The 12 Types of Rays of the Secondary Fan of a Cubic

The computational results listed below prove some of the results in the paper

Eric Katz, Hannah Markwig, Thomas Markwig: The j-invariant of a Plane Tropical Cubic. Preprint 2007.

For an explantion of the terms used in the following, for some of the notation, like val.(j), and for a thorough explanation of the results we refer to this paper.

We consider a general plane cubic given by the polynomial

$$f = u_{30}x^3 + u_{21}x^2y + u_{12}xy^2 + u_{03}y^3 + u_{20}x^2 + u_{11}xy + u_{02}y^2 + u_{10}x + u_{01}y + u_{00}$$

and the associated secondary fan which is the secondary fan of the marked polytope $(Q_3, \mathcal{A}_3)$, where $\mathcal{A}_3 = \{(3,0),(2,1),(1,2),(0,3),(2,0),(1,1),(0,2),(1,0),(0,1),(0,0)\}$ is the set of marked points of Q_3 and Q_3 is the convex hull of $\mathcal{A}_3$.

In the above mentioned paper the rays of the secondary fan have been classified as *lifts*, *folds*, and *pinwheels*, and we list them below up to symmetry. Moreover, we give in each of the cases a particular polynomial representative and compute for this one the *generalised cycle length* and the *generic valuation of the j-invariant.*

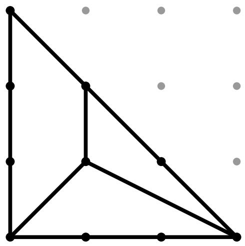
It is out aim to see that the negative of the one of the functions coincide with the other one on each ray of the secondary fan of $\mathcal{A}_3$. However, since they are linear on a cone of the secondary fan of $\mathcal{A}_3$, since they coincide on the the linearity space space anyhow and since a ray is generated by the linearity space plus one additional vector, it suffices to show that they coincide for a particular representative of a ray in order to see that they coincide on the whole ray. Thus our investigation proves this claim for all rays that we consider. However, since all other rays are obtained from these by integral unimodular affine transformations, which respects the generic valuation of the j-invariant as well as the generalised cycle length, we get the result indeed for all rays of the secondary fan of $\mathcal{A}_3$.

Note that in the marked subdivisions on the following pages black lattice points correspond to markings, while grey lattice points are not marked.

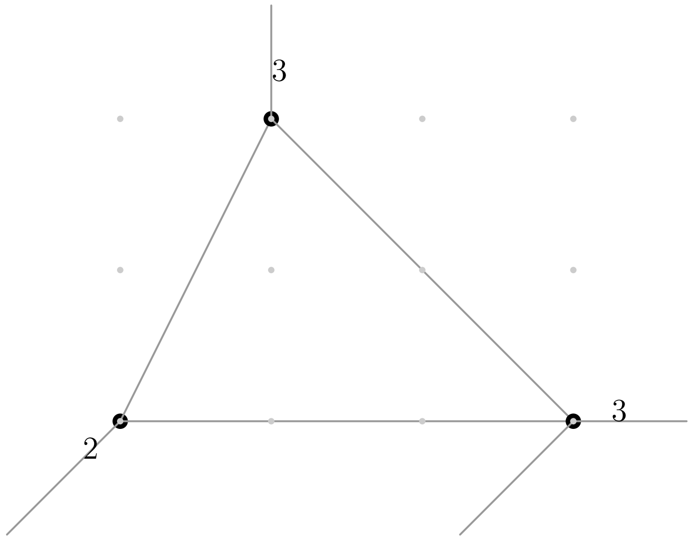
1. Ray: a pinwheel given by the equation:

$$x^3 + t \cdot x^2y + t^2 \cdot xy^2 + t^6 \cdot y^3 + x^2 + xy + t^4 \cdot y^2 + x + t^2 \cdot y + 1.$$

The corresponding Newton polygon is:



The dual tropical curve looks like:



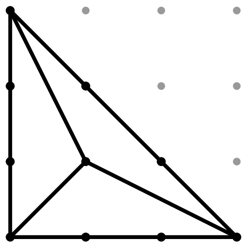
The generic valuation of the j-invariant is: -6

The generalised cycle length is: 6

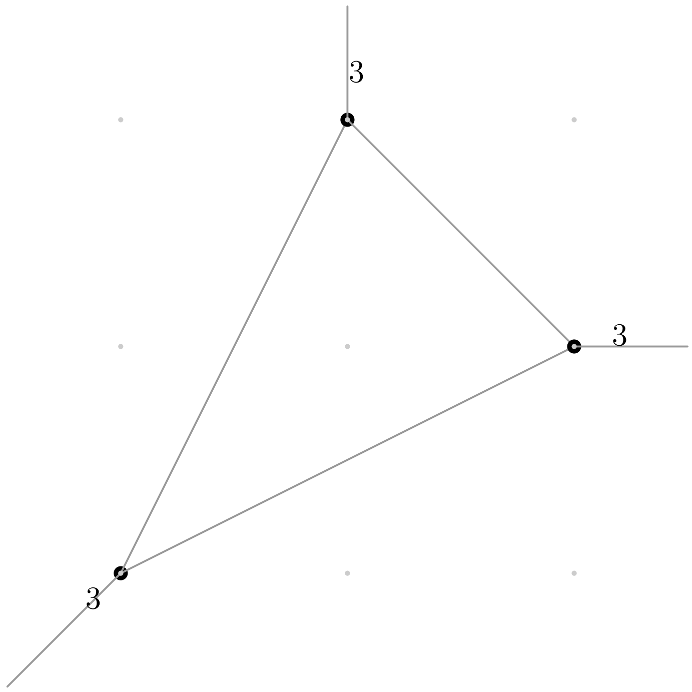
2. Ray: a pinwheel given by the equation:

$$x^3 + 3 \cdot t \cdot x^2 y + t^2 \cdot xy^2 + t^3 \cdot y^3 + x^2 + xy + t^2 \cdot y^2 + x + t \cdot y + 1.$$

The corresponding Newton polygon is:



The dual tropical curve looks like:



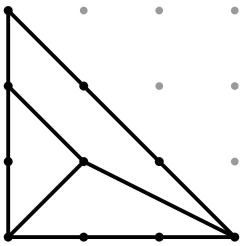
The generic valuation of the j-invariant is: -3

The generalised cycle length is: 3

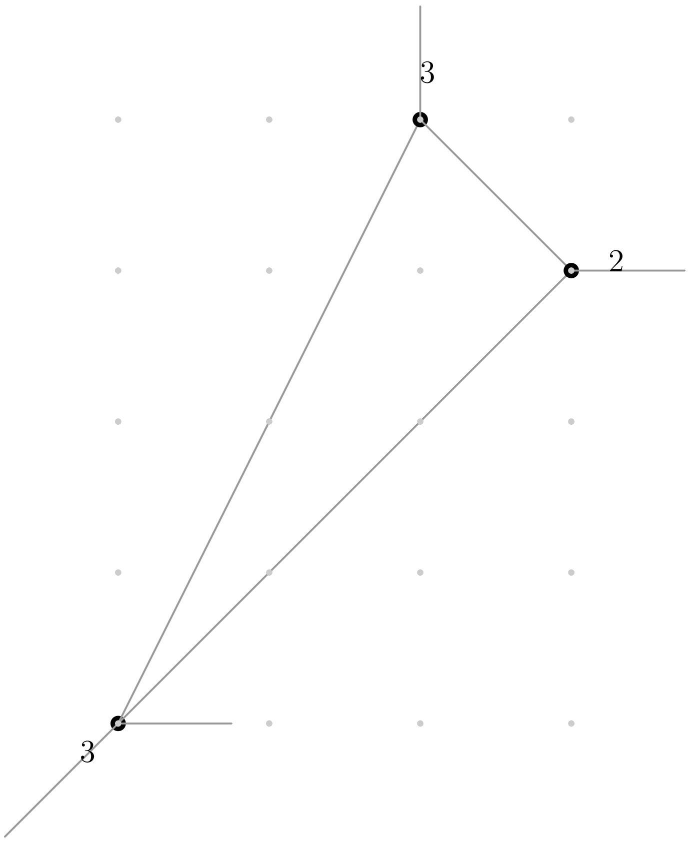
3. Ray: a pinwheel given by the equation:

$$x^3 + t^2 \cdot x^2y + t^4 \cdot xy^2 + t^6 \cdot y^3 + x^2 + xy + t^2 \cdot y^2 + x + t \cdot y + 1.$$

The corresponding Newton polygon is:



The dual tropical curve looks like:



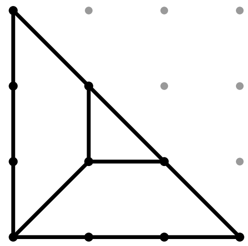
The generic valuation of the j-invariant is: -6

The generalised cycle length is: 6

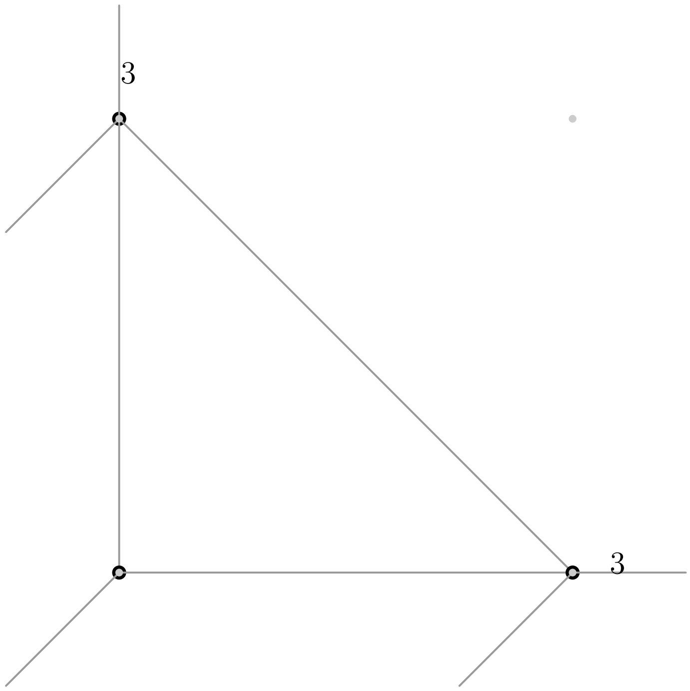
4. Ray: a pinwheel given by the equation:

$$x^3 + x^2y + t \cdot xy^2 + t^3 \cdot y^3 + x^2 + xy + t^2 \cdot y^2 + x + t \cdot y + 1.$$

The corresponding Newton polygon is:



The dual tropical curve looks like:



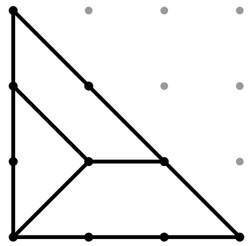
The generic valuation of the j-invariant is: -3

The generalised cycle length is: 3

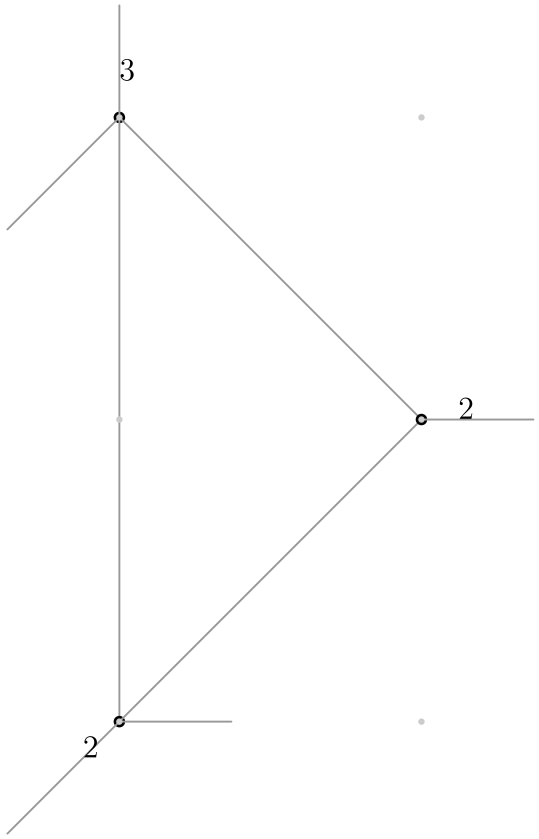
5. Ray: a pinwheel given by the equation:

$$x^3 + x^2y + t^2 \cdot xy^2 + t^4 \cdot y^3 + x^2 + xy + t^2 \cdot y^2 + x + 2 \cdot t \cdot y + 1.$$

The corresponding Newton polygon is:



The dual tropical curve looks like:



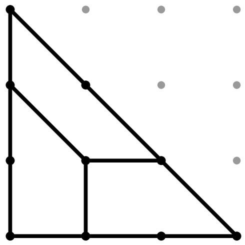
The generic valuation of the j-invariant is: -4

The generalised cycle length is: 4

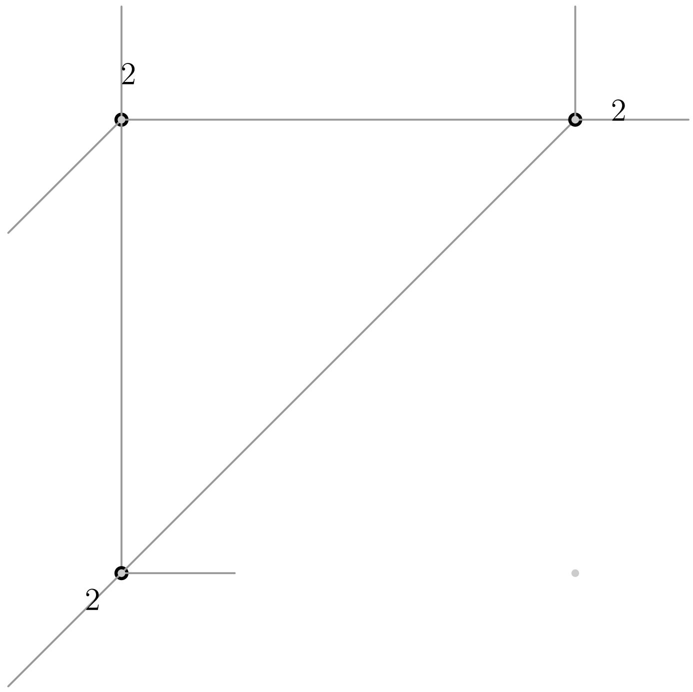
6. Ray: a pinwheel given by the equation:

$$t^2 \cdot x^3 + t \cdot x^2 y + t \cdot xy^2 + t \cdot y^3 + t \cdot x^2 + xy + y^2 + 3 \cdot x + y + 1.$$

The corresponding Newton polygon is:



The dual tropical curve looks like:



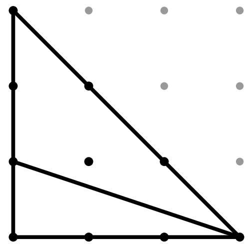
The generic valuation of the j-invariant is: -3

The generalised cycle length is: 3

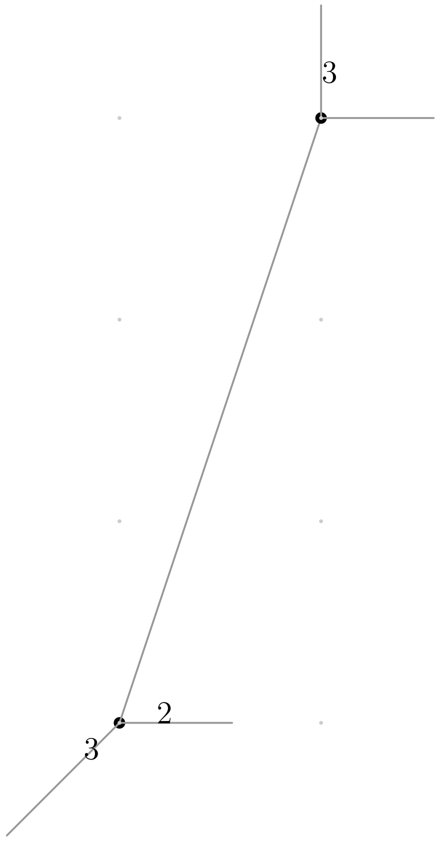
7. Ray: a fold given by the equation:

$$x^3 + x^2y + xy^2 + y^3 + t \cdot x^2 + xy + y^2 + t^2 \cdot x + y + t^3.$$

The corresponding Newton polygon is:



The dual tropical curve looks like:



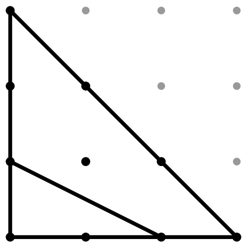
The generic valuation of the j-invariant is: 0

The generalised cycle length is: 0

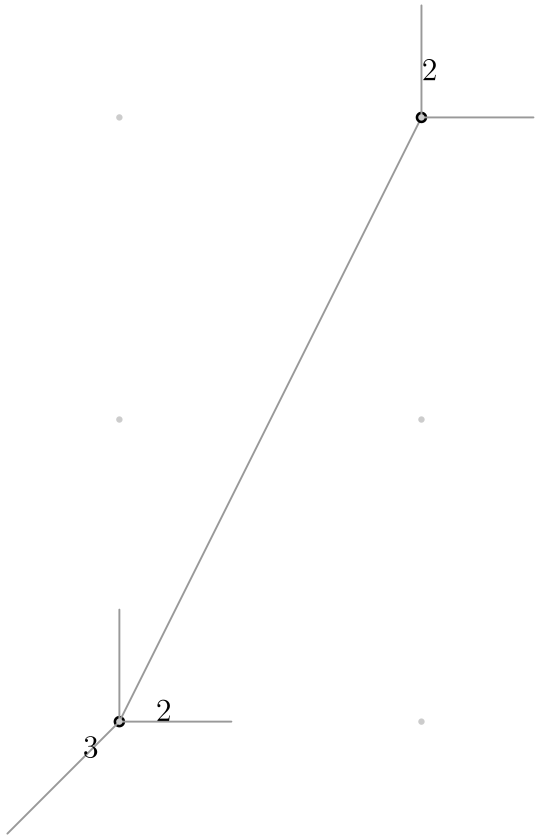
8. Ray: a fold given by the equation:

$$x^3 + x^2y + xy^2 + y^3 + x^2 + xy + y^2 + t \cdot x + y + t^2.$$

The corresponding Newton polygon is:



The dual tropical curve looks like:



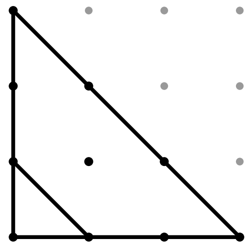
The generic valuation of the j-invariant is: 0

The generalised cycle length is: 0

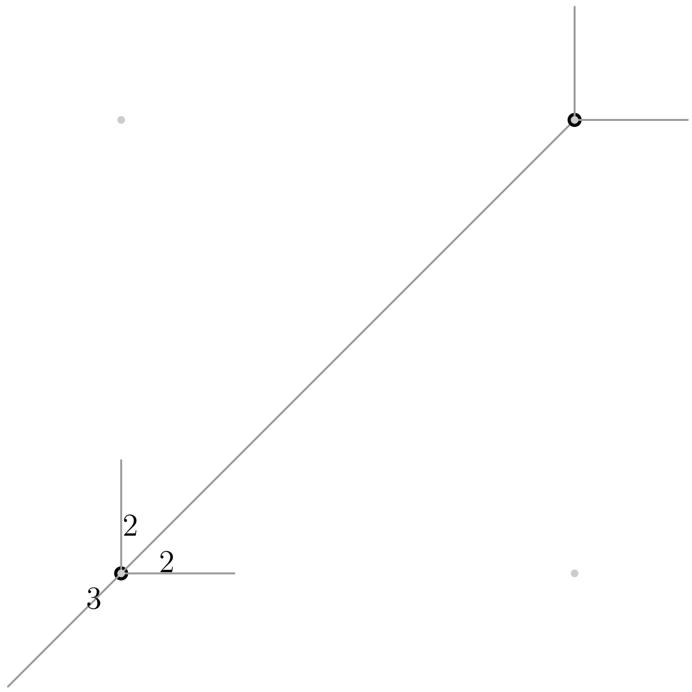
9. Ray: a fold given by the equation:

$$x^3 + x^2y + xy^2 + y^3 + x^2 + xy + y^2 + x + y + t.$$

The corresponding Newton polygon is:



The dual tropical curve looks like:



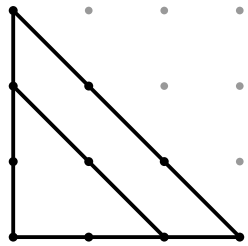
The generic valuation of the j-invariant is: 0

The generalised cycle length is: 0

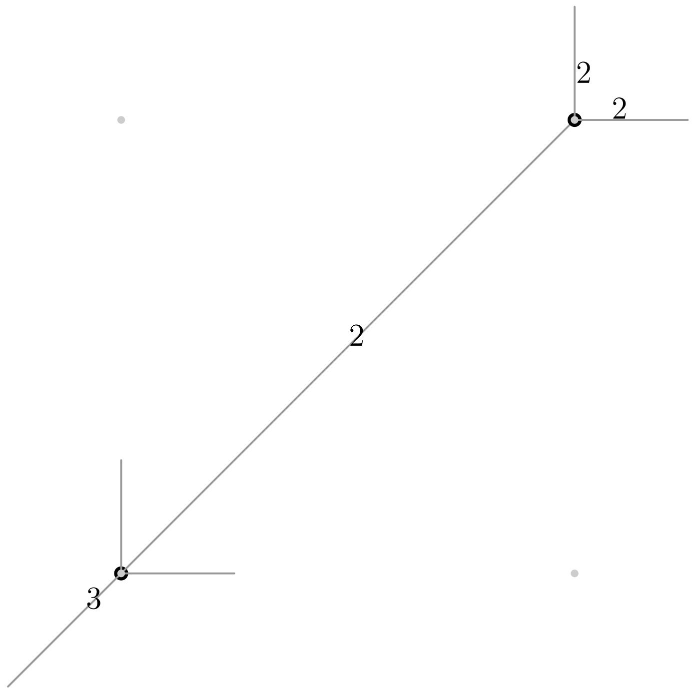
10. Ray: a fold given by the equation:

$$x^3 + x^2y + xy^2 + y^3 + x^2 + xy + y^2 + t \cdot x + t \cdot y + t^2.$$

The corresponding Newton polygon is:



The dual tropical curve looks like:



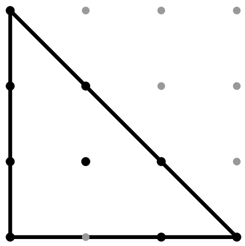
The generic valuation of the j-invariant is: -2

The generalised cycle length is: 2

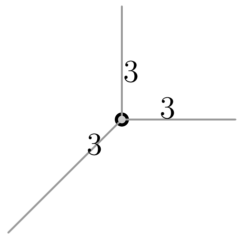
11. Ray: a lift given by the equation:

$$x^3 + x^2y + xy^2 + y^3 + x^2 + xy + y^2 + t \cdot x + y + 1.$$

The corresponding Newton polygon is:



The dual tropical curve looks like:



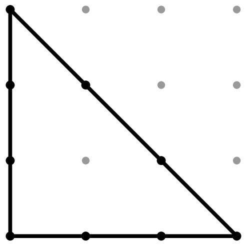
The generic valuation of the j-invariant is: 0

The generalised cycle length is: 0

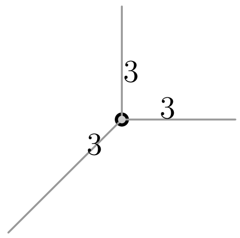
12. Ray: a lift given by the equation:

$$x^3 + x^2y + xy^2 + y^3 + x^2 + t \cdot xy + y^2 + x + y + 1.$$

The corresponding Newton polygon is:



The dual tropical curve looks like:



The generic valuation of the j-invariant is: 0

The generalised cycle length is: 0