Eberhard Karls Universität Tübingen Sommersemester 2019
Fachbereich Mathematik
PD Dr. Jörg Zintl

Algebraische Kurven

Übungsaufgaben zum 9. Tutorium am 10.07.2019

Aufgabe 31.
Let $C \subset \mathbb{P}^2$ be a smooth projective plane curve. Let $D \in \mathrm{Div}(C)$ be a divisor of degree $\deg(D) = 0$. Show that $L(D) \neq \{0\}$ if and only if D is linearly equivalent to the trivial divisor 0.

Aufgabe 32.
Let $C \subset \mathbb{P}^2$ be a smooth projective plane curve of genus g. Let $D \in \mathrm{Div}(C)$ be a divisor of degree $\deg(D) = 2g - 2$ such that $\ell(D) = g$. Show that D is a canonical divisor of C.

Aufgabe 33.
Let $C \subset \mathbb{P}^2$ be a smooth projective plane curve. Let $P_1, \ldots, P_n \in C$ be pairwise distinct points. Show that there exists a rational function $f \in k(C)$, which has a pole (of some order) in every point P_i, for $i = 1, \ldots, n$, but is regular everywhere else.

Aufgabe 34. (Keine Abgabe, Präsenzübung)
a) Consider a polynomial $f \in \mathbb{C}[X]$ of degree $d > 0$. Since $k(\mathbb{P}^1) \cong \mathbb{C}(X)$ we can view f as a rational function on $\mathbb{P}^1$. Show that there exists a unique point $P \in \mathbb{P}^1$ such that $f \in L(dP)$.

b) Let $d \in \mathbb{N}_{>0}$ and $P_i = (x_i : y_i) \in \mathbb{P}^1$ for $i = 1, \ldots, d$. For simplicity, we assume $y_1 = \ldots = y_d = 1$. Put $D := P_1 + \ldots + P_d \in \mathrm{Div}(\mathbb{P}^1)$. Show that $\dim_k L(D) = 1 + d$ by proving the isomorphism

$$L(D) \cong \{f \in k(\mathbb{P}^1) : f = \tfrac{g}{h} \text{ for some } g \in \mathbb{C}[X] \text{ with } \deg(g) \leq d\},$$

where $h(X) := (x - x_1) \cdot \ldots \cdot (x - x_d)$.

Aufgabe 35. (Keine Abgabe, Präsenzübung)
Let $C \subset \mathbb{P}^2$ be the smooth projective plane cubic curve given by $C := V(Y^2Z - X^3 + XZ^2)$. Use the point $P_0 := (0 : 1 : 0) \in C$ to define a group structure "$\oplus$" on C. Show that for any $f \in L(2 \cdot P_0)$ holds

$$f(P) = f(-P) \quad \text{for all } P \in C.$$

Abgabe der Lösungen zu Aufgaben 31 bis 33 am 10.07.2019 in der Übung.