Proseminar Curves and Surfaces
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Handout

The Isoperimetric Inequality

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This handout is part of the Proseminar Curves and Surfaces supervised by Prof. Dr. Franz Pedit and Dr. Allison Tanguay at *Eberhard-Karls Universitaet Tuebingen* in *Wintersemester 2012/2013*. It will discuss Topic 4: The Isoperimetric Inequality.

After a short introduction about the History of the Isoperimetric Problem which resulted in the Isoperimetric Inequality, we will formulate the requirements and the theorem. We will then give a geometric 'proof' by Jakob Steiner to show an easily comprehensive approach, followed by a rigorous proof by Erhard Schmidt. This proof may be more technical and not as elegant as other proofs but it can be given with basic mathematical tools from Linear Algebra and Analysis.

For further reading and other approaches please refer to [Blaschke].

Contents

1 History

To emphasize the significance of the discussed topic, we will start with the history of the Isoperimetric Problem/Inequality which mainly has been taken from [Siegel] and [Wiegert].

The so-called isoperimetric problem dates back to antique literature and geometry, giving physical insight into nature phenomena and answering questions such as why bees build hives with cells that are hexagonal in shape.

Literary history is dating back the problem to Vergil's Aeneid and his tale of the foundation of the city of Carthage. Vergil reports that Queen Dido from Phoenicia was obliged to flee from her bloodthirsty brother to North Africa. Once there, she made a deal with a local chieftain: In return for her fortune she would get as much land as she could isolate with the skin of a single ox. The deal was agreed upon and an ox was sacrificed. Queen Dido broke the skin of the ox down into very thin strips of leather, tied them together and constructed a huge semicircle which, together with the natural boundary of the sea, turned out to be way bigger than anyone would have expected. Upon this land, Carthage was established. Apparently, the Queen knew the isoperimetric inequality and understood how to apply this knowledge to gain the best possible solution to her problem.

The history of geometric proofs goes back to the ancient Greeks and was recorded by Pappus of Alexandria in the fourth century CE. He credited the isoperimetric results to Zenodorus who lived during the second century BCE and had originally not dealt with circles but with rectilinear figures. However, according to modern standards their proofs were incomplete since they apparently did not study the irregular cases. These always turn out to have less area than the more natural figures they examined and should have nevertheless been taken into account in order to present any rigorous proof. Archimedes also studied the problem but his work, like the original writing of Zenodorus, has been lost.

In modern days Steiner realized that the Greek arguments were incomplete and established a better way to proof the inequality by means of showing how any figure that does not have a circular boundary can be transformed into a new one with the same perimeter but greater area (we see his idea later). However, Weierstrass, by a new formalized mathematical system, showed that the proof could still not be considered a rigorous proof.

It was Weierstrass himself who did the first rigorous proof as a corollary of his Theory of Calculus of Several Variables in 1870.

For other approaches (e.g. Hurwitz who used the theory of fourier series) you may refer to [Blaschke].

2 The Isoperimetric Inequality

2.1 Formulation

Theorem (The Isoperimetric Inequality)
Let $c(t) = (x(t), y(t))$ be a simple, closed, positively oriented and regular parameterised C^1 curve with t $\in [a, b]$. Denote the area enclosed in the above defined curve c(t) with A.

For a given length l of $c(t) = (x(t), y(t))$, we then have

$$l^2 - 4\pi A \geq 0$$

or equivalently

$$A \leq \frac{l^2}{4\pi}$$

with equality iff c(t) is a circle.

2.2 Proof by Steiner

Let c(t) be as described above. First, we will show geometrically that for a given length l, the biggest area enclosed by c(t) is the area of a circle. This idea is by Jakob Steiner and is taken from [Froehlich] and [Hopf].

1. The area A must be convex

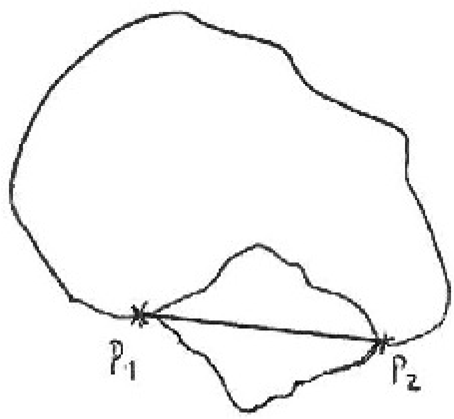

Figure 1: Reflecting for Non-Convex Areas

   If the area is not convex, we can always find two points P_1 and P_2 on c(t) which have a connecting line outside of A. Use this line as a mirror axis to reflect c(t) between the two points to the other side of the connecting line. We see that l has not changed, however A has become bigger. Thus A has to be convex.

2. A consists of two parts of the same size
   Now, choose two points on c(t), call them Q_1 and Q_2. While Q_1 can be chosen freely, Q_2 has to be the point that bisects c(t) into two parts $c_1(t)$ and $c_2(t)$, with $c_1(t)$ having exactly the same length as $c_2(t)$. This means that walking along c(t) starting at Q_1, Q_2 will appear exactly after the length $\frac{l}{2}$. By drawing a line from Q_1 to Q_2, we obtain two new areas A_1 and A_2, with $A_1 + A_2 =$ A (note: A_1 and A_2 have to be equal). Again, we can see that if they are not, then WLOG $A_1 > A_2$. In this case we can again use the connecting line between Q_1 and Q_2 to reflect A_1 to the other side of our connecting line, getting a bigger area A without changing the length l of c(t).

3. The curves $c_1(t)$ and $c_2(t)$ denote semi-circles

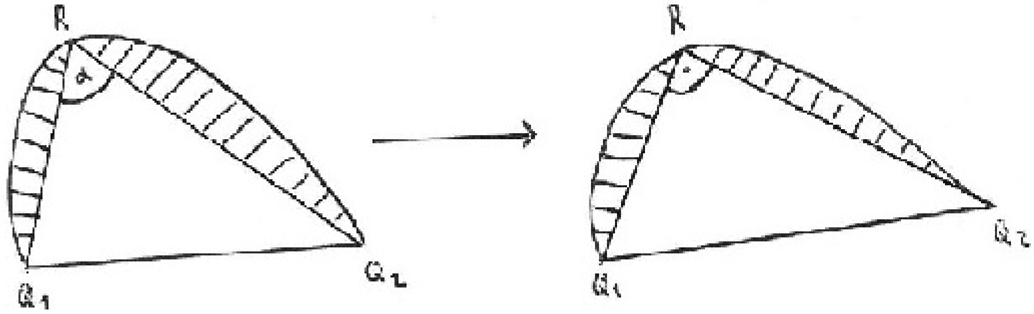

Figure 2: Changing the Angle to Maximise the Area

Again, look at Q_1 and Q_2. We showed above that $A_1 = A_2$, therefore it is enough to show WLOG that $c_1(t)$ is a semi-circle. Suppose $c_1(t)$ is not a semi-circle. Then there exists a point R on $c_1(t)$ such that the angle at R (which we call α) is not equal to 90° (note: This is Thales' Theorem). Now leave the length of Q_1R and RQ_2 fixed but allow α to be changed (and therefore Q_1 and Q_2 are moved, of course changing the connecting line between the two points, too) in order to maximise the area of the triangle Q_1RQ_2. This operation leaves the shaded areas as well as the length of $c_1(t)$ untouched but for $\alpha = 90°$ we have a bigger area for Q_1RQ_2 than before. This however is a contradiction to our assumption. Thus $c_1(t)$ has to be a semi-circle.

Note: While this construction is intuitively accessible, it cannot be seen as a complete proof of the Isoperimetric Inequality. It merely gives one way to construct the biggest area enclosed in a curve with fixed length l, namely the circle. Yet, we have not shown the existence of a (unique) solution, so we have to see this 'proof' as incomplete.

2.3 Preparation for Proof by E. Schmidt

We will prepare the proof by Erhard Schmidt using the following lemmata:

Lemma (i) - (Calculation of A)
We can determine A by the following formula:

$$\begin{aligned} A &= -\int_a^b y(t)x'(t)\,dt \\ &= \int_a^b y'(t)x(t)\,dt \\ &= \frac{1}{2}\int_a^b (y(t)'x(t) - y(t)x'(t))\,dt \end{aligned}$$

Proof
The first equality is Green's Theorem, which is a special case of Stokes' Theorem in the plane. (Stokes' Theorem should have been proved in Analysis III.)

The second equality follows from the Fundamental Theorem of Integration and Differentiation. Since c(t) is a closed curve, parameterised by arc-length with t $\in$ [a,b], we have that c(a) = c(b):

$$\begin{aligned} \int_a^b y'x\,dt &= \int_a^b (xy)'\,dt - \int_a^b x'y\,dt \\ &= [x(b)y(b) - x(a)y(a)] - \int_a^b x'y\,dt \\ &= -\int_a^b x'y\,dt \end{aligned}$$

Finally, the last equality follows immediately from the second one:

$$\begin{aligned} \int_a^b y'x\,dt &= \frac{1}{2}\int_a^b y'x\,dt + \frac{1}{2}\int_a^b y'x\,dt \\ &= \frac{1}{2}\int_a^b y'x\,dt - \frac{1}{2}\int_a^b yx'\ dt \\ &= \frac{1}{2}\int_a^b (y'x - yx')\,dt \end{aligned}$$

which concludes this Lemma.

Lemma (ii)
We will use the following inequality/equality later in our proof. For better readability and structure it is shown now and later, we will only quote the result.
Let x, y, z be functions of t and x, y, z $\in C^1$, that is x', y', z' exist and are continuous. Then we have:

$$(xy' - zx')^2 \leq (x^2 + z^2) * ((x')^2 + (y')^2)$$

with equality $\Leftrightarrow (xx' + zy')^2 = 0 \Leftrightarrow zy' = -xx'$

Proof

$$(x^2 + z^2) * ((x')^2 + (y')^2) - (xy' - zx')^2$$

$$= x^2(x')^2 + x^2(y')^2 + z^2(x')^2 + z^2(y')^2 - (x^2(y')^2 - 2xy'zx' + z^2(x')^2)$$

$$= x^2(x')^2 + 2xy'zx' + z^2(y')^2$$

$$= (xx' + zy')^2 \geq 0$$

□

For equality it has to hold

$$\begin{aligned}(xx' + zy')^2 = 0 &\Leftrightarrow xx' + zy' = 0\\ &\Leftrightarrow zy' = -xx'\end{aligned}$$

Revision (i) - (Inequality of Geometric/Arithmetic Mean)
Recall the Inequality of Geometric/Arithmetic Mean from foundation courses:

Let a, b $\in \mathbb{R}^+$, then the following inequality holds

$$\sqrt{ab} \leq \frac{a+b}{2}$$

with equality $\Leftrightarrow a = b$

Revision (ii) - (Properties of Special Parameterised Curves)
Remember from previous talks and foundation courses that

c(t)=(x(t),y(t)) is parameterised by arc-length $\Leftrightarrow$ |c'(t)| = 1 $\forall$t
$\Leftrightarrow |(x')^2 + (y')^2| = 1$

c(t)=(x(t),y(t)) is a circle with radius r $> 0 \Leftrightarrow x^2 + y^2 = r^2$

2.4 Proof by E. Schmidt

This proof was given by E. Schmidt in 1939 and the main concepts have been taken from [Carmo] and [Hopf]:

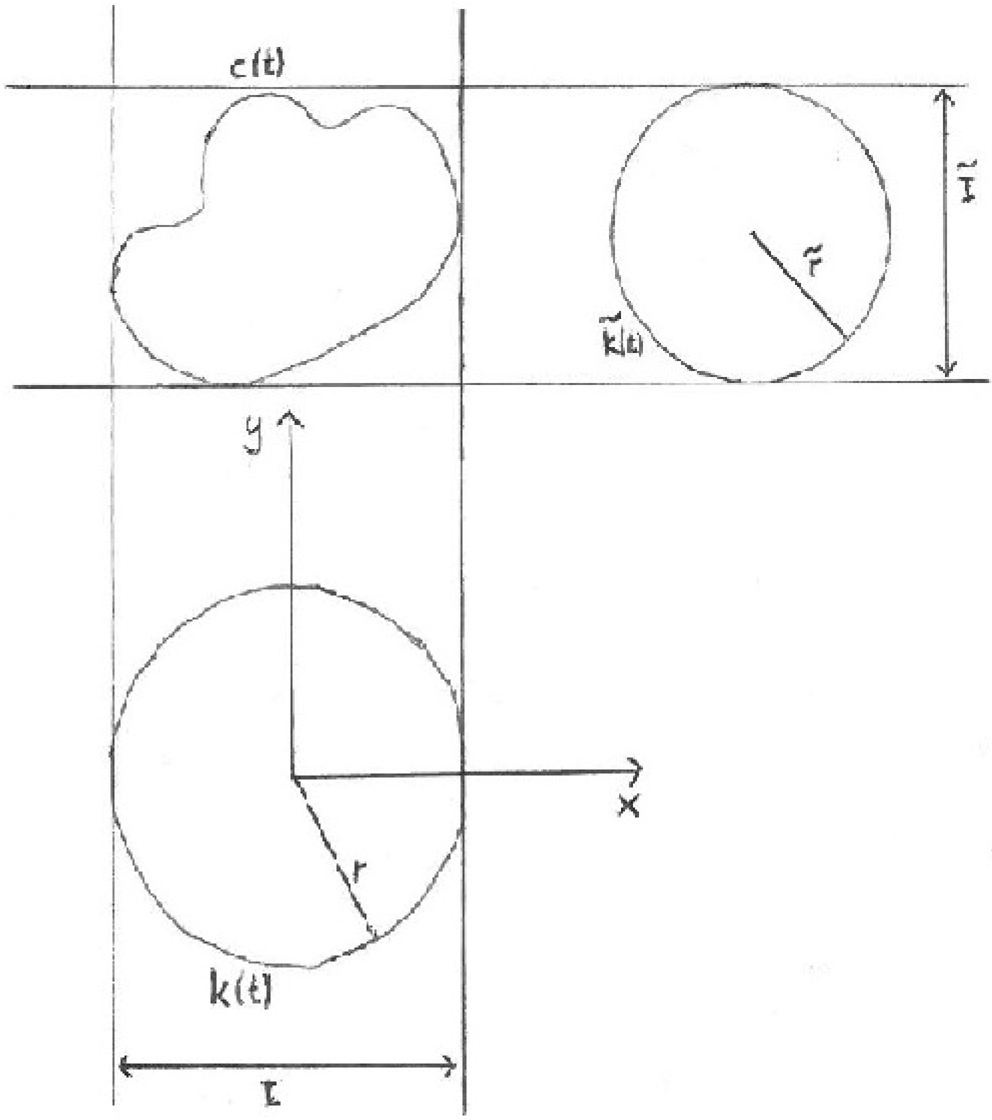

Figure 3: Construction of I, $\tilde{I}$ with $c(t)$, $k(t)$ and $\tilde{k}(t)$

Proof (The Isoperimetric Inequality by E. Schmidt)
Let c(t) = (x(t), y(t)) be an arc-length parameterised and positively oriented curve as defined above. Then we can find an interval $I = [-r, +r]$ such that $x(t) \in I$ (by setting two parallel lines touching $c(t)$ and bounding I as seen in Figure 3). Set x(t) in such a way that it starts on one bound of I WLOG x(0)=+r and x(p_1) = −r. Now define a circle $k(t) = (x(t), z(t))$ with radius r and the same $x(t)$ as above in $c(t)$, meaning $x(t)$ of $k(t)$

has the same parameterisation as in $c(t)$ (You can set $z(t)$ as $z(t) = +\sqrt{r^2 - x(t)^2}$ for $0 < t < p_1$ and $z(t) = -\sqrt{r^2 - x(t)^2}$ for $p_1 < t < l$ to ensure this parameterisation of the circle $k(t)$). Then $k(t)$ is entirely in I, too. Denote A as the area enclosed in $c(t)$ and B as the area enclosed by $k(t)$. Then (by Lemma (i)):

$$A = \int_0^l y'(t)x(t)\,dt; \qquad B = -\int_0^l z(t)x'(t)\,dt = \pi r^2$$

We can add A and B to get:

$$\begin{aligned}
A + B = A + \pi r^2 &= \int_0^l (y'x - zx')\,dt \\
&\leq \int_0^l \sqrt{(y'x - zx')^2}\,dt && (1) \\
&\leq \int_0^l \sqrt{\underbrace{(x^2 + z^2)}_{=r^2 \ by\ R(iv)} * \underbrace{((x')^2 + (y')^2)}_{=1\ by\ R(ii)}}\,dt && \text{(by L(ii))} \quad (2) \\
&= \int_0^l r\,dt = lr
\end{aligned}$$

Use Revision (i) (setting a=A and b=πr^2) and the calculation above to get the following inequalities:

$$\sqrt{A}\sqrt{\pi r^2} \leq \frac{A + \pi r^2}{2} \leq \frac{lr}{2}$$

$$\begin{aligned}
&\Rightarrow \sqrt{A}\sqrt{\pi r^2} \leq \frac{lr}{2} \\
&\Rightarrow 4\pi A r^2 \leq l^2 r^2 \\
&\Rightarrow 0 \leq l^2 - 4\pi A && (3)
\end{aligned}$$

To get equality in the Isoperimetric Inequality, we have to have equalities instead of inequalities in all of the above calculations. To get equality in (3) we know by Revision (i) that since a = b it has to be $A = \pi r^2$ and thus $l = 2\pi r$ without restriction on the orientation of r. Equality in (1) and (2) implies that

$$(xy' - zx')^2 = (x^2 + z^2) * ((x')^2 + (y')^2)$$

which, by Lemma (ii), tells us that $-xx' = zy'$. Substituting this again into the left side of the above equation gives us

$$
\begin{aligned}
&\Rightarrow (xy' - zx')^2 = \underbrace{(x^2 + z^2)}_{=r^2 \ by \ R(ii)} * \underbrace{((x')^2 + (y')^2)}_{=1 \ by \ R(ii)} \\
&\Rightarrow x^2(y')^2 - 2zx'xy' + z^2(x')^2 = r^2 \\
&\Rightarrow x^2(y')^2 + 2(x')^2x^2 + z^2(x')^2 = r^2 \\
&\Rightarrow x^2 \underbrace{((x')^2 + (y')^2)}_{=1 \ by \ R(ii)} + (x')^2 * (x^2 + \underbrace{z^2}_{=r^2 - x^2 \ by \ R(ii)}) = r^2 \\
&\Rightarrow x^2 + (x')^2 r^2 = r^2 \\
&\Rightarrow x^2 = r^2(\underbrace{1 - (x')^2}_{=(y')^2 \ by \ R(ii)}) \\
&\Rightarrow x = \pm ry'
\end{aligned}
$$

To finish this proof, we show that $y = \pm rx'$. To do this we go back to the beginning of our proof. We now find another Interval $\tilde{I} = [-\tilde{r}, \tilde{r}]$ (by setting the parallel lines perpendicular to the ones bounding the Intervall I). Again, we can put a circle $\tilde{c}(t)$ in $\tilde{I}$ but this time $\tilde{c}(t)$ is parameterised as $\tilde{c}$(t) = (w(t),y(t)). We have to redo the complete proof now for the same A as above but $\tilde{B} = \pi\tilde{r}^2$. Requiring equalities everywhere as we did before, this leads us to A $= \pi\tilde{r}^2$ but since we used the same A as above, we get $\tilde{r} = r$ as well as $-xw' = yy'$ which finally results (after doing the calculations again) in $y = \pm rx'$.

Eventually, adding x^2 and y^2 we get:

$$x^2 + y^2 = r^2 \underbrace{((x')^2 + (y')^2)}_{=1 \ by \ R(ii)} = r^2$$

□

By Revision (ii) then c(t) is a circle.

3 References

[Blaschke] Blaschke, Wilhelm. *Die Grundlehren der Mathematischen Wissenschaften.* Berlin: Springer Verlag, 1950. Print.

[Carmo] Carmo, do Manfredo P. *Differentialgeometrie von Kurven und Flächen.* Trans. Michael Grüter. Wiesbaden: Vieweg Verlag, 1993. Print.

[Froehlich] Froehlich, Steffen. *Konvexe Kurven und das Isoperimetrische Problem.* FU Berlin, 2009. Web. 7 Feb 2013. <`http://page.mi.fu-berlin.de/sfroehli/ss2007/vorlesung03.pdf`>

[Hopf] Hopf, Heinz. *Selected Chapters of Geometry.* Trans. H. Samelson ETH Zuerich, 1940. Web. 7 Feb 2013. <`http://www.math.cornell.edu/~hatcher/Other/hopf-samelson.pdf`>

[Siegel] Siegel, Alan. *A Historical Review of the Isoperimetric Theorem in 2-D, and its Place in Elementary Plane Geometry.* Courant Institute of Mathematical Sciences New York University. Web. 7 Feb 2013. <`http://www.cs.nyu.edu/faculty/siegel/SCIAM.pdf`>

[Wiegert] Wiegert, Jennifer. ’The Sagacity of Circles: A History of the Isoperimetric Problem.’ *MathDL.* MAA, 2006. Web. 7 Feb 2013. <`http://mathdl.maa.org/mathDL/46/?pa=content&sa=viewDocument&nodeId=1186&pf=1`>