sum_{i=1}^m a_i \leq \begin{cases} \frac{2\alpha}{(D-K_\Sigma)^2}\cdot\big(\deg(X_S)\big)^2, & \text{if } \sum_{i=1}^m b_i = 0,\\ \frac{8}{(D-K_\Sigma)^2}\cdot\big(\deg(X_S)\big)^2, & \text{else.}\end{cases}$

If $\sum_{i=1}^m b_i = 0$, then the same consideration as in Step 2a shows

$$\deg(X_S) \geq (b - 2g_1 + 2)\cdot\sum_{i=1}^m a_i > 0,$$

and thus

$$\tfrac{(D-K_\Sigma)^2}{2\alpha}\cdot\sum_{i=1}^m a_i \leq (b-2g_1+2)^2\cdot\left(\sum_{i=1}^m a_i\right)^2 \leq \big(\deg(X_S)\big)^2.$$

If $\sum_{i=1}^m b_i \neq 0$, then by Step 2b

$$\sum_{i=1}^m a_i \leq \sum_{i=1}^m a_i\cdot\sum_{i=1}^m b_i \leq \tfrac{8}{(D-K_\Sigma)^2}\cdot\big(\deg(X_S)\big)^2.$$

**Step 2d:** $\sum_{i=1}^m b_i \leq \begin{cases} \frac{2}{\alpha\cdot(D-K_\Sigma)^2}\cdot\big(\deg(X_S)\big)^2, & \text{if } \sum_{i=1}^m a_i = 0,\\ \frac{8}{(D-K_\Sigma)^2}\cdot\big(\deg(X_S)\big)^2, & \text{else.}\end{cases}$

This is proved in the same way as Step 2c.

**Step 3:** $h^1(\Sigma, \mathcal{J}_{X_0}(D)) \leq 2\sum_{i=1}^m a_i \sum_{i=1}^m b_i + (2g_1-2)\sum_{i=1}^m a_i + (2g_2-2)\sum_{i=1}^m b_i + m.$

The following sequence of inequalities is due to Lemma 3.2 and the fact that $\Delta_i.\Delta_j \geq 0$ for any $i, j \in \{1, \ldots, m\}$:

$$\begin{aligned} h^1(\Sigma, \mathcal{J}_{X_0}(D)) &\leq \sum_{i=1}^m \Big(\Delta_i \cdot \big(K_\Sigma + \sum_{k=1}^i \Delta_k\big) + 1\Big) \\ &= K_\Sigma \cdot \sum_{i=1}^m \Delta_i + \sum_{1\leq k\leq i\leq m} \Delta_i.\Delta_k + m \\ &\leq K_\Sigma \cdot \sum_{i=1}^m \Delta_i + \big(\sum_{i=1}^m \Delta_i\big)^2 + m \\ &= (2g_1-2)\cdot \sum_{i=1}^m a_i + (2g_2-2)\cdot\sum_{i=1}^m b_i + 2\cdot\sum_{i=1}^m a_i \cdot \sum_{i=1}^m b_i + m. \end{aligned}$$

**Step 4:** We find the estimation $\sum_{i=1}^m \Big(h^0(\Sigma, \mathcal{O}_\Sigma(\Delta_i)) - 1\Big) \leq \beta$, where

$$\beta = \begin{cases} \sum_{i=1}^m a_i \cdot \sum_{i=1}^m b_i + \sum_{i=1}^m b_i, & \text{if } g_1 = 1, g_2 = 0, \\ \sum_{i=1}^m a_i \cdot \sum_{i=1}^m b_i - m, & \text{if } g_1 = 1, g_2 = 1, \exists\, i_0 : a_{i_0} b_{i_0} > 0, \\ \sum_{i=1}^m a_i + \sum_{i=1}^m b_i - m, & \text{if } g_1 = 1, g_2 = 1, \forall\, i : a_i b_i = 0, \\ \sum_{i=1}^m a_i \cdot \sum_{i=1}^m b_i + \sum_{i=1}^m a_i + \sum_{i=1}^m b_i, & \text{else.} \end{cases}$$

In general by Corollary G.9 $h^0(\Sigma, \mathcal{O}_\Sigma(\Delta_i)) \leq a_i b_i + a_i + b_i + 1$, while if $g_1 = 1, g_2 = 0$ by Lemma G.5 we have $h^0(\Sigma, \mathcal{O}_\Sigma(\Delta_i)) = a_i b_i + b_i + 1$. It thus only remains to consider the case $g_1 = g_2 = 1$.

Applying Lemma G.15 we get

$$\sum_{i=1}^m h^0(\Sigma, \mathcal{O}_\Sigma(\Delta_i)) = \sum_{a_i, b_i > 0} a_i b_i + \sum_{a_i = 0} b_i + \sum_{b_i = 0} a_i.$$

If always either $a_i$ or $b_i$ is zero, we are done. Otherwise there exists some $i_0 \in \{1, \ldots, m\}$ such that $a_{i_0} \neq 0 \neq b_{i_0}$. Then looking at the right hand side we see

$$\sum_{i=1}^m h^0(\Sigma, \mathcal{O}_\Sigma(\Delta_i)) \leq \sum_{a_i, b_i > 0} a_i b_i + a_{i_0} \cdot \sum_{a_i = 0} b_i + b_{i_0} \cdot \sum_{b_i = 0} a_i \leq \sum_{i=1}^m a_i \cdot \sum_{i=1}^m b_i.$$

**Step 5:** Finish the proof.

Using Step 3 and Step 4, and taking $m \leq \sum_{i=1}^m a_i + b_i$ into account, we get $h^1(\Sigma, \mathcal{J}_{X_0}(D)) + \sum_{i=1}^m \Big(h^0(\Sigma, \mathcal{O}_\Sigma(\Delta_i)) - 1\Big) \leq \beta'$, where $\beta'$ may be chosen as

$$\beta' = \begin{cases} 3\cdot\sum_{i=1}^m a_i \cdot \sum_{i=1}^m b_i, & \text{if } g_1 = 0, g_2 = 0, \\ 3\cdot\sum_{i=1}^m a_i \cdot \sum_{i=1}^m b_i + \sum_{i=1}^m a_i, & \text{if } g_1 = 1, g_2 = 0, \\ 3\cdot\sum_{i=1}^m a_i \cdot \sum_{i=1}^m b_i + 2g_1 \cdot \sum_{i=1}^m a_i + 2g_2\cdot\sum_{i=1}^m b_i, & \text{if } g_1 \geq 2, g_2 \geq 0. \end{cases}$$

For the case $g_1 = g_2 = 1$ we take a closer look. We find at once the following upper bounds $\beta''$ for $h^1\big(\Sigma, \mathcal{J}_{X_0}(D)\big) + \sum_{i=1}^m \Big(h^0\big(\Sigma, \mathcal{O}_\Sigma(\Delta_i)\big) - 1\Big)$

$$\beta'' = \begin{cases} 3 \cdot \sum_{i=1}^m a_i \cdot \sum_{i=1}^m b_i, & \text{if } \exists\, i_0 \;:\; a_{i_0} b_{i_0} \neq 0, \\ 2 \cdot \sum_{i=1}^m a_i \cdot \sum_{i=1}^m b_i + \sum_{i=1}^m a_i + \sum_{i=1}^m b_i, & \text{if } \forall\, i : a_i b_i = 0. \end{cases}$$

Considering now the cases $\sum_{i=1}^m a_i \neq 0 \neq \sum_{i=1}^m b_i$, $\sum_{i=1}^m a_i = 0$ and $\sum_{i=1}^m b_i = 0$, we can replace these by

$$\beta'' \leq \beta' = \begin{cases} 4 \cdot \sum_{i=1}^m a_i \cdot \sum_{i=1}^m b_i, & \text{if } \sum_{i=1}^m a_i \neq 0 \neq \sum_{i=1}^m b_i, \\ \sum_{i=1}^m a_i, & \text{if } \sum_{i=1}^m b_i = 0, \\ \sum_{i=1}^m b_i, & \text{if } \sum_{i=1}^m a_i = 0. \end{cases}$$

Applying now the results of Step 2 in all cases we get

$$\begin{aligned} h^1\big(\Sigma, \mathcal{J}_{X_0}(D)\big) + \sum_{i=1}^m \Big(h^0\big(\Sigma, \mathcal{O}_\Sigma(\Delta_i)\big) - 1\Big) \;&\leq\; \beta' \;\leq\; \tfrac{1}{\gamma\cdot(D-K_\Sigma)^2} \cdot \big(\deg(X_S)\big)^2 \\ = \; \tfrac{1}{\gamma\cdot(D-K_\Sigma)^2} \cdot \Big(\sum_{z\in\Sigma} \deg(X_{S,z})\Big)^2 \;&\leq\; \tfrac{\#X_S}{\gamma\cdot(D-K_\Sigma)^2} \cdot \sum_{z\in\Sigma} \deg(X_{S,z})^2 \\ \leq \; \tfrac{\#X_S}{\gamma\cdot(D-K_\Sigma)^2} \cdot \sum_{z\in\Sigma} \deg(X_{0,z})^2 \;&\underset{(4)}{<}\; \#X_S. \end{aligned}$$

□

**3.5 Remark**

Lemma 3.4, and hence Theorem 2.3 could easily be generalised to other surfaces $\Sigma$ with irreducible curves $C_1, C_2 \subset \Sigma$ such that

$$\mathrm{NS}(\Sigma) = C_1\mathbb{Z} \oplus C_2\mathbb{Z} \text{ with intersection matrix } \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \tag{3.22}$$

once we have an estimation similar to

$$h^0(\Sigma, aC_1 + bC_2) \leq ab + a + b + 1$$

for an effective divisor $aC_1 + bC_2$.

With a number of small modifications we are even able to adapt it in the following lemma in the case of geometrically ruled surfaces with non-positive invariant $e$ although the intersection pairing looks more complicated.

The problem with arbitrary geometrically ruled surfaces is the existence of the section with negative self-intersection, once the invariant $e > 0$, since then the proof of Lemma 3.1 no longer works.

In the following lemma we use the notation of Section G.a.

**3.6 Lemma**

*Let $\pi : \Sigma \to C$ be a geometrically ruled surface with invariant $e \leq 0$ and $g = g(C)$, and let $D \in \mathrm{Div}(\Sigma)$ such that $D \sim_a aC_0 + bF$ with $a \geq 2$, $b > 2g - 2 + \frac{ae}{2}$, and if $g = 0$ then $b \geq 2$. Suppose moreover that $X_0 \subset \Sigma$ is a zero-dimensional scheme satisfying (1)–(3) from Lemma 3.1 and*

(4) $\sum_{z\in\Sigma}\big(\deg(X_{0,z})\big)^2 \;<\; \gamma\cdot(D-K_\Sigma)^2,$

*where $\gamma$ may be taken from the following table with $\alpha=\frac{a+2}{b+2-2g-\frac{ae}{2}}>0$.*

| $g$ | $e$ | $\gamma$ |
|---|---|---|
| $0$ | $0$ | $\frac{1}{24}$ |
| $1$ | $0$ | $\frac{1}{\max\{24,2\alpha\}}$ |
| $1$ | $-1$ | $\frac{1}{\max\left\{\min\left\{30+\frac{16}{\alpha}+4\alpha,40+9\alpha\right\},\frac{13}{2}\alpha\right\}}$ |
| $\geq 2$ | $0$ | $\frac{1}{\max\{24+16g,4g\alpha\}}$ |
| $\geq 2$ | $<0$ | $\frac{1}{\max\left\{\min\left\{24+16g-9e\alpha,18+16g-9e\alpha-\frac{16}{e\alpha}\right\},4g\alpha-9e\alpha\right\}}$ |

*Then, using the notation of Lemma 3.1 and setting*[6] $X_S=\bigcup_{i=1}^m X_i^0$,

$$h^1\big(\Sigma,\mathcal{J}_{X_0}(D)\big)+\sum_{i=1}^m\Big(h^0\big(\Sigma,\mathcal{O}_\Sigma(\Delta_i)\big)-1\Big)<\#X_S.$$

**Proof:** Remember that the Néron–Severi group of $\Sigma$ is generated by a section $C_0$ of $\pi$ and a fibre $F$ with intersection pairing given by $\left(\begin{smallmatrix}-e & 1\\ 1 & 0\end{smallmatrix}\right)$. Then $K_\Sigma\sim_a -2C_0+(2g-2-e)\cdot F$ and we fix the notation:

$$\Delta_i\sim_a a_iC_0+b_iF.$$

Note that by Lemma G.1 we have

$$a_i\geq 0 \quad\text{and}\quad b_i':=b_i-\tfrac{e}{2}a_i\geq 0.$$

Finally we set $\kappa_1=a+2$ and $\kappa_2=b+2-2g-\frac{ae}{2}$ and get

$$(D-K_\Sigma)^2=-e\cdot(a+2)^2+2\cdot(a+2)\cdot(b+2+e-2g)=2\cdot\kappa_1\cdot\kappa_2. \tag{3.23}$$

**Step 1:** By Lemma G.1 $\Sigma$ satisfies the assumption (*) of Lemma 3.1, and by the assumptions on $a$ and $b$ we know that $D-K_\Sigma$ is ample and $D+K_\Sigma$ is nef, that is, (0) in Lemma 3.1 is fulfilled.

**Step 2a:** $\frac{\kappa_1}{4}\cdot\sum_{i=1}^m b_i'+\frac{\kappa_2}{4}\cdot\sum_{i=1}^m a_i\leq\deg(X_S)$.

Let us notice first that the strict inequality "<" in Lemma 3.1 (e) for ample divisors $H$ comes down to "≤" for nef divisors $H$. We may apply this for $H=C_0+\frac{e}{2}\cdot F$ and $H=F$ and deduce the following inequalities:

$$0\leq\left(D-K_\Sigma-\sum_{k=1}^i\Delta_k-\Delta_i\right).\big(C_0+\tfrac{e}{2}F\big)=\kappa_2-\sum_{k=1}^i b_k'-b_i', \tag{3.24}$$

[6] Remember that $\#X_S$ is the number of points in the support of $X_S$.

and

$$0 \le \left(D - K_\Sigma - \sum_{k=1}^{i} \Delta_k - \Delta_i\right).F = \kappa_1 - \sum_{k=1}^{i} a_k - a_i. \tag{3.25}$$

For the following consideration we choose $i_0, j_0 \in \{1, \dots, m\}$ such that $a_{i_0} \ge a_i$ for all $i = 1, \dots, m$ and $b'_{j_0} \ge b'_j$ for all $j = 1, \dots, m$. Then

$$\kappa_1 \ge \sum_{k=1}^{i_0} a_k + a_{i_0} \ge 2a_{i_0} \ge 2a_i \tag{3.26}$$

and

$$\kappa_2 \ge \sum_{k=1}^{j_0} b'_k + b'_{j_0} \ge 2b'_{j_0} \ge 2b'_j \tag{3.27}$$

for all $i, j = 1, \dots, m$, and we finally get

$$\begin{aligned}
\deg(X_S) &= \textstyle\sum_{i=1}^m \deg\left(X_i^0\right) \ \ge_{3.1\,(c)} \ \sum_{i=1}^m \left(D - K_\Sigma - \sum_{k=1}^i \Delta_k\right).\Delta_i \\
&= (D - K_\Sigma).\textstyle\sum_{i=1}^m \Delta_i - \sum_{1 \le k \le i \le 1} \Delta_k.\Delta_i \\
&= (D - K_\Sigma).\textstyle\sum_{i=1}^m \Delta_i - \frac{1}{2}\sum_{i=1}^m \Delta_i^2 - \frac{1}{2}\left(\sum_{i=1}^m \Delta_i\right)^2 \\
&= \kappa_1 \cdot \textstyle\sum_{i=1}^m b'_i + \kappa_2 \cdot \sum_{i=1}^m a_i - \sum_{i=1}^m a_i b'_i - \sum_{i=1}^m a_i \cdot \sum_{i=1}^m b'_i \\
&\ge_{(3.24)\,/\,(3.25)} \ \textstyle\frac{\kappa_1}{2} \cdot \sum_{i=1}^m b'_i + \frac{\kappa_2}{2} \cdot \sum_{i=1}^m a_i + \frac{1}{2} \cdot \left(\sum_{i=1}^m a_i + a_m\right) \cdot \sum_{i=1}^m b'_i \\
&\qquad \textstyle + \frac{1}{2} \cdot \left(\sum_{i=1}^m b'_i + b'_m\right) \cdot \sum_{i=1}^m a_i - \sum_{i=1}^m a_i b'_i - \sum_{i=1}^m a_i \cdot \sum_{i=1}^m b'_i \\
&= \textstyle\frac{\kappa_1}{2} \cdot \sum_{i=1}^m b'_i + \frac{\kappa_2}{2} \cdot \sum_{i=1}^m a_i + \frac{a_m}{2} \cdot \sum_{i=1}^m b'_i + \frac{b'_m}{2} \cdot \sum_{i=1}^m a_i - \sum_{i=1}^m a_i b'_i \\
&\ge_{(3.26)\,/\,(3.27)} \ \textstyle\frac{\kappa_1}{4} \cdot \sum_{i=1}^m b'_i + \frac{1}{4}\sum_{i=1}^m 2a_i b'_i + \frac{\kappa_2}{4} \cdot \sum_{i=1}^m a_i + \frac{1}{4}\sum_{i=1}^m 2a_i b'_i \\
&\qquad \textstyle + \frac{a_m}{2} \cdot \sum_{i=1}^m b'_i + \frac{b'_m}{2} \cdot \sum_{i=1}^m a_i - \sum_{i=1}^m a_i b'_i \\
&\ge \textstyle\frac{\kappa_1}{4} \cdot \sum_{i=1}^m b'_i + \frac{\kappa_2}{4} \cdot \sum_{i=1}^m a_i.
\end{aligned}$$

**Step 2b:** $\sum_{i=1}^m a_i \cdot \sum_{i=1}^m b'_i \le \frac{8}{(D-K_\Sigma)^2} \cdot \left(\deg(X_S)\right)^2$.

Using Step 2a and taking (3.23) into account, we deduce

$$\left(\deg(X_S)\right)^2 \ge \left(\frac{\kappa_2}{4} \cdot \sum_{i=1}^m a_i + \frac{\kappa_1}{4} \cdot \sum_{i=1}^m b'_i\right)^2 \ge \frac{4 \cdot \kappa_1 \cdot \kappa_2}{16} \cdot \sum_{i=1}^m a_i \cdot \sum_{i=1}^m b'_i.$$

**Step 2c:** $\left(\sum_{i=1}^m a_i\right)^2 \le \frac{32\alpha}{(D-K_\Sigma)^2}\left(\deg(X_S)\right)^2$ and $\left(\sum_{i=1}^m b'_i\right)^2 \le \frac{32}{\alpha \cdot (D-K_\Sigma)^2}\left(\deg(X_S)\right)^2$.

This follows from the following inequality with the aid of Step 2a and (3.23),

$$\begin{aligned}
\left(\deg(X_S)\right)^2 &\ge \left(\tfrac{\kappa_2}{4} \cdot \textstyle\sum_{i=1}^m a_i\right)^2 + \left(\tfrac{\kappa_1}{4} \cdot \textstyle\sum_{i=1}^m b'_i\right)^2 \\
&\ge \tfrac{2 \cdot \kappa_1 \cdot \kappa_2}{32\alpha} \cdot \left(\textstyle\sum_{i=1}^m a_i\right)^2 + \tfrac{2 \cdot \kappa_1 \cdot \kappa_2 \cdot \alpha}{32} \cdot \left(\textstyle\sum_{i=1}^m b'_i\right)^2.
\end{aligned}$$

**Step 2d:** $\sum_{i=1}^{m} a_i \le \begin{cases} \frac{2\alpha}{(D-K_\Sigma)^2} \cdot \big(\deg(X_S)\big)^2, & \text{if } \sum_{i=1}^m b_i' = 0, \\ \frac{8}{(D-K_\Sigma)^2} \cdot \big(\deg(X_S)\big)^2, & \text{else.} \end{cases}$

If $\sum_{i=1}^m b_i' = 0$, i. e. $b_i' = 0$ for all $i = 1, \ldots, m$, then the same consideration as in Step 2a shows

$$\deg(X_S) \ge \kappa_2 \cdot \sum_{i=1}^{m} a_i,$$

and thus

$$\frac{(D-K_\Sigma)^2}{2\alpha} \cdot \sum_{i=1}^{m} a_i \le \kappa_2^2 \cdot \left(\sum_{i=1}^{m} a_i\right)^2 \le \big(\deg(X_S)\big)^2.$$

If $\sum_{i=1}^m b_i' \neq 0$, then by Step 2b

$$\sum_{i=1}^{m} a_i \le \sum_{i=1}^{m} a_i \cdot \sum_{i=1}^{m} b_i' \le \frac{8}{(D-K_\Sigma)^2} \cdot \big(\deg(X_S)\big)^2.$$

**Step 3:** $h^1\big(\Sigma, \mathcal{J}_{X_0}(D)\big) \le 2 \cdot \sum_{i=1}^{m} a_i \cdot \sum_{i=1}^{m} b_i' + (2g-2) \cdot \sum_{i=1}^{m} a_i - 2 \cdot \sum_{i=1}^{m} b_i' + m.$

By Lemma 3.2 and since $\Delta_i.\Delta_j \ge 0$ for any $i, j \in \{1, \ldots, m\}$ we have:

$$\begin{aligned} h^1\big(\Sigma, \mathcal{J}_{X_0}(D)\big) &\le \sum_{i=1}^m \Big(\Delta_i \cdot \big(K_\Sigma + \textstyle\sum_{k=1}^i \Delta_k\big) + 1\Big) \\ &\le K_\Sigma \cdot \textstyle\sum_{i=1}^m \Delta_i + \big(\sum_{i=1}^m \Delta_i\big)^2 + m \\ &= (2g-2) \cdot \textstyle\sum_{i=1}^m a_i - 2 \cdot \sum_{i=1}^m b_i' + 2 \cdot \sum_{i=1}^m a_i \cdot \sum_{i=1}^m b_i' + m. \end{aligned}$$

**Step 4a:** If $e = 0$, we find the estimation

$$\sum_{i=1}^{m} \Big(h^0\big(\Sigma, \mathcal{O}_\Sigma(\Delta_i)\big) - 1\Big) \le \begin{cases} \sum_{i=1}^{m} a_i \cdot \sum_{i=1}^{m} b_i' + \sum_{i=1}^{m} b_i' - m, & \text{if } g = 1, \sum_{i=1}^m b_i' \neq 0, \\ \sum_{i=1}^{m} a_i \cdot \sum_{i=1}^{m} b_i' + \sum_{i=1}^{m} b_i' = 0, & \text{if } g = 1, \sum_{i=1}^m b_i' = 0, \\ \sum_{i=1}^{m} a_i \cdot \sum_{i=1}^{m} b_i' + \sum_{i=1}^{m} a_i + \sum_{i=1}^{m} b_i', & \text{for } g \text{ arbitrary.} \end{cases}$$

We note that in this case $b_i = b_i'$ and that $b_i' = 0$ thus implies $a_i > 0$. By Corollary G.5 we have

$$h^0\big(\Sigma, \mathcal{O}_\Sigma(\Delta_i)\big) \le \begin{cases} a_i b_i' + b_i', & \text{if } g = 1, b_i' > 0, \\ a_i b_i' + b_i' + 1 = 1, & \text{if } g = 1, b_i' = 0, \\ a_i b_i' + a_i + b_i' + 1, & \text{else.} \end{cases}$$

The results for $g$ arbitrary respectively $g = 1$ and $\sum_{i=1}^m b_i' = 0$ thus follow right away. If, however, some $b_{i_0}' > 0$, then

$$\sum_{i \neq j} a_i b_j \ge b_{i_0} \sum_{i \neq i_0} a_i \ge \#\big\{b_i' \mid b_i' = 0\big\}$$

and hence

$$h^0\big(\Sigma,\mathcal{O}_\Sigma(\Delta_i)\big)\le\sum_{i=1}^m a_ib_i'+\sum_{i=1}^m b_i'+\#\big\{b_i'\mid b_i'=0\big\}$$

$$=\sum_{i=1}^m a_i\cdot\sum_{i=1}^m b_i'+\sum_{i=1}^m b_i'+\#\big\{b_i'\mid b_i'=0\big\}-\sum_{i\neq j}a_ib_j'\le\sum_{i=1}^m a_i\cdot\sum_{i=1}^m b_i'+\sum_{i=1}^m b_i'.$$

**Step 4b:** If $e<0$, we give several upper bounds for $\beta=\sum_{i=1}^m\Big(h^0\big(\Sigma,\mathcal{O}_\Sigma(\Delta_i)\big)-1\Big)$:

$$\beta\le\begin{cases}\frac{1}{2}\sum_{i=1}^m a_i\sum_{i=1}^m b_i'+\frac{1}{2}\left(\sum_{i=1}^m b_i'\right)^2+\frac{1}{8}\left(\sum_{i=1}^m a_i\right)^2+\frac{1}{4}\sum_{i=1}^m a_i+\frac{1}{2}\sum_{i=1}^m b_i', & \text{if } g=1,\\ \sum_{i=1}^m a_i\sum_{i=1}^m b_i'+\sum_{i=1}^m a_i+\sum_{i=1}^m b_i'-\frac{9e}{32}\left(\sum_{i=1}^m a_i\right)^2, & \text{for } g \text{ arbitrary.}\\ \frac{1}{4}\sum_{i=1}^m a_i\sum_{i=1}^m b_i'+\sum_{i=1}^m a_i+\sum_{i=1}^m b_i'-\frac{9e}{32}\left(\sum_{i=1}^m a_i\right)^2-\frac{1}{2e}\left(\sum_{i=1}^m b_i'\right)^2, & g \text{ arbitrary.}\end{cases}$$

If $g$ is arbitrary, the claim follows since by Corollary G.5 we have

$$h^0\big(\Sigma,\mathcal{O}_\Sigma(\Delta_i)\big)\le a_ib_i'+a_i+b_i'+1-\tfrac{9e}{32}\cdot a_i^2$$

and

$$h^0\big(\Sigma,\mathcal{O}_\Sigma(\Delta_i)\big)\le\tfrac{1}{4}\cdot a_ib_i'+a_i+b_i'+1-\tfrac{9e}{32}\cdot a_i^2-\tfrac{1}{2e}\cdot b_i'^2.$$

If $g=1$, then $e=-1$ and $b'=b+\frac{a}{2}$. We may once more apply Corollary G.5 and see that in any case

$$h^0\big(\Sigma,\mathcal{O}_\Sigma(\Delta_i)\big)\le a_ib_i+b_i+1+\tfrac{a_i(a_i+1)}{2}+\tfrac{b_i(b_i-1)}{2}$$

$$=\tfrac{1}{2}\cdot a_ib_i'+\tfrac{1}{2}\cdot b_i'^2+\tfrac{1}{8}\cdot a_i^2+\tfrac{1}{4}\cdot a_i+\tfrac{1}{2}\cdot b_i'+1,$$

which finishes the case $g=1$.

**Step 5:** In this last step we gather the information from the previous investigations and finish the proof considering a bunch of different cases.

Using Step 3 and Step 4 and taking $\sum_{i=1}^m a_i+b_i'\le m$ into account, we get the following upper bounds for $\beta'=h^1\big(\Sigma,\mathcal{J}_{X_0}(D)\big)+\sum_{i=1}^m\Big(h^0\big(\Sigma,\mathcal{O}_\Sigma(\Delta_i)\big)-1\Big)$

$$\beta'\le\begin{cases}3\sum_{i=1}^m a_i\sum_{i=1}^m b_i'+2g\sum_{i=1}^m a_i, & \text{if } e=0,\\ 3\sum_{i=1}^m a_i\sum_{i=1}^m b_i'+2g\sum_{i=1}^m a_i-\frac{9e}{32}\left(\sum_{i=1}^m a_i\right)^2, & \text{if } e<0,\\ \frac{9}{4}\sum_{i=1}^m a_i\sum_{i=1}^m b_i'+2g\sum_{i=1}^m a_i-\frac{9e}{32}\left(\sum_{i=1}^m a_i\right)^2-\frac{1}{2e}\left(\sum_{i=1}^m b_i'\right)^2, & \text{if } e<0,\\ 3\sum_{i=1}^m a_i\sum_{i=1}^m b_i', & \text{if } e=0, g=1, \sum_{i=1}^m b_i'\neq 0,\\ m\le\sum_{i=1}^m a_i, & \text{if } e=0, g=1, \sum_{i=1}^m b_i'=0,\\ \frac{5}{2}\sum_{i=1}^m a_i\sum_{i=1}^m b_i'+\frac{1}{2}\left(\sum_{i=1}^m b_i'\right)^2+\frac{1}{8}\left(\sum_{i=1}^m a_i\right)^2+\frac{5}{4}\sum_{i=1}^m a_i, & \text{if } e<0, g=1.\end{cases}$$

Applying now the results of Step 2b-2d we get

$$\frac{\beta'\cdot(D-K_\Sigma)^2}{\big(\deg(X_S)\big)^2} \leq \begin{cases} 24, & \text{if } g=0,\\ 24, & \text{if } g=1, e=0, \sum_{i=1}^m b_i'\neq 0,\\ 2\alpha, & \text{if } g=1, e=0, \sum_{i=1}^m b_i'=0,\\ \min\left\{30+\frac{16}{\alpha}+4\alpha, 40+9\alpha\right\}, & \text{if } g=1, e<0, \sum_{i=1}^m b_i'\neq 0,\\ \frac{13}{2}\alpha, & \text{if } g=1, e<0, \sum_{i=1}^m b_i'=0,\\ 24+16g, & \text{if } g\geq 2, e=0, \sum_{i=1}^m b_i'\neq 0,\\ 4g\alpha, & \text{if } g\geq 2, e=0, \sum_{i=1}^m b_i'=0,\\ \min\left\{24+16g-9e\alpha, 18+16g-9e\alpha-\frac{16}{e\alpha}\right\}, & \\ & \text{if } g\geq 2, e<0, \sum_{i=1}^m b_i'\neq 0,\\ 4g\alpha-9e\alpha, & \text{if } g\geq 2, e<0, \sum_{i=1}^m b_i'=0. \end{cases}$$

Hence we have

$$\frac{\beta'\cdot(D-K_\Sigma)^2}{\big(\deg(X_S)\big)^2} \leq \begin{cases} 24, & \text{if } g=0,\\ \max\{24, 2\alpha\}, & \text{if } g=1, e=0,\\ \max\left\{\min\left\{30+\frac{16}{\alpha}+4\alpha, 40+9\alpha\right\}, \frac{13}{2}\alpha\right\}, & \text{if } g=1, e<0,\\ \max\{24+16g, 4g\alpha\} & \text{if } g\geq 2, e=0,\\ \max\left\{\min\left\{24+16g-9e\alpha, 18+16g-9e\alpha-\frac{16}{e\alpha}\right\}, 4g\alpha-9e\alpha\right\}, & \\ & \text{if } g\geq 2, e<0. \end{cases}$$

We thus finally get

$$\begin{aligned} h^1\big(\Sigma,\mathcal{J}_{X_0}(D)\big)+\textstyle\sum_{i=1}^m\Big(h^0\big(\Sigma,\mathcal{O}_\Sigma(\Delta_i)\big)-1\Big) &= \beta' \leq \frac{1}{\gamma\cdot(D-K_\Sigma)^2}\cdot\big(\deg(X_S)\big)^2\\ &= \frac{1}{\gamma\cdot(D-K_\Sigma)^2}\cdot\big(\textstyle\sum_{z\in\Sigma}\deg(X_{S,z})\big)^2 \leq \frac{\#X_S}{\gamma\cdot(D-K_\Sigma)^2}\cdot\textstyle\sum_{z\in\Sigma}\deg(X_{S,z})^2\\ &\leq \frac{\#X_S}{\gamma\cdot(D-K_\Sigma)^2}\cdot\textstyle\sum_{z\in\Sigma}\deg(X_{0,z})^2 \;<_{(4)}\; \#X_S. \end{aligned}$$

□

It remains to show, that the inequality which we derived in the above cases cannot hold.

**3.7 Lemma**

*Let* $D\in\mathrm{Div}(\Sigma)$, $\mathcal{S}_1,\ldots,\mathcal{S}_r$ *be topological or analytical singularity types. Suppose that* $V_{|D|}^{irr,fix}(\mathcal{S}_1,\ldots,\mathcal{S}_r)$ *is non-empty.*

*Then there exists no curve*[7] $C \in V^{irr}_{|D|}(\mathcal{S}_1,\ldots,\mathcal{S}_r)\setminus\overline{V^{irr,fix}_{|D|}(\mathcal{S}_1,\ldots,\mathcal{S}_r)}$ *such that for the zero-dimensional scheme* $X_0 = X^{es}_{fix}(C)$ *respectively* $X_0 = X^{ea}_{fix}(C)$ *there exist curves* $\Delta_1,\ldots,\Delta_m \subset \Sigma$ *and zero-dimensional locally complete intersections* $X^0_i \subseteq X_{i-1}$ *for* $i = 1,\ldots,m$*, where* $X_i = X_{i-1} : \Delta_i$ *for* $i = 1,\ldots,m$ *such that* $X_S = \bigcup_{i=1}^m X^0_i$ *satisfies*

$$h^1\big(\Sigma,\mathcal{J}_{X_0}(D)\big) + \sum_{i=1}^m \Big(h^0\big(\Sigma,\mathcal{O}_\Sigma(\Delta_i)\big)-1\Big) < \#X_S. \tag{3.28}$$

**Proof:** Throughout the proof we use the notation $V^{irr} = V^{irr}_{|D|}(\mathcal{S}_1,\ldots,\mathcal{S}_r)$ and $V^{irr,fix} = V^{irr,fix}_{|D|}(\mathcal{S}_1,\ldots,\mathcal{S}_r)$.

Suppose there exists a curve $C \in V^{irr}\setminus\overline{V^{irr,fix}}$ satisfying the assumption of the Lemma, and let $V^*$ be the irreducible component of $V^{irr}$ containing $C$. Moreover, let $C_0 \in V^{irr,fix}$.

We consider in the following the morphism

$$\Phi = \Phi_D(\mathcal{S}_1,\ldots,\mathcal{S}_r) : V_{|D|}(\mathcal{S}_1,\ldots,\mathcal{S}_r) \to \mathrm{Sym}^r(\Sigma) =: \mathcal{B}$$

from Definition I.2.16.

**Step 1:** $h^0\big(\Sigma,\mathcal{J}_{X^*_{fix}(C_0)/\Sigma}(D)\big) = h^0\big(\Sigma,\mathcal{J}_{X^*_{fix}(C)/\Sigma}(D)\big) - h^1\big(\Sigma,\mathcal{J}_{X^*_{fix}(C)/\Sigma}(D)\big)$.

By the choice of $C_0$ we have

$$0 = H^1\big(\Sigma,\mathcal{J}_{X^*_{fix}(C_0)/\Sigma}(D)\big) \to H^1\big(\Sigma,\mathcal{O}_\Sigma(D)\big) \to H^1\big(\Sigma,\mathcal{O}_{X^*_{fix}(C_0)}(D)\big) = 0,$$

and thus $D$ is non-special, i. e. $h^1\big(\Sigma,\mathcal{O}_\Sigma(D)\big) = 0$. But then

$$\begin{aligned} h^0\big(\Sigma,\mathcal{J}_{X^*_{fix}(C_0)/\Sigma}(D)\big) &= h^0\big(\Sigma,\mathcal{O}_\Sigma(D)\big) - \deg\big(X^*_{fix}(C_0)\big) \\ &= h^0\big(\Sigma,\mathcal{O}_\Sigma(D)\big) - \deg\big(X^*_{fix}(C)\big) \\ &= h^0\big(\Sigma,\mathcal{J}_{X^*_{fix}(C)/\Sigma}(D)\big) - h^1\big(\Sigma,\mathcal{J}_{X^*_{fix}(C)/\Sigma}(D)\big). \end{aligned}$$

**Step 2:** $h^1\big(\Sigma,\mathcal{J}_{X^*_{fix}(C)}(D)\big) \geq \mathrm{codim}_{\mathcal{B}}\Big(\Phi\big(V^*\big)\Big)$.

Suppose the contrary, that is $\dim\Big(\Phi\big(V^*\big)\Big) < \dim(\mathcal{B}) - h^1\big(\Sigma,\mathcal{J}_{X^*_{fix}(C)/\Sigma}(D)\big)$, then by Step 1, Remark I.2.15 and Equation (I.2.3) in Remark I.2.17

$$\begin{aligned} \dim\big(V^*\big) &\leq \dim\Big(\Phi\big(V^*\big)\Big) + \dim\Big(\Phi^{-1}\big(\Phi(C)\big)\Big) \\ &< \dim(\mathcal{B}) - h^1\big(\Sigma,\mathcal{J}_{X^*_{fix}(C)/\Sigma}(D)\big) + h^0\big(\Sigma,\mathcal{J}_{X^*_{fix}(C)/\Sigma}(D)\big) - 1 \\ &= 2\cdot r + h^0\big(\Sigma,\mathcal{J}_{X^*_{fix}(C_0)/\Sigma}(D)\big) - 1 = \dim\big(V^{irr,fix}\big). \end{aligned}$$

However, any irreducible component of $V^{irr}$ has at least the expected dimension $\dim\big(V^{irr,fix}\big)$, which gives a contradiction.

**Step 3:** $\mathrm{codim}_{\mathcal{B}}\Big(\Phi\big(V^*\big)\Big) \geq \#X_S - \sum_{i=1}^m \dim|\Delta_i|_l$.

[7] For a subset $U \subseteq V$ of a topological space $V$ we denote by $\overline{U}$ the closure of $U$ in $V$.

The existence of the subschemes $X_i^0 \subseteq X^*_{fix}(C) \cap \Delta_i$ imposes at least $\#X_i^0 - \dim|\Delta_i|_l$ conditions on $X^*_{fix}(C)$ and increases thus the codimension of $\Phi\big(V^*\big)$ by the same number. To see this, let $k_i = \dim|\Delta_i|_l$ and let $H^0(\Sigma, \Delta_i)$ be spanned by the linearly independent sections $s_{i,0}, \ldots, s_{i,k_i}$. Then for $k_i$ general points $p_1, \ldots, p_{k_i}$ in $\Sigma$ the linear system of equations $a_0 s_{i,0}(p_j) + \ldots + a_{k_i} s_{i,k_i}(p_j) = 0$, $j = 1, \ldots, k_i$, has a one-dimensional solution set, hence there is a unique curve $C$ in $|\Delta_i|_l$ through $p_1, \ldots, p_{k_i}$. But then for the remaining $k_i' = \#X_i^0 - k_i$ points which must lie on $C$ as well there is only one degree of freedom left instead of two. Hence the dimension of $\mathcal{B}^*$ is at most the dimension of $\mathcal{B}$, which is $2r$, lowered by $\sum_{i=1}^m k_i'$.

**Step 4:** Derive a contradiction.

Collecting the results we derive the following contradiction:

$$h^1\big(\Sigma, \mathcal{J}_{X^*_{fix}(C)}(D)\big) \geq_{\text{Step 2}} \text{codim}_{\mathcal{B}}\Big(\Phi\big(V^*\big)\Big)$$
$$\geq_{\text{Step 3}} \#X_S - \textstyle\sum_{i=1}^m \dim|\Delta_i|_l >_{(3.28)} h^1\big(\Sigma, \mathcal{J}_{X^*_{fix}(C)}(D)\big).$$

□

The following two lemmata provide conditions which ensure that $V^{irr,reg}$ and $V^{irr,fix}$ share some dense subset $V_U^{gen}$, and thus that $V^{irr,reg}$ is dense in $V^{irr,fix}$.

**3.8 Lemma**

*Let $\mathcal{S}_1, \ldots, \mathcal{S}_r$ be topological or analytical singularity types, let $D \in \text{Div}(\Sigma)$ and let $V^{irr} = V^{irr}_{|D|}(\mathcal{S}_1, \ldots, \mathcal{S}_r)$.*

*There exists a very general subset $U \subset \Sigma^r$ such that*[8] $V_U^{gen} = V^{gen}_{|D|,U}(\mathcal{S}_1, \ldots, \mathcal{S}_r) = \big\{C \in V^{irr} \;\big|\; \underline{z} \in U, (C, z_i) \sim S_i, i = 1, \ldots, r\big\}$ *is dense in* $V^{irr,fix}_{|D|}(\mathcal{S}_1, \ldots, \mathcal{S}_r)$.

**Proof:** This follows from Remark I.2.17 (b). □

**3.9 Lemma**

*With the notation of Lemma 3.8 we assume that*

(a) $(D - K_\Sigma)^2 \geq 2 \cdot \sum_{i=1}^k \big(\nu^*(\mathcal{S}_i) + 2\big)^2$,

(b) $(D - K_\Sigma).B > \max\big\{\nu^*(\mathcal{S}_i) + 1 \;\big|\; i = 1, \ldots, r\big\}$ *for any irreducible curve* $B$ *with* $B^2 = 0$ *and* $\dim|B|_a > 0$, *and*

(c) $D - K_\Sigma$ *is nef.*

*Then there exists a very general subset* $U \subset \Sigma^r$ *such that* $V_U^{gen} \subseteq V^{irr,reg}_{|D|}(\mathcal{S}_1, \ldots, \mathcal{S}_r)$.

**Proof:** By Theorem II.1.1 we know that there is a very general subset $U \subset \Sigma^r$ such that for $\underline{z} \in U$ and $\underline{\nu} = \big(\nu^*(\mathcal{S}_1) + 1, \ldots, \nu^*(\mathcal{S}_r) + 1\big)$ we have

$$h^1\big(\Sigma, \mathcal{J}_{X(\underline{\nu};\underline{z})/\Sigma}(D)\big) = 0.$$

[8]Here $\sim$ means either topological equivalence $\sim_t$ or contact equivalence $\sim_c$.

However, if $C \in V^{irr}$ and $\underline{z} \in U$ with $(C, z_i) \sim \mathcal{S}_i$, then by the definition of $\nu^*(\mathcal{S}_i)$ we have

$$\mathcal{J}_{X(\underline{\nu};\underline{z})/\Sigma} \hookrightarrow \mathcal{J}_{X(C)/\Sigma},$$

and hence the vanishing of $H^1\big(\Sigma, \mathcal{J}_{X(\underline{\nu};\underline{z})/\Sigma}(D)\big)$ implies $h^1\big(\Sigma, \mathcal{J}_{X(C)/\Sigma}(D)\big) = 0$, i. e. $C \in V^{irr,reg}$. □

In Section 4 we would like to combine the irreducibility results with the existence results from Chapter III. Due to the following lemma this basically comes down to show that for a suitable very ample line bundle $L$ we have (cf. Corollary III.2.4)

$$\gamma \cdot (D - K_\Sigma)^2 \le \tfrac{1}{2} \cdot (D - K_\Sigma - L)^2. \tag{3.29}$$

**3.10 Lemma**

*Let $\mathcal{S}$ be a topological or analytical singularity type, then*

$$\big(e^*(\mathcal{S}) + 1\big)^2 \le \big(\tau^*(\mathcal{S}) + 2\big)^2.$$

**Proof:** Let us first consider the case that $\mathcal{S}$ is a simple singularity. Then

$$e^*(\mathcal{S}) = e^s(\mathcal{S}) = e^a(\mathcal{S}), \quad \mu(\mathcal{S}) = \tau(\mathcal{S}) = \tau^{es}(\mathcal{S}) = \tau^*(\mathcal{S}), \quad r(\mathcal{S}) \le \text{mult}(\mathcal{S}) \le 3$$

and thus

$$2\delta(\mathcal{S}) = \mu(\mathcal{S}) + r(\mathcal{S}) - 1 \le \mu(\mathcal{S}) + 2 = \tau^*(\mathcal{S}) + 2.$$

But then, once $\tau^*(\mathcal{S}) \ge 5$, we have

$$\big(e^*(\mathcal{S}) + 1\big)^2 \le \tfrac{625}{48} \cdot \delta(\mathcal{S}) \le \tfrac{625}{48 \cdot 2 \cdot 7} \cdot \big(\tau^*(\mathcal{S}) + 2\big)^2 \le \big(\tau^*(\mathcal{S}) + 2\big)^2,$$

while for $\tau^*(\mathcal{S}) \le 4$, that is for $\mathcal{S} \in \{A_1, A_2, A_3, A_4, D_4\}$ we know $e^*(\mathcal{S})$ precisely and the inequality is fulfilled as well. (Cf. Remark III.2.2.)

We may thus suppose that $\mathcal{S}$ is not simple, i. e. $\text{mod}(\mathcal{S}) \ge 1$.

Let us first consider the case of a topological singularity type. A simple calculation shows that for $\tau^{es}(\mathcal{S}_i) \ge 9$ we always have

$$\tfrac{625}{48} \cdot \tau^{es}(\mathcal{S}_i) \le \big(\tau^{es}(\mathcal{S}) + 2\big)^2.$$

Moreover, from Remark III.2.2 and Remark I.2.3 we thus deduce

$$\big(e^s(\mathcal{S}) + 1\big)^2 \le \tfrac{625}{48} \cdot \delta(\mathcal{S}) \le \tfrac{625}{48} \cdot \tau^{es}(\mathcal{S}_i) \le \big(\tau^{es}(\mathcal{S}) + 2\big)^2.$$

However, there is only one non-simple topological singularity type with $\tau^{es}(\mathcal{S}_i) \le 8$, namely $X_9$, that is four lines through one point (cf. Remark I.2.5). But then

$$\delta(\mathcal{S}_i) = \tfrac{\mu(\mathcal{S}_i) + r(\mathcal{S}_i) - 1}{2} = 6,$$

and thus again

$$\big(e^s(\mathcal{S}) + 1\big)^2 \le \tfrac{625}{48} \cdot \delta(\mathcal{S}) \le \tfrac{625 \cdot 6}{48} \le 100 = \big(\tau^{es}(\mathcal{S}) + 2\big)^2.$$

We, therefore may turn to the case of non-simple analytical singularity types. Since $\mathcal{S}$ is not simple, we know that $\text{mult}(\mathcal{S}) \geq 3$ and $\mu(\mathcal{S}) \geq \tau(\mathcal{S}) \geq 9$. Moreover, we have

$$2\tau(\mathcal{S}) \geq 2\tau^*(\mathcal{S}) \geq \kappa(\mathcal{S}) = \mu(\mathcal{S}) + \text{mult}(\mathcal{S}) - 1,$$

and thus

$$\mu(\mathcal{S}) \leq 2\tau(\mathcal{S}) - \text{mult}(\mathcal{S}) + 1 \leq 2\tau(\mathcal{S}) - 2.$$

Thus, once $\tau(\mathcal{S}) \geq 13$, we have due to Remark III.2.2

$$\big(e^a(\mathcal{S}) + 1\big)^2 \leq 9\mu(\mathcal{S}) \leq 18\tau(\mathcal{S}) - 18 \leq \big(\tau(\mathcal{S}) + 2\big)^2,$$

and we may suppose that $9 \leq \tau(\mathcal{S}) \leq 12$.

Let us first consider the case that $\text{mod}(\mathcal{S}) \in \{1, 2\}$. Then

$$\mu(\mathcal{S}) \leq \tau(\mathcal{S}) + \text{mod}(\mathcal{S}) \leq \tau(\mathcal{S}) + 2,$$

and hence

$$\big(e^a(\mathcal{S}) + 1\big)^2 \leq 9\mu(\mathcal{S}) \leq 9 \cdot \big(\tau(\mathcal{S}) + 2\big) \leq \big(\tau(\mathcal{S}) + 2\big)^2.$$

It thus remains to consider the case $\text{mod}(\mathcal{S}) \geq 3$, and therefore we also have $\mu(\mathcal{S}) \geq 16$ (cf. Remark I.2.5). If $\text{mod}(\mathcal{S}) = 3$, then $\tau(\mathcal{S}) \geq \mu(\mathcal{S}) - \text{mod}(\mathcal{S}) \geq 16 - 3 = 13$ and we are done. Thus indeed $\text{mod}(\mathcal{S}) \geq 4$.

We do the remaining part be considering the cases $\text{mult}(\mathcal{S}) \geq 5$, $\text{mult}(\mathcal{S}) = 4$ and $\text{mult}(\mathcal{S}) = 3$ seperately.

If $\text{mult}(\mathcal{S}) \geq 5$, then

$$\tau(\mathcal{S}) \geq \frac{\mu(\mathcal{S}) + \text{mult}(\mathcal{S}) - 1}{2} \geq \tfrac{16+5-1}{2} = 10,$$

and therefore

$$\big(e^a(\mathcal{S}) + 1\big)^2 \leq 9\mu(\mathcal{S}) \leq 9 \cdot \big(2\tau(\mathcal{S}) - 4\big) \leq \big(\tau(\mathcal{S}) + 2\big)^2.$$

If $\text{mult}(\mathcal{S}) = 4$, then by Remark I.2.5 $\mu(\mathcal{S}) \geq 22$ and therefore we are done by the following inequality:

$$\tau(\mathcal{S}) \geq \frac{\mu(\mathcal{S}) + \text{mult}(\mathcal{S}) - 1}{2} \geq \tfrac{22+4-1}{2} = 12\tfrac{1}{2}.$$

If, finally $\text{mult}(\mathcal{S}) = 3$, then $\mu(\mathcal{S}) \geq 28$ and by the same reasoning we see that $\tau(\mathcal{S}) \geq 15$ and this finishes the proof. □

## 4. Examples

### 4.a. The Classical Case - $\Sigma = \mathbb{P}^2_{\mathbb{C}}$

Since any curve in $\mathbb{P}^2_{\mathbb{C}}$ is non-special and since the Picard number is of course one, the assumptions of Theorem 2.1 are fulfilled. In view of Remark 2.2 the theorem thus reads in this situation.

**2.1a Theorem**

*Let* $d \geq 3$*,* $L \subset \mathbb{P}^2_{\mathbb{C}}$ *a line, and let* $\mathcal{S}_1, \ldots, \mathcal{S}_r$ *be topological or analytical singularity types.*

*Suppose that*

$$\sum_{i=1}^{r} \big(\tau^*(\mathcal{S}_i) + 2\big)^2 < \tfrac{90}{289} \cdot (d+3)^2. \tag{2.4a}$$

*Then* $V^{irr}_{|dL|}(\mathcal{S}_1, \ldots, \mathcal{S}_r)$ *is non-empty, irreducible and T-smooth.*

**Proof:** It remains to show that $V^{irr}_{|dL|}(\mathcal{S}_1, \ldots, \mathcal{S}_r)$ is not empty and T-smooth. However, since $d \geq 3$ we have

$$\tfrac{90}{289} \cdot (d+3) \leq \tfrac{1}{2} \cdot (d+2) \leq d^2 + 6d$$

and, in view of Lemma 3.10, Corollary III.2.4 applies, as does Theorem IV.1.1a. □

Many authors were concerned with the question in the case of nodes and cusps, or of nodes and one more complicated singularity, or simply of ordinary multiple points – cf. e. g. [**Sev21, BrG81, ArC83, Har85b, Kan89a, Kan89b, Ran89, Shu91b, Shu91a, Bar93a, Shu94, Shu96b, Shu96a, Wal96, GLS98a, GLS98b, Los98, Bru99, GLS00**]. Using particularly designed techniques they get of course better results than we may expect to.

The best general results in this case can be found in [**GLS00**] (see also [**Los98**] Corollary 6.1). Given a plane curve of degree $d$, omitting nodes and cusps, they get

$$\sum_{i=1}^{r} \big(\tau^*(\mathcal{S}_i) + 2\big)^2 \leq \tfrac{9}{10} \cdot d^2$$

as main irreducibility condition. The coefficients differ by a factor of about 3.

## 4.b. Geometrically Ruled Surfaces

Throughout this section we use the notation of Section G.a. In particular, $\pi : \Sigma \to C$ is geometrically ruled surface with $g = g(C)$.

In [**Ran89**] and in [**GLS98a**] the case of nodal curves on the Hirzebruch surface $\mathbb{F}_1$ is treated, since this is just $\mathbb{P}^2_{\mathbb{C}}$ blown up in one point. $\mathbb{F}_1$ is an example of a geometrically ruled surface with invariant $e = 1 > 0$, a case which we cannot treat with the above methods, due to the section with self-intersection $-1$.

So far, we are only able to give a general result for geometrically ruled surfaces with invariant $e \leq 0$ – see Theorem 2.4. In the case where $g = 0$, i. e. where $\Sigma$ is rational, $\mathbb{F}_0 = \mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$ is the only surface for which we get any result; and if $g = 1$, by [**Har77**] V.2.15 we get apart form the product $C \times \mathbb{P}^1_{\mathbb{C}}$ precisely two other surfaces – one with invariant $e = 0$ and one with $e = -1$. For $g \geq 2$ the classes become larger.

**2.4b Theorem**

*Let* $\pi : \Sigma \to C$ *be a geometrically ruled surface with* $e \leq 0$.

*Let* $\mathcal{S}_1, \ldots, \mathcal{S}_r$ *be topological or analytical singularity types, and let* $D \in \mathrm{Div}(\Sigma)$ *such that* $D \sim_a aC_0 + bF$ *with* $a \geq \max\{2, \nu^*(\mathcal{S}_i) \mid i = 1, \ldots, r\}$*, and, if* $g = 0$*,* $b \geq \max\{2, \nu^*(\mathcal{S}_i) \mid i = 1, \ldots, r\}$*, while if* $g > 0$*,* $b > 2g - 2 + \frac{ae}{2}$.

*Suppose that*

$$\sum_{i=1}^{r} \big(\tau^*(\mathcal{S}_i) + 2\big)^2 < \gamma \cdot (D - K_\Sigma)^2, \tag{2.6b}$$

*where* $\gamma$ *may be taken from the following table with* $\alpha = \frac{a+2}{b+2-2g-\frac{ae}{2}} > 0$.

| $g$ | $e$ | $\gamma$ | $\gamma$, *if* $\alpha = 1$ |
|---|---|---|---|
| $0$ | $0$ | $\frac{1}{24}$ | $\frac{1}{24}$ |
| $1$ | $0$ | $\frac{1}{\max\{24, 2\alpha\}}$ | $\frac{1}{24}$ |
| $1$ | $-1$ | $\frac{1}{\max\left\{\min\left\{30+\frac{16}{\alpha}+4\alpha, 40+9\alpha\right\}, \frac{13}{2}\alpha\right\}}$ | $\frac{1}{49}$ |
| $\geq 2$ | $0$ | $\frac{1}{\max\{24+16g, 4g\alpha\}}$ | $\frac{1}{24+16g}$ |
| $\geq 2$ | $< 0$ | $\frac{1}{\max\left\{\min\left\{24+16g-9e\alpha, 18+16g-9e\alpha-\frac{16}{e\alpha}\right\}, 4g\alpha-9e\alpha\right\}}$ | |

*Then either* $V_{|D|}^{irr}(\mathcal{S}_1, \ldots, \mathcal{S}_r)$ *is empty or irreducible of the expected dimension.*

**Addendum 1:** *If we moreover suppose that* $a \geq e^*(\mathcal{S}_i)$ *and* $b \geq \max\{2g - 2 + \frac{ae}{2} + \frac{3}{2} \cdot (l - \frac{e}{2}), 2g - 2 + \frac{ae}{2} + (l + \frac{ae}{2}) + e^*(\mathcal{S}_i) + 1\}$*,* $l = 1$*, if* $g = 0$*, and else* $l \geq 2$ *such that such that* $L = C_0 + lF$ *is very ample, then* $V_{|D|}^{irr}(\mathcal{S}_1, \ldots, \mathcal{S}_r)$ *is non-empty.*

**Addendum 2:** *If* $g = 0$*,* $e = 0$*,* $\frac{24}{47} \cdot a \leq b \leq \frac{47}{24} \cdot a$*, or if* $g = 1$*,* $e = 0$*,* $b \leq \frac{23}{12} \cdot a + \frac{23}{6}$*, then* $V_{|D|}^{irr}(\mathcal{S}_1, \ldots, \mathcal{S}_r)$ *is also T-smooth.*

**Proof:** First of all we note that for $g > 0$ we have

$$\big(\tau^*(\mathcal{S}_i) + 2\big)^2 < \tfrac{1}{2\alpha} \cdot (D - K_\Sigma)^2 = \big(b + 2 - 2g - \tfrac{ae}{2}\big)^2,$$

and thus

$$b + 2 - 2g - \tfrac{ae}{2} > \tau^*(\mathcal{S}_i) + 2 > \nu^*(\mathcal{S}_i) + 1.$$

Therefore, the conditions of Theorem 2.4 are fulfilled.

Addendum 2 follows from Corollary IV.2.2 and Corollary IV.2.5, since by the assumptions on $a$ and $b$ we have for $g = 0 = e$

$$\tfrac{1}{24} \cdot (D - K_\Sigma) = \tfrac{1}{12} \cdot (a + 2) \cdot (b + 2) \leq 4ab + 4a + 4b - a^2 - b^2$$

respectively for $g = 1$ and $e = 0$ we have $b \in \left[1, a + \sqrt{a^2 + 4a} + 2\right[$ and

$$\tfrac{1}{24} \cdot (D - K_\Sigma) = \tfrac{1}{12} \cdot (ab + 2b) \leq 2ab + 4b - b^2.$$

It remains to prove Addendum 1 with the aid of Corollary III.2.4.

The assumptions on $b$ can be reformulated as

$$b - 2g + 2 - \tfrac{ae}{2} \geq \tfrac{3}{2} \cdot \big(l - \tfrac{e}{2}\big) \tag{4.1}$$

and

$$(D - L - K_\Sigma).C_0 = b - 2g + 2 - ae - l \geq e^*(\mathcal{S}_i) + 1.$$

Since moreover $(D - L - K_\Sigma).F = a + 1 > e^*(\mathcal{S}_i)$, Condition (III.2.4) is satisfied, and at the same time

$$\begin{aligned}(D - L - K_\Sigma).L &= (D - L - K_\Sigma).C_0 + l \cdot (D - L - K_\Sigma).F \\ &\geq e^*(\mathcal{S}_i) + 1 + l \cdot \big(e^*(\mathcal{S}_i) + 1\big) \geq 2 \cdot e^*(\mathcal{S}_i) + 2 > e^*(\mathcal{S}_i) + 2,\end{aligned}$$

which implies Condition (III.2.5). From (4.1) we deduce that

$$b - 2g + 2 + e - l \geq \tfrac{a+1}{2} \cdot e,$$

and hence, since also $a+1 \geq 0$, $D-L-K_\Sigma$ is nef. It remains to verify Condition (III.2.3), which in view of Lemma 3.10 and $\gamma \leq \frac{1}{24}$ comes down to

$$\tfrac{1}{24} \cdot (D - K_\Sigma)^2 \leq \tfrac{1}{2}(D - K_\Sigma - L)^2.$$

We note that due to $a \geq 2$ we have

$$b - 2g + 2 - \tfrac{ae}{2} \leq \tfrac{a+2}{4} \cdot \big(b - 2g + 2 - \tfrac{ae}{2}\big) = \tfrac{1}{8} \cdot (D - K_\Sigma)^2,$$

and by (4.1) we get

$$(a + 1) \cdot \big(l - \tfrac{e}{2}\big) \leq \tfrac{2}{3} \cdot (a + 1) \cdot \big(b - 2g + 2 - \tfrac{ae}{2}\big) \leq \tfrac{1}{3} \cdot (D - K_\Sigma)^2.$$

These two results then give

$$\tfrac{1}{2}(D-K_\Sigma-L)^2 = \tfrac{1}{2}(D-K_\Sigma)^2 - \Big(b-2g+2-\tfrac{ae}{2}+(a+1)\cdot\big(l-\tfrac{e}{2}\big)\Big) \geq \tfrac{1}{24}\cdot(D-K_\Sigma)$$

which finishes the proof. □

In the case $g = 0$, that is when $\Sigma \cong \mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$, we are in the lucky situation that the constant $\gamma$ does not at all depend on the chosen divisor $D$, while in the case $g \geq 1$ the ratio of $a$ and $b$ is involved in $\gamma$. This means that an asymptotical behaviour can only be examined if the ratio remains unchanged.

If $\Sigma$ is a product $C \times \mathbb{P}^1_{\mathbb{C}}$ the constant $\gamma$ here is the same as in Section 4.c.

Because of the importance of the case $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$ we would like to formulate the result for this surface separatly once more in a slightly weaker form, replacing $\nu^*(\mathcal{S}_i)$ and $e^*(\mathcal{S}_i)$ by $\tau^*(\mathcal{S}_i) + 1$, which is an upper bound for both of them.

**4.1 Corollary**

*Let* $\Sigma \cong \mathbb{F}_0 = \mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$, $\mathcal{S}_1, \ldots, \mathcal{S}_r$ *be topological or analytical singularity types and* $D \in \operatorname{Div}(\Sigma)$ *such that* $D \sim_l aC_0 + bF$ *with* $a, b \geq \tau^*(\mathcal{S}_i) + 1$ *for* $i = 1, \ldots, r$.

*Suppose that*

$$\sum_{i=1}^{r} \big(\tau^*(\mathcal{S}_i) + 2\big)^2 \ < \ \tfrac{1}{24} \cdot (D - K_\Sigma)^2 = \tfrac{1}{12} \cdot (a + 2) \cdot (b + 2). \tag{2.6b}$$

*Then* $V^{irr}_{|D|}(\mathcal{S}_1,\ldots,\mathcal{S}_r)$ *is non-empty, irreducible and of the expected dimension.*
**Addendum:** *If, moreover,* $\frac{24}{47}\cdot a \le b \le \frac{47}{24}\cdot a$*, then* $V^{irr}_{|D|}(\mathcal{S}_1,\ldots,\mathcal{S}_r)$ *is T-smooth.*

## 4.c. Products of Curves

Throughout this section we use the notation of Section G.b. In particular, $\Sigma = C_1 \times C_2$ where $C_1$ and $C_2$ are smooth projective curves over $\mathbb{C}$ of genera $g_1$ and $g_2$ respectively.

For a generic choice of $C_1$ and $C_2$ the Néron–Severi group $\mathrm{NS}(\Sigma)$ is two-dimensional by Proposition G.12. Thus Theorem 2.3 gives the following result for a generic product surface.

**2.3c Theorem**
*Let* $C_1$ *and* $C_2$ *be two smooth projective curves of genera* $g_1$ *and* $g_2$ *respectively with* $g_1 \ge g_2 \ge 0$*, such that for* $\Sigma = C_1 \times C_2$ *the Néron–Severi group is* $\mathrm{NS}(\Sigma) = C_1\mathbb{Z} \oplus C_2\mathbb{Z}$.

*Let* $\mathcal{S}_1,\ldots,\mathcal{S}_r$ *be topological or analytical singularity types, and let* $D \in \mathrm{Div}(\Sigma)$ *such that* $D \sim_a aC_1 + bC_2$ *with*

$$a \ge \begin{cases} \max\left\{2, \nu^*(\mathcal{S}_i) \;\middle|\; i=1,\ldots,r\right\}, & \textit{if } g_2 = 0 \\ 2g_2 - 1, & \textit{else}, \end{cases}$$

*and*

$$b \ge \begin{cases} \max\left\{2, \nu^*(\mathcal{S}_i) \;\middle|\; i=1,\ldots,r\right\}, & \textit{if } g_1 = 0 \\ 2g_1 - 1, & \textit{else}. \end{cases}$$

*Suppose that*

$$\sum_{i=1}^{r} \left(\tau^*(\mathcal{S}_i) + 2\right)^2 \;<\; \gamma \cdot (D - K_\Sigma)^2, \tag{2.5c}$$

*where* $\gamma$ *may be taken from the following table with* $\alpha = \frac{a-2g_2+2}{b-2g_1+2} > 0$.

| $g_1$ | $g_2$ | $\gamma$ | $\gamma$, *if* $\alpha = 1$ |
|---|---|---|---|
| $0$ | $0$ | $\frac{1}{24}$ | $\frac{1}{24}$ |
| $1$ | $0$ | $\frac{1}{\max\{32,2\alpha\}}$ | $\frac{1}{32}$ |
| $\ge 2$ | $0$ | $\frac{1}{\max\{24+16g_1,4g_1\alpha\}}$ | $\frac{1}{24+16g_1}$ |
| $1$ | $1$ | $\frac{1}{\max\left\{32,2\alpha,\frac{2}{\alpha}\right\}}$ | $\frac{1}{32}$ |
| $\ge 2$ | $\ge 1$ | $\frac{1}{\max\left\{24+16g_1+16g_2,4g_1\alpha,\frac{4g_2}{\alpha}\right\}}$ | $\frac{1}{24+16g_1+16g_2}$ |

*Then either* $V^{irr}_{|D|}(\mathcal{S}_1,\ldots,\mathcal{S}_r)$ *is empty or irreducible of the expected dimension.*

**Addendum 1:** *If we moreover suppose that*

$$a > \begin{cases} \max\left\{e^*(\mathcal{S}_i) - 2 \;\middle|\; i = 1, \ldots, r\right\}, & \text{if } g_2 = 0 \\ 2g_2 - 2 + \frac{3l}{2}, & \text{else,} \end{cases}$$

*and*

$$b > \begin{cases} \max\left\{e^*(\mathcal{S}_i) - 2 \;\middle|\; i = 1, \ldots, r\right\}, & \text{if } g_1 = 0 \\ 2g_1 - 2 + \frac{3l}{2}, & \text{else,} \end{cases}$$

*where* $l$ *is a positive integer such that* $L = lC_1 + lC_2$ *respectively* $L = C_1 + lC_2$, *if* $g_2 = 0$, *is very ample, then* $V^{irr}_{|D|}(\mathcal{S}_1, \ldots, \mathcal{S}_r)$ *is non-empty.*
**Addendum 2:** *If* $g_2 \geq 2$, $a = p \cdot (g_2 - 1)$, $b = p \cdot (g_1 - 1)$, *i. e.* $D \sim_a \frac{p}{2} \cdot K_\Sigma$, *and if we suppose that* $p > \frac{87}{176} \cdot \left(\tau^*(\mathcal{S}_i) + 1\right)$, *then* $V^{irr}_{|D|}(\mathcal{S}_1, \ldots, \mathcal{S}_r)$ *is also T-smooth.*

**Proof:** First we note that for $g_1 > 0$ we have

$$\left(\tau^*(\mathcal{S}_i) + 2\right)^2 < \frac{1}{2\alpha} \cdot (D - K_\Sigma)^2 = \left(b + 2 - 2g_1\right)^2,$$

and thus

$$b + 2 - 2g_1 > \tau^*(\mathcal{S}_i) + 2 \geq \nu^*(\mathcal{S}_i) + 2.$$

Similarly, if $g_2 > 0$, then $a + 2 - 2g_2 > \nu^*(\mathcal{S}_i) + 2$. Therefore, the conditions of Theorem 2.4 are fulfilled.

For Addendum 2 we may apply Theorem IV.1.1c and the fact that

$$\gamma \cdot \frac{(D-K_\Sigma)^2}{(g_1-1)\cdot(g_2-1)} \leq \frac{1}{44} \cdot (p-2)^2 \leq \frac{1}{44} \cdot p^2 \leq 2 \cdot \Big(p^2 - 2 \cdot p \cdot \left(\tau^*(\mathcal{S}_i) + 1\right)\Big).$$

It remains to prove Addendum 1 with the aid of Corollary III.2.4. For this we note that by Lemma 3.10 we have $e^*(\mathcal{S}_i) < \tau^*(\mathcal{S}_i) + 2$. But then Due to the assumptions on $a$ and $b$ and the above considerations, we have that $D - L - K_\Sigma$ is nef and that the intersection of $D - L - K_\Sigma$ with any irreducible curve of self-intersection zero, that is with the fibres $C_1$ and $C_2$, is greater than $e^*(\mathcal{S}_i)$. Moreover,

$$D.L - 2g(L) + 2 = (D - L - K_\Sigma).L \geq \frac{l}{3} \cdot (a - 2g_2 + 2 + b - 2g_1 + 2) > e^*(\mathcal{S}_i) + 2.$$

It therefore remains to show that Condition (III.2.3) is satisfied, which in view of Lemma 3.10 and $\gamma \leq \frac{1}{24}$ amounts to showing

$$\frac{1}{12} \cdot (a - 2g_2 + 2) \cdot (b - 2g_1 + 2) \leq (a - 2g_2 + 2 - l) \cdot (b - 2g_1 + 2 - l).$$

This, however, is again fulfilled due to the assumptions on $a$ and $b$. □

Once more, only in the case $\Sigma \cong \mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$ we get a constant $\gamma$ which does not depend on the chosen divisor $D$, while in the remaining cases the ratio of $a$ and $b$ is involved in $\gamma$. This means that an asymptotical behaviour can only be examined if the ratio remains unchanged.

## 4.d. Products of Elliptic Curves

If in Section 4.c the curves $C_1$ and $C_2$ are chosen to be both elliptic curves, then "$C_1$ and $C_2$ are generic" means precisely that they are not isogenous (cf. p. 194). We thus get the following theorem.

**2.3d Theorem**
*Let $\Sigma = C_1 \times C_2$, where $C_1$ and $C_2$ are two non-isogenous elliptic curves, let $\mathcal{S}_1, \ldots, \mathcal{S}_r$ be topological or analytical singularity types, and let $D \in \text{Div}(\Sigma)$ such that $D \sim_a aC_1 + bC_2$ with $a, b \geq 5$.*

*Suppose that*

$$\sum_{i=1}^{r} \left(\tau^*(\mathcal{S}_i) + 2\right)^2 \;<\; \min\left\{\tfrac{1}{32}, \tfrac{a}{2b}, \tfrac{b}{2a}\right\} \cdot D^2. \tag{2.5d}$$

*Then $V_{|D|}^{\text{irr}}(\mathcal{S}_1, \ldots, \mathcal{S}_r)$ is non-empty, irreducible and T-smooth.*

**Proof:** We note that due to Theorem IV.1.1d $V_{|D|}^{\text{irr}}(\mathcal{S}_1, \ldots, \mathcal{S}_r)$ is T-smooth, and thus we are done by Theorem 2.3c. □

## 4.e. Complete Intersection Surfaces

Throughout this section we use the notations of Section G.d.

In Proposition G.20 we show that a smooth complete intersection surface, always satisfies Condition (ii) of Theorem 2.1 for hypersurface-sections. Thus, a complete intersection with Picard number one satisfies the assumptions of Theorem 2.1.

**2.1e Theorem**
*Let $\mathbb{P}^2_{\mathbb{C}} \not\cong \Sigma \subset \mathbb{P}^N_{\mathbb{C}}$ be a smooth complete intersection of type $(d_1, \ldots, d_{N-2})$, let $H \subset \Sigma$ be a hyperplane section and suppose that the Picard number of $\Sigma$ is one.*

*Let $d > \kappa = \sum_{i=1}^{N-2} d_i - N - 1 \geq 0$, $n = H^2 = d_1 \cdots d_{N-2}$ and let $\mathcal{S}_1, \ldots, \mathcal{S}_r$ be topological or analytical singularity types.*

*Suppose that*

$$\sum_{i=1}^{r} \left(\tau^*(\mathcal{S}_i) + 2\right)^2 < \frac{18 \cdot \Big((\kappa+3)\cdot n + 2\chi(\mathcal{O}_\Sigma)\Big) \cdot n^2}{\Big((3\kappa+11)\cdot n + 6\chi(\mathcal{O}_\Sigma)\Big)^2} \cdot (d - \kappa)^2. \tag{2.4e}$$

*Then either $V_{|dH|}^{\text{irr}}(\mathcal{S}_1, \ldots, \mathcal{S}_r)$ is empty or irreducible of the expected dimension.*

**Proof:** It remains to show $\kappa = -N - 1 + \sum_{i=1}^{N-2} d_i \geq 0$, then in particular $dH - K_\Sigma = (d - \kappa) \cdot H$ is ample and $dH + K_\Sigma = (d + \kappa) \cdot H$ is nef.

Since $\kappa \geq 2N - 4 - N - 1 = N - 5$ anyway, the critical situations are $N = 3$ with $d_1 \leq 3$, and $N = 4$ with $d_1 = d_2 = 2$. In these cases the surface $\Sigma$ is either $\mathbb{P}^2_{\mathbb{C}}$ or rational with a Picard number larger than one (see p. 200 and [**Har77**] Ex. V.4.13). This finishes the claim. □

By Proposition G.20 we even do know a formula for $\chi(\mathcal{O}_\Sigma)$, however, the formula would just become nastier.

### 4.f. Surfaces in $\mathbb{P}^3_{\mathbb{C}}$

For a generic surface in $\mathbb{P}^3_{\mathbb{C}}$ we know that the Picard number is one, since then any curve is a hypersurface section by a Theorem of Noether (see Section G.d). Being a complete intersection, the results of Section 4.e apply for generic hypersurfaces in $\mathbb{P}^3_{\mathbb{C}}$. However, in this case the invariants involved in the formulation are much simpler.

**2.1f Theorem**
*Let $\Sigma \subset \mathbb{P}^3_{\mathbb{C}}$ be a smooth hypersurface of degree $n \geq 4$, let $H \subset \Sigma$ be a hyperplane section, and suppose that the Picard number of $\Sigma$ is one.*

*Let $d \geq n+6$ and let $\mathcal{S}_1, \ldots, \mathcal{S}_r$ be topological or analytical singularity types.*

*Suppose that*

$$\sum_{i=1}^{r} \big(\tau^*(\mathcal{S}_i)+2\big)^2 < \frac{6\cdot\big(n^3-3n^2+8n-6\big)\cdot n^2}{\big(n^3-3n^2+10n-6\big)^2} \cdot (d+4-n)^2. \tag{2.4f}$$

*Then $V^{irr}_{|dH|}(\mathcal{S}_1, \ldots, \mathcal{S}_r)$ is non-empty and irreducible of the expected dimension.*
**Addendum:** *If we, moreover, assume that $d \geq n\cdot\big(\tau^*(\mathcal{S}_i)+1\big)$, then $V^{irr}_{|dH|}(\mathcal{S}_1, \ldots, \mathcal{S}_r)$ is T-smooth.*

**Proof:** For the Addendum we note that $\frac{6\cdot(n^3-3n^2+8n-6)\cdot n^2}{(n^3-3n^2+10n-6)^2} \leq 2$ for $n \geq 4$, and hence

$$nd^2 - d\cdot(n-4)\cdot n\cdot\big(\tau^*(\mathcal{S}_i)+1\big) \geq 4d^2 \geq \frac{6\cdot\big(n^3-3n^2+8n-6\big)\cdot n^2}{\big(n^3-3n^2+10n-6\big)^2}\cdot(d+4-n)^2,$$

and the result thus follows from Theorem IV.1.1e.

It remains to show that $V^{irr}_{|dH|}(\mathcal{S}_1, \ldots, \mathcal{S}_r)$ is non-empty, using Corollary III.2.4. With $L = H$ we have $D - L - K_\Sigma = (d-n+3)\cdot H$, and since $d \geq n+6$ the divisor $D - L - K_\Sigma$ is nef. We thus have to verify the Conditions (III.2.3) and (III.2.5) in Corollary III.2.4. However, once (III.2.3) is fulfilled, then

$$e^*(\mathcal{S}_i)+1 \leq \sqrt{n}\cdot(d-n+4) < 2\cdot\sqrt{n}\cdot(d-n+3)-1 \leq n\cdot(d-n+3)-1 = D.L-2g(L)+1$$

for $i = 1, \ldots, r$, that is, (III.2.5) is satisfied as well.

Taking Lemma 3.10 into account (III.2.3) comes down to showing that

$$\frac{6\cdot\big(n^3-3n^2+8n-6\big)\cdot n}{\big(n^3-3n^2+10n-6\big)^2}\cdot n\cdot(d-n+4)^2 \leq \frac{1}{2}\cdot n\cdot(d-n+3)^2.$$

We claim that the function

$$\gamma : [4,\infty) \to \mathbb{R} : x \mapsto \frac{6\cdot\big(x^3-3x^2+8x-6\big)\cdot x}{\big(x^3-3x^2+10x-6\big)^2}$$

is monotonously decreasing. In order to see this, it suffices to show that the derivative

$$\gamma'(x) = \frac{-6\cdot(2x^6-9x^5+21x^4-24x^3+36x-36)}{(x^3-3x^2+10x-6)^3} < 0$$

for all $x \in [4,\infty)$, or equivalently that $2x^6-9x^5+21x^4-24x^3+36x-36$ is strictly positive in the interval $[4,\infty)$. However, since $x \geq 4$ we get

$$\big(2x^6 - 9x^5 + 6x^4\big) + \big(15x^4 - 24x^3\big) + \big(36x - 36\big) > 0$$

since each summand is strictly positive.

It follows that $\gamma(n) \leq \gamma(4) = \frac{252}{625}$ for all $n \geq 4$, and it remains to show that

$$\tfrac{252}{625} \cdot (d - n + 4)^2 \leq \tfrac{1}{2} \cdot (d - n + 3)^2,$$

which is a fairly easy calculation, since $d - n + 3 \geq 9$. □

Note that a slightly more sophisticated investigation would have allowed to replace the restriction $d \geq n + 6$ by $d \geq \max\{n-1, 10\}$.

Calculating the invariants in (2.4f) for the examples of reducible families of irreducible nodal curves on surfaces in $\mathbb{P}^3_{\mathbb{C}}$ given in the introduction on page 119 we end up with

$$\begin{aligned}\frac{6\cdot\big(n^3-3n^2+8n-6\big)\cdot n^2}{\big(n^3-3n^2+10n-6\big)^2} &\cdot (d+4-n)^2 \\ &< \tfrac{6}{n-3}\cdot(d+4-n)^2 < \tfrac{6}{n-3}\cdot\big(d^2+(4-n)\cdot d+2\big) \\ &< \tfrac{9n}{2}\cdot\big(d^2+(4-n)\cdot d+2\big) = 9r = \sum_{i=1}^{r}\big(\tau^*(A_1)+2\big)^2,\end{aligned}$$

that is, our result fits with these families.

## 4.g. Generic K3-Surfaces

A generic K3-surface has Picard number one (cf. Section G.e). Moreover, if $\Sigma$ is a K3-surface with $\mathrm{NS}(\Sigma) = L \cdot \Sigma$ and if $C \sim_l dL$ is a curve in $\Sigma$, then by Lemma E.1 $C$ is ample. Thus by the Kodaira Vanishing Theorem we have $h^1(\Sigma, C) = h^1(\Sigma, K_\Sigma + C) = 0$. Hence the surface satisfies the assumptions of Theorem 2.1.

**2.1g Theorem**

*Let $\Sigma$ be a smooth K3-surface with $\mathrm{NS}(\Sigma) = L \cdot \mathbb{Z}$ and set $n = L^2$.*

*Let $d \geq 19$, $D \sim_a dL$ and let $\mathcal{S}_1, \ldots, \mathcal{S}_r$ be topological or analytical singularity types.*

*Suppose that*

$$\sum_{i=1}^{r}\big(\tau^*(\mathcal{S}_i)+2\big)^2 < \tfrac{54n^2+72n}{(11n+12)^2}\cdot d^2\cdot n. \qquad (2.4g)$$

*Then $V^{irr}_{|D|}(\mathcal{S}_1, \ldots, \mathcal{S}_r)$ is irreducible and T-smooth.*

**Proof:** Apart from T-smoothness, it remains to prove that $V^{irr}_{|dL|}(\mathcal{S}_1,\ldots,\mathcal{S}_r)$ is non-empty and as in the case of surfaces in $\mathbb{P}^3_{\mathbb{C}}$ for this it suffices to show that (III.2.3) and (III.2.5) in Corollary III.2.4 are satisfied, where once again the first condition implies the latter one due to

$$e^*(\mathcal{S}_i)+1 \le \sqrt{n}\cdot d < 2\cdot\sqrt{n}\cdot(d-1)-1 \le n\cdot(d-1)-1 = D.L-2g(L)+1.$$

Taking Lemma 3.10 into account it thus remains to show

$$\tfrac{54n^2+72n}{(11n+12)^2}\cdot d^2\cdot n \le \tfrac{1}{2}\cdot(d-1)^2\cdot n.$$

We consider the function

$$\gamma:[0,\infty)\to\mathbb{R}: x\mapsto \tfrac{54x^2+72x}{(11x+12)^2},$$

which converges to $\frac{54}{121}$ for $x\mapsto\infty$ and is actually bounded from above by this number. It thus suffices to show

$$\tfrac{54}{121}\cdot d^2 \le \tfrac{1}{2}\cdot(d-1)^2,$$

which is satisfied, since $d\ge 19$.

T-smoothness follows from Theorem IV.1.1f, since $\frac{54n^2+72n}{(11n+12)^2}<1$. □

# Appendix

## A. Very General Position

It is our first aim to show that if there is a curve passing through points $z_1, \ldots, z_r \in \Sigma$ in very general position with multiplicities $n_1, \ldots, n_r$, then it can be equimultiply deformed in its algebraic system in a good way - i. e. suitable for Lemma II.2.3.

For the convenience of the reader we recall the definition of a *very general subset* from the Introduction.

**A.1 Definition**
Let $X$ be any Zariski topological space.

We say a subset $U \subseteq X$ is *very general* if it is an at most countable intersection of open dense subsets of $X$.

Some statement is said to hold for points $z_1, \ldots, z_r \in X$ (or $\underline{z} \in X^r$) *in very general position* if there is a suitable very general subset $U \subseteq X^r$, contained in the complement of the closed subvariety $\bigcup_{i \neq j} \{\underline{z} \in X^r \mid z_i = z_j\}$ of $X^r$, such that the statement holds for all $\underline{z} \in U$.

In the following we suppose that $\Sigma \subseteq \mathbb{P}^n_{\mathbb{C}}$ is an embedded smooth projective surface. We will denote by $\text{Hilb}_\Sigma$ the Hilbert scheme of curves on $\Sigma$, and for $h \in \mathbb{Q}[x]$ by $\text{Hilb}^h_\Sigma$ the Hilbert scheme of curves on $\Sigma$ with Hilbert polynomial $h$. The latter is a projective variety, and has in particular only finitely many connected components. If $B \subset \Sigma$ is a curve with Hilbert polynomial $h$, then we denote $|B|_a$ the connected component of the reduction of $\text{Hilb}^h_\Sigma$ containing $B$. (Cf. **[Mum66]** Chapter 15.)

**A.2 Lemma**
*Let* $B \subset \Sigma$ *be a curve, and* $\underline{n} \in \mathbb{N}_0^r$. *Then*

$$V_{B,\underline{n}} = \left\{\underline{z} \in \Sigma^r \;\middle|\; \exists\, C \in |B|_a \,:\, \text{mult}_{z_i}(C) \geq n_i \;\forall i = 1, \ldots, r\right\}$$

*is a closed subset of* $\Sigma^r$.

**Proof:**

**Step 1:** Show first that for $n \in \mathbb{N}_0$

$$X_{B,n} := \left\{(C, z) \in H \times \Sigma \mid \text{mult}_z(C) \geq n\right\}$$

is a closed subset of $H \times \Sigma$, where $H := |B|_a$.

Being the reduction of a connected component of the Hilbert scheme $\text{Hilb}_\Sigma$, $H$ is a projective variety endowed with a universal family of curves, giving rise to the following diagram of morphisms

$$\begin{array}{ccccc} \mathcal{C} = \bigcup_{C\in H}\{C\}\times C & \hookrightarrow & H\times\Sigma & \xrightarrow[\mathrm{pr}_\Sigma]{} & \Sigma \\ & \searrow & \downarrow{\scriptstyle \mathrm{pr}_H} & & \\ & & H, & & \end{array}$$

where $\mathcal{C}$ is an effective Cartier divisor on $H\times\Sigma$ with $\mathcal{C}_{|\{C\}\times\Sigma} = C$.[9]

Let $s\in H^0\big(H\times\Sigma,\mathcal{O}_{H\times\Sigma}(\mathcal{C})\big)$ be a global section defining $\mathcal{C}$. Then[10]

$$X_{B,n} = \big\{\eta=(C,z)\in H\times\Sigma \;\big|\; s_\eta\in(\mathfrak{m}_{\Sigma,z}^n+\mathfrak{m}_{H,C})\cdot\mathcal{O}_{H\times\Sigma,\eta}\big\}.$$

We may consider a finite open affine covering of $H\times\Sigma$ of the form $\{H_i\times U_j \mid i\in I, j\in J\}$, $H_i\subset H$ and $U_j\subset\Sigma$ open, such that $\mathcal{C}$ is locally on $H_i\times U_j$ given by one polynomial equation, say[11]

$$s_{i,j}(\underline{a},\underline{b})=0,\ \text{for } \underline{a}\in H_i,\ \underline{b}\in U_j.$$

It suffices to show that $X_{B,n}\cap(H_i\times U_j)$ is closed in $H_i\times U_j$ for all $i,j$.

However, for $\eta=(C,z)=(\underline{a},\underline{b})\in H_i\times U_j$ we have

$$s_\eta\in\big(\mathfrak{m}_{\Sigma,z}^n+\mathfrak{m}_{H,C}\big)\cdot\mathcal{O}_{H\times\Sigma,\eta}$$

if and only if

$$s_{i,j}(\underline{a},\underline{b})=0 \text{ and}$$
$$\frac{\partial^\alpha s_{i,j}}{\partial\underline{b}^\alpha}(\underline{a},\underline{b})=0,\ \text{for all } |\alpha|\le n-1,$$

where $\alpha$ is a multi index. Thus,

$$X_{B,n}\cap(H_i\times U_j)=\left\{(\underline{a},\underline{b})\in H_i\times U_j \;\middle|\; s_{i,j}(\underline{a},\underline{b})=0=\frac{\partial^\alpha s_{i,j}}{\partial\underline{b}^\alpha}(\underline{a},\underline{b}),\ \forall\,|\alpha|\le n-1\right\}$$

is a closed subvariety of $H_i\times U_j$, since $s_{i,j}$ and the $\frac{\partial^\alpha s_{i,j}}{\partial\underline{b}^\alpha}(\underline{a},\underline{b})$ are polynomial expressions in $\underline{a}$ and $\underline{b}$.

**Step 2:** $V_{B,\underline{n}}$ is a closed subset of $\Sigma^r$.

By Step 1 for $i=1,\ldots,r$ the set

$$X_{B,\underline{n},i} := \big\{(\underline{z},C)\in\Sigma^r\times H \;\big|\; \mathrm{mult}_{z_i}(C)\ge n_i\big\}\cong\Sigma^{r-1}\times X_{B,n_i}$$

is a closed subset of $\Sigma^r\times H\cong\Sigma^{r-1}\times H\times\Sigma$. Considering now

$$\begin{array}{ccc} X_{B,\underline{n}} := \bigcap_{i=1}^r X_{B,\underline{n},i} & \hookrightarrow & \Sigma^r\times H \\ & {\scriptstyle\rho}\searrow & \downarrow \\ & & \Sigma^r, \end{array}$$

[9]For the definition of an algebraic family of curves see [**Har77**] Ex. V.1.7.

[10]$\mathfrak{m}_{\Sigma,z}^n=\big(\mathfrak{m}_{\Sigma,z}^n+\mathfrak{m}_{H,C}\big)\cdot\mathcal{O}_{H\times\Sigma,\eta}/\mathfrak{m}_{H,C}\cdot\mathcal{O}_{H\times\Sigma,\eta}$ and $C=\mathcal{C}_{|\{C\}\times\Sigma}$ is locally in $z$ given by the image of $s_\eta$ in $\mathcal{O}_{\Sigma,z}=\mathcal{O}_{H\times\Sigma,\eta}/\mathfrak{m}_{H,C}\cdot\mathcal{O}_{H\times\Sigma,\eta}$.

[11]The $\underline{a}$ and $\underline{b}$ denote coordinates on the affine ambient spaces of $H_i\subseteq\mathbb{A}^{N_i}$ respectively $U_j\subseteq\mathbb{A}^{M_j}$.

we find that $V_{B,\underline{n}} = \rho(X_{B,\underline{n}})$, being the image of a closed subset under a morphism between projective varieties, is a closed subset of $\Sigma^r$ (cf. **[Har77]** Ex. II.4.4). $\square$

**A.3 Corollary**

*Then the complement of the set*

$$V = \bigcup_{B \in \mathrm{Hilb}_\Sigma} \bigcup_{\underline{n} \in \mathbb{N}_0^r} \bigcup_{V_{B,\underline{n}} \neq \Sigma^r} V_{B,\underline{n}}$$

*in* $\Sigma^r$ *is very general, where* $\mathrm{Hilb}_\Sigma$ *is the Hilbert scheme of curves on* $\Sigma$.

*In particular, there is a very general subset* $U \subseteq \Sigma^r$ *such that if for some* $\underline{z} \in U$ *there is a curve* $B \subset \Sigma$ *with* $\mathrm{mult}_{z_i}(B) = n_i$ *for* $i = 1, \ldots, r$*, then for any* $\underline{z}' \in U$ *there is a curve* $B' \in |B|_a$ *with* $\mathrm{mult}_{z_i'}(B') \geq n_i$.

**Proof:** Fixing some embedding $\Sigma \subseteq \mathbb{P}^n_{\mathbb{C}}$ and $h \in \mathbb{Q}[x]$, $\mathrm{Hilb}^h_\Sigma$ is a projective variety and has thus only finitely many connected components. Thus the Hilbert scheme $\mathrm{Hilb}_\Sigma$ has only a countable number of connected components, and we have only a countable number of different $V_{B,\underline{n}}$, where $B$ runs through $\mathrm{Hilb}_\Sigma$ and $\underline{n}$ through $\mathbb{N}_0^r$. By Lemma A.2 the sets $V_{B,\underline{n}}$ are closed, hence their complements $\Sigma^r \setminus V_{B,\underline{n}}$ are open. But then

$$U = \Sigma^r \setminus V = \bigcap_{B \in \mathrm{Hilb}_\Sigma} \bigcap_{\underline{n} \in \mathbb{N}_0^r} \bigcap_{V_{B,\underline{n}} \neq \Sigma^r} (\Sigma^r \setminus V_{B,\underline{n}})$$

is an at most countable intersection of open dense subsets of $\Sigma^r$, and is hence very general. $\square$

If $\Sigma$ is regular, i. e. the irregularity $q(\Sigma) = h^1(\Sigma, \mathcal{O}_\Sigma) = 0$, algebraic and linear equivalence coincide, and thus $\mathrm{Hilb}^h_\Sigma = |D|_l$, if $D$ is any divisor with Hilbert polynomial $h$. This makes the proofs given above a bit simpler.

**A.4 Lemma**

*Given* $\underline{n} \in \mathbb{N}_0^r$ *and* $D \in \mathrm{Pic}(\Sigma)$*, the set*

$$V_{D,\underline{n}} = \left\{ \underline{z} \in \Sigma^r \;\middle|\; \exists\, C \in |D|_l \,:\, \mathrm{mult}_{z_i}(C) \geq n_i \right\}$$

*is a closed subset of* $\Sigma^r$.

**Proof:** Fix an affine covering $\Sigma = U_1 \cup \ldots \cup U_k$ of $\Sigma$, and a basis $s_0, \ldots, s_n$ of $H^0\big(\Sigma, \mathcal{O}_\Sigma(D)\big)$.

It suffices to show that $V_{D,\underline{n}} \cap \big(U_{j_1} \times \ldots \times U_{j_r}\big)$ is closed in $U_{j_1} \times \ldots \times U_{j_r}$ for all $\underline{j} = (j_1, \ldots, j_r) \in \{1, \ldots, k\}^r$, since those sets form an open covering of $V_{D,\underline{n}}$.

Consider the set

$$V_{\alpha,i,\underline{j}} = \left\{ (\underline{z}, \underline{a}) \in U_{j_1} \times \ldots \times U_{j_r} \times \mathbb{P}^n_{\mathbb{C}} \;\middle|\; a_0(D^\alpha s_0)(z_i) + \ldots + a_n(D^\alpha s_n)(z_i) = 0 \right\},$$

where $\alpha$ is a multi index and $D^\alpha$ denotes the corresponding differential operator. This is a closed subvariety of $U_{j_1} \times \ldots \times U_{j_r} \times \mathbb{P}^n_{\mathbb{C}}$.[12] Thus, also the set

$$V_{\underline{j}} = \bigcap_{i=1,\ldots,r} \; \bigcap_{|\alpha| \leq n_i - 1} V_{\alpha,i,\underline{j}}$$

is a closed subset of $U_{j_1} \times \ldots \times U_{j_r} \times \mathbb{P}^n_{\mathbb{C}}$. Considering now the projection to $U_{j_1} \times \ldots \times U_{j_r}$,

$$\begin{array}{ccc} V_{\underline{j}} & \hookrightarrow & U_{j_1} \times \ldots \times U_{j_r} \times \mathbb{P}^n_{\mathbb{C}} \\ & \overset{\rho}{\searrow} & \downarrow \\ & & U_{j_1} \times \ldots \times U_{j_r}, \end{array}$$

we find that $\mathrm{Im}(\rho) = V \cap \left(U_{j_1} \times \ldots \times U_{j_r}\right)$ is closed in $U_{j_1} \times \ldots \times U_{j_r}$ (cf. **[Har92]** Theorem 3.12). $\square$

**A.5 Corollary**

*Let $\Sigma$ be regular. Then the complement of the set*

$$V = \bigcup_{D \in \mathrm{Pic}(\Sigma)} \; \bigcup_{\underline{n} \in \mathbb{N}_0^r} \; \bigcup_{V_{D,\underline{n}} \neq \Sigma^r} V_{D,\underline{n}}$$

*in $\Sigma^r$ is very general.*

*In particular, there is a very general subset $U \subseteq \Sigma^r$ such that if for some $\underline{z} \in U$ there is a curve $B \subset \Sigma$ with $\mathrm{mult}_{z_i}(B) = n_i$ for $i = 1, \ldots, r$, then for any $\underline{z}' \in U$ there is a curve $B' \in |B|_l$ with $\mathrm{mult}_{z_i'}(B') \geq n_i$.*

**Proof:** By Lemma A.4 the sets $V_{D,\underline{n}}$ are closed, hence their complement $\Sigma^r \setminus V_{D,\underline{n}}$ is open. Since $\Sigma$ is regular, $\mathrm{Pic}(\Sigma) \cong \mathrm{NS}(\Sigma)$ is a finitely generated abelian group, hence countable.[13] But then

$$U = \Sigma^r \setminus V = \bigcap_{D \in \mathrm{Pic}(\Sigma)} \; \bigcap_{\underline{n} \in \mathbb{N}_0^r} \; \bigcap_{V_{D,\underline{n}} \neq \Sigma^r} \left(\Sigma^r \setminus V_{D,\underline{n}}\right)$$

is an at most countable intersection of open dense subsets of $\Sigma^r$ and is hence very general. $\square$

In the proof of Theorem II.1.1 we use at some place the result of Corollary A.6. We could instead use Corollary A.3. However, since the results are quite nice and simple to prove we just give them.

**A.6 Corollary**

(a) The number of curves $B$ in $\Sigma$ with $\dim |B|_a = 0$ is at most countable.

[12] On $U_j$ the $s_i$ are given as quotients of polynomials where the denominator is not vanishing on $U_j$. Thus we may assume w. l. o. g. that $s_i$ is represented by a polynomial, and hence the above equation is polynomial in $z_i$ and even linear in the $a_i$.

[13] See **[IsS96]** Chapters 3.2 and 3.3. From the exponential sequence it follows that $\mathrm{NS}(\Sigma) \cong \mathrm{Pic}(\Sigma)/\mathrm{Pic}^0(\Sigma)$ with $\mathrm{Pic}^0(\Sigma) \cong \mathbb{C}^q/\mathbb{Z}^{2q}$, $q = q(\Sigma)$. Thus, $\mathrm{Pic}(\Sigma)$ is countable if and only if $\Sigma$ is regular.

(b) The number of exceptional curves in $\Sigma$ (i. e. curves with negative self intersection) is at most countable.

(c) There is a very general subset $U$ of $\Sigma^r$, $r \geq 1$, such that for $\underline{z} \in U$ no $z_i$ belongs to a curve $B \subset \Sigma$ with $\dim |B|_a = 0$, in particular to no exceptional curve.

**Proof:**

(a) By definition $|B|_a$ is a connected component of $\mathrm{Hilb}_\Sigma$, whose number is at most countable (see proof of Lemma A.3). If in addition $\dim |B|_a = 0$, then $|B|_a = \{B\}$ which proves the claim.

(b) Curves of negative self-intersection are not algebraically equivalent to any other curve (cf. **[IsS96]** . p. 153)

(c) Follows from (a).

□

**A.7 Example (Kodaira)**
Let $z_1, \ldots, z_9 \in \mathbb{P}^2_{\mathbb{C}}$ be in very general position[14] and let $\Sigma = \mathrm{Bl}_{\underline{z}}\left(\mathbb{P}^2_{\mathbb{C}}\right)$ be the blow up of $\mathbb{P}^2_{\mathbb{C}}$ in $\underline{z} = (z_1, \ldots, z_9)$. Then $\Sigma$ contains infinitely many irreducible smooth rational $-1$-curves, i. e. exceptional curves of the first kind.

**Proof:** It suffices to find an infinite number of irreducible curves $C$ in $\mathbb{P}^2_{\mathbb{C}}$ such that

$$d^2 - \sum_{i=1}^{9} m_i^2 = -1, \tag{A.1}$$

and

$$p_a(C) - \sum_{i=1}^{9} \frac{m_i(m_i-1)}{2} = 0, \tag{A.2}$$

where $m_i = \mathrm{mult}_{z_i}(C)$ and $d = \deg(C)$, since the expression in (A.1) is the self intersection of the strict transform $\widetilde{C} = \mathrm{Bl}_{\underline{z}}(C)$ of $C$ and (A.2) gives its arithmetical genus.[15] In particular $\widetilde{C}$ cannot contain any singularities, since they would contribute to the arithmetical genus, and, being irreducible anyway, $\widetilde{C}$ is an exceptional curve of the first kind.

[14] To be precise, no three of the nine points should be collinear, and after any finite number of quadratic Cremona transformations centred at the $z_i$ (respectively the newly obtained centres) still no three should be collinear. Thus the admissible tuples in $(\mathbb{P}^2_{\mathbb{C}})^9$ form a very general set, cf. **[Har77]** Ex. V.4.15.

[15]

$$\begin{aligned} p_a(\widetilde{C}) &= 1 + \frac{K_\Sigma.\widetilde{C}+\widetilde{C}^2}{2} = 1 + \frac{\left(K_{\mathbb{P}^2_{\mathbb{C}}} + \sum E_i\right).\left(C - \sum m_i E_i\right) + \left(C - \sum m_i E_i\right)^2}{2} \\ &= 1 + \frac{K_{\mathbb{P}^2_{\mathbb{C}}}.C + C^2}{2} - \sum \frac{m_i(m_i-1)}{2} = p_a(C) - \sum \frac{m_i(m_i-1)}{2}. \end{aligned}$$

We are going to deduce the existence of these curves with the aid of quadratic Cremona transformations.

**Claim:** If for some $d > 0$ and $m_1, \ldots, m_9 \geq 0$ with $3d - \sum_{i=1}^{9} m_i = 1$ there is an irreducible curve $C \in \left|\mathcal{J}_{X(\underline{m};\underline{z})}(d)\right|_l$, then $T(C) \in \left|\mathcal{J}_{X(\underline{m}';\underline{z}')}(d+a)\right|_l$ is an irreducible curve, where

- $\{i, j, k\} \subset \{1, \ldots, 9\}$ are such that $m_i + m_j + m_k < d$,
- $T : \mathbb{P}^2_{\mathbb{C}} \dashrightarrow \mathbb{P}^2_{\mathbb{C}}$ is the quadratic Cremona transformation at $z_i, z_j, z_k$,
- $z'_\nu = \begin{cases} z_\nu, & \text{if } \nu \neq i, j, k, \\ T(\overline{z_\lambda z_\mu}), & \text{if } \{\nu, \lambda, \mu\} = \{i, j, k\}, \end{cases}$
- $m'_\nu = \begin{cases} m_\nu, & \text{if } \nu \neq i, j, k, \\ m_\nu + a, & \text{else, and} \end{cases}$
- $a = d - (m_i + m_j + m_k)$.

Note that, $3 \cdot (d + a) - \sum_{i=1}^{9} m'_i = 1$, i. e. we may iterate the process since the hypothesis of the claim will be preserved.

Since $3d > \sum_{i=1}^{9} m_i$, there must be a triple $(i, j, k)$ such that $d > m_i + m_j + m_k$. Let us now consider the following diagram

$$\begin{array}{ccc} & \Sigma = \mathrm{Bl}_{z_i,z_j,z_k}(\mathbb{P}^2_{\mathbb{C}}) = \mathrm{Bl}_{z'_i,z'_j,z'_k}(\mathbb{P}^2_{\mathbb{C}}) & \\ \swarrow^{\pi} & & \searrow^{\pi'} \\ \mathbb{P}^2_{\mathbb{C}} & \overset{T}{\dashrightarrow} & \mathbb{P}^2_{\mathbb{C}}, \end{array}$$

and let us denote the exceptional divisors of $\pi$ by $E_i$ and those of $\pi'$ by $E'_i$. Moreover, let $\widetilde{C} = \mathrm{Bl}_{z_i,z_j,z_k}(C)$ be the strict transform of $C$ under $\pi$ and let $\widetilde{T(C)} = \mathrm{Bl}_{z'_i,z'_j,z'_k}\big(T(C)\big)$ be the strict transform of $T(C)$ under $\pi'$. Then of course $\widetilde{C} = \widetilde{T(C)}$, and $T(C)$, being the projection $\pi'\big(\widetilde{C}\big)$ of the strict transform $\widetilde{C}$ of the irreducible curve $C$, is of course an irreducible curve. Note that the condition $d > m_i + m_j + m_k$ ensures that $\widetilde{C}$ is not one of the curves which are contracted. It thus suffices to verify

$$\deg\big(T(C)\big) = d + a,$$

and

$$m'_i = \mathrm{mult}_{z'_i}\big(T(C)\big) = \begin{cases} m_\nu, & \text{if } \nu \neq i, j, k, \\ m_\nu + a, & \text{else.} \end{cases}$$

Since outside the lines $\overline{z_i z_j}$, $\overline{z_i z_k}$, and $\overline{z_k z_j}$ the transformation $T$ is an isomorphism and since by hypothesis none of the remaining $z_\nu$ belongs to one of these lines we clearly have $m'_\nu = m_\nu$ for $\nu \neq i, j, k$. Moreover, we have

$$\begin{aligned} m'_i &= \widetilde{T(C)}.E'_i = \widetilde{C}.\mathrm{Bl}_{z_i,z_j,z_k}(\overline{z_j z_k}) \\ &= (\pi^*C - \textstyle\sum_{\nu=i,j,k} m_i E_i).(\pi^*\overline{z_j z_k} - E_j - E_k) \\ &= C.\overline{z_j z_k} - m_j - m_k = d - m_j - m_k = m_i + a. \end{aligned}$$

Analogously for $m_j'$ and $m_k'$.

Finally we find

$$\begin{aligned}\deg\big(T(C)\big) &= T(C).\overline{z_i'z_j'} = \pi'^*T(C).\pi'^*\overline{z_i'z_j'}\\ &= \big(\widetilde{T(C)} + \textstyle\sum_{\nu=i,j,k} m_\nu' E_\nu'\big).\big(E_k + E_i' + E_j'\big)\\ &= \widetilde{C}.E_k + \textstyle\sum_{\nu=i,j,k} m_\nu' E_\nu'.E_k\\ &= m_k + m_i' + m_j' = d + a.\end{aligned}$$

This proves the claim.

Let us now show by induction that for any $d > 0$ there is an irreducible curve $C$ of degree $d' \geq d$ satisfying (A.1) and (A.2). For $d = 1$ the line $C = \overline{z_1 z_2}$ through $z_1$ and $z_2$ gives the induction start. Given some suitable curve of degree $d' \geq d$ the above claim then ensures that through points in very general position there is an irreducible curve of higher degree satisfying (A.1) and (A.2), since $a = d - (m_1 + m_2 + m_3) > 0$. Thus the induction step is done. □

The example shows that a smooth projective surface $\Sigma$ may indeed carry an infinite number of exceptional curves - even of the first kind. According to Nagata ([**Nag60**] Theorem 4a, p. 283) the example is due to Kodaira. For further references on the example see [**Har77**] Ex. V.4.15, [**BeS95**] Example 4.2.7, or [**Fra41**]. [**IsS96**] p. 198 Example 3 shows that also $\mathbb{P}^2_{\mathbb{C}}$ blown up in the nine intersection points of two plane cubics carries infinitely many exceptional curves of the first kind.

## B. Curves of Self-Intersection Zero

### B.1 Proposition

*Suppose that* $B \subset \Sigma$ *is an irreducible curve with* $B^2 = 0$ *and* $\dim |B|_a \geq 1$*, then*

(B.1) $|B|_a$ *is an irreducible reduced projective curve, and*

(B.2) *there is a fibration* $f : \Sigma \to H$ *whose fibres are just the elements of* $|B|_a$*, and* $H$ *is the normalisation of* $|B|_a$*.*

We are proving the proposition in several steps.

### B.2 Proposition

*Let* $f : Y' \to Y$ *be a finite flat morphism of noetherian schemes with* $Y$ *irreducible such that for some point* $y_0 \in Y$ *the fibre* $Y'_{y_0} = f^{-1}(y_0) = Y' \times_Y$ Spec $\big(k(y_0)\big)$ *is a single reduced point.*[16]

*Then the structure map* $f^\# : \mathcal{O}_Y \longrightarrow f_*\mathcal{O}_{Y'}$ *is an isomorphism, and hence so is* $f$.

[16]The assumption "reduced" is necessary, since finite flat morphisms may very well have only non-reduced fibres. Consider the ring homomorphism $\varphi : A = \mathbb{C}[x] \to \mathbb{C}[x,y]/(y^2) = B : x \mapsto x$, making $B \cong A \oplus y \cdot A$ into a free and hence flat $A$-module with only non-reduced fibres (= double-points).

**Proof:** Since there is at least one connected reduced fibre $Y'_{y_0}$, semicontinuity of flat, proper[17] morphisms in the version **[GrD67]** IV.12.2.4 (vi) implies that there is an open dense subset $U \subseteq Y$ such that $Y'_y$ is connected and reduced, hence a single reduced point, $\forall\, y \in U$. ($U$ dense in $Y$ is due to the fact that $Y$ is irreducible.)

Thus the assumptions are stable under restriction to open subschemes of $Y$, and since the claim that we have to show is local on $Y$, we may assume that $Y = \mathrm{Spec}(A)$ is affine. Moreover, $f$ being finite, thus affine, we have $Y' = \mathrm{Spec}(B)$ is also affine.

Since $f$ is flat it is open and hence dominates the irreducible affine variety $Y$ and, therefore, induces an inclusion of rings $A \hookrightarrow B$. It now suffices to show:

**Claim:** $A \hookrightarrow B$ is an isomorphism.

By assumption there exists a $y = P \in \mathrm{Spec}(A) = Y$ such that $Y'_y = f^{-1}(y) = \mathrm{Spec}(B_P/PB_P)$ is a single point with reduced structure. In particular we have for the multiplicity of $Y'_y = \mathrm{Spec}(B_P/PB_P)$ over $\{y\} = \mathrm{Spec}(A_P/PA_P)$

$$1 = \mu(Y'_y) = \mathrm{length}_{A_P/PA_P}(B_P/PB_P),$$

(cf. **[Har77]** p. 51 for the definition of the multiplicity) which implies that

$$A_P/PA_P \hookrightarrow B_P/PB_P$$

is an isomorphism. Hence by Nakayama's Lemma also

$$A_P \hookrightarrow B_P$$

is an isomorphism, that is, $B_P$ is free of rank 1 over $A_P$. $B$ being locally free[18] over $A$, with $A/\sqrt{0}$ an integral domain, thus fulfils

$$A_Q \hookrightarrow B_Q$$

is an isomorphism for all $Q \in \mathrm{Spec}(A)$, and hence the claim follows (cf. **[AtM69]** Proposition 3.9). $\square$

**B.3 Remark**

Some comments from Bernd Kreußler.

(a) Let $Y'$ be the standard example of a non-separated variety, the affine line over $k$ with the origin doubled, and let $f : Y' \to Y = \mathbb{A}^1_k$ be the projection to $Y$. Then $f$ is quasi-finite, but NOT finite, since the preimage of the affine $Y$ is not affine. However, $f_*\mathcal{O}_{Y'} = \mathcal{O}_Y$ (see **[Har77]** II.2.3.5/6), but $f$ is certainly not an isomorphism.

(b) By **[Har77]** III.11.3 for $f : Y' \to Y$ projective between noetherian schemes the condition $f_*\mathcal{O}_{Y'} = \mathcal{O}_Y$ implies that the fibres of $f$ are connected. The converse is also true, if the fibres are connected, the $f_*\mathcal{O}_{Y'} = \mathcal{O}_Y$.

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[17] By **[Har77]** Ex. II.4.1 $f$ is proper, since $f$ is finite.

[18] Since $A \hookrightarrow B$ is flat!

(c) See (a) for an example, that the condition “projective” for the morphism is not obsolete.

(d) Let $f : Y' \to Y$ be the blow up of a point of $\mathbb{P}^2_k$, then the fibres are connected, and hence $f_*\mathcal{O}_{Y'} = \mathcal{O}_Y$. $f$ is generically finite, but not finite and certainly no isomorphism.

(e) If $f : X \to \mathrm{Spec}(A)$ is a morphism such that $f^\# : A \to \Gamma(X, \mathcal{O}_X)$ is an isomorphism, then $f$ need not be an isomorphism - despite the fact that $\mathrm{Hom}\big(X, \mathrm{Spec}(A)\big)$ and $\mathrm{Hom}\big(A, \Gamma(X, \mathcal{O}_X)\big)$ are naturally bijective. E. g. $\mathbb{P}^1_k$ and $f$ the constant map to $\mathrm{Spec}(k)$. Then $\Gamma(X, \mathcal{O}_X) = k$ and thus $f^\# = \mathrm{id}$ is an isomorphism, but $f$ is certainly not! If $X$ is also affine, then however $f$ is an isomorphism if and only if $f^\#$ is so.

(f) If $A \to B$ is a ring homomorphism and $B$ is locally free of rank $1$ as an $A$-module, and if for some $P \in \mathrm{Spec}(A)$ the localisation $A_P \to B_P$ is an isomorphism, then the homomorphism itself is an isomorphism. If, however, $A \to B$ is only an $A$-module homomorphism, this need not be true. E. g. let $X$ be a smooth curve, e. g. $\mathbb{P}^1$, and $x \in X$ a point; let $\mathcal{J}_{x/X} := \mathcal{O}_X(-x)$ be the ideal sheaf of $x$, in the case of $\mathbb{P}^1$ it's just $\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(-1)$. This is a locally free sheaf, and $x$ is a divisor on the smooth curve $X$. This gives an exact sequence $0 \to \mathcal{J}_{x/X} \to \mathcal{O}_X \to \mathcal{O}_x \to 0$. For the stalks over $y \neq x$ the map from $L \to \mathcal{O}_X$ is of course an isomorphism, but not for $y = x$.

(g) Let $T = \mathbb{A}^1_k$, $X_t = V(x^2 - txy) \subset \mathbb{P}^1_k$ with homogeneous coordinates $x, y$ on $\mathbb{P}^1_k$. Then for $t_0 = 0$ the fibre $X_{t_0}$ is connected, namely a non-reduced fat point, while for $t \neq 0$ the fibre $X_t$ consists of two single points, and is thus not connected. The condition $X_{t_0}$ reduced in the Corollary B.5 is hence vital.

**B.4 Proposition (Principle of Connectedness)**
*Let $X$ and $Y$ be noetherian schemes, $Y$ connected, and let $\pi : X \to Y$ be a flat projective morphism such that for some $y_0 \in Y$ the fibre $X_{y_0} = \pi^{-1}(y_0)$ is reduced and connected.*

*Then for all $y \in Y$ the fibre $X_y = \pi^{-1}(y)$ is connected.*

**Proof:** Stein Factorisation (cf. [**GrD67**] III.4.3.3 or [**Har77**] III.11.5) gives a factorisation of $\pi$ of the form

$$\pi : X \xrightarrow{\pi'} Y' = \mathrm{Spec}(\pi_*\mathcal{O}_X) \xrightarrow{f} Y,$$

with

(1) $\pi'$ connected (i. e. its fibres are connected),

(2) $f$ finite,

(3) $f_*\mathcal{O}_{Y'} = \pi_*\mathcal{O}_X$ locally free over $\mathcal{O}_Y$, since $\pi$ is flat, and

(4) $Y'_{y_0} = f^{-1}(y_0)$ is connected and reduced, i. e. a single reduced point.

Because of (1) it suffices to show that $f$ is connected; and we claim, moreover, that the fibres of $f$ are reduced as well.

Considering points in the intersections of the finite number of irreducible components of $Y$ we can reduce to the case $Y$ irreducible. Since $f$ is finite (3) is equivalent to saying that $f$ is flat (cf. **[AtM69]** Proposition 3.10). Hence $f$ fulfils the assumptions of Proposition B.2, and we conclude that $\mathcal{O}_Y = f_*\mathcal{O}_{Y'}$ and the proposition follows from **[Har77]** III.11.3.[19]

Alternatively, from **[GrD67]** IV.15.5.9 (ii) it follows that there is an open dense subset $U \subseteq Y$ such that $X_y$ is connected for all $y \in U$. Since, moreover, by the same theorem the number of connected components of the fibres is a lower semi-continuous function on $Y$ the special fibres cannot have more connected components than the general ones, that is, all fibres are connected. □

**B.5 Corollary (Principle of Connectedness)**
*Let $\{X_t\}_{t\in T}$ be a flat family of closed subschemes $X_t \subseteq \mathbb{P}^n_{k(t)}$, where $T$ is a connected noetherian scheme. Suppose that $X_{t_0}$ is connected and reduced for some point $t_0 \in T$. Then $X_t$ is connected for all $t \in T$.*

**Proof:** The $X_t$ are the fibres of a flat projective morphism

$$\begin{array}{ccc} \mathcal{X} & \hookrightarrow & \mathbb{P}^n_T \\ & \searrow^{\pi} & \downarrow \\ & & T, \end{array}$$

where $\mathcal{X}$ is a closed subscheme of $\mathbb{P}^n_T$. The result thus follows from Proposition B.4.[20] □

**B.6 Lemma**
*Under the hypotheses of Proposition B.1 let $C \in |B|_a$ then $C$ is connected.*

**Proof:** Consider the universal family

$$\begin{array}{ccc} |B|_a \times \Sigma & \hookleftarrow & \bigcup_{C\in|B|_a} \{C\} \times C =: S \\ & \searrow_{pr_{|B|_a}} & \downarrow \pi \text{ flat} \\ & & |B|_a \end{array} \qquad \text{(B.3)}$$

over the connected projective scheme $|B|_a \subseteq \text{Hilb}_\Sigma$.

[19]Compare the result with **[GrD67]** III.4.3.10, which deals with the case of $X$ also being integral, but weakens the hypothesis on $\pi$ to proper and universally open. Also the assumption on the fibre is reduced to $R(Y)$ being algebraically closed in $R(X)$ - in the case where the characteristic of $R(Y)$ is zero. Compare also **[GrD67]** IV.12.2.4 (vi) and III.15.5.9, which both deal with an arbitrary number of connected components.

[20]Compare the result with **[Har77]** Ex. III.11.4.

Since the projection $\pi$ is a flat projective morphism, and since the fibre $\pi^{-1}(B) = \{B\} \times B$ is connected and reduced, the result follows from Proposition B.4. □

**B.7 Lemma**
*Under the hypotheses of Proposition B.1 let* $C \in |B|_a$ *with* $B \subseteq C$*, then* $C = B$.

**Proof:** Suppose $B \subsetneqq C$, then the Hilbert polynomials of $B$ and $C$ are different in contradiction to $B \sim_a C$. □

**B.8 Lemma**
*Under the hypotheses of Proposition B.1 let* $C \in |B|_a$ *with* $C \neq B$*, then* $C \cap B = \emptyset$.

**Proof:** Since $B$ is irreducible by Lemma B.7 $B$ and $C$ do not have a common component. Suppose $B \cap C = \{p_1, \ldots, p_r\}$, then $B^2 = B.C \geq r > 0$ in contradiction to $B^2 = 0$. □

**B.9 Proposition (Zariski's Lemma)**
*Under the hypotheses of Proposition B.1 let* $C = \sum_{i=1}^{r} n_i C_i \in |B|_a$*, where the* $C_i$ *are pairwise different irreducible curves and* $n_i > 0$ *for* $i = 1, \ldots, r$.

*Then the intersection matrix* $Q = (C_i.C_j)_{i,j=1,\ldots,r}$ *is negative semi-definite, and, moreover,* $C$*, considered as an element of the vector space* $\bigoplus_{i=1}^{r} \mathbb{Q} \cdot C_i$*, generates the annihilator of* $Q$.

*In particular,* $D^2 \leq 0$ *for all curves* $D \subseteq C$*, and, moreover,* $D^2 = 0$ *if and only if* $D = C$.

**Proof:** By Lemma B.6 $C$ is connected. We are going to apply [**BPV84**] I.2.10, and thus we have to verify three conditions.

(a') $C.C_i = B.C_i = 0$ for all $i = 1, \ldots, r$ by Lemma B.8. Thus $C$ is an element of the annihilator of $Q$ with $n_i > 0$ for all $i = 1, \ldots, r$.

(b) $C_i.C_j \geq 0$ for all $i \neq j$.

(c) Since $C$ is connected there is no non-trivial partition $I \cup J$ of $\{1, \ldots r\}$ such that $C_i.C_j = 0$ for all $i \in I$ and $j \in J$.

Thus [**BPV84**] I.2.10 implies that $-Q$ is positive semi-definite. □

**B.10 Lemma**
*Under the hypotheses of Proposition B.1 let* $C, C' \in |B|_a$ *be two distinct curves, then* $C \cap C' = \emptyset$.

**Proof:** Suppose $C = A + D$ and $C' = A + D'$ such that $D$ and $D'$ have no common component.

We have

$$0 = B^2 = (A + D)^2 = (A + D')^2 = (A + D).(A + D'),$$

and thus

$$(A + D)^2 + (A + D')^2 = 2(A + D).(A + D'),$$

which implies that

$$D^2 + D'^2 = 2D.D',$$

where each summand on the left hand side is less than or equal to zero by Proposition B.9, and the right hand side is greater than or equal to zero, since the curves $D$ and $D'$ have no common component. We thus conclude

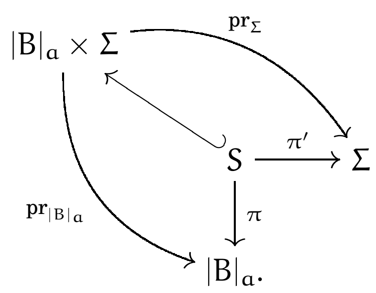
$$D^2 = D'^2 = D.D' = 0.$$

But then again Proposition B.9 implies that $D = C$ and $D' = C'$, that is, $C$ and $C'$ have no common component.

Suppose $C \cap C' = \{p_1, \ldots, p_r\}$, then $B^2 = C.C' \geq r > 0$ would be a contradiction to $B^2 = 0$. Hence, $C \cap C' = \emptyset$. □

**B.11 Lemma**

*Under the hypotheses of Proposition B.1 consider once more the universal family (B.3) together with its projection onto* $\Sigma$*,*




(B.4)

*Then* $S$ *is an irreducible projective surface,* $|B|_a$ *is an irreducible curve, and* $\pi'$ *is surjective.*

**Proof:**

**Step 1:** $S$ is an irreducible projective surface and $\pi'$ is surjective.

The universal property of $|B|_a$ implies that $S$ is an effective Cartier divisor of $|B|_a \times \Sigma$, and thus in particular projective of dimension at least $2 \leq \dim |B|_a + \dim(\Sigma) - 1$. Since $\pi'$ is projective, its image is closed in $\Sigma$ and of dimension 2, hence it is the whole of $\Sigma$, since $\Sigma$ is irreducible.

By Lemma B.10 the fibres of $\pi'$ are all single points, and thus, by **[Har92]** Theorem 11.14, $S$ is irreducible. Moreover, $\dim(S) = \dim(\Sigma) + \dim(\text{fibre}) = 2$.

**Step 2:** $\dim |B|_a = \dim(|B|_a \times \Sigma) - \dim(\Sigma) = \dim(S) + 1 - 2 = 1$.

**Step 3:** $|B|_a$ is irreducible.

Let $V$ be any irreducible component of $|B|_a$ of dimension one, then we have a universal family over $V$ and the analogue of Step 1 for $V$ shows that the curves in $V$ cover $\Sigma$. But then by Lemma B.10 there can be no further curve in $|B|_a$, since any further curve would necessarily have a non-empty intersection with one of the curves in $V$. □

**B.12 Lemma**
*Let's consider the following commutative diagram of projective morphisms*

$$\begin{array}{ccc} S & \xrightarrow{\pi'} & \Sigma \\ {\scriptstyle\pi}\big\downarrow & \nwarrow & \big\uparrow{\scriptstyle\varphi'} \\ |B|_a & \xleftarrow[\varphi]{} & S_{red} \end{array} \tag{B.5}$$

*The map* $\varphi' : S_{red} \longrightarrow \Sigma$ *is birational.*[21]

**Proof:** Since $S_{red}$ and $\Sigma$ are irreducible and reduced, and since $\varphi'$ is surjective, we may apply [**Har77**] III.10.5, and thus there is an open dense subset $U \subseteq S_{red}$ such that $\varphi'_{|} : U \to \Sigma$ is smooth. Hence, in particular $\varphi'_{|}$ is flat and the fibres are single reduced points.[22] Since $\varphi'_{|} : U \to \varphi'(U)$ is projective and quasi-finite, it is finite (cf. [**Har77**] Ex. III.11.2), and it follows from Proposition B.2 that $\varphi'_{|}$ is an isomorphism onto its image, i. e. $\varphi'$ is birational. □

**B.13 Lemma**
*If* $\psi : \Sigma \dashrightarrow S_{red}$ *denotes the rational inverse of the map* $\varphi'$ *in (B.5), then* $\psi$ *is indeed a morphism, i. e.* $\varphi'$ *is an isomorphism.*

**Proof:** By Lemma B.10 the fibres of $\varphi'$ over the possible points of indeterminacy of $\varphi'$ are just points, and thus the result follows from [**Bea83**] Lemma II.9. □

**B.14 Lemma**
*The map* $g : \Sigma \to |B|_a$ *assigning to each point* $p \in \Sigma$ *the unique curve* $C \in |B|_a$ *with* $p \in C$ *is a morphism, and is thus a fibration whose fibres are the curves in* $|B|_a$.

**Proof:** We just have $g = \varphi \circ \psi$. □

**Proof of Proposition B.1:** Let $\nu : H \to |B|_a$ be the normalisation of the irreducible curve $|B|_a$. Then $H$ is a smooth irreducible curve.

Moreover, since $\Sigma$ is irreducible and smooth, and since $g : \Sigma \to |B|_a$ is surjective, $g$ factorises over $H$, i. e. we have the following commutative diagram

$$\begin{array}{ccc} \Sigma & \xrightarrow{g} & |B|_a \\ {\scriptstyle\exists f}\big\downarrow & \nearrow{\scriptstyle\nu} & \\ H. & & \end{array}$$

Then $f$ is the desired fibration. □

[21] The proof uses that the characteristic of the ground field is zero, even though one might perhaps avoid this.

[22] By definition $\varphi'_{|}$ is étale, and hence the completed local rings of the fibres are isomorphic to the completed local rings of their base points and hence regular. But then the local rings themselves are regular and thus reduced.

## C. Some Facts used for the Proof of the Lemma of Geng Xu

In this section we are, in particular, writing down some identifications of certain sheaves respectively of their global sections. Doing this we try to be very formal. However, in a situation of the kind $X \overset{i}{\hookrightarrow} Y \overset{\pi}{\longrightarrow} Z$ we do not distinguish between $\mathcal{O}_X$ and $i_*\mathcal{O}_X$, or between $\pi$ and the restriction of $\pi$ to $X$.

**C.1 Lemma**

*Let* $\varphi(x, y, t) = \sum_{i=0}^{\infty} \varphi_i(x, y) \cdot t^i \in \mathbb{C}\{x, y, t\}$ *with* $\varphi(x, y, t) \in (x, y)^m$ *for every fixed* $t$ *in some small disc* $\Delta$ *around* $0$. *Then* $\varphi_i(x, y) \in (x, y)^m$ *for every* $i \in \mathbb{N}_0$.

**Proof:** We write the power series as $\varphi = \sum_{\alpha+\beta=0}^{\infty} \left( \sum_{i=0}^{\infty} c_{\alpha,\beta,i} \cdot t^i \right) \cdot x^\alpha y^\beta$.
$\varphi(x, y, t) \in (x, y)^m$ for every $t \in \Delta$ implies

$$\sum_{i=0}^{\infty} c_{\alpha,\beta,i} \cdot t^i = 0 \ \ \forall \ \alpha + \beta < m \text{ and } t \in \Delta.$$

The identity theorem for power series in $\mathbb{C}$ then implies that

$$c_{\alpha,\beta,i} = 0 \ \ \forall \ \alpha + \beta < m \text{ and } i \geq 0.$$

□

**C.2 Lemma**

*Let* $X$ *be a noetherian scheme,* $i : C \hookrightarrow X$ *a closed subscheme,* $\mathcal{F}$ *a sheaf of modules on* $C$*, and* $\mathcal{G}$ *a sheaf of modules on* $X$*. Then*

(C.1) $\quad i_*\mathcal{F} \cong i_*\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_C,$

(C.2) $\quad H^0(C, \mathcal{F}) = H^0(X, i_*\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_C),$

(C.3) $\quad \mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{O}_C \cong i_*i^*(\mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{O}_C)$, *and*

(C.4) $\quad H^0\big(C, i^*(\mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{O}_C)\big) = H^0(X, \mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{O}_C).$

**Proof:**

(C.1) For $U \subseteq X$ open, we define

$$\begin{array}{ccccc} \Gamma(U, i_*\mathcal{F}) & \to & \Gamma(U, i_*\mathcal{F}) \otimes_{\Gamma(U,\mathcal{O}_X)} \Gamma(U, \mathcal{O}_C) & \subseteq & \Gamma(U, i_*\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_C) \\ s & \mapsto & s \otimes 1. & & \end{array}$$

This morphism induces on the stalks the isomorphism

$$i_*\mathcal{F}_x = \left\{ \begin{array}{ccc} \mathcal{F}_x, \text{ (if } x \in C) & = & \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{X,x}/I_{C,x} \\ 0, \text{ (else)} & = & 0 \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{X,x}/I_{C,x} \end{array} \right\} \cong i_*\mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{C,x},$$

where $I_{C,x}$ is the ideal defining $C$ in $X$ locally at $x$.

(C.2) The identification (C.1) together with **[Har77]** III.2.10 gives:

$$H^0(C, \mathcal{F}) = H^0(X, i_*\mathcal{F}) = H^0(X, i_*\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_C).$$

(C.3) The adjoint property of $i_*$ and $i^*$ together with $i^*i_* \cong \mathrm{id}$ gives rise to the following isomorphisms:

$$\mathbf{End}\left(i_*i^*(\mathcal{G}\otimes\mathcal{O}_C)\right) \cong \mathbf{Hom}\left(i^*i_*i^*(\mathcal{G}\otimes\mathcal{O}_C), i^*(\mathcal{G}\otimes\mathcal{O}_C)\right)$$
$$\cong \mathbf{End}\left(i^*(\mathcal{G}\otimes\mathcal{O}_C)\right) \cong \mathbf{Hom}\left(\mathcal{G}\otimes\mathcal{O}_C, i_*i^*(\mathcal{G}\otimes\mathcal{O}_C)\right).$$

That means, that the identity morphism on $i_*i^*(\mathcal{G}\otimes\mathcal{O}_C)$ must correspond to an isomorphism from $\mathcal{G}\otimes\mathcal{O}_C$ to $i_*i^*(\mathcal{G}\otimes\mathcal{O}_C)$ via these identifications.

(C.4) follows from (C.3) and once more [**Har77**] III.2.10.

□

**C.3 Corollary**

*In the situation of Lemma II.2.4 we have:*

(C.5) $$H^0\big(C, \pi_*\mathcal{O}_{\widetilde{C}}(E)\otimes_{\mathcal{O}_C}\mathcal{O}_C(C)\big) = H^0\big(\Sigma, \pi_*\mathcal{O}_{\widetilde{C}}(E)\otimes_{\mathcal{O}_\Sigma}\mathcal{O}_C(C)\big),$$ *and*

(C.6) $$H^0\big(C, \pi_*\mathcal{O}_{\widetilde{\Sigma}}(E)\otimes_{\mathcal{O}_\Sigma}\mathcal{O}_C(C)\big) = H^0\big(\Sigma, \pi_*\mathcal{O}_{\widetilde{\Sigma}}(E)\otimes_{\mathcal{O}_\Sigma}\mathcal{O}_C(C)\big).$$

**Proof:** We denote by $j : \widetilde{C} \hookrightarrow \widetilde{\Sigma}$ and $i : C \hookrightarrow \Sigma$ respectively the given embeddings.

(C.5) By (C.2) in Lemma C.2 we have:

$$H^0\big(C, \pi_*\mathcal{O}_{\widetilde{C}}(E)\otimes_{\mathcal{O}_C}\mathcal{O}_C(C)\big) = H^0\Big(\Sigma, i_*\big(\pi_*\mathcal{O}_{\widetilde{C}}(E)\otimes_{\mathcal{O}_C}\mathcal{O}_C(C)\big)\otimes_{\mathcal{O}_\Sigma}\mathcal{O}_C\Big).$$

By the projection formula this is just equal to:

$$H^0\Big(\Sigma, \big(i_*\pi_*\mathcal{O}_{\widetilde{C}}(E)\otimes_{\mathcal{O}_\Sigma}\mathcal{O}_\Sigma(C)\big)\otimes_{\mathcal{O}_\Sigma}\mathcal{O}_C\Big) = H^0\big(\Sigma, \pi_*j_*\mathcal{O}_{\widetilde{C}}(E)\otimes_{\mathcal{O}_\Sigma}\mathcal{O}_C(C)\big)$$
$$=_{\mathrm{def}} H^0\big(\Sigma, \pi_*\mathcal{O}_{\widetilde{C}}(E)\otimes_{\mathcal{O}_\Sigma}\mathcal{O}_C(C)\big).$$

(C.6) Using (C.4) in Lemma C.2 we get:

$$H^0\big(C, \pi_*\mathcal{O}_{\widetilde{\Sigma}}(E)\otimes_{\mathcal{O}_\Sigma}\mathcal{O}_C(C)\big) =_{\mathrm{def}} H^0\Big(C, i^*\big(\pi_*\mathcal{O}_{\widetilde{\Sigma}}(E)\otimes_{\mathcal{O}_\Sigma}\mathcal{O}_C(C)\big)\Big) =$$
$$H^0\big(\Sigma, \pi_*\mathcal{O}_{\widetilde{\Sigma}}(E)\otimes_{\mathcal{O}_\Sigma}\mathcal{O}_C(C)\big).$$

□

**C.4 Lemma**

*Let $\mathcal{F}$ be any coherent sheaf on $\Sigma$. Then the kernel of the natural map*

$$\bigotimes_{i=1}^r \mathcal{J}_{X(m_i;z_i)/\Sigma}\otimes\mathcal{F} \xrightarrow{\ \delta\ } \mathcal{J}_{X(\underline{m};\underline{z})/\Sigma}\cdot\mathcal{F}$$

*has support contained in $\{z_1, \ldots, z_r\}$.*

*Moreover, if $\mathcal{F}$ is locally free, then $\delta$ is an isomorphism.*

**Proof:** We have $\mathrm{Ker}(\delta)_z = \mathrm{Ker}(\delta_z)$ and for $z \notin \{z_1, \ldots, z_r\}$ the map $\delta_z$ is given by

$$\bigotimes_{i=1}^r \mathcal{O}_{\Sigma,z}\otimes\mathcal{F}_z \longrightarrow \mathcal{O}_{\Sigma,z}\cdot\mathcal{F}_z$$
$$f_1\otimes\cdots\otimes f_r\otimes g \longmapsto f_1\cdots f_r\cdot g,$$

and is thus an isomorphism.

Suppose now that $\mathcal{F}$ is locally free and that $z \in \{z_1, \ldots, z_r\}$. Then $\delta_z$ is given by

$$\begin{aligned} \mathcal{O}_{\Sigma,z_i} \otimes \cdots \otimes \mathfrak{m}^{m_i}_{\Sigma,z_i} \otimes \cdots \otimes \mathcal{O}_{\Sigma,z_i} \otimes \mathcal{O}_{\Sigma,z_i} &\longrightarrow \mathfrak{m}^{m_i}_{\Sigma,z_i} \\ f_1 \otimes \cdots \otimes f_r \otimes g &\longmapsto f_1 \cdots f_r \cdot g, \end{aligned}$$

which again is an isomorphism. □

**C.5 Lemma**

*With the notation of Lemma II.2.4 we show that* $\operatorname{supp}\big(\operatorname{Ker}(\gamma)\big) \subseteq \{z_1, \ldots, z_r\}$.

**Proof:** Since $\pi : \widetilde{\Sigma} \setminus \big(\bigcup_{i=1}^r E_i\big) \longrightarrow \Sigma \setminus \{z_1, \ldots, z_r\}$ is an isomorphism, we have for any sheaf $\mathcal{F}$ of $\mathcal{O}_{\widetilde{\Sigma}}$-modules and $y \in \widetilde{\Sigma} \setminus \big(\bigcup_{i=1}^r E_i\big)$:

$$(\pi_*\mathcal{F})_{\pi(y)} = \lim_{\pi(y)\in V} \mathcal{F}(\pi^{-1}(V)) = \lim_{y\in U} \mathcal{F}(U) = \mathcal{F}_y.$$

In particular,

$$\big(\pi_*\mathcal{O}_{\widetilde{\Sigma}}(E) \otimes_{\mathcal{O}_\Sigma} \mathcal{O}_C(C)\big)_{\pi(y)} \cong \mathcal{O}_{\widetilde{\Sigma},y} \otimes_{\mathcal{O}_{\Sigma,\pi(y)}} \mathcal{O}_{C,\pi(y)} \cong \mathcal{O}_{\Sigma,\pi(y)} \otimes_{\mathcal{O}_{\Sigma,\pi(y)}} \mathcal{O}_{C,\pi(y)},$$

and

$$\big(\pi_*\mathcal{O}_{\widetilde{C}}(E) \otimes_{\mathcal{O}_\Sigma} \mathcal{O}_C(C)\big)_{\pi(y)} \cong \mathcal{O}_{\widetilde{C},y} \otimes_{\mathcal{O}_{\Sigma,\pi(y)}} \mathcal{O}_{C,\pi(y)} \cong \mathcal{O}_{C,\pi(y)} \otimes_{\mathcal{O}_{\Sigma,\pi(y)}} \mathcal{O}_{C,\pi(y)}.$$

Moreover, the morphism $\gamma_{\pi(y)}$ becomes under these identifications just the morphism given by $a \otimes \overline{b} = 1 \otimes \overline{ab} \mapsto \overline{a} \otimes \overline{b} = 1 \otimes \overline{ab}$, which is injective. Thus, $0 = \operatorname{Ker}(\gamma_{\pi(y)}) = \operatorname{Ker}(\gamma)_{\pi(y)}$, and $\pi(y) \notin \operatorname{supp}(\operatorname{Ker}(\gamma))$. □

**C.6 Lemma**

*Let* $X$ *be an irreducible noetherian scheme,* $\mathcal{F}$ *a coherent sheaf on* $X$*, and* $s \in H^0(X, \mathcal{F})$ *such that* $\dim\big(\operatorname{supp}(s)\big) < \dim(X)$. *Then* $s \in H^0\big(X, \operatorname{Tor}(\mathcal{F})\big)$.

**Proof:** The multiplication by $s$ gives rise to the following exact sequence:

$$0 \longrightarrow \operatorname{Ker}(\cdot s) \longrightarrow \mathcal{O}_X \stackrel{\cdot s}{\longrightarrow} \mathcal{F}.$$

Since $\mathcal{O}_X$ and $\mathcal{F}$ are coherent, so is $\operatorname{Ker}(\cdot s)$, and hence $\operatorname{supp}\big(\operatorname{Ker}(\cdot s)\big)$ is closed in $X$. Now,

$$\begin{aligned} \operatorname{supp}(\operatorname{Ker}(\cdot s)) &= \{z \in X \mid \exists\, 0 \neq r_z \in \mathcal{O}_{X,z} : r_z \cdot s_z = 0\} \\ &= \{z \in X \mid s_z \in \operatorname{Tor}(\mathcal{F}_z)\}. \end{aligned}$$

But then the complement $\{z \in X \mid s_z \notin \operatorname{Tor}(\mathcal{F}_z)\}$ is open and is contained in $\operatorname{supp}(s)$ (since $s_z = 0$ implies that $s_z \in \operatorname{Tor}(\mathcal{F}_z)$), and is thus empty since $X$ is irreducible and $\operatorname{supp}(s)$ of lower dimension. □

## D. Some Facts on Divisors on Curves

**D.1 Remark**

Let $C = C_1 \cup \ldots \cup C_k$ be a reduced curve on a smooth projective surface $\Sigma$ over $\mathbb{C}$, where the $C_i$ are irreducible, and let $\mathcal{L}$ be a line bundle on $C$. Then we define the *degree* of $\mathcal{L}$ with the aid of the normalisation $\nu : C' \to C$. We have $H^2(C, \mathbb{Z}) \cong \bigoplus_{i=1}^k H^2(C_i', \mathbb{Z}) = \mathbb{Z}^k$, and thus the image of $\mathcal{L}$ in $H^2(C, \mathbb{Z})$, which is the first Chern class of $\mathcal{L}$, can be viewed as a vector $(l_1, \ldots, l_k)$ of integers, and we may define the degree of $\mathcal{L}$ by

$$\deg(\mathcal{L}) := l_1 + \cdots + l_k.$$

In particular, if $C$ is irreducible, we get:

$$\deg(\mathcal{L}) = \deg(\nu^*\mathcal{L}) = c_1(\nu^*\mathcal{L}).$$

Since $H^0(C, \mathcal{L}) \neq 0$ implies that $H^0(C', \nu^*\mathcal{L}) \neq 0$, and since the existence of a non-vanishing global section of $\nu^*\mathcal{L}$ on the smooth curve $C'$ implies that the corresponding divisor is effective, we get the following lemma. (cf. **[BPV84]** Section II.2)

**D.2 Lemma**

*Let $C$ be an irreducible reduced curve on a smooth projective surface $\Sigma$, and let $\mathcal{L}$ be a line bundle on $C$. If $H^0(C, \mathcal{L}) \neq 0$, then $\deg(\mathcal{L}) \geq 0$.*

When studying divisors on geometrically ruled surfaces or on products of curves, we need the following estimation of the number of independent sections of a divisor on a curve.

**D.3 Lemma**

*Let $C$ be a smooth projective curve over $\mathbb{C}$ of genus $g$ and let $\mathfrak{c} \in \mathrm{Div}(C)$ with $\deg(\mathfrak{c}) \geq 0$. Then $h^0(C, \mathfrak{c}) \leq \deg(\mathfrak{c}) + 1$.*

**Proof:** Let us suppose that $\mathfrak{c}$ is effective, since otherwise $h^0(C, \mathfrak{c}) \leq 1$ anyway. Then the inclusion

$$\mathcal{O}_C(K_C - \mathfrak{c}) \stackrel{+\mathfrak{c}}{\hookrightarrow} \mathcal{O}_C(K_C)$$

implies that $h^0(C, K_C - \mathfrak{c}) \leq h^0(C, K_C) = g$. Hence by the Riemann-Roch Formula we get

$$h^0(C, \mathfrak{c}) = \deg(\mathfrak{c}) + 1 - g + h^0(C, K_C - \mathfrak{c}) \leq \deg(\mathfrak{c}) + 1.$$

□

## E. Some Facts on Divisors on Surfaces

In this section we gather some simple facts on divisors on smooth surfaces which are used throughout the paper.

**E.1 Lemma**

*If $\mathrm{NS}(\Sigma) = L \cdot \mathbb{Z}$, then either $L$ or $-L$ is ample.*

**Proof:** Since $\Sigma$ is projective, there exists some very ample divisor $H$, and by assumption $H \sim_a h \cdot L$ for some $0 \neq h \in \mathbb{Z}$. W. l. o. g. $h > 0$. Thus

$$L^2 = \frac{1}{h^2} \cdot H^2 > 0,$$

and for any irreducible curve $C \subset \Sigma$

$$L.C = \frac{1}{h} \cdot H.C > 0.$$

Therefore, by the Nakai-Moishezon Criterion $L$ is ample. □

**E.2 Lemma**

*Let $\Sigma$ be any smooth projective surface and let $A, B \in \mathrm{Div}(\Sigma)$ be such that $A$ is effective, reduced and $-A^2 > A.B \geq 0$. Then*

(a) $h^0(\Sigma, A + B) = h^0(\Sigma, B)$.

(b) *If* $(B - K_\Sigma).A > 0$, *then* $h^1(\Sigma, A + B) = h^1(\Sigma, B)$.

**Proof:**

(a) We note that the divisor $\mathcal{O}_A(A + B)$ has degree $A.(A+B) < 0$ and is thus not effective, i. e. $h^0\big(A, \mathcal{O}_A(A + B)\big) = 0$.
Considering the exact sequence

$$0 \longrightarrow \mathcal{O}_\Sigma(B) \xrightarrow{+A} \mathcal{O}_\Sigma(A + B) \longrightarrow \mathcal{O}_A(A + B) \longrightarrow 0, \qquad \text{(E.1)}$$

the statement is implied by its long exact cohomology sequence

$$0 \longrightarrow H^0(\Sigma, B) \longrightarrow H^0(\Sigma, A + B) \longrightarrow H^0\big(A, \mathcal{O}_A(A + B)\big) = 0.$$

(b) By assumption $0 > A.(K_\Sigma - B) = \deg\big(\mathcal{O}_A(K_\Sigma - B)\big) = \deg\big(\mathcal{O}_A(K_A) \otimes \mathcal{O}_A(-A - B)\big)$. Hence

$$0 = h^0\big(A, \mathcal{O}_A(K_A) \otimes \mathcal{O}_A(-A - B)\big) = h^1\big(A, \mathcal{O}_A(A + B)\big).$$

Hence once more the long exact cohomology sequence of (E.1) finishes the proof:

$$0 \longrightarrow H^1(\Sigma, B) \longrightarrow H^1(\Sigma, A + B) \longrightarrow H^1\big(A, \mathcal{O}_A(A + B)\big) = 0.$$

□

**E.3 Remark**

Note, that the condition *reduced* in Lemma E.2 is necessary.

First of all, if $E \subset \Sigma$ is a smooth rational curve with $E^2 = -1$, then the restriction sequence

$$0 \to \mathcal{O}_E(-E) \to \mathcal{O}_{2E} \to \mathcal{O}_E \to 0$$

tensored by $E$ shows that

$$h^0\big(2E, \mathcal{O}_{2E}(E)\big) = h^0(E, \mathcal{O}_E) = 1$$

even though $\deg\big(\mathcal{O}_{2E}(E)\big) = 2E.E = -2 < 0$. Therefore, the argument given in the proof does not work.

Moreover, suppose the $\Sigma$ is the blow-up of $\mathbb{P}^2_{\mathbb{C}}$ in a single point with $E$ the exceptional divisor and $H$ the strict transform of a line. Let $A = 2E$ and $B = H - E$, then $-A^2 = 4 > 2 = A.B$. That is, the prerequisites of part a. of the lemma are fulfilled. However,

$$h^0(\Sigma, A+B) = h^0(\Sigma, H+E) = h^0(\Sigma, H) = 3 \neq 2 = h^0(\Sigma, H-E) = h^0(\Sigma, B).$$

Note also, that if $A = nC$ for some irreducible curve $C$ such that $-C^2 > C.B$ then part a. of Lemma E.2 still holds true. □

We reformulate the Hodge Index Theorem from [**BPV84**] in the following well known equivalent form which suits our purposes better.

**E.4 Theorem (Hodge Index Theorem)**
*Let* $H, D \in \mathrm{Div}(\Sigma) \otimes \mathbb{Q}$ *be* $\mathbb{Q}$*-divisors with* $H^2 > 0$. *Then* $H^2 \cdot D^2 \leq (H.D)^2$ *with equality if and only if* $D \sim_n \alpha H$ *for some* $\alpha \in \mathbb{Q}$.

**Proof:** We set $\alpha = -\frac{H.D}{H^2} \in \mathbb{Q}$. Then $H.(D + \alpha H) = 0$ and by the standard Hodge Index Theorem (cf. [**BPV84**] Corollary 2.15) we have

$$0 \geq (D + \alpha H)^2 = H^2 \cdot \alpha^2 + 2 \cdot H.D \cdot \alpha + D^2.$$

We define a polynomial function

$$\varphi : \mathbb{R} \to \mathbb{R} : t \mapsto H^2 \cdot t^2 + 2 \cdot H.D \cdot t + D^2.$$

Since the leading term $H^2$ is positive, there is some value $\beta \in \mathbb{R}$ such that $\varphi(\beta) > 0$. However, $\varphi(\alpha) \leq 0$, and thus by the Intermediate Value Theorem $\varphi$ has at least one zero. This, in turn, implies that the discriminant of the polynomial $\varphi$ is non-negative, i. e.

$$4 \cdot (D.H)^2 - 4 \cdot D^2 \cdot H^2 \geq 0.$$

Moreover, equality holds precisely if $\varphi$ has just one zero and is thus always non-negative, in particular we must have $0 = \varphi(\alpha) = H.(D + \alpha H)$. But then by the standard Hodge Index Theorem we get $D + \alpha H \sim_n 0$, i. e. $D \sim_n -\alpha H$. □

Finally we state the Theorem of Bertini in a version which is suitable for our purposes and which can be found in [**Wae73**] § 47, Satz 3 and Satz 4, and we deduce some corollaries.

**E.5 Theorem (Bertini)**
*Let* $\Sigma$ *be a smooth projective surface,* $D \in \mathrm{Div}(\Sigma)$ *a divisor and* $\mathcal{D} \subseteq |D|_l$ *a linear system without fixed part. Suppose that every curve in* $\mathcal{D}$ *is reducible.*

(a) *Two general curves in* $\mathcal{D}$ *intersect only in the base points of* $\mathcal{D}$.

(b) *There exists an irreducible* one-*dimensional algebraic family* $\mathcal{F}$ *of curves in* $\Sigma$*, whose general element is irreducible, such that the irreducible components of any curve in* $\mathcal{D}$ *belong to* $\mathcal{F}$.

(c) *There is a very general subset* $U$ *of* $\Sigma$ *such that through every point of* $U$ *there passes a unique curve in* $\mathcal{F}$.

(d) *The base locus of $\mathcal{F}$ and $\mathcal{D}$ coincide. In particular, if the base locus is non-empty, the elements of $\mathcal{D}$ are all singular.*

**Proof:** Part (a), (b) and (c) are just Satz 3 and Satz 4 in § 47 of [**Wae73**]. For Part (d) we note that if $p$ is in the base locus of $\mathcal{F}$, then every curve in $\mathcal{F}$ passes through $p$ and thus by Part (b) also every curve in $\mathcal{D}$ passes through $p$, i. e. $p$ belongs to the base locus of $\mathcal{D}$. Now suppose that $p$ was a point in the base locus of $\mathcal{D}$ which does not belong to every curve in $\mathcal{F}$. The set $\mathcal{F}_p$ of curves in $\mathcal{F}$ passing through $p$ is a strict, closed subvariety of $\mathcal{F}$, and since $\mathcal{F}$ is irreducible of dimension one, $\mathcal{F}_p = \{C_1, \ldots, C_r\}$ is a finite set. Since $\mathcal{D}$ has no fixed component and is irreducible as a linear system, the subsystem $\mathcal{D}_{C_i}$ of curves in $\mathcal{D}$ having $C_i$ as component is a closed subsystem of lower dimension. But then the union $\bigcup_{i=1}^r \mathcal{D}_{C_i}$ is not all of $\mathcal{D}$, in particular there is a curve $C \in \mathcal{D}$ containing none of the $C_i$. By Part (b) $C$ is composed of curves in $\mathcal{F}$, and thus $C$ does not contain the point $p$ in contradiction to $p$ being a basepoint of $\mathcal{D}$. Thus, every base point of $\mathcal{D}$ is also a base point of $\mathcal{F}$.

If the base locus is non-empty and every element of $\mathcal{D}$ has at least two irreducible components, then they will intersect in the base locus and the element of $\mathcal{D}$ is thus singular. □

**E.6 Remark**
Let $\Sigma \subset \mathbb{P}^n_{\mathbb{C}}$ be a (not necessarily) smooth projective surface and let $z \in \Sigma$ be fixed. We consider the Fano variety $F_1(\Sigma)$ of lines in $\mathbb{P}^n_{\mathbb{C}}$ contained in $\Sigma$. This is a closed subvariety of the Grassmannian $\mathbb{G}(1, n)$. (Cf. [**Har92**] Ex. 6.19.)

Now, let's restrict to the closed subvariety of lines in $F_1(\Sigma)$ containing $z$ and consider the associated universal family

$$\widehat{F}_z = \{(l, p) \in F_1(\Sigma) \times \Sigma \mid z, p \in l\} \longrightarrow \Sigma : (l, p) \mapsto p.$$

The image, say $F_z$, in $\Sigma$ is closed since the projection is a projective morphism. Thus it is either all of $\Sigma$, or a finite union of lines – possibly none.

$F_z$ is all of $\Sigma$ if $\Sigma$ is linear in $\mathbb{P}^n_{\mathbb{C}}$, and it might happen when $z$ is a singular point, e. g. $\Sigma = \{x_0x_1 - x_2^2 = 0\} \subset \mathbb{P}^3_{\mathbb{C}}$ and $z = (0 : 0 : 0 : 1)$.

**E.7 Lemma**
*Let $\Sigma \subseteq \mathbb{P}^n_{\mathbb{C}}$ a non-linear projective surface. If $z \in \Sigma$ is not singular, then a general secant line through $z$ is not contained in $\Sigma$. In particular, $F_z$ is a finite union of lines.*

**Proof:** Since $z \in \Sigma$ is regular, $\mathcal{O}_{\Sigma,z} \cong \mathcal{O}_{\mathbb{P}^n_{\mathbb{C}},z}/(f_1, \ldots, f_{n-2})$, where $(f_1, \ldots, f_{n-2})$ is a regular sequence in $\mathcal{O}_{\mathbb{P}^n_{\mathbb{C}},z}$. For generic linear forms $l, l' \in \mathbb{C}[x_0, \ldots, x_n]_1$ through the point $z$, the sequence $(f_1, \ldots, f_{n-2}, l, l')$ will then be regular. This in particular means that the linear $n-2$-space $H = V(l, l')$ intersects $\Sigma$ in $z$ transversally, i. e. with intersection multiplicity 1 and $z$ is an isolated point of $H \cap \Sigma$. Since $\Sigma$ is not linear $H$ intersects $\Sigma$ in at least one more point $z'$ by the

Theorem of Bézout, and then $\overline{zz'} \not\subset \Sigma$. This proves the first assertion, and by Remark E.6 we also know that $F_z$ is a finite union of lines. □

**E.8 Corollary**

*Let $\Sigma \subset \mathbb{P}^n_{\mathbb{C}}$ be a smooth projective surface embedded via the very ample divisor $L$. For $z \in \Sigma$ we denote by $B_z$ the set $F_z$ if $\Sigma$ is not $\mathbb{P}^2_{\mathbb{C}}$ and the empty set otherwise.*

(a) *For $z \in \Sigma$ and $z^* \in \Sigma \setminus B_z$ then a general hyperplane section through $z$ and $z^*$ is smooth and irreducible.*

(b) *Given distinct points $z_1, \ldots, z_r \in \Sigma$, the set $U$ of points $z \in \Sigma$ such that for a point $z^* \in \Sigma \setminus B_z$ there is a smooth and irreducible hyperplane section through $z$ and $z^*$ containing at most one of the $z_j$ for $j = 1, \ldots, r$ is open and dense in $\Sigma$.*

**Proof:** If $\Sigma = \mathbb{P}^2_{\mathbb{C}}$ then in Part (a) and (b) a line through $z$ and $z^*$ will do. We may thus assume that $\Sigma$ is not the projective plane and is thus not a linear subspace of $\mathbb{P}^n_{\mathbb{C}}$.

(a) Consider the linear system $\mathcal{L}_{z,z^*}$ of hyperplane sections in $|L|_l$ which pass through $z$ and $z^*$. The only base points of the linear system are the points on the line $\overline{zz^*}$, which by assumption is a finite number of points.
By the Theorem of Bertini (cf. **[Har77]** II.8.18 and III.10.9.2) the general element of $\mathcal{L}_{z,z^*}$ will be smooth outside the base locus. However, since the local rings of $\Sigma$ at the finite number of base points are regular, i. e. $\mathcal{O}_{\mathbb{P}^n_{\mathbb{C}},z'}$ modulo some regular sequence $(f_1, \ldots, f_{n-2})$, a general linear form $l \in \mathbb{C}[x_0, \ldots, x_n]_1$ will be a regular element in all of them. That means a general hyperplane section will also be smooth in the finite number of base points.
It remains to show that the general hyperplane section will also be irreducible. Suppose the contrary, that is that all hyperplane sections are reducible. By the Theorem of Bertini E.5 then all elements of $\mathcal{L}_{z,z^*}$ would be singular, since the base locus is non-empty.

(b) Let $U = \Sigma \setminus \bigcup_{i \neq j} \overline{z_i z_j}$. Then for $z \in U$ and $z^* \in \Sigma \setminus B_z$ a general hyperplane section through $z$ and $z^*$ will be irreducible and smooth. However, by the choice of $z$ the points $z_i$ and $z_j$ do not both belong to the line $\overline{zz^*}$, hence a general hyperplane section through $z$ and $z^*$ will not contain both $z_i$ and $z_j$.

□

**E.9 Corollary**

*Let $L$ be very ample over $\mathbb{C}$ on the smooth projective surface $\Sigma$.*

(a) *There is a general subset $U \subset \Sigma \times \Sigma$ such that for $(z, z^*) \in U$, there is a smooth connected curve through $z$ and $z^*$ in $|L|_l$. Indeed, a general curve in $|L|_l$ through $z$ and $z^*$ will be so.*

(b) *Given distinct points* $z_1, \ldots, z_r \in \Sigma$*, the set* $U$ *of points* $z \in \Sigma$ *such that for a general point* $z^* \in \Sigma$ *there is a smooth connected curve through* $z$ *and* $z^*$ *in* $|L|_l$ *containing at most one of the* $z_j$ *for* $j = 1, \ldots, r$ *is open and dense in* $\Sigma$*.*

**Proof:** Considering the embedding $\Sigma \subseteq \mathbb{P}^n_{\mathbb{C}}$ defined by $L$ the curves in $|L|_l$ are in one-to-one correspondence with the hyperplane sections. The result thus follows from Lemma E.8. □

**E.10 Remark**

To be more precise, for two points $z, z^* \in \Sigma$ the linear system $|L|_l$ in Lemma E.9 does not contain a smooth and irreducible curve $C$ trough $z$ and $z^*$ if and only if the linear system has a fix component through $z$ and $z^*$. Having embedded the surface $\Sigma$ into $\mathbb{P}^n_{\mathbb{C}}$ via $L$ this means that the secant line through $z$ and $z^*$ lies in $\Sigma$.

If we consider e. g. $\Sigma = \mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$ with $L = \mathcal{O}_\Sigma(1,1)$, then for two points on the lines $\{p\} \times \mathbb{P}^1_{\mathbb{C}}$ respectively on $\mathbb{P}^1_{\mathbb{C}} \times \{p\}$ there is no smooth curve in $|L|_l$ through the two points.

## F. The Hilbert Scheme $\mathrm{Hilb}^n_\Sigma$

$\mathrm{Hilb}^n_\Sigma$ denotes the Hilbert scheme of zero-dimensional schemes of degree $n$.

**F.1 Lemma**

*Let* $D \in \mathrm{Div}(\Sigma)$ *be a divisor and* $X \subset \Sigma$ *be a zero-dimensional scheme of degree* $n$ *with* $h^1\big(\Sigma, \mathcal{J}_{X/\Sigma}(D)\big) = 0$*. Then there is an open, dense, irreducible neighbourhood* $U = U(X) \subseteq \mathrm{Hilb}^n_\Sigma$ *of* $X$ *such that* $h^1\big(\Sigma, \mathcal{J}_{Y/\Sigma}(D)\big) = 0$ *for any* $Y \in U$*.*

**Proof:** We consider the universal family of $\mathrm{Hilb}^n_\Sigma$:

$$\begin{array}{ccccc} \mathcal{X} & \hookrightarrow & \mathrm{Hilb}^n_\Sigma \times \Sigma & \xrightarrow{\mathrm{pr}_\Sigma} & \Sigma, \\ {\scriptstyle \pi}\downarrow & \swarrow{\scriptstyle \mathrm{pr}} & & & \\ \mathrm{Hilb}^n_\Sigma & & & & \end{array} \tag{F.1}$$

where pr and $\mathrm{pr}_\Sigma$ denote the canonical projections. In particular $\pi$ and pr are flat, that is $\mathcal{O}_{\mathrm{Hilb}^n_\Sigma \times \Sigma}$ and $\mathcal{O}_{\mathcal{X}}$ are flat $\mathcal{O}_{\mathrm{Hilb}^n_\Sigma}$-modules via pr and $\pi$ respectively. From the exact sequence

$$0 \to \mathcal{J}_{\mathcal{X}} \to \mathcal{O}_{\mathrm{Hilb}^n_\Sigma \times \Sigma} \to \mathcal{O}_{\mathcal{X}} \to 0$$

it thus follows that the ideal sheaf $\mathcal{J}_{\mathcal{X}}$ of $\mathcal{X}$ is flat over $\mathcal{O}_{\mathrm{Hilb}^n_\Sigma}$. Moreover, by Lemma F.3 also the sheaf $\mathcal{J}_{\mathcal{X}}(D) := \mathcal{J}_{\mathcal{X}} \otimes_{\mathcal{O}_{\mathrm{Hilb}^n_\Sigma \times \Sigma}} \mathrm{pr}^*_\Sigma \mathcal{O}_\Sigma(D)$ is flat over $\mathcal{O}_{\mathrm{Hilb}^n_\Sigma}$. The Semicontinuity Theorem (cf. **[Har77]** III.12.8) therefore implies that the function

$$\alpha : \mathrm{Hilb}^n_\Sigma \to \mathbb{Z} : Y \mapsto \dim_{k(Y)} H^1\big(\mathrm{pr}^{-1}(Y), \mathcal{J}_{\mathcal{X}}(D)_Y\big)$$

is upper semi continuous.

Now, for $Y \in \mathrm{Hilb}^n_\Sigma$ the restriction $\mathrm{pr}_{\Sigma|} : \mathrm{pr}^{-1}(Y) = \{Y\} \times \Sigma \to \Sigma$ is an isomorphism, and this isomorphism induces

$$Y \cong \{Y\} \times Y = \pi^{-1}(Y) \hookrightarrow \mathrm{pr}^{-1}(Y) = \{Y\} \times \Sigma \cong \Sigma$$

and $\mathcal{J}_{\mathcal{X},Y} \cong \mathrm{pr}_{\Sigma *}\mathcal{J}_{\mathcal{X},Y} = \mathcal{J}_{Y/\Sigma}$. With the aid of the projection formula we deduce

$$\big(\mathcal{J}_{\mathcal{X}}(D)\big)_Y \cong \mathrm{pr}_{\Sigma *}\left(\mathcal{J}_{\mathcal{X}} \otimes_{\mathcal{O}_{\mathrm{Hilb}^n_\Sigma \times \Sigma}} \mathrm{pr}^*_\Sigma \mathcal{O}_\Sigma(D)\right) \cong \mathrm{pr}_{\Sigma *}\mathcal{J}_{\mathcal{X}} \otimes_{\mathcal{O}_\Sigma} \mathcal{O}_\Sigma(D) \cong \mathcal{J}_{Y/\Sigma}(D).$$

Thus $\dim_{k(Y)} H^1\big(\mathrm{pr}^{-1}(Y), \mathcal{J}_{\mathcal{X}}(D)_Y\big) = h^1\big(\Sigma, \mathcal{J}_{Y/\Sigma}(D)\big)$, and the upper semicontinuity of $\alpha$ implies that the set $U = \big\{Y \in \mathrm{Hilb}^n_\Sigma \mid h^1\big(\Sigma, \mathcal{J}_{Y/\Sigma}(D)\big) = 0\big\}$ is a non-empty open subset of $\mathrm{Hilb}^n_\Sigma$. Since the latter is smooth and connected, hence irreducible, $U$ is dense and irreducible. □

**F.2 Lemma**
*Let $\varphi : B \to A$ and $\psi : C \to A$ be morphisms of rings. If the $A$-module $M$ is flat over $B$ and if $N$ is a free $C$-module, then $M \otimes_C N$ is a flat $B$-module.*

**Proof:** Let $0 \to K' \to K \to K'' \to 0$ be an exact sequence of $B$-modules. Since $M$ is flat over $B$, we have

$$0 \to K' \otimes_B M \to K \otimes_B M \to K'' \otimes_B M \to 0$$

is an exact sequence of $B$-modules, and thus via $\varphi$ also of $A$-modules, but then via $\psi$ of $C$-modules. Since $N$ is free, hence flat, over $C$ we moreover have

$$0 \to K' \otimes_B M \otimes_C N \to K \otimes_B M \otimes_C N \to K'' \otimes_B M \otimes_C N \to 0$$

is an exact sequence of $C$-modules, and thus it is exact as a sequence of $B$-modules. □

**F.3 Lemma**
*Let $f : X \to Y$ and $g : X \to Z$ be two morphisms of schemes. If the $\mathcal{O}_X$-module $\mathcal{F}$ is flat over $\mathcal{O}_Y$ and if $\mathcal{G}$ is a locally free $\mathcal{O}_Z$-module, then $\mathcal{F} \otimes_{\mathcal{O}_X} g^*\mathcal{G}$ is a flat $\mathcal{O}_Y$-module.*

**Proof:** Let $x \in X$ be given. We set $y = f(x)$, $z = g(x)$, $A = \mathcal{O}_{X,x}$, $B = \mathcal{O}_{Y,y}$, $C = \mathcal{O}_{Z,z}$, $\varphi = f^\# : B \to A$, $\psi = g^\# : C \to A$, $M = \mathcal{F}_x$ and $N = \mathcal{G}_z$. By assumption $M$ is flat over $B$ and $N$ is free over $C$. Moreover,

$$\big(\mathcal{F} \otimes_{\mathcal{O}_X} g^*\mathcal{G}\big)_x = \big(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X \otimes_{g^{-1}\mathcal{O}_Z} g^{-1}\mathcal{G}\big)_x = \mathcal{F}_x \otimes_{\mathcal{O}_{Z,z}} \mathcal{G}_z = M \otimes_C N.$$

Thus the statement follows from Lemma F.2 □

## G. Examples of Surfaces

### G.a. Geometrically Ruled Surfaces

Let $\Sigma = \mathbb{P}(\mathcal{E}) \xrightarrow{\pi} C$ be a geometrically ruled surface with normalised bundle $\mathcal{E}$ (in the sense of [**Har77**] V.2.8.1). The Néron–Severi group of $\Sigma$ is

$$\mathrm{NS}(\Sigma) = C_0\mathbb{Z} \oplus F\mathbb{Z},$$

with intersection matrix

$$\begin{pmatrix} -e & 1 \\ 1 & 0 \end{pmatrix},$$

where $F \cong \mathbb{P}^1_{\mathbb{C}}$ is a fibre of $\pi$, $C_0$ a section of $\pi$ with $\mathcal{O}_\Sigma(C_0) \cong \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$, $g = g(C)$ the genus of $C$, $\mathfrak{e} = \Lambda^2\mathcal{E}$ and $e = -\deg(\mathfrak{e}) \geq -g$.[23] For the canonical divisor we have

$$K_\Sigma \sim_a -2C_0 + (2g - 2 - e) \cdot F,$$

where $g = g(C)$ is the genus of the base curve $C$.

In [**Har77**] V.2.18, V.2.20 and V.2.21 there one finds numerical criteria for an algebraic divisor class $aC_0 + bF$ to be ample respectively to contain irreducible curves. The following lemma on the nefness of $aC_0 + bF$ is an easy deduction of these statements.

**G.1 Lemma**

*Let* $C \sim_a aC_0 + bF$ *be given.*

(a) *If* $e \geq 0$*, then* $C$ *is nef if and only if* $a \geq 0$ *and* $b \geq ae$.

(b) *If* $e \leq 0$*, then* $C$ *is nef if and only if* $a \geq 0$ *and* $b \geq \frac{ae}{2}$.

(c) *If* $e \leq 0$*, then any curve* $C \subset \Sigma$ *is nef.*

**Proof:** For (a) and (b) we note that a divisor is nef if and only if it belongs to the closure of the ample cone in $NS(\Sigma) \otimes \mathbb{R}$ (cf. [**IsS96**] p. 162 and p. 175). The results thus follow from [**Har77**] V.2.20 b. and V.2.21 b.

It suffices to show (c) for irreducible curves, since the sum of nef divisors is nef. If, however, $C$ is irreducible, then by [**Har77**] V.2.20 a. and V.2.21 a. the divisor is nef in view of (a) and (b).

□

In Chapter II we have to examine special irreducible curves on $\Sigma$. Thus the following Lemma is of interest.

**G.2 Lemma**

*Let* $B \in |aC_0 + bF|_a$ *be an irreducible curve with* $B^2 = 0$ *and* $\dim |B|_a \geq 1$*. Then we are in one of the following cases*

(G.1) $a = 0,\ b = 1,$ *and* $B \sim_a F$,

(G.2) $e = 0,\ a \geq 1,\ b = 0,$ *and* $B \sim_a aC_0$, *or*

(G.3) $e < 0,\ a \geq 2,\ b = \frac{ae}{2} < 0,$ *and* $B \sim_a aC_0 + \frac{ae}{2} \cdot F$.

*Moreover, if* $a = 1$*, then* $\Sigma \cong C_0 \times \mathbb{P}^1_{\mathbb{C}}$.

[23] By [**Nag70**] Theorem 1 there is some section $D \sim_a C_0 + bF$ with $g \geq D^2 = 2b - e$. Since $D$ is irreducible, by [**Har77**] V.2.20/21 $b \geq 0$, and thus $-g \leq e$.

**Proof:** Since $B$ is irreducible, we have

$$0 \leq B.F = a \quad \text{and} \quad 0 \leq B.C_0 = b - ae. \tag{G.4}$$

If $a = 0$, then $|B|_a = |bF|_a$, but since the general element of $|B|_a$ is irreducible, $b$ has to be one, and we are in case (G.1).

We, therefore, may assume that $a \geq 1$. Since $B^2 = 0$ we have

$$0 = B^2 = 2a\left(b - \frac{ae}{2}\right), \quad \text{hence} \quad b = \frac{ae}{2}. \tag{G.5}$$

Combining this with (G.4) we get $e \leq 0$.

Moreover, if $e = 0$, then of course $b = 0$, while, if $e < 0$, then $a \geq 2$ by **[Har77]** V.2.21, since otherwise $b$ would have to be non-negative. This brings us down to the cases (G.2) and (G.3).

It remains to show, that $B.F = a = 1$ implies $\Sigma \cong C_0 \times \mathbb{P}^1_{\mathbb{C}}$. But by assumption the elements of $|B|_a$ are disjoint sections of the fibration $\pi$. Thus, by Lemma G.3, $\Sigma \cong C \times \mathbb{P}^1_{\mathbb{C}}$. □

**G.3 Lemma**

*If $\pi : \Sigma \to C$ has three disjoint sections, then $\Sigma$ is isomorphic to $C \times \mathbb{P}^1_{\mathbb{C}}$ as a ruled surface, i. e. there is an isomorphism $\alpha : \Sigma \to C \times \mathbb{P}^1_{\mathbb{C}}$ such that the following diagram is commutative:*

$$\begin{array}{ccc} \Sigma & \xrightarrow{\alpha} & C \times \mathbb{P}^1_{\mathbb{C}} \\ & {\scriptstyle \pi}\searrow \quad \swarrow{\scriptstyle \mathrm{pr}} & \\ & C. & \end{array}$$

**Proof:** $\pi$ is a locally trivial $\mathbb{P}^1_{\mathbb{C}}$-bundle, thus $C$ is covered by a finite number of open affine subsets $U_i \subset C$ with trivialisations

$$\begin{array}{ccc} \pi^{-1}(U_i) & \xrightarrow[\varphi_i]{\cong} & U_i \times \mathbb{P}^1_{\mathbb{C}} \\ & {\scriptstyle \pi}\searrow \quad \swarrow{\scriptstyle \mathrm{pr}} & \\ & U_i, & \end{array}$$

which are linear on the fibres.

The three disjoint sections on $\Sigma$, say $S_0$, $S_1$, and $S_\infty$, give rise to three sections $S^i_0$, $S^i_1$, and $S^i_\infty$ on $U_i \times \mathbb{P}^1_{\mathbb{C}}$. For each point $z \in U_i$ there is a unique linear projectivity on the fibre $\{z\} \times \mathbb{P}^1_{\mathbb{C}}$ mapping the three points $p_{0,z} = S^i_0 \cap \left(\{z\} \times \mathbb{P}^1_{\mathbb{C}}\right)$, $p_{1,z} = S^i_1 \cap \left(\{z\} \times \mathbb{P}^1_{\mathbb{C}}\right)$, and $p_{\infty,z} = S^i_\infty \cap \left(\{z\} \times \mathbb{P}^1_{\mathbb{C}}\right)$ to the standard basis $0 \equiv \big(z, (1 : 0)\big)$, $1 \equiv \big(z, (1 : 1)\big)$, and $\infty \equiv \big(z, (0 : 1)\big)$ of $\mathbb{P}^1_{\mathbb{C}} \cong \{z\} \times \mathbb{P}^1_{\mathbb{C}}$. If $p_{0,z} = \big(z, (x_0 : y_0)\big)$, $p_{1,z} = \big(z, (x_1 : y_1)\big)$, and $p_{\infty,z} = \big(z, (x_\infty : y_\infty)\big)$, the projectivity is given by the matrix

$$A = \begin{pmatrix} \frac{(x_0 y_1 - y_0 x_1) y_\infty}{y_0 y_1 x_\infty^2 - y_0 x_1 x_\infty y_\infty - x_0 y_1 x_\infty y_\infty + x_0 x_1 y_\infty} & \frac{(x_0 y_1 - y_0 x_1) x_\infty}{y_0 y_1 x_\infty^2 - y_0 x_1 x_\infty y_\infty - x_0 y_1 x_\infty y_\infty + x_0 x_1 y_\infty} \\ \frac{y_0}{x_0 y_\infty - y_0 x_\infty} & \frac{x_0}{x_0 y_\infty - y_0 x_\infty} \end{pmatrix},$$

whose entries are rational functions in the coordinates of $p_{0,z}$, $p_{1,z}$, and $p_{\infty,z}$. Inserting for the coordinates local equations of the sections, $A$ finally gives rise to an isomorphism of $\mathbb{P}^1_{\mathbb{C}}$-bundles, i. e. a morphism which is a linear isomorphism on the fibres,

$$\alpha_i : U_i \times \mathbb{P}^1_{\mathbb{C}} \to U_i \times \mathbb{P}^1_{\mathbb{C}}$$

mapping the sections $S^i_0$, $S^i_1$, and $S^i_\infty$ to the trivial sections.

The transition maps

$$U_{ij} \times \mathbb{P}^1_{\mathbb{C}} \xrightarrow{\alpha_{i|}^{-1}} U_{ij} \times \mathbb{P}^1_{\mathbb{C}} \xrightarrow{\varphi_{i|}^{-1}} \pi^{-1}(U_{ij}) \xrightarrow{\varphi_{j|}} U_{ij} \times \mathbb{P}^1_{\mathbb{C}} \xrightarrow{\alpha_{j|}} U_{ij} \times \mathbb{P}^1_{\mathbb{C}} \ ,$$

with $U_{ij} = U_i \cap U_j$, are linear on the fibres and fix the three trivial sections. Thus they must be the identity maps, which implies that the $\alpha_i \circ \varphi_i$, $i = 1, \ldots, r$, glue together to an isomorphism of ruled surfaces:

$$\begin{array}{ccc} \Sigma & \xrightarrow{\ \alpha\ } & C \times \mathbb{P}^1_{\mathbb{C}} \\ & {\scriptstyle \pi}\searrow \quad \swarrow{\scriptstyle \mathrm{pr}} & \\ & C. & \end{array}$$

See also [**IsS96**] p. 229. □

Knowing the algebraic equivalence classes of irreducible curves in $\Sigma$ which satisfy the assumptions in Condition (II.1.5) we can give a better formulation of the vanishing theorem in the case of geometrically ruled surfaces.

In order to do the same for the existence theorems, we have to study very ample divisors on $\Sigma$. These, however, depend very much on the structure of the base curve $C$,[24] and the general results which we give may be not the best possible. Only in the case $C = \mathbb{P}^1_{\mathbb{C}}$ we can give a complete investigation.

**G.4 Remark**

The geometrically ruled surfaces with base curve $\mathbb{P}^1_{\mathbb{C}}$ are, up to isomorphism, just the Hirzebruch surfaces $\mathbb{F}_e = \mathbb{P}\big(\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}} \oplus \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(-e)\big)$, $e \geq 0$. Note that $\mathrm{Pic}(\mathbb{F}_e) = \mathrm{NS}(\mathbb{F}_e)$, that is, algebraic equivalence and linear equivalence coincide. Moreover, by [**Har77**] V.2.18 a divisor class $L = \alpha C_0 + \beta F$ is very ample over $\mathbb{C}$ if and only if $\alpha > 0$ and $\beta > \alpha e$. The conditions throughout the existence theorems turn out to be optimal if we work with $L = C_0 + (e+1) \cdot F$, while for other choices of $L$ they become more restrictive.[25]

[24] Cf. [**Har77**] V.2.22.2, Ex. V.2.11 and Ex. V.2.12.

[25] Let $L' = \alpha C_0 + \beta F$, then $D - L' - K_{\mathbb{F}_e} = (a + 1 - \alpha) \cdot C_0 + (b + 1 + e - \beta) \cdot F$, and thus the optimality of the conditions follows from

(III.1.8b) $(D - L' - K_{\mathbb{F}_e})^2 = (a + 1 - \alpha) \cdot \big(2 \cdot (b + 1 + e - \beta) - (a + 1 - \alpha) \cdot e\big) \leq a\big((2b - ae) + (\alpha e + e + 2 - 2\beta)\big) \leq a \cdot (2b - ae) = (D - L - K_{\mathbb{F}_e})^2$,

(III.1.9b) $(D - L' - K_{\mathbb{F}_e}).F = a + 1 - \alpha \leq a = (D - L - K_{\mathbb{F}_e}).F$, and for $e = 0$, $(D - L' - K_{\mathbb{F}_e}).C_0 = b + 1 - \beta \leq b = (D - L - K_{\mathbb{F}_e}).C_0$, and

(III.1.11b) $b + 1 + e - \beta \geq e(a + 1 - \alpha)$ implies $b \geq b + e\alpha + 1 - \beta \geq ae$.

In the case $C \not\cong \mathbb{P}^1_{\mathbb{C}}$, we may choose an integer $l \geq \max\{e+1, 2\}$ such that the algebraic equivalence class $|C_0 + lF|_a$ contains a very ample divisor $L$, e. g. $l = e + 3$ will do, if $C$ is an elliptic curve.[26] In particular, $l \geq 2$ as soon as $\Sigma \not\cong \mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$.

With the above choice of L we have $g(L) = 1 + \frac{L^2 + L.K_\Sigma}{2} = 1 + \frac{(-e+2l)+(e-2l+2g-2)}{2} = g$, and hence the generic curve in $|L|_l$ is a smooth curve whose genus equals the genus of the base curve.

Finally, in several situations we need a thorough knowledge on the cohomology groups of a divisor on a surface. We therefore prove the following lemmata.

### G.5 Lemma

*Let* $D \sim_l aC_0 + \mathfrak{b}F$ *with* $\mathfrak{b} \in \mathrm{Div}(C)$ *of degree* $b = \deg(\mathfrak{b})$, *let* $a' = \max\{k \mid a \geq k \geq 0, b \geq ke\}$, $a'' = \min\{k \mid a \geq k \geq 0, b \geq ke\}$, $a^* = \min\{k \mid a \geq k \geq 0, b \leq ke + 2g - 2\}$ *and* $a^{**} = \max\{k \mid a \geq k \geq 0, b \leq ke + 2g - 2\}$.[27]

(a) *Let* $a < 0$, *then* $h^0(\Sigma, D) = 0$.

(b) *Let* $a \geq 0$. *If* $e \geq 0$, *then*

$$\begin{aligned} h^0(\Sigma, D) = h^0(\Sigma, a'C_0 + \mathfrak{b}F) &\leq \sum_{k=0}^{a'} h^0\big(C, \mathcal{O}_C(k \cdot \mathfrak{e} + \mathfrak{b})\big) \\ &\leq a'b + a' + b + 1 - e \cdot \tfrac{a'(a'+1)}{2}, \end{aligned}$$

*while if* $e \leq 0$, *then*

$$\begin{aligned} h^0(\Sigma, D) &\leq \sum_{k=a''}^{a} h^0\big(C, \mathcal{O}_C(k \cdot \mathfrak{e} + \mathfrak{b})\big) \\ &\leq \Big(ab + a + b + 1 - e \cdot \tfrac{a(a+1)}{2}\Big) - \Big(a''b + a'' - e \cdot \tfrac{a''(a''-1)}{2}\Big). \end{aligned}$$

(c) *Let* $a \geq 0$. *If* $e \geq 0$, *then*

$$h^1(\Sigma, D) \leq \sum_{k=a^*}^{a} h^1\big(C, \mathcal{O}_C(k \cdot \mathfrak{e} + \mathfrak{b})\big),$$

*while if* $e \leq 0$, *then*

$$h^1(\Sigma, D) \leq \sum_{k=0}^{a^{**}} h^1\big(C, \mathcal{O}_C(k \cdot \mathfrak{e} + \mathfrak{b})\big).$$

(d) *If* $\mathcal{E}$ *is decomposable and* $a \geq 0$, *then for any* $i \in \mathbb{Z}$ *and* $a \geq 0$

$$h^i(\Sigma, D) = \sum_{k=0}^{a'} h^i\big(C, \mathcal{O}_C(k \cdot \mathfrak{e} + \mathfrak{b})\big).$$

---

[26] $l$ will be the degree of a suitable very ample divisor $\mathfrak{d}$ on $C$. Now $\mathfrak{d}$ defines an embedding of $C$ into some $\mathbb{P}^N_{\mathbb{C}}$ such that the degree of the image $C'$ is just $\deg(\mathfrak{d})$. Therefore, $\deg(\mathfrak{d}) \geq 2$, unless $C'$ is linear (cf. [**Har77**] Ex. I.7.6), which implies $C \cong \mathbb{P}^1_{\mathbb{C}}$.

[27] If a minimum does not exist, we set its value to $a+1$, and likewise we set a non-existing maximum to $-1$.

(e) *If* $e = 0$ *and* $D$ *nef, then* $a' = a$ *and* $a'' = 0$.

(f) *Let* $g(C) = 1$ *and* $a \geq 0$. *If* $e \geq 0$, *then*

$$h^0(\Sigma, D) \leq a'b + b + 1 - e \cdot \tfrac{a'(a'+1)}{2},$$

*and if in addition* $b > a'e$, *then even*

$$h^0(\Sigma, D) \leq a'b + b - e \cdot \tfrac{a'(a'+1)}{2}.$$

*If*[28] $e = -1$, *then*

$$h^0(\Sigma, D) \leq \begin{cases} ab + b + \frac{a(a+1)}{2}, & \text{if } b > 0, \\ ab + \frac{a(a+1)}{2} + \frac{b(b+1)}{2}, & \text{if } 0 \geq b \geq -\frac{a}{2} \text{ and } \mathfrak{b} \not\sim_l b \cdot \mathfrak{e}, \\ ab + 1 + \frac{a(a+1)}{2} + \frac{b(b+1)}{2}, & \text{if } 0 \geq b \geq -\frac{a}{2} \text{ and } \mathfrak{b} \sim_l b \cdot \mathfrak{e}, \\ 0, & \text{if } -\frac{a}{2} > b. \end{cases}$$

(g) *If* $e < 0$ *and* $a \geq 0$, *then setting* $b' = b - \frac{ae}{2}$

$$h^0(\Sigma, D) \leq ab' + a + b' + 1 - \tfrac{1}{2e} \cdot \left(b + 1 + \tfrac{e}{2}\right)^2.$$

*Moreover, if* $D$ *is effective, then*

$$h^0(\Sigma, D) \leq ab' + a + b' + 1 - \tfrac{1}{2e} \cdot \left(b' + \tfrac{3ea}{4}\right)^2$$

*and*

$$h^0(\Sigma, D) \leq ab' + a + b' + 1 - \tfrac{9e}{32} \cdot a^2.$$

**Proof:**

(a) If $a < 0$, then $0 > a = D.F$, which is impossible for any curve.

(b) We consider the exact sequence

$$0 \to \mathcal{O}_\Sigma\big((a-1)C_0 + \mathfrak{b}F\big) \to \mathcal{O}_\Sigma(aC_0 + \mathfrak{b}F) \to \mathcal{O}_{C_0}(aC_0 + \mathfrak{b}F) \to 0. \quad \text{(G.6)}$$

Taking into account that $\mathcal{O}_{C_0}(aC_0 + \mathfrak{b}F) \cong \mathcal{O}_C(a\mathfrak{e} + \mathfrak{b})$ the long exact cohomology sequence of (G.6) gives

$$h^0(\Sigma, aC_0 + \mathfrak{b}F) \leq h^0\big(\Sigma, (a-1)C_0 + \mathfrak{b}F\big) + h^0\big(C, \mathcal{O}_C(a\mathfrak{e} + \mathfrak{b})\big).$$

By induction on $a$, starting with $a = 0$, we thus get

$$h^0(\Sigma, D) \leq \sum_{k=0}^{a} h^0\big(C, \mathcal{O}_C(k \cdot \mathfrak{e} + \mathfrak{b})\big).$$

For the induction basis $a = 0$ we note that by **[Har77]** Ex. III.8.3 and Ex. III.8.4 and by the projection formula we have for all $i > 0$

$$R^i\pi_*(\mathfrak{b}F) = R^i\pi_*(\mathcal{O}_\Sigma \otimes \pi^*\mathfrak{b}) = R^i\pi_*\mathcal{O}_\Sigma \otimes \mathfrak{b} = 0,$$

and thus by **[Har77]** Ex. III.8.1,

$$h^i(\Sigma, \mathfrak{b}F) = h^i(C, \mathfrak{b}) \quad \text{for all} \quad i \geq 0. \quad \text{(G.7)}$$

[28]If $g = 1$ and $e < 0$, then $e = -1$.

If $e \geq 0$ and $k > a'$ respectively if $e \leq 0$ and $k < a''$, then $\deg(k\mathfrak{e} + \mathfrak{b}) = -ke + b < 0$ and

$$h^0\big(C_0, \mathcal{O}_{C_0}(kC_0 + \mathfrak{b}F)\big) = h^0\big(C, \mathcal{O}_C(k\mathfrak{e} + \mathfrak{b})\big) = 0.$$

Thus, if $e \geq 0$ we have

$$h^0(\Sigma, D) \leq \sum_{k=0}^{a'} h^0\big(C, \mathcal{O}_C(k \cdot \mathfrak{e} + \mathfrak{b})\big),$$

and if $e \leq 0$

$$h^0(\Sigma, D) \leq \sum_{k=a''}^{a} h^0\big(C, \mathcal{O}_C(k \cdot \mathfrak{e} + \mathfrak{b})\big).$$

By Lemma D.3 for $e \geq 0$ and $k \leq a'$ respectively $e \leq 0$ and $k \geq a''$ we have $h^0\big(C, \mathcal{O}_C(k \cdot \mathfrak{e} + \mathfrak{b})\big) \leq b - ke + 1$. Thus if $e \geq 0$

$$\sum_{k=0}^{a'} h^0\big(C, \mathcal{O}_C(k \cdot \mathfrak{e} + \mathfrak{b})\big) \leq a'b + a' + b + 1 - e \cdot \tfrac{a'(a'+1)}{2},$$

and

$$\sum_{k=a''}^{a} h^0\big(C, \mathcal{O}_C(k \cdot \mathfrak{e} + \mathfrak{b})\big)$$
$$\leq \Big(ab + a + b + 1 - e \cdot \tfrac{a(a+1)}{2}\Big) - \Big(a''b + a'' - e \cdot \tfrac{a''(a''-1)}{2}\Big).$$

It remains to show that for $e \geq 0$ and $a > a'$,

$$h^0(\Sigma, aC_0 + \mathfrak{b}F) = h^0\big(\Sigma, a'C_0 + \mathfrak{b}F\big).$$

Since $a > a'$, $\deg(a\mathfrak{e} + \mathfrak{b}) = -ae + b < 0$, the long exact cohomology sequence of (G.6) implies

$$h^0(\Sigma, aC_0 + \mathfrak{b}F) = h^0\big(\Sigma, (a-1)C_0 + \mathfrak{b}F\big).$$

The result thus follows by descending induction on $a$ for $a > a'$.

(c) The long exact cohomology sequence of (G.6) gives

$$H^1\big(\Sigma, (a-1)C_0 + \mathfrak{b}F\big) \longrightarrow H^1\big(\Sigma, aC_0 + \mathfrak{b}F\big) \longrightarrow H^1\big(C_0, \mathcal{O}_{C_0}(aC_0 + \mathfrak{b}F)\big).$$

We thus get

$$h^1(\Sigma, aC_0 + \mathfrak{b}F) \leq h^1\big(\Sigma, (a-1)C_0 + \mathfrak{b}F\big) + h^1\big(C, \mathcal{O}_C(a\mathfrak{e} + \mathfrak{b})\big),$$

and by induction on $a$, starting with $a = 0$ – see (G.7) –, we find

$$h^1(\Sigma, D) \leq \sum_{k=0}^{a} h^1\big(C, \mathcal{O}_C(k \cdot \mathfrak{e} + \mathfrak{b})\big).$$

However, if $e \geq 0$ and $k < a^*$ respectively if $e \leq 0$ and $k > a^{**}$, then $\deg(k\mathfrak{e} + \mathfrak{b}) = b - ke > 2g - 2$ and thus $h^1(C, k\mathfrak{e} + \mathfrak{b}) = 0$.

(d) See **[FuP00]** Lemma 2.10. Replacing the section $C_0$ by a section in $|C_0 - \mathfrak{e}F|_a$, which only exists in the decomposable case, they do basically the same considerations as in (a) – however, they get surjectivity at the important positions in the long exact cohomology sequence.

(e) If $e = 0$ and $D$ nef, then by Lemma G.1 we know that $a, b \geq 0$ and thus $a' = a$ and $a'' = 0$.

(f) Let us consider the case $e \geq 0$ first. By part (a) we know $h^0(\Sigma, D) \leq \sum_{k=0}^{a'} h^0\big(C, \mathcal{O}_C(k \cdot \mathfrak{e} + \mathfrak{b})\big)$. Moreover, we have

$$\deg(k \cdot \mathfrak{e} + \mathfrak{b}) = b - ke \begin{cases} > 0, & \text{if } k = 0, \ldots, a' - 1, \\ & \text{and if } k = a' \text{ and } b > a'e, \\ \geq 0, & \text{if } k = a'. \end{cases}$$

By Lemma G.14 we thus get

$$h^0\big(C, \mathcal{O}_C(k \cdot \mathfrak{e} + \mathfrak{b})\big) \leq \begin{cases} b - ke, & \text{if } k = 0, \ldots, a' - 1, \\ & \text{and if } k = a' \text{ and } b > a'e, \\ b - ke + 1, & \text{if } k = a'. \end{cases}$$

Thus the claim for $e \geq 0$ follows. Let now $e = -1$. The claim then is implied by the following considerations.
If $b > 0$, then obviously $b > -k = ek$ for all $k = 0, \ldots, a$ and thus $a'' = 0$. Moreover, as above we see $\deg(k \cdot \mathfrak{e} + \mathfrak{b}) = b + k > 0$ for all $k = 0, \ldots, a$ in this case, and thus $h^0\big(C, \mathcal{O}_C(k \cdot \mathfrak{e} + \mathfrak{b})\big) = b + k$ for all $k$.
If $0 \geq b \geq -a$, then $a'' = -b$ and $\deg(k \cdot \mathfrak{e} + \mathfrak{b}) = b + k > 0$ for $k = -b + 1, \ldots, a$. Therefore we get $h^0\big(C, \mathcal{O}_C(k \cdot \mathfrak{e} + \mathfrak{b})\big) = b + k$ for all $k = -b + 1, \ldots, a$. If, moreover, $\mathfrak{b} - b \cdot \mathfrak{e} \not\sim_l 0$, then $h^0\big(C, \mathcal{O}_C(-b \cdot \mathfrak{e} + \mathfrak{b})\big) = 0$, while if $\mathfrak{b} - b \cdot \mathfrak{e} \sim_l 0$, then $h^0\big(C, \mathcal{O}_C(-b \cdot \mathfrak{e} + \mathfrak{b})\big) = 1$.
Finally, if $-\frac{a}{2} > b$ then $|D|_l$ is empty by **[Har77]** V.2.21.

(g) Since $e$ is negative, any of the expressions $-\frac{1}{2e} \cdot \big(b+1+\frac{e}{2}\big)^2$, $-\frac{1}{2e} \cdot \big(b'+\frac{3ea}{4}\big)^2$ and $-\frac{9e}{32} \cdot a^2$ is non-negative. Moreover, for $b \geq 0$ we have $a'' = 0$ and thus $h^0(\Sigma, D) = ab' + a + b' + 1$. The inequalities are thus fulfilled for $b \geq 0$.
It remains to verify them for $b < 0$, that is, for $b < 0$ we have to show that $-\Big(a''b + a'' - e \cdot \frac{a'' \cdot (a''-1)}{2}\Big)$ is bounded from above by any of these three expressions.
Let $b < 0$ be given, and consider the function

$$\varphi : \mathbb{R} \to \mathbb{R} : t \mapsto \tfrac{e}{2} \cdot t^2 - \big(b + 1 + \tfrac{e}{2}\big) \cdot t.$$

The derivative $\frac{\partial\varphi}{\partial t}(t) = e \cdot t - \left(b + 1 + \frac{e}{2}\right)$ has its unique zero at $t = \frac{1}{e} \cdot \left(b + 1 + \frac{e}{2}\right)$ and $\varphi$ thus takes its maximum there. That is,

$$-\left(a''b + a'' - e \cdot \frac{a'' \cdot (a''-1)}{2}\right) = \varphi(a'') \leq \varphi\left(\frac{1}{e} \cdot \left(b + 1 + \frac{e}{2}\right)\right) = -\frac{1}{2e} \cdot \left(b + 1 + \frac{e}{2}\right)^2.$$

This proves the first inequality.

If $D$ is effective and $b < 0$, then by **[Har77]** V.2.21 $a \geq 2$ and $0 > b \geq \frac{ae}{2}$. But then $b' + \frac{3ea}{4} = b + \frac{ea}{4} < b + 1 + \frac{e}{2} < 0$ and therefore

$$-\frac{1}{2e} \cdot \left(b + 1 + \frac{e}{2}\right)^2 < -\frac{1}{2e} \cdot \left(b' + \frac{3ea}{4}\right)^2,$$

which gives the second inequality. And finally, since $b' = b - \frac{ae}{2} \geq 0$ in this situation, we get $\frac{3ea}{4} \leq b' + \frac{3ea}{4} < 0$ and thus

$$-\frac{1}{2e} \cdot \left(b' + \frac{3ea}{4}\right)^2 \leq -\frac{9e}{32} \cdot a^2.$$

□

**G.6 Lemma**

*Let* $\Sigma = \mathbb{F}_e$ *and let* $D = aC_0 + bF$*,* $a' = \max\{k \mid a \geq k, b \geq ke\}$ *and* $a^* = \max\{k \mid -a - 2 \geq k, b + 2 + e \leq ke\}$.

$$\text{(a)} \quad h^1(\Sigma, D) = \begin{cases} 0, & \textit{if } a = -1 \textit{ or } (a \geq 0 \textit{ and } b \geq ae - 1) \\ & \textit{or } (a \leq -2 \textit{ and } b \leq -(a+1) \cdot e - 1), \\ e \cdot \frac{a \cdot (a+1)}{2} - ab - a - b - 1 \neq 0, & \textit{if } a \geq 0 \textit{ and } b < 0 \\ & \textit{and } (a, b) \neq (0, -1), \\ e \cdot \frac{l \cdot (l+1)}{2} - l \cdot (k+1) \neq 0, & \textit{if } a > 0 \textit{ and } \\ & 0 \leq b = (a - l) \cdot e + k \leq ae - 2 \\ & \textit{with } 1 < l \leq a, 0 \leq k < e, \\ h^1(\Sigma, K_\Sigma - D), & \textit{if } a \leq -2. \end{cases}$$

$$\text{(b)} \quad h^0(\Sigma, D) = \begin{cases} 0, & \textit{if } a < 0 \textit{ or } b < 0, \\ ab + a + b + 1 - e \cdot \frac{a \cdot (a+1)}{2}, & \textit{if } 0 \leq a \textit{ and } ae \leq b, \\ a'b + a' + b + 1 - e \cdot \frac{a'(a'+1)}{2}, & \textit{if } 0 \leq a \textit{ and } 0 \leq b \leq ae. \end{cases}$$

$$\text{(c)} \quad h^2(\Sigma, D) = \begin{cases} 0 & \textit{if } a > -2 \textit{ or } b > -2 - e, \\ ab + a + b + 1 - e \cdot \frac{a \cdot (a+1)}{2} & \textit{if } a \leq -2, b + 2 + e \leq ae, \\ a^*b + a^* + b + 1 - e \cdot \frac{a^*(a^*+1)}{2} & \textit{if } a \leq -2 \textit{ and } \\ & ae < b + 2 + e \leq 0. \end{cases}$$

**Proof:**

(a) By the Theorem of Riemann-Roch we have

$$\chi(D) = \frac{D^2 - D.K_\Sigma}{2} + \chi(\mathcal{O}_\Sigma) = ab + a + b + 1 - e \cdot \tfrac{a(a+1)}{2}.$$

The result is thus an application of part (b) and (c).

(b) This follows from Lemma G.5 (d) taking into account that for a divisor $\mathcal{O}_{\mathbb{P}^1_\mathbb{C}}(d)$ on $\mathbb{P}^1_\mathbb{C}$ of non-negative degree we have $h^0\big(\mathbb{P}^1_\mathbb{C}, \mathcal{O}_{\mathbb{P}^1_\mathbb{C}}(d)\big) = d + 1$.

(c) By Serre duality we have

$$h^2(\Sigma, D) = h^0(\Sigma, K_\Sigma - D) = h^0\big(\Sigma, (-a-2)\cdot C_0 + (-b-2-e)\cdot F\big).$$

The result thus follows by simple calculations from part (b).

□

## G.b. Products of Curves

Let $C_1$ and $C_2$ be two smooth projective curves of genera $g_1 \geq 0$ and $g_2 \geq 0$ respectively. The surface $\Sigma = C_1 \times C_2$ is naturally equipped with two fibrations $\mathrm{pr}_i : \Sigma \to C_i$, $i = 1, 2$, and by abuse of notation we denote two generic fibres $\mathrm{pr}_2^{-1}(p_2) = C_1 \times \{p_2\}$ resp. $\mathrm{pr}_1^{-1}(p_1) = \{p_1\} \times C_2$ again by $C_1$ resp. $C_2$.

**G.7 Lemma**

*Let $D \in \mathrm{Div}(\Sigma)$ be a divisor such that $D \sim_l \mathrm{pr}_2^* \mathfrak{a} + \mathrm{pr}_1^* \mathfrak{b}$ with $\mathfrak{a} \in \mathrm{Div}(C_2)$ of degree $a$ and $\mathfrak{b} \in \mathrm{Div}(C_1)$ of degree $b$.*

(a) $D \sim_a aC_1 + bC_2$.

(b) *If $\mathfrak{a}$ is very ample on $C_2$ and $\mathfrak{b}$ is very ample on $C_1$, then $\mathrm{pr}_2^* \mathfrak{a} \otimes \mathrm{pr}_1^* \mathfrak{b}$ is very ample on $\Sigma$.*

(c) *If $a \geq 2g_2 + 1$ and $b \geq 2g_1 + 1$, then $\mathrm{pr}_2^* \mathfrak{a} \otimes \mathrm{pr}_1^* \mathfrak{b}$ is very ample on $\Sigma$.*

(d) *$D$ is ample if and only if $a, b > 0$.*

(e) *$D$ is nef if and only if $a, b \geq 0$.*

(f) *$D$ is effective if and only if $a, b \geq 0$ and $(a, b) \neq (0, 0)$.*

(g) $K_\Sigma \cong \mathrm{pr}_2^* K_{C_2} + \mathrm{pr}_1^* K_{C_1} \sim_a (2g_2 - 2)C_1 + (2g_1 - 2)C_2$.

**Proof:**

(a) Obviously $D \sim_l \mathrm{pr}_2^* \mathfrak{a} + \mathrm{pr}_1^* \mathfrak{b} \sim_a a'C_1 + b'C_2$ for some $a', b' \in \mathbb{Z}$. But then $a' = D.C_2 = \deg\big(\mathcal{O}_{C_2}(D)\big) = \deg(\mathfrak{a})$ and $b' = D.C_1 = \deg\big(\mathcal{O}_{C_1}(D)\big) = \deg(\mathfrak{b})$.

(b) $\mathfrak{a}$ and $\mathfrak{b}$ induce embeddings of $C_2$ and $C_1$ into some projective spaces $\mathbb{P}^n_\mathbb{C}$ and $\mathbb{P}^m_\mathbb{C}$ respectively. Thus $\mathrm{pr}_2^* \mathfrak{a} \otimes \mathrm{pr}_1^* \mathfrak{b}$ induces the embedding into the product of these spaces and is the pullback of the very ample line bundle $\mathcal{O}_{\mathbb{P}^n_\mathbb{C}}(1) \otimes \mathcal{O}_{\mathbb{P}^m_\mathbb{C}}(1)$ which induces the Segre embedding of $\mathbb{P}^n_\mathbb{C} \times \mathbb{P}^m_\mathbb{C}$.

(c) By **[Har77]** IV.3.2 then $\mathfrak{a}$ and $\mathfrak{b}$ are very ample, and we are done with (b).

(d) If $D$ is ample, then $a = D.C_2 > 0$ and $b = D.C_1 > 0$. On the other hand, if $a, b > 0$ we may choose some $n > 0$ such that $na \geq 2g_2 + 1$ and $nb \geq 2g_1 + 1$, and thus $nD$ is very ample, which implies that $D$ itself is ample.

(e) If $D$ is nef, then $a = D.C_2 \geq 0$ and $b = D.C_1 \geq 0$.
Now let $a, b \geq 0$. Since any irreducible curve which is not a fibre of $\mathrm{pr}_i$ is intersected by some fibre of $\mathrm{pr}_i$, the fibres of $\mathrm{pr}_i$ are nef, $i = 1, 2$. Hence, $D$ is nef if $a = 0$ or $b = 0$. If, however, both are strictly positive, then $D$ is even ample, hence in particular nef.

(f) If $D$ is effective, then $0 < D.(nC_1 + mC_2) = ma + nb$ for all $n, m > 0$. Hence, neither $a$ nor $b$ can be negative. The converse is obvious.

(g) See **[Har77]** Exercise II.8.3 and Example IV.1.3.3.

□

In many situations we are interested in the cohomology groups of a divisor of the type $\mathrm{pr}_1^* \mathfrak{a} + \mathrm{pr}_2^* \mathfrak{b}$. The Künneth-Formula implies the following useful results.

**G.8 Lemma**
*Let* $\mathfrak{a} \in \mathrm{Div}(C_2)$ *and* $\mathfrak{b} \in \mathrm{Div}(C_1)$ *and let* $i \in \mathbb{Z}$, *then*

$$h^i(\Sigma, \mathrm{pr}_2^* \mathfrak{a} + \mathrm{pr}_1^* \mathfrak{b}) = \sum_{k=0}^{i} h^k(C_2, \mathfrak{a}) \cdot h^{i-k}(C_1, \mathfrak{b}).$$

*In particular,* $h^0(\Sigma, \mathrm{pr}_2^* \mathfrak{a} + \mathrm{pr}_1^* \mathfrak{b}) = h^0(C_2, \mathfrak{a}) \cdot h^0(C_1, \mathfrak{b})$.

**Proof:** The Künneth-Formula (cf. **[Dan96]** I.1.7) gives

$$H^i(\Sigma, \mathrm{pr}_2^* \mathfrak{a} + \mathrm{pr}_1^* \mathfrak{b}) = \sum_{k=0}^{i} H^k(C_2, \mathfrak{a}) \otimes H^{i-k}(C_1, \mathfrak{b}),$$

and thus the result follows. □

**G.9 Corollary**
*Let* $D \in \mathrm{Div}(\Sigma)$ *be a divisor such that* $D \sim_l \mathfrak{a}C_1 + \mathfrak{b}C_2$ *with* $\mathfrak{a} \in \mathrm{Div}(C_2)$ *of degree* $a \geq 0$ *and* $\mathfrak{b} \in \mathrm{Div}(C_1)$ *of degree* $b \geq 0$. *Then:*

$$h^0(\Sigma, \mathrm{pr}_2^* \mathfrak{a} + \mathrm{pr}_1^* \mathfrak{b}) \leq (a + 1) \cdot (b + 1).$$

**Proof:** By Lemma G.8 it suffices to show that for a divisor $\mathfrak{c} \in \mathrm{Div}(C)$ on a smooth projective curve $C$ we have $h^0(C, \mathfrak{c}) \leq \deg(\mathfrak{c}) + 1$. This, however, follows from Lemma D.3. □

**G.10 Corollary**
*Suppose that* $g_2 = 0$ *and* $g_1 = 1$, *and let* $D \in \mathrm{Div}(\Sigma)$ *be a divisor such that* $D \sim_l \mathrm{pr}_2^* \mathfrak{a} + \mathrm{pr}_1^* \mathfrak{b}$ *with* $\mathfrak{a} \in \mathrm{Div}(C_2)$ *of degree* $a \geq 0$ *and* $\mathfrak{b} \in \mathrm{Div}(C_1)$ *of degree* $b \geq 1$. *Then:*

$$h^0(\Sigma, \mathrm{pr}_2^* \mathfrak{a} + \mathrm{pr}_1^* \mathfrak{b}) = (a + 1) \cdot b.$$

**Proof:** This follows by Lemma G.8, taking into account that $h^0(C_2, \mathfrak{a}) = a + 1$ (cf. Lemma G.6) and $h^0(C_1, \mathfrak{b}) = b$ (cf. Lemma G.14). □

**G.11 Remark**

We are going to show that for a generic choice of the curves $C_1$ and $C_2$ the Néron–Severi group $\mathrm{NS}(\Sigma) = C_1\mathbb{Z} \oplus C_2\mathbb{Z}$ of $\Sigma$ is two-dimensional[29] with intersection matrix

$$(C_i.C_j)_{i,j} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Thus, the only irreducible curves $B \subset \Sigma$ with selfintersection $B^2 = 0$ are the fibres $C_1$ and $C_2$, and for any irreducible curve $B \sim_a aC_1 + bC_2$ the coefficients $a$ and $b$ must be non-negative.

Supposed that one of the curves is rational, the surface is geometrically ruled and the Picard number of $\Sigma$ is two. Whereas, if both $C_1$ and $C_2$ are of strictly positive genus, this need no longer be the case as we will see in Remark G.18. Thus the following proposition is the best we may expect.

**G.12 Proposition**

*For a generic choice of smooth projective curves $C_1$ and $C_2$ the Néron–Severi group of $\Sigma = C_1 \times C_2$ is $\mathrm{NS}(\Sigma) \cong C_1\mathbb{Z} \oplus C_2\mathbb{Z}$.*

*More precisely, fixing $g_1$ and $g_2$ there is a very general subset $U \subseteq M_{g_1} \times M_{g_2}$ such that for any $(C_1, C_2) \in U$ the Picard number of $C_1 \times C_2$ is two, where $M_{g_i}$ denotes the moduli space of smooth projective curves of genus $g_i$, $i = 1, 2$.*

**Proof:** As already mentioned, if either $g_1$ or $g_2$ is zero, then we may take $U = M_{g_1} \times M_{g_2}$.

Suppose that $g_1 = g_2 = 1$. Given an elliptic curve $C_1$ there is a countable union $V$ of proper subvarieties of $M_1$ such that for any $C_2 \in M_1 \setminus V$ the Picard number of $C_1 \times C_2$ is two - namely, if $\tau_1$ and $\tau_2$ denote the periods as in Section G.c, then we have to require that there exists no invertible integer matrix $\left(\begin{smallmatrix} z_1 & z_2 \\ z_3 & z_4 \end{smallmatrix}\right)$ such that $\tau_2 = \frac{z_4 - z_3\tau_1}{z_2 - z_1\tau_1}$. (Compare also **[GrH94]** p. 286.)

We, therefore, may assume that $g_1 \geq 2$ and $g_2 \geq 1$. The claim then follows from Lemma G.13, which is due to Denis Gaitsgory. □

**G.13 Lemma (Denis Gaitsgory)**

*Let $C_2$ be any smooth projective curve of genus $g_2 \geq 1$. Then for any $g_1 \geq 2$ there is a very general subset $U$ of the moduli space $M_{g_1}$ of smooth projective curves of genus $g_1$ such that the Picard number of $C_1 \times C_2$ is two for any $C_1 \in U$.*

**Proof:** We note that a curve $B \subset \Sigma = C_1 \times C_2$ with $C_1 \not\sim_a B \not\sim_a C_2$ induces a non-trivial morphism $\mu_B : C_1 \to \mathrm{Pic}(C_2) : p \mapsto \mathrm{pr}_{2*}\left(\mathrm{pr}_1^*(p)\right)$. It thus makes

[29] In the case that $C_1$ and $C_2$ are elliptic curves, generic means precisely, that they are not isogenous - see Section G.c.

sense to study the moduli problem of (non-trivial) maps from curves of genus $g_1$ into $\mathrm{Pic}(C_2)$.

More precisely, let $k \in \mathbb{N}$ and let $0 \neq \beta \in H_2\big(\mathrm{Pic}_k(C_2), \mathbb{Z}\big) = \mathbb{Z}^{2g_2}$ be given, where $\mathrm{Pic}_k(C_2)$ is the Picard variety of divisors of degree $k$ on $C_2$. Following the notation of **[FuP97]** we denote by $M_{g_1,0}\big(\mathrm{Pic}_k(C_2), \beta\big)$ the moduli space of pairs $(C_1, \mu)$, where $C_1$ is a smooth projective curve of genus $g_1$ and $\mu : C_1 \to \mathrm{Pic}_k(C_2)$ a morphism with $\mu_*\big([C_1]\big) = \beta$. We then have the canonical morphism

$$F_{k,\beta} : M_{g_1,0}\big(\mathrm{Pic}_k(C_2), \beta\big) \to M_{g_1} : (C_1, \mu) \mapsto C_1,$$

just forgetting the map $\mu$, and the proposition reduces to the following claim.

**Claim:** For no choice of $k \in \mathbb{N}$ and $0 \neq \beta \in H_2\big(\mathrm{Pic}_k(C_2), \mathbb{Z}\big)$ the morphism $F_{k,\beta}$ is dominant.

Let $\mu : C_1 \to \mathrm{Pic}_k(C_2)$ be any morphism with $\mu_*\big([C_1]\big) = \beta$. Then $\mu$ is not a contraction and the image of $C_1$ is a projective curve in the abelian variety $\mathrm{Pic}_k(C_2)$. Moreover, we have the following exact sequence of sheaves

$$0 \longrightarrow \mathcal{T}_{C_1} \xrightarrow{d\mu} \mu^*\mathcal{T}_{\mathrm{Pic}_k(C_2)} = \mathcal{O}_{C_1}^{g_2} \longrightarrow \mathcal{N}_\mu := \mathrm{coker}(d\mu) \longrightarrow 0. \tag{G.8}$$

Since $d\mu$ is a non-zero inclusion, its dual

$$d\mu^\vee : (\mu^*\mathcal{T}_{\mathrm{Pic}_k(C_2)})^\vee = \mathcal{O}_{C_1}^{g_2} \to \Omega_{C_1} = \omega_{C_1}$$

is not zero on global sections, that is

$$\begin{array}{ccc} H^0\big(d\mu^\vee\big) : H^0\big(C_1, \mathcal{O}_{C_1}^{g_2}\big) & \longrightarrow & H^0(C_1, \omega_{C_1}) \\ \| & & \| \\ \mathrm{Hom}_{\mathcal{O}_{C_1}}\big(\mathcal{O}_{C_1}^{g_2}, \mathcal{O}_{C_1}\big) & & \mathrm{Hom}_{\mathcal{O}_{C_1}}\big(\mathcal{T}_{C_1}, \mathcal{O}_{C_1}\big) \end{array}$$

is not the zero map.

Since $g_1 \geq 2$ we have $h^0(C_1, \omega_{C_1}) = 2g_1 - 2 > 0$, and thus $\omega_{C_1}$ has global sections. Therefore, the induced map $H^0\big(C_1, \omega_{C_1} \otimes \mathcal{O}_{C_1}^{g_2}\big) \to H^0(C_1, \omega_{C_1} \otimes \omega_{C_1})$ is not the zero map, which by Serre duality gives that the map

$$H^1(d\mu) : H^1(C_1, \mathcal{T}_{C_1}) \to H^1\big(C_1, \mu^*\mathcal{T}_{\mathrm{Pic}_k(C_2)}\big)$$

from the long exact cohomology sequence of (G.8) is not zero. Hence the coboundary map

$$\delta : H^0(C_1, \mathcal{N}_\mu) \to H^1(C_1, \mathcal{T}_{C_1})$$

cannot be surjective. According to **[Har98]** p. 96 we have

$$\delta = dF_{k,\beta} : \mathcal{T}_{M_{g_1,0}(\mathrm{Pic}_k(C_2),\beta)} = H^0(C_1, \mathcal{N}_\mu) \longrightarrow \mathcal{T}_{M_{g_1}} = H^1(C_1, \mathcal{T}_{C_1}).$$

But if the differential of $F_{k,\beta}$ is not surjective, then $F_{k,\beta}$ itself cannot be dominant. $\square$

## G.c. Products of Elliptic Curves

Let $C_1 = \mathbb{C}/\Lambda_1$ and $C_2 = \mathbb{C}/\Lambda_2$ be two elliptic curves, where $\Lambda_i = \mathbb{Z} \oplus \tau_i\mathbb{Z} \subset \mathbb{C}$ is a lattice and $\tau_i$ is in the upper half plane of $\mathbb{C}$. We denote the natural group structure on each of the $C_i$ by $+$ and the neutral element by $0$.

Our interest lies in the study of the surface $\Sigma = C_1 \times C_2$. This surface is naturally equipped with two fibrations $\mathrm{pr}_i : \Sigma \to C_i$, $i = 1, 2$, and by abuse of notation we denote the fibres $\mathrm{pr}_2^{-1}(0) = C_1 \times \{0\}$ resp. $\mathrm{pr}_1^{-1}(0) = \{0\} \times C_2$ again by $C_1$ resp. $C_2$. The group structures on $C_1$ and $C_2$ extend to $\Sigma$ so that $\Sigma$ itself is an abelian variety. Moreover, for $p = (p_1, p_2) \in \Sigma$ the mapping $\tau_p : \Sigma \to \Sigma : (a, b) \mapsto (a + p_1, b + p_2)$ is an automorphism of abelian varieties. Due to these translation morphisms we know that for any curve $B \subset \Sigma$ the algebraic family of curves $|B|_a$ covers the whole of $\Sigma$, and in particular $\dim |B|_a \geq 1$. This also implies $B^2 \geq 0$.

Since $\Sigma$ is an abelian surface, $\mathrm{NS}(\Sigma) = \mathrm{Num}(\Sigma)$, $K_\Sigma = 0$, and the Picard number $\rho = \rho(\Sigma) \leq 4$ (cf. **[LaB92]** 4.11.2 and Ex. 2.5). But the Néron–Severi group of $\Sigma$ contains the two independent elements $C_1$ and $C_2$, so that $\rho \geq 2$. The general case[30] is indeed $\rho = 2$, however $\rho$ might also be larger (see Example G.17), in which case the additional generators may be chosen to be graphs of surjective morphisms from $C_1$ to $C_2$ (cf. **[IsS96]** 3.2 Example 3). This shows:

$\rho(\Sigma) = 2$ *if and only if* $C_1$ *and* $C_2$ *are not isogenous.*

The following lemma provides a better knowledge on the dimension of the global sections of a divisor on an elliptic curve, and leads to better results in the irreduciblity and smoothness theorems, when products of elliptic curves are studied.

**G.14 Lemma**

*Let* $C$ *be a smooth elliptic curve over* $\mathbb{C}$ *and let* $\mathfrak{a}$ *be a divisor on* $C$ *of degree* $a = \deg(\mathfrak{a})$. *Then:*

(a) $h^0(C, \mathfrak{a}) = \chi(\mathfrak{a}) = \deg(\mathfrak{a})$ *and* $h^1(C, \mathfrak{a}) = 0$*, if* $a > 0$.

(b) $h^0(C, \mathfrak{a}) = h^1(C, \mathfrak{a}) = 0$*, if* $a = 0$ *and* $\mathfrak{a} \not\sim_l \mathcal{O}_C$.

(c) $h^0(C, \mathfrak{a}) = h^1(C, \mathfrak{a}) = 1$*, if* $a = 0$ *and* $\mathfrak{a} \sim_l \mathcal{O}_C$.

**Proof:**

(a) Since $\deg(-\mathfrak{a}) = -a < 0$ and since $K_C$ is trivial, $h^1(C, \mathfrak{a}) = h^0(C, -\mathfrak{a}) = 0$. Thus the Theorem of Riemann-Roch gives

$$h^0(C, \mathfrak{a}) = \chi(\mathfrak{a}) = \deg(\mathfrak{a}) + g(C) - 1 = \deg(\mathfrak{a}).$$

[30]The abelian surfaces with $\rho \geq 2$ possessing a principle polarisation are parametrised by a countable number of surfaces in a three-dimensional space, and the Picard number of such an abelian surface is two unless it is contained in the intersection of two or three of these surfaces (cf. **[IsS96]** 11.2). See also **[GrH94]** p. 286, and Proposition G.12.

(b) and (c) follow from **[Har77]** IV.1.2. □

**G.15 Lemma**

*Let* $D \subset \Sigma$ *be an effective divisor such that* $D \sim_l \mathrm{pr}_1^* \mathfrak{a} + \mathrm{pr}_2^* \mathfrak{b}$*, where* $\mathfrak{a} \in \mathrm{Div}(C_1)$ *with* $a = \deg(\mathfrak{a}) \geq 0$ *and* $\mathfrak{b} \in \mathrm{Div}(C_2)$ *with* $b = \deg(\mathfrak{b}) \geq 0$.

(a) *If* $a, b > 0$*, then* $h^0(\Sigma, D) = ab$ *and* $h^1(\Sigma, D) = 0$.

(b) $h^0(\Sigma, D) = h^1(\Sigma, D) \leq \begin{cases} a, & \text{if } a > 0, b = 0 \\ b, & \text{if } a = 0, b > 0. \end{cases}$

(c) $h^2(\Sigma, D) = 0$.

**Proof:** We first of all note, that $a, b \geq 0$ and $(a, b) \neq (0, 0)$ by Lemma G.7. Since D is effective, $0 = h^0(\Sigma, -D) = h^2(\Sigma, K_\Sigma + D) = h^2(\Sigma, D)$.

By Lemma G.8 we know that

$$h^0(\Sigma, D) = h^0(C_2, \mathfrak{a}) \cdot h^0(C_1, \mathfrak{b})$$

and

$$h^1(\Sigma, D) = h^1(C_2, \mathfrak{a}) \cdot h^0(C_1, \mathfrak{b}) + h^0(C_2, \mathfrak{a}) \cdot h^1(C_1, \mathfrak{b}).$$

The result thus follows from Lemma G.14. □

**G.16 Lemma**

*Let* $B \subset \Sigma$ *be an irreducible curve,* $B \not\sim_a C_k$*,* $k = 1, 2$*, and* $\{i, j\} = \{1, 2\}$.

(a) *If* $B^2 = 0$*, then* $B$ *is smooth,* $g(B) = 1$*, and* $\mathrm{pr}_{i|} : B \to C_i$ *is an unramified covering of degree* $B.C_j$.

(b) *If* $B^2 = 0$*, then* $\#\big(B \cap \tau_p(C_i)\big) = B.C_j$ *for any* $p \in \Sigma$*, and if* $q, q' \in B$*, then* $\tau_{q-q'}(B) = B$.

(c) *If* $B^2 = 0$*, then the base curve* $H$ *in the fibration* $\pi : \Sigma \to H$ *with fibre* $B$*, which exists according to Proposition B.1, is an elliptic curve.*

(d) *If* $B.C_i = 1$*, then* $B^2 = 0$ *and* $C_j \cong B$.

(e) *If* $B.C_i = 1 = B.C_j$*, then* $C_1 \cong C_2$.

(f) *If* $B$ *is the graph of a morphism* $\alpha : C_i \to C_j$*, then* $B.C_j = 1$ *and* $B^2 = 0$.

**Proof:**

(a) The adjunction formula gives

$$p_a(B) = 1 + \frac{B^2 + K_\Sigma.B}{2} = 1.$$

Since $|C_2|_a$ covers the whole of $\Sigma$ and $B \not\sim_a C_2$, the two irreducible curves $B$ and $C_2$ must intersect properly, that is, $B$ is not a fibre of $\mathrm{pr}_1$. But then the mapping $\mathrm{pr}_{1|} : B \to C_1$ is a finite surjective morphism of degree $B.C_2$. If $B$ was a singular curve its normalisation would have to have arithmetical genus 0 and the composition of the normalisation with $\mathrm{pr}_{1|}$

would give rise to a surjective morphism from $\mathbb{P}^1_{\mathbb{C}}$ to an elliptic curve, contradicting Hurwitz's formula. Hence, $B$ is smooth and $g(B) = p_a(B) = 1$. We thus may apply the formula of Hurwitz to $\mathrm{pr}_{1|}$ and the degree of the ramification divisor $R$ turns out to be

$$\deg(R) = 2 \cdot \Big(g(B) - 1 + \big(g(C_1) - 1\big) \cdot \deg(\mathrm{pr}_{1|})\Big) = 0.$$

The remaining case is treated analogously.

(b) W. l. o. g. $i = 2$. For $p = (p_1, p_2) \in \Sigma$ we have $\tau_p(C_2) = \mathrm{pr}_1^{-1}(p_1)$ is a fibre of $\mathrm{pr}_1$, and since $\mathrm{pr}_{1|}$ is unramified, $\#\big(B \cap \tau_p(C_2)\big) = \deg\big(\mathrm{pr}_{1|}\big) = B.C_2$. Suppose $q, q' \in B$ with $\tau_{q-q'}(B) \neq B$. Then $q = \tau_{q-q'}(q') \in B \cap \tau_{q-q'}(B)$, and hence $B^2 = B.\tau_{q-q'}(B) > 0$, which contradicts the assumption $B^2 = 0$.

(c) Since $\chi(\Sigma) = 0$, **[FrM94]** Lemma I.3.18 and Proposition I.3.22 imply that $g(H) = p_g(\Sigma) = h^0(\Sigma, K_\Sigma) = 1$.

(d) W. l. o. g. $B.C_2 = 1$. Let $0 \neq p \in C_2$. We claim that $B \cap \tau_p(B) = \emptyset$, and hence $B^2 = B.\tau_p(B) = 0$.
Suppose $(a, b) \in B \cap \tau_p(B)$, then there is an $(a', b') \in B$ such that $(a, b) = \tau_p(a', b') = (a', b' + p)$, i. e. $a = a'$ and $b = b' + p$. Hence, $(0, b), (0, b') \in \tau_{-a}(B) \cap C_2$. But, $\tau_{-a}(B).C_2 = B.C_2 = 1$, and thus $b' = b = b' + p$ in contradiction to the choice of $p$.
$C_1 \cong B$ via $\mathrm{pr}_{1|}$ follows from (a).

(e) By (d) we have $C_1 \cong B \cong C_2$.

(f) $\mathrm{pr}_{i|} : B \to C_i$ is an isomorphism (cf. **[GrD67]** I.5.1.4), and has thus degree one. But $\deg(\mathrm{pr}_{i|}) = B.C_j$. Thus we are done with (d).

□

**G.17 Example**

(a) Let $C_1 = C_2 = C = \mathbb{C}/\Lambda$ with $\Lambda = \mathbb{Z} \oplus \tau\mathbb{Z}$, and $\Sigma = C_1 \times C_2 = C \times C$. The Picard number $\rho(\Sigma)$ is then either three or four, depending on whether the group $\mathrm{End}_0(C)$ of endomorphisms of $C$ fixing $0$ is just $\mathbb{Z}$ or larger. Using **[Har77]** Theorem IV.4.19 and Exercise IV.4.11 we find the following classification.
**Case 1:** $\exists\, d \in \mathbb{N}$ such that $\tau \in \mathbb{Q}\big[\sqrt{-d}\big]$, i. e. $\mathbb{Z} \subsetneqq \mathrm{End}_0(C)$.
Then $\rho(\Sigma) = 4$ and $\mathrm{NS}(\Sigma) = C_1\mathbb{Z} \oplus C_2\mathbb{Z} \oplus C_3\mathbb{Z} \oplus C_4\mathbb{Z}$ where $C_3$ is the diagonal in $\Sigma$, i. e. the graph of the identity map from $C$ to $C$, and $C_4$ is the graph of the morphism $\alpha : C \to C : p \mapsto (b\tau) \cdot p$ of degree $|b\tau|^2$, where $0 \neq b \in \mathbb{N}$ minimal with $b \cdot (\tau + \overline{\tau}) \in \mathbb{Z}$ and $b\tau\overline{\tau} \in \mathbb{Z}$. Setting

$a := C_3.C_4 \geq 1$, the intersection matrix is

$$(C_j.C_k)_{j,k=1,\dots,4} = \begin{pmatrix} 0 & 1 & 1 & |b\tau|^2 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & a \\ |b\tau|^2 & 1 & a & 0 \end{pmatrix}.$$

If e. g. $\tau = i$, then $C_4 = \{(c, ic) \mid c \in C\}$ and

$$(C_j.C_k)_{j,k=1,\dots,4} = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}.$$

**Case 2:** $\nexists d \in \mathbb{N}$ such that $\tau \in \mathbb{Q}[\sqrt{-d}]$, i. e. $\mathbb{Z} = \mathrm{End}_0(C)$.
Then $\rho(\Sigma) = 3$ and $\mathrm{NS}(\Sigma) = C_1\mathbb{Z} \oplus C_2\mathbb{Z} \oplus C_3\mathbb{Z}$ where again $C_3$ is the diagonal in $\Sigma$. The intersection matrix in this case is

$$(C_j.C_k)_{j,k=1,2,3} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

(b) Let $C_1 = \mathbb{C}/\Lambda_1$ and $C_2 = \mathbb{C}/\Lambda_2$ with $\Lambda_1 = \mathbb{Z} \oplus \tau_1\mathbb{Z}$, $\tau_1 = i$, and $\Lambda_2 = \mathbb{Z} \oplus \tau_2\mathbb{Z}$, $\tau_2 = \frac{1}{2}i$. Then $C_1 \not\cong C_2$.[31]
We consider the surjective morphisms $\alpha_j : C_1 \to C_2$, $j = 3, 4$, induced by multiplication with the complex numbers $\alpha_3 = 1$ and $\alpha_4 = i$ respectively. Denoting by $C_j$ the graph of $\alpha_j$, we claim, $C_1.C_3 = \deg(\alpha_3) = 2$ and $C_1.C_4 = \deg(\alpha_4) = 2$. $\alpha_j$ being an unramified covering, we can calculate its degree by counting the preimages of $0$. If $p = [a + ib] \in \mathbb{C}/\Lambda_1 = C_1$ with $0 \leq a, b < 1$, then

$$\begin{aligned} \alpha_3(p) = 0 &\Leftrightarrow a + ib = \alpha_3 \cdot (a + ib) \in \Lambda_2 \\ &\Leftrightarrow \exists\, r, s \in \mathbb{Z} \;:\; a = r \text{ and } b = \tfrac{1}{2} \cdot s \\ &\Leftrightarrow a = 0 \text{ and } b \in \{0, \tfrac{1}{2}\}. \end{aligned}$$

and

$$\begin{aligned} \alpha_4(p) = 0 &\Leftrightarrow ia - b = \alpha_4 \cdot (a + ib) \in \Lambda_2 \\ &\Leftrightarrow \exists\, r, s \in \mathbb{Z} \;:\; -b = r \text{ and } a = \tfrac{1}{2} \cdot s \\ &\Leftrightarrow b = 0 \text{ and } a \in \{0, \tfrac{1}{2}\}. \end{aligned}$$

[31] Suppose $C_1 \cong C_2$, then there are integers $a, b, c, d \in \mathbb{Z}$ with $ad - bc = \pm 1$ such that $\frac{1}{2}i = \frac{ai+b}{ci+d}$ (cf. **[Har77]** Theorem IV.15B). But this leads to $ai + b = \frac{1}{2}di - \frac{1}{2}c$, i. e. $2a = d$ and $-2b = c$. Inserting these relations in the determinant equation we get $\pm 1 = 2a^2 + 2b^2 = 2(a^2 + b^2)$ which would say that two divides one.

Moreover, the graphs $C_3$ and $C_4$ intersect only in the point $(0, 0)$ and the intersection is obviously transversal, so $C_3.C_4 = 1$.

Thus $\Sigma = C_1 \times C_2$ is an example for a product of non-isomorphic elliptic curves with $\rho(\Sigma) = 4$, $\mathrm{NS}(\Sigma) = C_1\mathbb{Z} \oplus C_2\mathbb{Z} \oplus C_3\mathbb{Z} \oplus C_4\mathbb{Z}$, and intersection matrix

$$(C_j.C_k)_{j,k=1,\dots,4} = \begin{pmatrix} 0 & 1 & 2 & 2 \\ 1 & 0 & 1 & 1 \\ 2 & 1 & 0 & 1 \\ 2 & 1 & 1 & 0 \end{pmatrix}.$$

(c) See [**HuR98**] p. 4 for examples $\Sigma = C_1 \times C_2$ with $\rho(\Sigma) = 3$ and intersection matrix

$$\begin{pmatrix} 0 & 1 & a \\ 1 & 0 & 1 \\ a & 1 & 0 \end{pmatrix}, \quad a \neq 1.$$

**G.18 Remark**

(a) If $C_1$ and $C_2$ are isogenous, then there are irreducible curves $B \subset \Sigma$ with $B.C_i$ arbitrarily large.

For this just note, that we have a curve $\Gamma \subset \Sigma$ which is the graph of an isogeny $\alpha : C_1 \to C_2$. Denoting by $n_{C_2} : C_2 \to C_2$ the morphism induced by the multiplication with $n \in \mathbb{N}$, we have a morphism $n_{C_2} \circ \alpha$ whose degree is just $n^2 \cdot \deg(\alpha)$. But the degree is the intersection number of the graph with $C_1$. The dual morphism of $n_{C_2} \circ \alpha$ has the the same degree, which then is the intersection multiplicity of its graph with $C_2$. (Cf. [**Har77**] Ex. IV.4.7.)

(b) If $C_1$ and $C_2$ are isogenous, then $\Sigma$ might very well contain smooth irreducible elliptic curves $B$ which are neither isomorphic to $C_1$ nor to $C_2$, and hence cannot be the graph of an isogeny between $C_1$ and $C_2$. But being an elliptic curve we have $B^2 = 0$ by the adjunction formula. If now $\mathrm{NS}(\Sigma) = \bigoplus_{i=1}^{\rho(\Sigma)} C_i\mathbb{Z}$, where the additional generators are graphs, then $B \sim_a \sum_{i=1}^{\rho(\Sigma)} n_iC_i$ with some $n_i < 0$. (Cf. [**LaB92**] Ex. 10.6.)

Throughout the remaining part of the subsection we will restrict our attention to the general case, that is that $C_1$ and $C_2$ are *not* isogenous. This makes the formulae look much nicer, since then $\mathrm{NS}(\Sigma) = C_1\mathbb{Z} \oplus C_2\mathbb{Z}$.

**G.19 Lemma**

*Let* $C_1$ *and* $C_2$ *be non-isogenous elliptic curves,* $D \in \mathrm{Div}(\Sigma)$ *with* $D \sim_a aC_1+bC_2$.

(a) $D$ *is nef if and only if* $a, b \geq 0$.

(b) $D$ *is ample if and only if* $a, b > 0$.

(c) $D$ *is very ample if and only if* $a, b \geq 3$.

(d) $D^2 = 0$ *if and only if* $a = 0$ *or* $b = 0$.

(e) *If* $D$ *is an irreducible curve, then we are in one of the following cases:*
   (1) $a = 0$ *and* $b = 1$,
   (2) $a = 1$ *and* $b = 0$,
   (3) $a, b > 0$.

   *Moreover if we are in one of the cases (1), (2), or*
   (3') $a, b > 1$,
   *then there is an irreducible curve algebraically equivalent to* $D$.

(f) *If* $D$ *is an irreducible curve and* $D^2 = 0$, *then either* $D \sim_a C_1$ *or* $D \sim_a C_2$.

**Proof:** (a) and (b) follow from Lemma G.7.

(c) In Lemma G.7 we proved that if $a, b \geq 3$ then $D$ is very ample. Conversely, if $a < 3$, then $D \cap C_2$ is a divisor of degree $D.C_2 = a < 3$ on the elliptic curve $C_2$ and hence not very ample (cf. [**Har77**] Example IV.3.3.3). But then $D$ is not very ample. Analogously if $b < 3$.

(d) $0 = D^2 = 2ab$ if and only if $a = 0$ or $b = 0$.

(e) Let us first consider the case that $D$ is irreducible.
If $a = 0$ or $b = 0$, then $D$ is algebraically equivalent to a multiple of a fibre of one of the projections $pr_i$, $i = 1, 2$. In this situation $D^2 = 0$ and thus the irreducible curve $D$ does not intersect any of the fibres properly. Hence it must be a union of several fibres, and being irreducible it must be a fibre. That is we are in one of the first two cases.
Suppose now that $a, b \neq 0$. Thus $D$ intersects $C_i$ properly, and $0 < D.C_1 = b$ and $0 < D.C_2 = a$.
It now remains to show that the mentioned algebraic systems contain irreducible curves, which is clear for the first two of them. Let therefore $a$ and $b$ be at least two. Then the linear system $|aC_1 + bC_2|_l$ contains no fixed component, and being ample by (v) its general element is irreducible according to [**LaB92**] Theorem 4.3.5.

(f) Follows from (d) and (e).

□

## G.d. Surfaces in $\mathbb{P}^3_{\mathbb{C}}$ and Complete Intersections

A smooth projective surface $\Sigma$ in $\mathbb{P}^3_{\mathbb{C}}$ is given by a single equation $f = 0$ with $f \in \mathbb{C}[w, x, y, z]$ homogeneous, and by definition the degree of $\Sigma$, say $n$, is just the degree of $f$ – for $n = 1$, $\Sigma \cong \mathbb{P}^2_{\mathbb{C}}$; for $n = 2$, $\Sigma \cong \mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$; and for $n = 3$, $\Sigma$ is isomorphic to $\mathbb{P}^2_{\mathbb{C}}$ blown up in six points in general position. Thus the Picard number $\rho(\Sigma)$, i. e. the rank of the Néron–Severi group, in these cases is 1, 2, and 7 respectively.[32] Note that these are also precisely the cases where $\Sigma$ is rational.

In general the Picard number $\rho(\Sigma)$ of a surface in $\mathbb{P}^3_{\mathbb{C}}$ may be arbitrarily large,[33] but the Néron–Severi group always contains a very special member, namely the class $H \in \mathrm{NS}(\Sigma)$ of a hyperplane section with $H^2 = n$. And the class of the canonical divisor is then just $(n-4)\cdot H$. Moreover, if the degree of $\Sigma$ is at least four, that is, if $\Sigma$ is not rational, then it is likely that $\mathrm{NS}(\Sigma) = H\mathbb{Z}$. More precisely, if $n \geq 4$, Noether's Theorem says that $\{\Sigma \mid \rho(\Sigma) = 1, \deg(\Sigma) = n\}$ is a very general subset of the projective space of smooth projective surfaces in $\mathbb{P}^3_{\mathbb{C}}$ of fixed degree $n$, i. e. it's complement is an at most countable union of lower dimensional subvarieties (cf. **[Har75]** Corollary 3.5 or **[IsS96]** p. 146).

In Chapter V we need to know that hypersurface sections on surfaces in $\mathbb{P}^3_{\mathbb{C}}$, or more generally on complete intersections, are always non-special. If $\Sigma = H_1 \cap \ldots \cap H_{N-2} \subseteq \mathbb{P}^N_{\mathbb{C}}$ is a complete intersection of hypersurfaces $H_i \subset \mathbb{P}^N_{\mathbb{C}}$ of degree $d_i$, $i = 1, \ldots, N-2$, with $d_1 \geq \ldots \geq d_{N-2} > 1$, we say $\Sigma$ is a *complete intersection of type* $(d_1, \ldots, d_{N-2})$.

**G.20 Proposition**

*Let* $\Sigma = H_1 \cap \ldots \cap H_{N-2} \subseteq \mathbb{P}^N_{\mathbb{C}}$ *be a smooth complete intersection of type* $(d_1, \ldots, d_{N-2})$*, and let* $H \subset \Sigma$ *be a hyperplane section.*

*Then* $K_\Sigma \sim_l \left(-N-1+\sum_{i=1}^{N-2} d_i\right)\cdot H$, $H^2 = d_1 \cdots d_{N-2}$ *and*[34] *for* $d \in \mathbb{Z}$

(a) $h^1(\Sigma, dH) = 0$,

(b) $h^0(\Sigma, dH) = \binom{N+d}{N} + \sum_{k=1}^{N-2} \sum_{1\leq j_1<\ldots<j_k\leq N-2} (-1)^k \cdot \binom{N+d-\sum_{l=1}^k d_{j_l}}{N}$, *and*

(c) $\chi(\mathcal{O}_\Sigma) = 1 + \binom{-1+\sum_{i=1}^{N-2} d_i}{N} + \sum_{k=1}^{N-2} \sum_{1\leq j_1<\ldots<j_k\leq N-2} (-1)^k \cdot \binom{-1+\sum_{i=1}^{N-2} d_i - \sum_{l=1}^k d_{j_l}}{N}$.

**Proof:** Denoting by $H^i$ a hyperplane section in $\Sigma_i = H_1 \cap \ldots \cap H_i$, $i = 0, \ldots, N-2$, we claim that we have indeed $K_{\Sigma_i} \sim_l \left(-N-1+\sum_{k=1}^{i} d_k\right)\cdot H^i$ and for $d \in \mathbb{Z}$

(a') $h^k\big(\Sigma_i, dH^i\big) = 0$ for all $0 < k < N-i$.

(b')
$$\begin{aligned} h^0\big(\Sigma_i, dH^i\big) &= h^0\big(\mathcal{O}_{\mathbb{P}^N_{\mathbb{C}}}(d)\big) + \sum_{k=1}^{i} \sum_{1\leq j_1<\ldots<j_k\leq i} (-1)^k \cdot h^0\Big(\mathcal{O}_{\mathbb{P}^N_{\mathbb{C}}}\big(d - \textstyle\sum_{l=1}^k d_{j_l}\big)\Big) \\ &= \binom{N+d}{N} + \sum_{k=1}^{i} \sum_{1\leq j_1<\ldots<j_k\leq i} (-1)^k \cdot \binom{N+d-\sum_{l=1}^k d_{j_l}}{N}. \end{aligned}$$

We do the proof by induction on $i$, where for $i = 0$ we are in the case $\Sigma_0 = \mathbb{P}^N_{\mathbb{C}}$ with $K_{\mathbb{P}^N_{\mathbb{C}}} \sim_l (-N-1)H^0$ and (a') follows from **[Har77]** III.5.1, while (b') is obvious. We may thus assume that $i \geq 1$.

[32] See e. g. **[Har77]** Example II.8.20.3 and Remark V.4.7.1.

[33] E. g. the $n$-th Fermat surface, given by $w^n + x^n + y^n + z^n = 0$ has Picard number $\rho \geq 3\cdot(n-1)\cdot(n-2)+1$, with equality if $\gcd(n,6) = 1$ (cf. **[Shi82]** Theorem 7, see also **[AoS83]** pp. 1f. and **[IsS96]** p. 146).

[34] If $m < n$, we set the binomial coefficient $\binom{m}{n} = 0$.

The inclusion $\Sigma_i \hookrightarrow \Sigma_{i-1}$ gives rise to the exact sequence

$$0 \longrightarrow \mathcal{O}_{\Sigma_{i-1}}\big(-d_i H^{i-1}\big) \xrightarrow{\cdot H_i} \mathcal{O}_{\Sigma_{i-1}} \longrightarrow \mathcal{O}_{\Sigma_i} \longrightarrow 0, \tag{G.9}$$

and by [**Har77**] II.8.20 $\omega_{\Sigma_i} \cong \omega_{\Sigma_{i-1}} \otimes \mathcal{O}_{\Sigma_i}\big(d_i H^i\big)$. But then by induction

$$K_{\Sigma_i} = \left(-N-1+\sum_{k=1}^{i} d_k\right) \cdot H^i.$$

The long exact cohomology sequence of (G.9) twisted by $dH^{i-1}$ then gives

$$0 \to H^0\big(\Sigma_{i-1},(d-d_i)H^{i-1}\big) \to H^0\big(\Sigma_{i-1}, dH^{i-1}\big) \longrightarrow H^0\big(\Sigma_i, dH^i\big) \longrightarrow H^1\big(\Sigma_{i-1},(d-d_i)H^{i-1}\big) \tag{G.10}$$

and for any $k \geq 1$

$$H^k\big(\Sigma_{i-1}, dH^{i-1}\big) \longrightarrow H^k\big(\Sigma_i, dH^i\big) \longrightarrow H^{k+1}\big(\Sigma_{i-1},(d-d_i)H^{i-1}\big). \tag{G.11}$$

(a') If $0 < k < N-i$, then $h^k\big(\Sigma_{i-1}, dH^{i-1}\big) = 0 = h^{k+1}\big(\Sigma_{i-1},(d-d_i)\cdot H^{i-1}\big)$ for any $d$ by induction. Thus by G.11 also $h^k\big(\Sigma_i, dH^i\big) = 0$.

(b') Since $h^1\big(\Sigma_i, dH^i\big) = 0$, by (G.10) and induction we get

$$\begin{aligned} h^0\big(\Sigma_i, dH^i\big) &= h^0\big(\Sigma_{i-1}, dH^{i-1}\big) + h^0\big(\Sigma_{i-1},(d-d_i)\cdot H^{i-1}\big) \\ &= h^0\big(\mathcal{O}_{\mathbb{P}^N_{\mathbb{C}}}(d)\big) + \sum_{k=1}^{i-1} \sum_{1\leq j_1<\ldots<j_k\leq i-1} (-1)^k \cdot h^0\Big(\mathcal{O}_{\mathbb{P}^N_{\mathbb{C}}}\big(d-\textstyle\sum_{l=1}^{k} d_{j_l}\big)\Big) \\ &\quad -h^0\big(\mathcal{O}_{\mathbb{P}^N_{\mathbb{C}}}(d-d_i)\big) - \sum_{k=1}^{i-1} \sum_{1\leq j_1<\ldots<j_k\leq i-1} (-1)^k \cdot h^0\Big(\mathcal{O}_{\mathbb{P}^N_{\mathbb{C}}}\big(d-d_i-\textstyle\sum_{l=1}^{k} d_{j_l}\big)\Big). \\ &= h^0\big(\mathbb{P}^N_{\mathbb{C}}, \mathcal{O}_{\mathbb{P}^N_{\mathbb{C}}}(d)\big) + \sum_{k=1}^{i} \sum_{1\leq j_1<\ldots<j_k\leq i} h^0\Big(\mathbb{P}^N_{\mathbb{C}}, \mathcal{O}_{\mathbb{P}^N_{\mathbb{C}}}\big(d-\textstyle\sum_{l=1}^{k} d_{j_l}\big)\Big). \end{aligned}$$

Knowing that for any $\delta \in \mathbb{Z}$ the dimension $h^0\big(\mathbb{P}^N_{\mathbb{C}}, \mathcal{O}_{\mathbb{P}^N_{\mathbb{C}}}(\delta)\big) = \binom{N+\delta}{N}$ we have proved (b').

(c) This follows immediately by (a), (b) and Serre duality, since

$$\chi(\mathcal{O}_\Sigma) = 1 + h^0\big(\Sigma, K_\Sigma\big).$$

Finally, we note that $H^2$ is just the degree of $\Sigma \subseteq \mathbb{P}^N_{\mathbb{C}}$, which is just $d_1 \cdots d_{N-2}$ by the Theorem of Bézout (cf. [**Har77**] I.7.7.). □

We note that complete intersections of type (4), (3,2) and (2,2,2) are the only smooth complete intersection surfaces which are K3-surfaces, since for no other choice of the $d_i$ and $N$ we can achieve that $\sum_{i=1}^{N-2} d_i - N - 1 = 0$.

Since a smooth hypersurface in $\mathbb{P}^3_{\mathbb{C}}$ is a complete intersection, we get the following corollary.

**G.21 Corollary**

*Let $\Sigma \subset \mathbb{P}^3_{\mathbb{C}}$ be a smooth hypersurface of degree $n$ in $\mathbb{P}^3_{\mathbb{C}}$, and let $H \subset \mathbb{P}^3_{\mathbb{C}}$ be any hyperplane section.*

*Then $K_\Sigma \sim_l (n-4) \cdot H$ and*[35] *for $d \in \mathbb{Z}$*

(a) $h^1(\Sigma, dH) = 0$,

(b) $h^0(\Sigma, dH) = \binom{d+3}{3} - \binom{d-n+3}{3}$, *and*

(c) $\chi(\mathcal{O}_\Sigma) = 1 + \binom{n-1}{3}$.

## G.e. K3-Surfaces

We note that if $\Sigma$ is a K3-surface then the Néron–Severi group $NS(\Sigma)$ and the Picard group $Pic(\Sigma)$ of $\Sigma$ coincide, i. e. $|D|_a = |D|_l$ for every divisor $D$ on $\Sigma$. The Picard number $\rho(\Sigma)$ of a K3-surface may take values between 1 and 20, and there are K3-surfaces $\Sigma$ with $\rho(\Sigma) = \rho$ for each $\rho \in \{1, \ldots, 20\}$. The generic case is $\rho(\Sigma) = 1$ – more precisely, the K3-surfaces with $\rho(\Sigma) = \rho$ are parametrised by points of an at most countable union of irreducible varieties of dimension $20 - \rho$ (cf. [**IsS96**] 12.5 Corollary 4). Moreover, if $NS(\Sigma) = L \cdot \mathbb{Z}$, then $\pi_\Sigma = \frac{1}{2} \cdot L^2$ may be any even positive number (cf. [**IsS96**] 12.5 Corollary 2). In general, we denote by $\pi_\Sigma = \min\left\{D^2 \mid D \in NS(\Sigma), D^2 > 0\right\}$. E. g. if $\Sigma$ is a double cover of $\mathbb{P}^2_{\mathbb{C}}$ branched along a sextic, then $\pi_\Sigma = 1$; if $\Sigma$ is quartic in $\mathbb{P}^3_{\mathbb{C}}$, then $\pi_\Sigma = 2$; if $\Sigma$ is a complete intersection of type (3,2) in $\mathbb{P}^4_{\mathbb{C}}$, then $\pi_\Sigma = 3$; if $\Sigma$ is a complete intersection of type (2,2,2) in $\mathbb{P}^5_{\mathbb{C}}$, then $\pi_\Sigma = 4$ (cf. [**IsS96**] p. 219). Moreover, the surfaces of this these types are each parametrised by a 19-dimensional variety (cf. [**Mér85**] 5.4), and thus by [**IsS96**] 12.5 Corollary 2 a generic member must have Picard number one.

An irreducible curve $B$ has self-intersection $B^2 = 0$ if and only if the arithmetical genus of $B$ is one. In that case $|B|_l$ is a pencil of elliptic curves without base points endowing $\Sigma$ with the structure of an elliptic fibration over $\mathbb{P}^1_{\mathbb{C}}$ (cf. [**Mér85**] or Proposition B.1). However, a generic K3-surface does not possess an elliptic fibration.

[35] If $m < n$, we set the binomial coefficient $\binom{m}{n} = 0$.

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# Index

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