FIRST EXPLICIT CONSTRAINED WILLMORE MINIMIZERS OF
NON-RECTANGULAR CONFORMAL CLASS

LYNN HELLER AND CHEIKH BIRAHIM NDIAYE

ABSTRACT. We study immersed tori in 3−space minimizing the Willmore energy in
their respective conformal class. We first construct a real analytic 2−dimensional
family of equivariant constrained Willmore tori parametrized by their conformal
structure (a, b), for a ∼ 0 and b ∼ 1. This family is then shown to minimize the
Willmore energy for elements with conformal class in an open neighborhood of
(0, b) in Teichmüller space for b ∼ 1, b ̸= 1, where the homogenous tori are known
to be constrained Willmore minimizers. As a byproduct of our argument, we show
for each prescribed rectangular conformal class b ∼ 1, b ̸= 1 that the minimal
Willmore energy ω(a, b) is real analytic and concave in a ∈ (0, a^b) for some a^b > 0.

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1. Introduction and statement of the results

In the 1960s Willmore [Wil] proposed to study the critical values and critical points of the bending energy

\[ W(f) = \int_M H^2 dA, \]

the average value of the squared mean curvature \( H \) of an immersion \( f : M \to \mathbb{R}^3 \) of a closed surface \( M \). In the definition of \( W \) we denote by \( dA \) the induced volume form and \( H := \frac{1}{2} \text{tr}(\text{II}) \) with \( \text{II} \) the second fundamental form of the immersion of \( f \). He showed that the absolute minimum of this functional is attained at round spheres with Willmore energy \( W = 4\pi \). Willmore also conjectured that the minimum over surfaces of genus 1 is attained at (a suitable stereographic projection of) the Clifford torus in the 3-sphere with \( W = 2\pi^2 \). It soon was noticed that the bending energy \( W \) (by then also known as the Willmore energy) is invariant under Möbius transformations of the target space – in fact, it is invariant under conformal changes of the metric in the target space, see [Bla, Ch]. Thus, it makes no difference for the study of the Willmore functional which constant curvature target space is chosen.

Bryant [Bry] characterized all Willmore spheres as Möbius transformations of genus 0 minimal surfaces in \( \mathbb{R}^3 \) with planar ends. The value of the bending energy on Willmore spheres is thus quantized to be \( W = 4\pi k \), with \( k \geq 1 \) the number of ends. With the exception of \( k = 2, 3, 5, 7 \) all values occur. The first examples of Willmore surfaces not Möbius equivalent to minimal surfaces were found by Pinkall [Pin] via lifting elastic curves \( \gamma \) with geodesic curvature \( \kappa \) on the 2-sphere, which are the critical points for the elastic energy

\[ E(\gamma) = \int_\gamma (\kappa^2 + 1) ds, \]

under the Hopf fibration to Willmore tori in the 3-sphere, where \( s \) is the arclength parameter of the curve. Later Ferus and Pedit [FerPed] classified all Willmore tori equivariant under a Möbius \( S^1 \)-action on the 3-sphere (for the definition of \( S^1 \)-action see Definition 2).

The Euler-Lagrange equation for the Willmore functional

\[ \Delta H + 2H(H^2 - K) = 0, \]

where \( K \) denotes the Gauss curvature of the surface \( f : M \to \mathbb{R}^3 \) and \( \Delta \) the Laplace-Beltrami operator of the surface, is a 4th order elliptic PDE for \( f \) since the mean curvature vector \( \vec{H} \) is the normal part of \( \Delta f \). Its analytic properties are prototypical for non-linear bi-Laplace equations. Existence of a minimizer for the Willmore functional \( W \) on the space of smooth immersions from 2-tori was shown by Simon [Sim]. Bauer and Kuwert [BauKuw] generalized this result to higher genus surfaces. After a number of partial results, e.g. [LiYau], [MonRos], [Ros], [Top] Marques and Neves [MarNev], using Almgren-Pitts min-max theory, gave a proof of the Willmore conjecture in 3–space in 2012.

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A more refined, and also richer, picture emerges when restricting the Willmore functional to the subspace of smooth immersions \( f : M \to \mathbb{R}^3 \) inducing a given conformal structure on \( M \). Thus, \( M \) now is a Riemann surface and we study the Willmore energy \( \mathcal{W} \) on the space of smooth conformal immersions \( f : M \to \mathbb{R}^3 \) whose critical points are called \((\text{conformally}) \ \text{constrained Willmore surfaces}\). The conformal constraint augments the Euler-Lagrange equation by \( \omega \in H^0(K_M^2) \) paired with the trace-free second fundamental form \( \hat{II} \) of the immersion

\[
\Delta H + 2H(H^2 - K) = <\omega, \hat{II}>,
\]

with \( H^0(K_M^2) \) denoting the space of holomorphic quadratic differentials. In the Geometric Analytic literature, the space \( H^0(K_M^2) \) is also referred to as \( S_{TT}^2(g_{\text{euc}}) \) the space of symmetric, covariant, transverse and traceless 2-tensors with respect to the euclidean metric \( g_{\text{euc}} \). Since there are no holomorphic (quadratic) differentials on a genus zero Riemann surface constrained Willmore spheres are the same as Willmore spheres. For higher genus surfaces this is no longer the case: constant mean curvature surfaces (and their Möbius transforms) are constrained Willmore, as one can see by choosing \( \omega := \hat{II} \) as the holomorphic Hopf differential in the Euler Lagrange equation \([1.1]\), but not Willmore unless they are minimal. Bohle \([\text{Boh}]\), using techniques developed in \([\text{BoLePePi}]\) and \([\text{BoPePi}]\), showed that all constrained Willmore tori have finite genus spectral curves and are described by linear flows on the Jacobians of those spectral curves\(^1\). Thus the complexity of the parametrization heavily depends on the genus of the spectral curve, called spectral genus, which gives the dimension of the Jacobian, i.e., the codimension of the linear flow. The simplest examples of constrained Willmore tori, which have spectral genus zero, are the tori of revolution in \( \mathbb{R}^3 \) with circular profiles - the homogenous tori. Those are stereographic images of products of circles of varying radii ratios in the 3-sphere and thus have constant mean curvature as surfaces in the 3-sphere. Starting at the Clifford torus, which has mean curvature \( H = 0 \) and a square conformal structure, these homogenous tori in the 3-sphere parametrized by their mean curvature \( H \) "converge" to a circle as \( H \to \infty \) and thereby sweeping out all rectangular conformal structures. Less trivial examples of constrained Willmore tori come from the Delaunay tori of various lobe counts (the \( n \)-lobed Delaunay tori) in the 3-sphere whose spectral curves have genus 1, see Figure \([\text{2}]\) and \([\text{KiScSc1}]\) for their definition.

Existence and regularity of a \( W^{2,2} \cap W^{1,\infty} \) minimizer \( f : M \to \mathbb{R}^3 \) in a given conformal class for a surface of any genus\(^2\) was shown by \([\text{KuwSch2}]\), \([\text{KuwLi}]\), \([\text{Riv2}]\) and \([\text{Sch}]\) under the assumption that the infimum Willmore energy in the conformal class is below \( 8\pi \). The latter assumption ensures that minimizers are embedded by the Li and Yau inequality \([\text{LiYau}]\). First explicit constrained Willmore minimizers were identified by Ndiaye and Schätzle \([\text{NdiSch1}]\, [\text{NdiSch2}]\) showing that for rectangular conformal classes in a neighborhood of the square class, the homogenous tori (whose spectral curves have genus 0) are the minimizers for the associated constrained Willmore problem in any codimension (the size of the neighborhood may depend on the

---

\(^1\)For the notion of spectral curves and the induced linear flows on the Jacobians see \([\text{BoLePePi}]\).

\(^2\)For the notion of \( W^{2,2} \cap W^{1,\infty} \) immersions see \([\text{KuwSch2}]\), \([\text{Riv}]\) or \([\text{KuwLi}]\).

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Figure 1. The vertical stalk represents the family of homogenous tori, starting with the Clifford torus at the bottom. Along this stalk are bifurcation points from which embedded Delaunay tori continue the homogenous family. The rectangles indicate the conformal types. The family of surfaces starting at the Clifford torus, bifurcating at the first branch point has Willmore energy below $8\pi$ and is conjectured to be the minimizer in their respective conformal class. Image by Nicholas Schmitt.

Those tori of revolution with circular profiles eventually have to fail to be minimizing in their conformal class for $H \gg 1$, since their Willmore energy can be made arbitrarily large and any rectangular torus can be conformally embedded into $\mathbb{R}^3$ (or $S^3$) with Willmore energy below $8\pi$, see [NdiSch2, KiScSc1]. Calculating the 2nd variation of the Willmore energy $W$ along homogenous tori Kuwert and Lorenz [KuwLor] showed that negative eigenvalues appear at those conformal classes whose rectangles have side length ratio $\sqrt{k^2 - 1}$ for $k \geq 2$. These are exactly the rectangular conformal classes from which the $k$-lobed Delaunay tori (of spectral genus 1) bifurcate. Any of the families starting from the Clifford torus, following homogenous tori to the $k$-th bifurcation point, and continuing with the $k$-lobed Delaunay tori sweeping out all rectangular classes (see Figure 1) "converge" to a neckless of spheres as conformal structure degenerates. The Willmore energy $W$ of the resulting family is strictly monotone and satisfies $2\pi^2 \leq W < 4\pi k$, see [KiScSc1, KiScSc2]. Thus for $k = 2$ the 2-lobed Delaunay tori imply that the infimum of the Willmore energy in rectangular conformal types is below $8\pi$ and hence there exist embedded constrained Willmore minimizers for every rectangular class by [KuwSch2]. It is conjectured that the minimizers for $W$ in rectangular conformal classes are given by

\[ W = \frac{2\pi^2}{k}. \]

\[ \text{For simplicity we call this family in the following the } k\text{-lobed Delaunay tori.} \]

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Figure 2. Two \((1, 2)\)-equivariant constrained Willmore torus (with intrinsic period 1). The tori lie in a 2-parameter family of surfaces deforming the Clifford torus. The family minimizes the Willmore functional in the respective conformal class for surfaces close enough to the Clifford torus. Images by Nick Schmitt.

In this paper we turn our attention to finding explicit constrained Willmore minimizer of non-rectangular conformal class. Unlike the rectangular case considered in [NdiSch1, NdiSch2] we have to find appropriate candidates of explicit type first.

Our main theorem is the following:

**Theorem 1** (Main Theorem).

For every \(b \sim 1\) and \(b \neq 1\) there exists \(a^b > 0\) and small such that for every \(a \in [0, a^b)\) the \((1, 2)\)-equivariant tori of intrinsic period 1 (see Figure 2) with conformal class \((a, b)\) are constrained Willmore minimizers. Moreover, for \(b \sim 1\) and \(b \neq 1\) fixed, the minimal Willmore energy map

\[
\omega(\cdot, b) : [0, a^b] \rightarrow \mathbb{R}_+ , \quad a \mapsto \omega(a, b)
\]

is real analytic on \((0, a^b)\) and continuous and concave on \([0, a^b]\).

As a byproduct of the arguments of the Theorem we have the following Corollary:

**Corollary 1.** For every \(b \sim 1\) and \(b \neq 1\) there exists \(a^b > 0\) small such that the minimization problem

\[
\text{Min}_b := \inf \{ W_\alpha(f) | f : T_{r_b} \rightarrow S^3 \text{ smooth immersion with} \\
0 \leq |\Pi^1(f)| \leq a^b \text{ and } \Pi^2(f) = b \}
\]

is attained at the homogenous tori \(f_{r_b}\) for all \(\alpha < \lim_{a \rightarrow} \alpha(a,b)\), where \(\alpha(a,b)\) is the Lagrange multiplier of the constrained Willmore minimizer of the Main Theorem.

The above Theorem and Corollary extends the result in [NdiSch1] which says that the homogenous tori minimizes the Willmore energy in their respective rectangular

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conformal class in a neighborhood of the square one. The main difference between [NdiSch1] to our case here is that homogenous tori as isothermic surfaces are degenerate in the Teichmüller space. Thus by relaxing the minimization problem, Ndiaye and Schätzle were able to restrict to a space where isothermic surfaces solve the relaxed Euler-Lagrange equation and become non-degenerate for the associated constraint. Hence they could use the existence and regularity result of [KuwSch2] and the compactness result of [NdiSch1] to obtain a family of minimizers of the constrained Willmore problem smoothly close to the Clifford torus. Further, combined with the implicit function theorem they show that this family is the unique 1–dimensional family of constrained Willmore tori deforming the Clifford torus which is then necessarily the family of homogenous tori.

This is in stark contrast to the case of non-rectangular conformal classes. In fact, rectangular minimizers close to the square class are isothermic and there is only one such family deforming the Clifford torus. While the non-rectangular candidates are necessarily non-isothermic (see Theorem 2) and it is well known within the integrable systems community that there exist various such families of constrained Willmore tori deforming the Clifford torus, as also discussed in Sections 4.2, 4.3. These families lie in the so-called associated family of constrained Willmore Hopf tori, see Definition 4. The constrained Willmore Hopf tori are given by the preimage of (constrained) elastic curves on $S^2$ under the Hopf fibration, and are isothermic if and only if they are homogenous [Hel2]. Moreover, in contrast to tori of revolution, every conformal structure on a torus can be realized by a constrained Willmore Hopf torus [Hel2]. It has been conjectured by Franz Pedit, Ulrich Pinkall and Martin U. Schmidt that constrained Willmore minimizers should be of Hopf type. Though we disprove this conjecture in this paper we show that the actual minimizer lie in the associated family of constrained Willmore Hopf tori, where the Hopf differential of the minimizer is just the one of the associated Hopf surface rotated by a phase. It turns out that the various families deforming the Clifford torus mentioned before can be analytically distinguished by looking at their limit Lagrange multiplier as they converge to the homogenous tori at rectangular conformal classes. This suggest that to determine the non-rectangular constrained Willmore minimizers we need more control on the abstract minimizers than in the Ndiaye-Schätzle case [NdiSch1], namely the identification of the limit Lagrange multipliers rather than just an upper bound.

The paper is organized as follows. In the second section we state the main observations leading to a strategy to prove the Main Theorem. It turns out that the degeneracy of an isothermic surface with respect to a penalized Willmore functional (i.e., the second variation has non-trivial kernel) is crucial for the existence of families deforming it. We also observe that the Lagrange multiplier is given by the derivative of Willmore energy with respect to the conformal class. These two properties provide sufficient information to characterize the possible limit Lagrange multipliers $(\alpha^b, \beta^b)$ for a family of constrained Willmore tori converging to a homogenous torus $T_{r_3}$, which we do in the third section. In the fourth to sixth section we prove the Main Theorem 1 consisting of three steps.
(1) **Candidates:**

We construct a real analytic 2-parameter family of candidate surfaces \( f_{(a,b)} \) parametrized by their conformal class \((a, b)\), with \( a \sim 0^+ \) and \( b \sim 1 \), satisfying the following properties

- \( f_{(0,b)} =: f_b \) are homogenous,
- \( f_{(a,b)} \) is non degenerate for \( a \neq 0 \), and \( f_{(a,b)} \to f_b \) smoothly as \( a \to 0 \),
- for every \( b \sim 1 \) fixed and \( a \neq 0 \), the corresponding Lagrange multipliers \( \alpha_{(a,b)} \) and \( \beta_{(a,b)} \) satisfy
  \[
  \alpha_{(a,b)} \nearrow \alpha^b \quad \text{and} \quad \beta_{(a,b)} \to \beta^b, \quad \text{as} \quad a \to 0.
  \]

(2) **Classification:**

We classify all solutions of the constrained Euler-Lagrange equation \( f \) such that \( f \) is close to a stable\(^4\) homogenous torus \( f^b (b \neq 1) \) in \( W^{4,2} \) and its Lagrange multiplier \((\alpha, \beta)\) is close to \((\alpha^b, \beta^b)\) via implicit function theorem and bifurcation theory. We obtain two branches of solutions \( f_{(a,b)}^\pm \) for fixed \( b \sim 1 \) and \( b \neq 1 \) with Lagrange multipliers \( \alpha_{+, (a,b)}^b \geq \alpha^b \) and \( \alpha_{-, (a,b)}^b \leq \alpha^b \) of which \( f_{+, (a,b)}^b \) has smaller Willmore energy than \( f_{+, (a,b)}^b \), since for \( f_{-, (a,b)}^b \) we show the \( \alpha - \) Lagrange multiplier converge from below.

(3) **Global to Local:**

We show that constrained Willmore minimizers in a conformal class \((a, b)\) for \( a \sim 1 \) and \( b \sim 1 \), \( b \neq 1 \) exist and that their Lagrange multipliers \( \alpha_{(a,b)} \) converge to \( \alpha^b \) as \( a \to 0 \), (and the surfaces converge to the homogenous tori). Thus these abstract minimizer for fixed \( b \sim 1 \) and \( b \neq 1 \) coincides with \( f_{(a,b)}^- \), the candidates surfaces constructed before.

We first use methods from integrable systems to construct the explicit examples with prescribed properties and then further develop Geometric Analytic techniques of \cite{NdSch} which shows that these examples are in fact constrained Willmore minimizers. Since the families of solutions constructed by integrable systems methods are real analytic we automatically obtain the regularity of the minimum Willmore energy as a function of the conformal type.

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2. **Strategy and main observations**

In this section we want to give the main insights and the strategy for the proof of the Main Theorem (Theorem [1]). We follow the notations used in \cite{KuwLor}.

\(^4\)By stability we mean that \( \delta^2(\mathcal{W} - \beta^b \Pi^2) > 0 \).
We identify the Teichmüller space with the upper half plane $\mathbb{H}$. Let 
\[ \Pi(f) = (\Pi^1(f), \Pi^2(f)) \]
be the projection map of an immersion $f : T^2 \rightarrow S^3$ to the Teichmüller space such that the Clifford torus $f^1 : T^2 \rightarrow S^3 \subset \mathbb{C}^2$ parametrized by 
\[ f^1(x, y) = \frac{1}{\sqrt{2}} \left( e^{ix\sqrt{2}}, e^{iy\sqrt{2}} \right) \]
is mapped to $\Pi(f_{Cliff}) = (0, 1)$. Then we can write the Euler-Lagrange equation for a constrained Willmore torus as 
\[(2.1)\]
\[ \delta W = <\omega, \hat{\Pi} > = \alpha \delta \Pi^1 + \beta \delta \Pi^2, \]
with Lagrange multipliers $\alpha$ and $\beta$. The surface is non-isothermic if and only if the Lagrange multipliers are uniquely determined (after choosing a base in $\mathbb{H}^2$). At the Clifford torus, and more generally, at homogenous tori we have $\delta \Pi^1 = 0$ and thus the $\alpha$–Lagrange multiplier can be arbitrarily chosen. As already discussed before, it is well known that there exist various families of constrained Willmore tori deforming a homogenous torus. These families can be distinguished by the limit of their $\alpha$–Lagrange multiplier as they converge smoothly to the homogenous torus. How these limit Lagrange multipliers relate to the Willmore energy of the families and the obstructions for such families to exist is summarized in the following Lemma. Though the proof of the Lemma is trivial, these observations gives the main intuition for the behavior of the minimum Willmore energy as a function of the conformal class.

**Lemma 1** (Main observation).
Let $a_0, b_0 \in \mathbb{R}$ be positive and small and $f^{(a,b)}$ with conformal type 
\[ (a, b) =: (\tilde{a}^2, b) \in [0, a_0) \times (1 - b_0, 1 + b_0) \]
a family of smooth immersions such that the map 
\[ (\tilde{a}, b) \mapsto f^{(a,b)} \in C^2([0, a_0) \times (1 - b_0, 1 + b_0), W^{4,2}), \]
with $\delta \Pi^1_{f^{(0,b)}} = 0$, but $\delta \Pi^1_{f^{(a,b)}} \neq 0$ for $a \neq 0$. Further, let $\alpha(a,b)$ and $\beta(a,b)$ be the corresponding Lagrange multipliers satisfying 
\[ (\tilde{a}, b) \mapsto \alpha(a,b), \beta(a,b) \in C^2([0, a_0) \times (1 - b_0, 1 + b_0), W^{4,2}), \]
and $\tilde{\omega}(a,b) := W(f^{(a,b)})$. Then we obtain 
\[ \frac{\partial \tilde{\omega}(a,b)}{\partial a} = \alpha(a,b) \text{ for } a \neq 0 \quad \text{and} \quad \lim_{a \rightarrow a_0} \frac{\partial \tilde{\omega}(a,b)}{\partial a} = \lim_{a \rightarrow a_0} \alpha(a,b) =: \tilde{\alpha}^b \quad \forall b, \]
\[ \frac{\partial \tilde{\omega}(a,b)}{\partial b} = \beta(a,b) \text{ for } a \neq 0 \quad \text{and} \quad \lim_{a \rightarrow a_0} \frac{\partial \tilde{\omega}(a,b)}{\partial b} = \lim_{a \rightarrow a_0} \beta(a,b) =: \tilde{\beta}^b \quad \forall b, \]
(3) \( \varphi^b := \partial_a f^{(a,b)}|_{a=0} \) satisfies
\[
\delta^2 \left( \mathcal{W} - \hat{a}^b \Pi^1 - \hat{b}^b \Pi^2 \right) (f^{(0,b)}, \varphi^b) = 0 \quad \forall b.
\]

Proof. The proof only uses the definition of the family, the constrained Euler-Lagrange equation and its derivatives. By assumption we have that \( \partial^k_a \partial^l_b f^{(a,b)} \) exist and is continuous for all \( k, l \in \mathbb{N} \). Since \( \partial_a = 2\sqrt{a} \partial_a \) for \( a \neq 0 \) we have that \( \partial_a f^{(a,b)} \) exist for \( a \neq 0 \) but \( \lim_{a \to 0} \partial_a f^{(a,b)} \) cannot exist due to the degeneracy of \( f^{(0,b)} \).

(1) Let \( \varphi := \partial f^{(a,b)} \partial_a \) for \( a \neq 0 \). Then \( \partial \varphi = \delta \mathcal{W}_{f^{(a,b)}}(\varphi) \) for \( a \neq 0 \) and hence by the constrained Euler-Lagrange equation we have:
\[
\frac{\partial \varphi}{\partial a}(a, b) = \alpha(a, b)\delta \Pi^1_{f^{(a,b)}}(\varphi) + \beta(a, b)\delta \Pi^2_{f^{(a,b)}}(\varphi), \quad \text{for } a \neq 0.
\]
Since \( \Pi(f^{(a,b)}) = (a, b) \), we obtain for \( a \neq 0 \) that \( \delta \Pi^1_{f^{(a,b)}}(\varphi) = 1 \) and \( \delta \Pi^2_{f^{(a,b)}}(\varphi) = 0 \) and therefore
\[
\frac{\partial \varphi}{\partial a}(a, b) = \alpha(a, b), \quad a \neq 0.
\]
Passing to the limit gives the first assertion.

(2) This follows completely analogously to (1).

(3) In this case we test the Euler-Lagrange equation by \( \varphi \) and obtain
\[
\delta \mathcal{W}_{f^{(a,b)}}(\varphi) = \alpha(a, b)\delta \Pi^1_{f^{(a,b)}}(\varphi) + \beta(a, b)\delta \Pi^2_{f^{(a,b)}}(\varphi).
\]
Now differentiating this equation with respect to \( a \) yields
\[
\delta^2 \mathcal{W}_{f^{(a,b)}}(\varphi, \varphi) = \alpha(a, b)\delta^2 \Pi^1_{f^{(a,b)}}(\varphi, \varphi) + \beta(a, b)\delta^2 \Pi^2_{f^{(a,b)}}(\varphi, \varphi)
+ \frac{\partial \alpha(a, b)}{\partial a} \delta \Pi^1_{f^{(a,b)}}(\varphi) + \frac{\partial \beta(a, b)}{\partial a} \delta \Pi^2_{f^{(a,b)}}(\varphi) \quad \text{for } a \neq 0.
\]
In order to pass to the limit, it is necessary to replace \( \varphi \) by \( \sqrt{a} \varphi \). This gives
\[
\delta^2 \mathcal{W}_{f^{(a,b)}}(\sqrt{a} \varphi, \sqrt{a} \varphi) = \alpha(a, b)\delta^2 \Pi^1_{f^{(a,b)}}(\sqrt{a} \varphi, \sqrt{a} \varphi)
+ \beta(a, b)\delta^2 \Pi^2_{f^{(a,b)}}(\sqrt{a} \varphi, \sqrt{a} \varphi)
+ \sqrt{a} \frac{\partial \alpha(a, b)}{\partial a} \delta \Pi^1_{f^{(a,b)}}(\sqrt{a} \varphi) + \sqrt{a} \frac{\partial \beta(a, b)}{\partial a} \delta \Pi^2_{f^{(a,b)}}(\sqrt{a} \varphi).
\]
By assumption \( \lim_{a \to 0} \sqrt{a} \frac{\partial \alpha(a, b)}{\partial a} = \lim_{a \to 0} \frac{\partial \alpha(a, b)}{\partial a} \) and \( \lim_{a \to 0} \sqrt{a} \frac{\partial \beta(a, b)}{\partial a} = \lim_{a \to 0} \frac{\partial \beta(a, b)}{\partial a} \).
Since \( \lim_{a \to 0} \delta \Pi^1_{f^{(a,b)}}(\sqrt{a} \varphi) = 0 \) and \( \delta \Pi^2_{f^{(a,b)}}(\sqrt{a} \varphi) = 0 \) as before, we obtain for \( a \to 0 \)
\[
\delta^2 \left( \mathcal{W}_{f^{(0,b)}} - a^b \Pi^1 - b^b \Pi^2 \right) |_{f^{(0,b)}}(\varphi^b, \varphi^b) = 0.
\]
\[\square\]

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The first assertion states that for a given family of constrained Willmore tori \( f^{(a,b)} \), as in the Lemma, their Lagrange multipliers can be computed from the derivative of the Willmore energy \( \tilde{\omega}(a, b) \). This suggests that a family of candidates for constrained Willmore minimizers in their respective conformal class \( f^{(a,b)} \) deforming the homogenous torus \( f^b \) with \( b \sim 1 \), which is known to be constrained minimizers for rectangular conformal class \((0, b)\) with the smallest possible limit Lagrange multiplier \( \alpha^b \). A necessary (and as we will later see a sufficient) condition for such a family to exist is given by the second statement of Lemma 1, namely the degeneracy of the second variation of the penalized Willmore functional \( \mathcal{W} - \alpha^b \Pi^1 - \beta^b \Pi^2 \). To abbreviate the notations we use the following definitions.

**Definition 1.** For \( \alpha, \beta \in \mathbb{R} \) we define

\[
\begin{align*}
\mathcal{W}_{\alpha,\beta}(f) &:= \mathcal{W}(f) - \alpha \Pi^1(f) - \beta \Pi^2(f) \\
\mathcal{W}_\alpha(f) &:= \mathcal{W}(f) - \alpha \Pi^1(f).
\end{align*}
\]

The discussion above suggest that the first step towards the proof of the main Theorem is to determine

\[
\alpha^b := \max \{ \alpha \mid \delta^2 \mathcal{W}_{\alpha,\beta} \geq 0 \}.
\]

**Remark 1.** The limit Lagrange multiplier \( \beta^b \) is uniquely determined as the \( \beta \)-Lagrange multiplier of the homogenous torus \( f^b \).

It is well known that the Clifford torus, and thus all homogenous tori smoothly close to the Clifford torus, is strictly stable (up to invariance) and therefore \( \alpha^b \) is positive. We check in the next section that it is finite. Further, since we show in Proposition 1 that the kernel is 1-dimensional for \( b \sim 1 \) and \( b \neq 1 \), up to invariance, the second statement of Lemma 1 implies that the kernel of the second variation of \( \mathcal{W}_{\alpha^b,\beta^b} \) at \( f^b \) is generated by the infinitesimal normal variation of the family \( f^{(a,b)} \) deforming the homogenous torus \( f^b \), up to reparametrization. Moreover, \( \varphi^b \in \delta^2 \mathcal{W}_{\alpha^b,\beta^b}(f^b) \) for \( b \neq 1 \) period one and is independent of the \( x \)-direction (as shown in Section 3) for a reparametrized homogenous torus as a \((1,2)\)-equivariant surface

\[
f^b : T_{r^b} \to S^3, \quad (x, y) \mapsto \left( r e^{i(x+\frac{a^b}{r^b}y)}, s e^{i(2x-\frac{r^b}{\sqrt{2}}y)} \right).
\]

Which means that the corresponding family \( f^{(a,b)} \) (as in Lemma 1) are infinitesimally \((1,2)\)-equivariant. Furthermore, in our case knowing the limit Lagrange multiplier \( \alpha^b \) is tantamount to knowing the infinitesimal normal variation.

Since for \( \alpha \in [0, \alpha^b) \) the second variation \( \delta^2 \mathcal{W}_{\alpha,\beta^b}(f_{r^b}) > 0 \) (up to invariance), there are clearly no 2-dimensional families deforming smoothly the homogenous torus with \( \lim_{a \to 0} \alpha(a, b) = \alpha \). Indeed the following Lemma shows this fact even in \( W^{4,2} \) topology with the same arguments as in [NdiSch1].

---

Equivariant surfaces are those with a 1-parameter family of isometric symmetries, we discuss these surfaces in Section 4.
Lemma 2. For \( a^b \) and \( b \sim 1 \) defined as before let \( \alpha < \alpha^b \). Then the homogenous tori \( f^b \) is the unique solution (up to invariance) of the equation

\[
\delta W_f = \alpha_f \delta \Pi_f^1 + \beta^b \delta \Pi_f^2
\]

with \( \alpha_f \sim \alpha \) and \( f \sim f_{r_b} \) in \( W^{1,2} \), \( \Pi^1(f) \geq 0 \) and \( \Pi^2(f) = b \).

At \( \alpha = \alpha^b \) (and \( b \sim 1 \)) the situation is very different. Using integrable systems theory we can construct a real analytic family of \((1,2)-\)equivariant constrained Willmore tori \( f_{\alpha(b)} \) parametrized by its conformal type \((a, b) \sim (0, 1)\) deforming smoothly the Clifford torus with \( f_{(0, b)} = f^b := f_{r_b} \), the homogenous torus with conformal type \((0, b)\), and \( \alpha_{(a,b)} \nrightarrow \alpha^b \) as \( a \to 0 \). In fact, we prove even more in Section 4.

Theorem 2. For \( a \sim 0^+ \) and \( b \sim 1 \) there exist a family of \((1,2)-\)equivariant constrained Willmore immersions

\[
f_{(a,b)} : T_{(a,b)}^2 \to S^3
\]

such that the map

\[
(\sqrt{a}, b) \mapsto f_{(a,b)} \in C^\infty ((0, a_0) \times [1, 1 + b_0), C^\infty_{Imm}) \cap C^2 ([0, a_0] \times [1, 1 + b_0), C^\infty_{Imm})
\]

and

\[
(a, b) \mapsto W(f_{(a,b)}) := \omega(a, b) \in C^2 ([0, a_0] \times [1, 1 + b_0))
\]

with the following properties

(1) \( \forall b, f_{(a,b)} \) converge smoothly to the homogenous torus \( f^b \) as \( a \to 0 \) given by

\[
f^b : T_0^2 \to S^3, (x, y) \mapsto \left( r e^{i(2x + \frac{2}{7}y)}, s e^{i(2x - 2\frac{2}{7}y)} \right)
\]

with \( r^2 + s^2 = 1 \) and \( b = 2^\frac{2}{7} \).

(2) \( f_{(a,b)} \) are non degenerate for \( a \neq 0 \) and satisfy

\[
\delta W(f_{(a,b)}) = \alpha_{(a,b)} \delta \Pi^1 + \beta_{(a,b)} \delta \Pi^2 \quad \text{for} \quad a \neq 0
\]

with Lagrange multipliers \((\alpha(a,b), \beta(a,b))\) such that \( \alpha_{(a,b)} \nrightarrow \alpha^b \) and \( \beta_{(a,b)} \to \beta^b \) as \( a \to 0 \).

Remark 2. By Lemma \( \square \), we obtain that

\[
\partial_{\sqrt{a} f_{(a,b)}}|_{a=0} = \varphi^b \in Ker(\delta^2 W_{\alpha^b, \beta^b}).
\]

Moreover Lemma \( \square \) also implies that for \( b \sim 1 \) fixed, the map \( a \mapsto W(f_{(a,b)}) \) is monotonically increasing and concave as \( a \to 0^+ \). In other words, there exist \( a^b > 0 \) and small such that for \( a \in [0, a^b) \)

\[
W_{\alpha^b}(f_{(a,b)}) < W_{\alpha^b}(f^b)
\]

and thus the homogenous tori \( f^b \) cannot be the minimizer of \( W_{\alpha^b} \) among immersions \( f \) with \( \Pi^2(f) = b \).
At \( f^b \) the second variation of \( W_{a^b, b^b} \) is degenerate. Thus an application of the implicit function theorem as in \[\text{NdiSch1}, \text{NdiSch2}\] to classify all solutions close to \( f^b \) in \( W^{4,2} \) is not possible. Since the kernel of \( \delta^2 W_{a^b, b^b}(f^b) \), for \( b \neq 1 \), is only 1–dimensional up to invariance, see Proposition 1, then the theory of bifurcation from simple eigenvalue can be used for the classification instead, if \( \delta^3 W_{a^b, b^b}(\varphi^b, \varphi^b, \varphi^b) \neq 0 \) which we show in Lemma 6. This imply the following classification result:

**Theorem 3.** For \( b \sim 1, b \neq 1 \) fixed and up to taking \( a^b \) in Remark 3 smaller, there exist two unique families of non degenerate solutions \( f_{\pm}^{(a, b)} \) for \( a \neq 0 \) to the constrained Euler-Lagrange equation (2.1) (up to invariance) parametrized by its conformal type \((a, b)\) with \( a \in [0, a^b) \), \( f_{\pm}^{(a, b)} \sim f^b \) in \( W^{4,2} \) as \( a \sim 0^+ \) and \( f_{\pm}^{(0, b)} = f^b \) satisfying

\[ \alpha^+(a, b) \searrow \alpha^b, \quad \alpha^-(a, b) \nearrow \alpha^b \]

and \( \beta(a, b) \rightarrow \beta^b \) as \( a \rightarrow 0 \), where \( \alpha^\pm(a, b) \) and \( \beta^\pm(a, b) \) are the Lagrange multipliers of the immersions \( f_{\pm}^{(a, b)} \). In particular, the only solution \( f \) of constrained equation with conformal type \( \Pi(f) = (0, b) \), \( \alpha = \alpha^b \) and \( \beta = \beta^b \) is the homogenous torus \( f_{\mathrm{rh}} \).

Since both branches start at a homogenous torus, which minimizes the Willmore energy in its conformal class, we obtain by Lemma 4 that the family of surfaces given by \( f_{-}^{(a, b)} \) has smaller energy, i.e.,

\[ W(f_{-}^{(a, b)}) < W(f_{+}^{(a, b)}). \]  

Further since our candidate surfaces in Theorem 2 has Lagrange multiplier \( \alpha_{(a, b)} \nearrow \alpha^b \) and smoothly converge to \( f^b \) as \( a \rightarrow 0 \) we conclude that \( f_{(a, b)} = f_{-}^{(a, b)} \) for \( a \in [0, a^b) \).

By minimality and (2.5) it remains to show that the abstract minimizers, which clearly exist by \[\text{KuwSch2}\], of the constrained Willmore problem for the conformal class \((a, b)\) with \( b \sim 1 \) and \( a \in [0, a^b) \) will have Lagrange multipliers \( \alpha(a, b) \rightarrow \alpha^b \), \( \beta(a, b) \rightarrow \beta^b \) and \( f(a, b) \sim f^b \) in \( W^{4,2} \) as \( a \rightarrow 0 \) to prove the main Theorem (Theorem 4). Then these abstract minimizers are covered by the classification in Theorem 3 and must therefore coincide with \( f_{-}^{(a, b)} \) and the candidate surfaces. The properties of the abstract minimizers are shown by considering the minimization of a relaxed problem as in the following theorem.

**Theorem 4.** For \( b \sim 1 \) fixed and up to taking \( a^b \) smaller we have that for all \( a \in [0, a^b) \) the minimization problem

\[ \text{Min}_{(a, b)} := \inf \{ W_{a^b}(f) \mid f : T_{rh} \rightarrow S^3 \text{ smooth immersion with} \]

\[ 0 \leq \| \Pi^1(f) \| \leq a \text{ and } \Pi^2(f) = b \} \]

is attained by a smooth and non degenerate (for \( a \neq 0 \)) constrained Willmore immersion \( f^{(a, b)} : T_{rh}^2 \rightarrow S^3 \) of conformal type \((a, b)\) with \( \alpha(a, b) \rightarrow \alpha^b \), \( \beta(a, b) \rightarrow \beta^b \) and \( f^{(a, b)} \sim f^b \) in \( W^{4,2} \) as \( a \rightarrow 0 \), where \( \alpha(a, b) \) and \( \beta(a, b) \) are the corresponding Lagrange multipliers of \( f^{(a, b)} \).

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The minimizers with respect to the penalized functional $\mathcal{W}_{\alpha, b}$ automatically minimize the plain constrained Willmore problem. We briefly discuss the main ingredients in for the proof of Theorem 4: By the work of Kuwert and Schätzle [KuwSch1] and Schätzle [Sch] we obtain the existence of the minimizers $f^{(a, b)}$. Because of Equation (2.4) and the classification (Theorem 3), the minimizers are always attained at the boundary $\Pi^1(f^{(a, b)}) = a$. Thus $\min_{(a, b)}$ is monotonically decreasing in $a$ from which we derive $\limsup_{a \to a^0} \alpha^{(a, b)} \leq \alpha^b$. Further, using Lemma 2 we show $\liminf_{a \to 0} \alpha^{(a, b)} \geq \alpha^b$. The remaining convergence $\beta^{(a, b)} \to \beta^b$ and $f^{(a, b)} \sim f^b$ in $W^{4,2}$ as $a \to 0$ follow from [NdiSch1].

3. Stability properties of the penalized Willmore energy

In the computations below we mostly follow [KuwLor] and thus we refer to that paper for details. To fix the notations, we consider immersions $f : \mathbb{C}/\Gamma \to (S^3, g_{S^3})$, where $\Gamma$ is a lattice $g_{S^3}$ is the round metric. Let $\text{Imm}(\mathbb{C}/\Gamma)$ denote the space of all such immersions and let $\text{Met}(\mathbb{C}/\Gamma)$ denote the space of all metrics on the torus $T^2 = \mathbb{C}/\Gamma$. Moreover, let $G : \text{Imm}(T^2) \to \text{Met}(T^2), \ f \mapsto f^*g_{S^3}$ be the map which assigns to every immersion its induced metric. We denote by $\pi$ the projection from the space of metrics to the Teichmüller space, which we model by the upper half plane $\mathbb{H}$ and with the notations above we can define $\Pi$ to be:

$$\Pi = \pi \circ G : \text{Imm}(T^2) \to \mathbb{H}.$$ 

As in [KuwLor] we parametrize the homogenous torus with conformal class $b = \frac{z}{r}$, and $r^2 + s^2 = 1$ as

$$(3.1) \quad f^b : T^2_b := \mathbb{C}/(2\pi r\mathbb{Z} + 2\pi i s\mathbb{Z}), \quad (x, y) \mapsto \left(re^{\frac{x}{r}}, se^{\frac{y}{s}}\right).$$

We want to compute the value of $\alpha^b$ which we recall to be

$$\alpha^b = \max \ \{ \alpha | \delta^2 \mathcal{W}_{\alpha, \beta^b} f^b \geq 0 \}.$$ 

From [KuwLor] we can derive that $\alpha^b$ is characterized by the fact that $\delta^2 \mathcal{W}_{\alpha, \beta^b} f^b \geq 0$ and there exist a non-trivial normal variation $\Phi$ of $f^b$ such that

$$\delta^2 \mathcal{W}_{\alpha^b, \beta^b} f^b(\Phi, \Phi) = 0, \quad \text{and} \quad \delta^2 \mathcal{W}_{\alpha, \beta^b} f^b(\Phi, \Phi) < 0,$$

for $\alpha > \alpha^b$. We will show that $\varphi$ is unique up to isometry of the ambient space and reparametrization of the surface $f^b$ for $b \neq 1$. We will also choose the orientation of $f^b$ and the variation $\Phi$ such that $\delta^2 \Pi^1 f^b \geq 0$.

While for $b = 1$ the $\alpha^1$ and the associated normal variations can be explicitly computed, the $\alpha^b$ for $b \neq 1$ does not have a nice explicit form. Nevertheless, we will show that the unique normal variation $\varphi$ characterizing $\alpha^b$ remain the same (in a appropriate sense) as in the $b = 1$ case. As already observed before, this is equivalent to the knowledge of $\alpha^b$. In fact, the normal variation $\lim_{a \to 0} \partial_a f(a, b)$ is the information
we use to show that the Lagrange multipliers of our candidates $f_{(a,b)}$ converge to the $\alpha^b$ as $a \to 0$, see Section 4.4.3.

We first restrict to the case $b = 1$ - the Clifford torus. We have $\beta^1 = 0$ thus we investigate for which $\alpha$ the Clifford torus $f^1$ is stable for the perturbed Willmore functional

$$W_\alpha = W - \alpha \Pi^1.$$ 

The second variation of the Willmore functional is well known. Thus we first concentrate on the computation of the second variation of $\Pi^1$. Another well known fact is $\delta \Pi^1(f^1) = 0$.

$$D_2^\Pi^1(f^1)(\Phi, \Phi) = D_2^\pi^1(G(f^1))(DG(f^1)\Phi, DG(f^1)\Phi) + D_1^\Pi^1(G(f^1))(D^2G(f^1)(\Phi, \Phi))$$

The first term is computed in Lemma 4 of [KuwLor] to be

$$D_1^\Pi^1(G(f^1))(D^2G(f^1)(\Phi, \Phi)) = -\frac{1}{\pi^2} \int_{T^2_1} <\nabla^2_1 \Phi, \Phi > d\mu_{\text{geuc}},$$

for normal variations $\Phi$.

It remains to compute the second term

$$D_2^\Pi^1(f^1)(\Phi, \Phi) = D_2^\Pi^1(G(f^1))(DG(f^1)\Phi, DG(f^1)\Phi).$$

By a straightforward computation (or by Lemma 2 of [KuwLor]) we have

$$DG(f^1)\Phi = -2 \int_{T^2_1} <I, \Phi > d\mu_{\text{geuc}},$$

where $I$ is the second fundamental form of the Clifford torus, which is trace free.

Let $u$ and $v \in S_2(T^2_1)$ be symmetric 2–forms satisfying

$$tr_{\text{geuc}} u = tr_{\text{geuc}} v = 0 \quad \text{and} \quad v \perp_{\text{euc}} S_2^{TT}(g_{\text{euc}}),$$

where $S_2^{TT}(g_{\text{euc}})$ is the space of symmetric, covariant, transverse traceless 2–tensors with standard basis $q^1$ and $q^2$ and $q^i(t)$ the corresponding basis of $g(t)$. Let $g(t) = g_{\text{euc}} + tu$ and $q^i(t) = q^i(g(t))$ such that $(q^i(t) - q^i) \perp_{\text{euc}} S_2^{TT}(g_{\text{euc}})$. Then we can expand $v$ by

$$v = v_i(t)q^i(t) + v^\perp(t), \quad \text{where} \quad v^\perp(t) \perp_{\text{euc}} S_2^{TT}(g_{\text{euc}}).$$

By assumption we have $v_i(0) = 0$ and thus

$$D_2^\Pi^1(g_{\text{euc}})(u, v) = \frac{d}{dt}D\pi(g(t)) \cdot v|_{t=0} = v_1'(0)D\pi(g_{\text{euc}}) \cdot q^1,$$

where

$$v_1'(0) = \frac{1}{4\pi^2} <v, (q^1)'(0) >_{L^2(g_{\text{euc}})}$$

as computed in [KuwLor].

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Let \( \eta := (q^{-1})'(0) \) and \( \eta^o = \eta_1 q^1 + \eta_2 q^2 \) be its traceless part, then by Lemma 6 of [KuwLor] we have

\[
\begin{align*}
(\text{div}_{\text{euc}} \eta^o)_1 &= (\text{div}_{\text{euc}} u)_2 \\
(\text{div}_{\text{euc}} \eta^o)_2 &= (\text{div}_{\text{euc}} u)_1.
\end{align*}
\]

(3.2)

For \( u = u_1 q^1 + u_2 q^2 \) we obtain,

\[
(\text{div}_{\text{euc}} u)_1 = \partial_2 u_1 - \partial_1 u_2, \quad (\text{div}_{\text{euc}} u)_2 = \partial_1 u_1 + \partial_2 u_2,
\]

and therefore the Equations in (3.2) become

\[
\begin{align*}
\partial_2 \eta_1 - \partial_1 \eta_2 &= \partial_1 u_1 + \partial_2 u_2 \\
\partial_1 \eta_1 + \partial_2 \eta_2 &= \partial_2 u_1 - \partial_1 u_2.
\end{align*}
\]

(3.3)

If we specialize to \( u = u_2 q_2 \) and \( v = v_2 q_2 \), which is the relevant case, this yields

\[
(D^2 \pi(g_{\text{euc}})(u, v))_1 = \frac{1}{4\pi^2} <v_2 q^2, \eta^o>_{L^2(g_{\text{euc}})},
\]

and we only need to concentrate on \( \eta_2 \).

Differentiating the equations in (3.3) and subtracting these form each other yields (with \( u_1 = 0 \))

\[
\Delta \eta_2 = -2\partial_1 \partial_2 u_2.
\]

(3.4)

In order to compute \( \eta_2 \) we restrict to normal variations \( \Phi = \varphi \tilde{n} \) for doubly periodic functions \( \varphi \) in a Fourier space, i.e., \( \varphi \) is a doubly periodic function on \( \mathbb{C} \) with respect to the lattice \( \sqrt{2} \pi \mathbb{Z} + \sqrt{2} \pi i \mathbb{Z} \). The Fourier space \( \mathcal{F}(T^2_1) \) of doubly periodic functions is the disjoint union of the constant functions and the 4-dimensional spaces \( \mathcal{A}_{kl}(T^2_1) \), \( (k, l) \in \mathbb{N} \setminus \{(0, 0)\} \) with basis

\[
\begin{align*}
\sin(\sqrt{2} k x) \cos(\sqrt{2} l y), & \quad \cos(\sqrt{2} k x) \sin(\sqrt{2} l y), \\
\cos(\sqrt{2} k x) \cos(\sqrt{2} l y), & \quad \sin(\sqrt{2} k x) \sin(\sqrt{2} l y).
\end{align*}
\]

(3.5)

We restrict to the case where \( \varphi = \varphi_{kl} \in \mathcal{A}_{kl}, (k, l) \in \mathbb{N}^2 \setminus \{(0, 0)\} \) in the following.

Then for \( u = v = \varphi_{kl} \tilde{n} \) we obtain that \( \eta^o = \frac{2}{k^2 + l^2} \partial_1 \partial_2 \varphi_{kl} \) solves equation (3.4). The integration constant is hereby chosen such that \( <\eta^o, q_1>_{L^2(g_{\text{euc}})} = 0 \).

Thus

\[
D^2 \pi^1(G(f^1))(u, v) = \frac{1}{2\pi^2 (k^2 + l^2)} \int_{T^2_1} (\partial_{12}^2 \varphi_{kl}) \varphi_{kl}.
\]

Put all calculations together we obtain

\[
D^2 \Pi^1_{f_1}(\Phi, \Phi) = -\frac{1}{\pi^2} \int_{T^2_1} (\partial_{12}^2 \varphi_{kl}) \varphi_{kl} + \frac{2}{\pi^2 (k^2 + l^2)} \int_{T^2_1} (\partial_{12}^2 \varphi_{kl}) \varphi_{kl}.
\]

**Remark 3.** The second variation for general normal variation \( \Phi = \left( \sum_{k, l \in \mathbb{N}^2} a_{k, l} \varphi_{k, l} \right) \tilde{n} \) is obtained by linearity. Terms for \( \varphi_{k, l} \) and \( \varphi_{m, n} \), where \( (k, l) \neq (m, n) \) vanishes. To determine stability of \( \mathcal{W}_n \) we can thus restrict ourself to case \( \Phi = \varphi_{k, l} \tilde{n} \).

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Clearly, if $D^2\Pi_{f^1}(\Phi, \Phi) \leq 0$, then $D^2W_\alpha(f^1)(\Phi, \Phi)$ is non negative for all $\alpha \geq 0$. Let
\begin{align*}
\varphi_{kl} &= a \sin(\sqrt{2}kx) \cos(\sqrt{2}ly) + b \cos(\sqrt{2}kx) \sin(\sqrt{2}ly) \\
&\quad + c \cos(\sqrt{2}kx) \cos(\sqrt{2}ly) + d \sin(\sqrt{2}kx) \sin(\sqrt{2}ly)
\end{align*}
for $k, l \in \mathbb{N} \setminus \{0\}$ and $a, b, c, d \in \mathbb{R}$. Then we have:
\begin{equation}
D^2\Pi_{f^1}(\Phi, \Phi) = \frac{1}{\pi^2}(2kl - \frac{4kl}{k^2+l^2}) < \Phi, \Phi > L^2
\end{equation}
if and only if $a = b$ and $c = d$.

The second variation of the Willmore functional at the Clifford torus (Lemma 3 [KuwLor]) is given by:
\begin{equation}
D^2W_{f^1}(\Phi, \Phi) = (\frac{1}{2} \Delta^2 + 3\Delta + 4)\Phi, \Phi > L^2
\end{equation}
(3.6)
\begin{equation}
= (2(k^2 + l^2)^2 - 6(k^2 + l^2) + 4) < \Phi, \Phi > L^2.
\end{equation}
(3.7)
Therefore we have
\begin{equation}
D^2W_{f^1}(\Phi, \Phi) = 0,
\end{equation}
if and only if $k = \pm 1$ and $l = \pm 1$, or $k = 0$ and $l = \pm 1$, or $k = \pm 1$ and $l = 0$.

Let $c := \frac{k}{l}$ and we assume without loss of generality that $c \geq 1$, then the second variation formulas (3.6) and (3.7) simplifies to:
\begin{align*}
D^2W_{f^1}(\Phi, \Phi) &= (2(c^2 + 1)^2l^4 - 6(c^2 + 1)l^2 + 4) < \Phi, \Phi > L^2 \\
D^2\Pi_{f^1}(\Phi, \Phi) &\leq \frac{1}{\pi^2}(2cl^2 - 4\frac{c}{c^2+1}) < \Phi, \Phi > L^2.
\end{align*}
Hence we obtain for $\tilde{\alpha} = \frac{1}{4\pi^2}c$
\begin{equation}
D^2W_\alpha(f^1)(\Phi, \Phi) \geq (2(c^2 + 1)^2l^4 - 6(c^2 + 1) + 8\tilde{\alpha}c)l^2 + 4 + 16\tilde{\alpha}\frac{c}{c^2+1}) < \Phi, \Phi > L^2.
\end{equation}
We still want to determine the range of $\alpha$ for which $W_\alpha$ is stable. At $\alpha = \alpha^b$ the second variation of $W_\alpha$ have zero directions in the normal part which is not a Möbius variation. Thus we need to determine the roots the polynomial
\begin{equation}
g_{\tilde{\alpha},c}(l) := (2(c^2 + 1)^2l^4 - 6(c^2 + 1) + 8\tilde{\alpha}c)l^2 + 4 + 16\tilde{\alpha}\frac{c}{c^2+1})
\end{equation}
The polynomial $g_{\tilde{\alpha}}$ is even and its leading coefficient is positive and its roots satisfy:
\begin{equation}
l^2 = \frac{2}{c^2+1}, \text{ or } l^2 = \frac{1}{c^2+1} + 4\tilde{\alpha}\frac{c}{(c^2+1)^2}.
\end{equation}
The values of $l \in \mathbb{N}$ for which $g_{\tilde{\alpha}}$ is negative lies exactly between the positive roots.

So we want to determine $\tilde{\alpha}$ such that the region of negativity of $g_{\tilde{\alpha},c}$ given by the interval between the two positive solutions $l_1(\tilde{\alpha}, c)$ and $l_2(\tilde{\alpha}, c)$ of (3.8) contain no other positive integer for all $c \in \mathbb{Q}_{\geq 1}$. We consider two different cases:
\begin{equation}
c = 1 \text{ and } c \neq 1.
\end{equation}
For $c = 1$ the four roots of $g_{\tilde{\alpha},1}$ are determined by:
\begin{equation}
l^2 = 1, \text{ or } l^2 = \frac{1}{2} + \tilde{\alpha}.
\end{equation}

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The case of $l^2 = 1$, i.e., $l = k = \pm 1$ corresponds to Möbius variations. We can rule out the existence of negative values of $g_{\tilde{\alpha},1}$ in the case of $c = 1$ if and only if the second $l^2 = \frac{1}{2} + \tilde{\alpha} \leq 4$, i.e., $\tilde{\alpha} \leq \frac{7}{2}$.

For $c \neq 1$, the first equation $l^2 = \frac{2}{c^2 + 1} < 1$ is never satisfied for a integer $l$. Thus we only need to consider the equation

$$l^2 = \frac{1}{c^2 + 1} + 4\tilde{\alpha} \left(\frac{c}{c^2 + 1}\right)^2.$$ 

To rule out negative directions for $D^2W_{\tilde{\alpha}}$ it is necessary and sufficient to have

$$\frac{1}{c^2 + 1} + 4\tilde{\alpha} \left(\frac{c}{c^2 + 1}\right)^2 \leq 1,$$

which leads to the condition on $\tilde{\alpha}$:

$$\tilde{\alpha} \leq \frac{1}{4}(c^3 + c).$$

The right hand side is monotonic in $c$. For $\tilde{\alpha} := \inf \left\{ \frac{1}{4}(c^3 + c) \mid c \in \mathbb{Q}_{>1} \right\}$, we have that there is a $c_0 \in \mathbb{Q}_{>1}$ such that $l = 1$ is a root of $g_{\tilde{\alpha},c_0}$. Then $c_0 = k \neq 1$ and hence $c_0 \geq 2$. At $c_0 = 2$ we obtain $\tilde{\alpha} = \frac{5}{2}$. Since $\frac{5}{2} < \frac{7}{2}$ which was the maximum $\tilde{\alpha}$ in the $c = 1$ case, we get that $\delta^2W_{\tilde{\alpha}} \geq 0$.

Further, at $\tilde{\alpha} = \frac{5}{2}$ the (non-Möbius) normal variations in the kernel of $\delta^2W_{\tilde{\alpha}}(f^1)$ are given by

$$\varphi_1 = \sin(2\sqrt{2}y) \cos(\sqrt{2}x) + \cos(2\sqrt{2}y) \sin(\sqrt{2}x) = \sin(\sqrt{2}(x + 2y))$$

and by symmetry of $k$ and $l$ (we have assumed $c \geq 1$):

$$\varphi_2 = \sin(2\sqrt{2}x) \cos(\sqrt{2}y) + \cos(2\sqrt{2}x) \sin(\sqrt{2}y) = \sin(\sqrt{2}(2x + y))$$

where $\tilde{\varphi}_i(x, y) = \varphi_i(x, y + \frac{\pi}{2})$, i.e., $\varphi_i$ and $\tilde{\varphi}_i$ differ only by a translation.

We have shown the following Lemma.

**Lemma 3.** At $b = 1$ we have that

$$\alpha^1 = \max \left\{ \alpha > 0 \mid \delta^2W_{\alpha} \geq 0 \right\}$$

is computed to be $\frac{5}{2}$.

The problem for $b = 1$ is that the kernel cannot reduced to be 1–dimensional (up to invaraince). The main reason is that linear combinations of both $\varphi_i$ cannot reduced to a translation and scaling of $\varphi_1$ only. This situation is different for $b \neq 1$, see Proposition [1] because for homogenous tori [3.1] the parameter directions $x$ and $y$ are no longer symmetric. For $b \neq 1$ we have that $\beta^b \neq 0$ and thus the second variation of $\Pi^2$ enters the game, since

$$\alpha^b = \max \left\{ \alpha > 0 \mid \delta^2W_{\alpha,\beta^b} \geq 0 \right\}.$$

Moreover, $A_{k,l}(T_1^2)$ is canonically isomorphic to $A_{k,l}(T_0^2)$ via

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(3.11) \[
\sin(k\sqrt{2}x)\cos(l\sqrt{2}y) \mapsto \sin\left(\frac{k\pi}{s}\right)\cos\left(\frac{ly}{s}\right),
\]
\[
\sin(k\sqrt{2}x)\sin(l\sqrt{2}y) \mapsto \sin\left(\frac{k\pi}{s}\right)\sin\left(\frac{ly}{s}\right),
\]
\[
\cos(k\sqrt{2}x)\sin(l\sqrt{2}y) \mapsto \cos\left(\frac{k\pi}{s}\right)\sin\left(\frac{ly}{s}\right),
\]
\[
\cos(k\sqrt{2}x)\cos(l\sqrt{2}y) \mapsto \cos\left(\frac{k\pi}{s}\right)\cos\left(\frac{ly}{s}\right).
\]

write as Lemma Because \(\delta^2\mathcal{W} \geq 0\) and \(\delta^2\Pi^1 \geq 0\) we have that if for \(\alpha_0\) fixed and \(\varphi^1 \in A_{k,l}(T^2_1)\)
\[
\delta^2\mathcal{W}_{\alpha_0}(f^1)(\varphi^1\vec{n}, \varphi^1\vec{n}) > 0
\]
then for \(b \sim 1\) close enough we also have
\[
\delta^2\mathcal{W}_{\alpha,\beta^b}(f^b)(\varphi^b\vec{n}_b, \varphi^b\vec{n}_b) > 0,
\]
for all \(\alpha \leq \alpha_0\) where \(\varphi^b\) is the image of \(\varphi^1\) under the canonical isomorphism and \(\vec{n}_b\) the normal vector of \(f^b\). By Lemma 4 and 7 of [KuwLor]
\[
D^2\Pi^2(f^b)(\Phi^b, \Phi^b) = \frac{1}{4\pi^2r^2} \int_{T^2_0} <\partial^2_{11} \Phi^b - \partial^2_{22} \Phi^b, \Phi^b > dA
\]
\[
+ \frac{r^2 - s^2}{4\pi^2r^2} \int_{T^2_0} |\Phi^b|^2 dA
\]
\[
- \frac{2(r^2-s^2) + c_r(k,l)}{4\pi^2r^2} \int_{T^2_0} |\Phi^b|^2 dA,
\]
where \(c_r(k,l) := \frac{k^2x^2 - t^2x^2}{ks^2 + t^2x^2}\) and \(\Phi^b \in A_{k,l}(T^2_b)\).

For \(b \sim 1\), i.e., \(r \sim \frac{1}{\sqrt{2}}\) this yields
\[
D^2\Pi^2(f^b)(\varphi^1_b\vec{n}_b, \varphi^1_b\vec{n}_b) > D^2\Pi^2(f^b)((\varphi^b_2\vec{n}_b, \varphi^b_2\vec{n}_b)),
\]
for \(\varphi^1_i\) are the images of \(\varphi_i \in \text{Ker} \delta^2\mathcal{W}_{\alpha^1}(f^1)\) under the canonical isomorphism and since for \(b > 1\), i.e., \(r < s\) and \(\beta^b > 0\) we obtain
\[
\delta^2\mathcal{W}_{\alpha^1,\beta^b}(\varphi^1_b\vec{n}, \varphi^1_b\vec{n}) < \delta^2\mathcal{W}_{\alpha^1,\beta^b}(\varphi^b_2\vec{n}, \varphi^b_2\vec{n}) < 0.
\]

Thus \(\alpha^b < \alpha^1\) and we obtain that for \(b \sim 1\) and \(b \neq 1\) the kernel of \(\mathcal{W}_{\alpha^b,\beta^b}\) is 2-dimensional and consists of \(\varphi^1_b\vec{n}\) and \(\varphi^1_b\vec{n}\) for \(b > 1\) or \(\varphi^b_2\vec{n}\) and \(\varphi^b_2\vec{n}\) for \(b < 1\). Both choices of \(b\) lead to Möbius invariant surfaces. We summarize the results in the following Lemma:

**Lemma 4.** For \(b \sim 1\), \(b > 1\) we have that \(\alpha^b\) is uniquely determined by the kernel of \(\delta^2\mathcal{W}_{\alpha^b,\beta^b}\) which is 2 dimensional and is spanned (up to invariance) by the normal variations
\[
\varphi^b_1\vec{n} = \sin(\sqrt{2}(\frac{x}{r} + \frac{2y}{s}))\vec{n} \quad \text{and} \quad \varphi^b_1\vec{n} = \cos(\sqrt{2}(\frac{x}{r} + \frac{2y}{s}))\vec{n}.
\]

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Now, for $b \sim 1$ consider the reparametrization of the homogenous torus as a $(2, -1)$-equivariant surface.

$$\tilde{f}^b : \mathbb{C}/(2\pi\mathbb{Z} + 2\pi\frac{2r^2 + 4sr}{4r^2 + s^2}\mathbb{Z}) \to S^3 \subset \mathbb{C}^2,$$

$$(\tilde{x}, \tilde{y}) \mapsto (re^{i2\tilde{x} + \frac{is\tilde{y}}{r}}, se^{-i\tilde{x} + \frac{2r\tilde{y}}{s}}).$$

Using these new coordinates the kernel of $\delta^2\mathcal{W}_{\alpha^b, \beta^b}$ for $b = \frac{s}{r} > 1$ is given by

$$\varphi^b = \sin((\frac{s}{r} + 4\frac{s}{r}^2)\tilde{y}), \quad \tilde{\varphi}^b = \cos((\frac{s}{r} + 4\frac{s}{r}^2)\tilde{y}).$$

Thus infinitesimally the $\tilde{x}$-direction of the surface is not changed by a deformation with normal variation $\varphi^b$, i.e., preserves the $(2, -1)$-equivariance. Since the space of $(2, -1)$-equivariant surfaces and $(1, 2)$-equivariant surfaces is isomorphic and differ only by the orientation of the surface and an isometry of $S^3$, we will consider $(1, 2)$-equivariant surfaces for convenience. Moreover, it is important to note that for all real numbers $c_1, c_2$ there exist $d_1, d_2 \in \mathbb{R}$ such that

$$c_1\varphi^b_1 + c_2\varphi^b_3 = c_1\sin((\frac{s}{r} + 4\frac{s}{r}^2)\tilde{y}) + c_2\cos((\frac{s}{r} + 4\frac{s}{r}^2)\tilde{y})$$

$$= d_1\sin((\frac{s}{r} + 4\frac{s}{r}^2)\tilde{y} + d_2) = d_1\varphi^b_1((\frac{s}{r} + 4\frac{s}{r}^2)\tilde{y} + d_2).$$

Since homogenous tori $\tilde{f}^b$ satisfy $\tilde{f}^b(\tilde{x}, \tilde{y} + d_2) = M\tilde{f}^b(\tilde{x}, \tilde{y})$, where $M$ is an isometry of $S^3$, we obtain the following proposition reducing the kernel dimension of $\delta^2\mathcal{W}_{\alpha^b, \beta^b}$ to 1 (up to invariance).

**Proposition 1.** Up to isometry of $S^3$ and reparametrization of $T^2_b$ the kernel of $\delta^2\mathcal{W}_{\alpha^b, \beta^b}$ that uniquely determine $\alpha^b$ is 1-dimensional for $b > 1$ and is given by

$$\varphi^b = \sin((\frac{s}{r} + 4\frac{s}{r}^2)\tilde{y}).$$

**Proof.** Let $f_t$ be a family of immersions from $T^2_b$ into $S^3$ with $f_0 = f^b$ with normal variation $\varphi := \partial_t f_t|_{t=0} \in Ker \delta^2\mathcal{W}_{\alpha^b, \beta^b}$, i.e., $\varphi = (c_1\varphi_1 + c_2\varphi_3)\vec{n}(\tilde{x}, \tilde{y})$, for real constants $c_1$ and $c_2$. Then there exist by Equation (3.13) real constants $d_1$ and $d_2$ satisfying $\varphi = (d_1\varphi(\tilde{y} + d_2))\vec{n}(\tilde{x}, \tilde{y})$. The aim is to show that there exist an isometry $M$ of $S^3$ and a reparametrization $\sigma$ of the underlying surface $T^2_b$ such that $\tilde{f}_t := M(f_t \circ \sigma)$ satisfy $\tilde{\varphi} := \partial_t \tilde{f}_t|_{t=0} = d_1 \cdot \varphi_1\vec{n}$. But by definition the homogenous tori there exist a $M$ such that $M(f^b) = f^b(\tilde{x}, \tilde{y} + d_2)$. Thus it induces a map, which we again denote by $M$ such that on the normal vector $\vec{n}_b$ of $f^b$ we have $M(\vec{n}(x, y)) = \vec{n}(\tilde{x}, \tilde{y} + d_2)$ and therefore $M(\varphi) = (d_1\varphi_1(\tilde{y} + d_2))\vec{n}(\tilde{x}, \tilde{y} + d_2)$. And we obtain for $\sigma : T^2_b \to T^2_b, (\tilde{x}, \tilde{y}) \mapsto (\tilde{x}, \tilde{y} - d_2)$ the desired property. \qed

4. Candidates for constrained Willmore minimizers

In this section we want to use Integrable systems methods to construct candidates for constrained Willmore minimizers $\tilde{f}(a,b)$ deforming homogenous tori $T^2_a$ for $a = 0$ and satisfying the following conditions:

1. the Lagrange multiplier $\alpha(a,b)$ of $f^{(a,b)}$ satisfy $\alpha(a,b) \not< \alpha^b$, as $a \to 0$,
2. $\alpha(a,b) < \alpha^b$ for $a > 0$

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proving Theorem[2] The appropriate candidates turn out to be equivariant and lie in the so called associated family of constrained Willmore Hopf tori. These are surfaces given by the preimage of (constrained) elastic curves on the 2—sphere under the Hopf fibration and the simplest examples that not CMC in a space form unless they are homogenous [Hel1]. We prove Theorem 2 in four steps in Section 4.4:

(1) Construct 2—parameter families of constrained Willmore Hopf cylinders de- forming homogenous tori and compute what properties the associated family inherits.

(2) Show there exist a 2—parameter family of (1, 2)—equivariant constrained Willmore tori lying in the associated family of the above defined constrained Willmore Hopf cylinders.

(3) Determine the infinitesimal normal variation \( \partial_\sqrt{a^f(a,b)} \) of the family of constrained Willmore (1, 2) tori at homogenous tori and thereby show that the limit Lagrange multiplier satisfies \( \lim_{a \to 0^+} \alpha(a,b) = \alpha^b \).

(4) Show that the Lagrange multiplier \( \alpha(a,b) \) converge to \( \alpha^b \) from below as \( a \to 0^+ \).

We change from the analysis notation to the integrable systems notation for this section only. Given the torus \( T^2_b := \mathbb{C}/(2r\pi \mathbb{Z} + 2s\pi i \mathbb{Z}) \), with \( b = \frac{r}{s} \) the space of trace free symmetric 2—tensors \( S^T_{2T} \) of \( T^2_b \) is identified with the space of holomorphic quadratic differentials \( H^0(K^2) \) using

\[
(dz)^2 = (dx \otimes dx - dy \otimes dy) + i(dx \otimes dy + dy \otimes dx).
\]

The imaginary part of \( (dz)^2 \) is denoted \( q_1 \) and the real part of \( (dz)^2 \) is denoted by \( q_2 \) in [KuwLor]. The \( q_i \) are normalized such that \( ||q_i||_{L^2} = 8rs\pi^2 \), thus we get \( q_i = 8rs\pi^2 \delta \Pi^i \). For the Clifford torus given by the parametrization

\[
f^1 : T_{\frac{1}{\sqrt{2}}} \to S^3 \subset \mathbb{C}^2, \quad (x, y) \mapsto \frac{1}{\sqrt{2}} \left( e^{i\sqrt{2}x}, e^{i\sqrt{2}y} \right)
\]

the Lagrange multipliers \( \lambda \) and \( \mu \) we consider now are no longer attached to the \( \Pi^1 \) and \( \Pi^2 \) component of the Teichmüller space. For the most important case below - the Clifford torus as a Hopf torus - they are given by \( \mu = \frac{1}{4\pi \beta} \) and \( \lambda = \frac{1}{4\pi \alpha} \). We start with fixing some notations and basic properties of equivariant constrained Willmore surfaces.

**Definition 2.** A map \( f : \mathbb{C} \to S^3 \) is called \( \mathbb{R} \)—equivariant, if there exist group homomorphisms

\[
M : \mathbb{R} \to \text{Möb}(S^3), \quad t \mapsto M_t,
\]

\[
\tilde{M} : \mathbb{R} \to \{\text{conformal transformations of } \mathbb{C}\}, \quad t \mapsto \tilde{M}_t,
\]

such that

\[
f \circ \tilde{M}_t = M_t \circ f, \text{ for all } t.
\]

Here \( \text{Möb}(S^3) \) is the group of Möbius transformations of \( S^3 \).

**Remark 4.** If the map \( f \) is doubly periodic, then the resulting surface is a torus. A necessary condition for doubly periodicity of \( f \) is that both \( M_t \) and \( \tilde{M}_t \) are periodic in \( t \), see [Hel]. The possible periodic 1—parameter subgroups \( M_t \) and \( \tilde{M}_t \) that can

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appear in the above definition can be easily classified, see for example [Hel]. Thus up to the choice of a holomorphic coordinate and isometries of $S^3$ we can assume with out loss of generality that an equivariant torus

$$f : \mathbb{C}/\Gamma \to S^3$$

for a lattice $\Gamma \subset \mathbb{C}$ is given by

$$(4.1) \quad f(x, y) = \begin{pmatrix} e^{imx} & 0 \\ 0 & e^{inx} \end{pmatrix} f(0, y),$$

for coprime integers $m$ and $n$ with $m \geq n$. In this case we call $f$ an $(m, n)$—equivariant surface and the curve given by $\gamma(y) = f(0, y)$ is called the profile curve which has to satisfy a certain closing condition to obtain a torus.

This notion of equivariant surfaces includes the well known examples of surfaces of revolution ($m = 1, n = 0$) and the Hopf cylinders ($m = 1, n = 1$), as for example discussed in [Hel1].

**Remark 5.** A $\mathbb{R}^-$—equivariant immersion $f : \mathbb{C} \to S^3$ such that $M(\mathbb{R})$ is not a periodic subgroup of M"ob$(S^3)$ but is close to a $S^1$—equivariant surface is still of the from $4.1$ (up to conjugacy) with $m, n \in \mathbb{R}$. This is due to the fact that whether a $M \in SO(4, 1)$ is conjugated to $SO(4)$ reduce to a restriction of the trace to be in a certain intervall.

In conformal geometry surfaces mapping into the conformal $S^3$ have two invariants which determine the surface up to Möbius transformations, see [BuPePi]. The first one is the conformal Hopf differential $q$. The second is the Schwarzian derivative $c$. In the equivariant case it can be easily computed that $q$ determines the Schwarzian derivative $c$ up to an complex integration constant by the Gauß-Codazzi equations, see [Hel2]. Thus we will only use $q$ in the following. In contrast to [BuPePi] we consider the conformal Hopf differential as a complex valued function by trivializing the canonical bundle $K_{\mathbb{C}/\Gamma}$ via $dz$.

**Definition 3.** Let $f : M \to S^3$ a conformal immersion. We call the function

$$q := \frac{\Pi \left( \frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}} \right)}{|df|}$$

the conformal Hopf differential of $f$.

**Remark 6.** For equivariant tori the conformal Hopf differential as well as the Schwarzian derivative depends only on the curve parameter and is periodic, see [Hel1].

**Example 1.** By definition we have for surfaces of revolution that $4q = \kappa$ is real valued with $\kappa$ is the curvature of the (arclength parametrized) profile curve $\gamma$ in the upper half-plane viewed as the hyperbolic plane. In the same way one can compute $4q = \kappa + \sqrt{G}i$ where $\kappa$ is its geodesic curvature of in a round $2$—sphere of curvature $G > 0$ for Hopf cylinders.
4.1. The equivariant constrained Willmore tori and their associated family.

For equivariant constrained Willmore tori we introduce the Euler Lagrange equation with respect to their Schwarzian derivative. This equation has an invariance which defines an associated family of constrained Willmore surfaces to a given solution. We start by recalling the result by [BuPePi] specified to the equivariant case:

**Theorem 5 ([BuPePi]).** Let \( f : T^2 \cong \mathbb{C}/\Gamma \to S^3 \) be a conformally parametrized equivariant immersion and \( q \) its conformal Hopf differential. Then \( f \) is constrained Willmore if and only if there exists a \( \mu + i\lambda \in \mathbb{C} \) such that \( q \) satisfies the equation:

\[
q'' + 8(|q|^2 + C)q - 8\rho q = 2\text{Re}((\mu + i\lambda)q),
\]

where \( \rho \) is a purely imaginary function, \( C \) a real constant and the derivative is taken with respect to the profile curve parameter.

**Remark 7.** The real part of equation (4.2) is the actual constrained Willmore Euler Lagrange equation. The imaginary part of the equation is the Codazzi equation and the equation on \( \rho \) is the Gauß equation. The Euler Lagrange equation for general surfaces can be found in [BuPePi]. For \((m,n)\)-equivariant tori the function \( \rho \) is given by \( \rho = \frac{imn}{4}H \), where \( H \) is the mean curvature of the immersion into \( S^3 \) and \( C = \frac{1}{4}(m^2 + n^3) \), as computed in [Hel].

Let \( f : T^2 \to S^3 \) be a conformally immersed constrained Willmore surface with conformal Hopf differential \( q \). Consider \( f \) as an immersion from \( \mathbb{C} \) into \( S^3 \) which is doubly periodic. By relaxing both periodicity conditions, i.e., by allowing general profile curves and real numbers for \( m \) and \( n \), we obtain for \( e^{i\theta} \in S^1 \) a circle worth of associated constrained Willmore surfaces \( f_\theta \) to \( f \), the so called constrained Willmore associated family as follows, see [BuPePi].

Let \( q \) be a solution of (4.2) and let \( q_\theta, e^{i\theta} \in S^1 \) be a family of complex functions given by

\[
q_\theta = q e^{2i\theta}
\]

Moreover, let

\[
C_\theta = C + \text{Re}((e^{4i\theta} - 1)(\mu - i\lambda)),
\rho_\theta = \rho + \text{Im}((e^{4i\theta} - 1)(\mu - i\lambda)),
\mu_\theta + i\lambda_\theta = e^{-4i\theta}(\mu + i\lambda).
\]

Then \( q_\theta \) satisfies equation (4.2) with parameters \( C_\theta, \lambda_\theta \) and function \( \rho_\theta \). In particular, the function \( q_\theta \) and \( \rho_\theta \) satisfies the Gauß-Codazzi equations for surfaces in \( S^3 \). Thus there exist a family of surfaces \( f_\theta \) with conformal Hopf differential \( q_\theta \) and mean curvature given by \( \rho_\theta \), see [BuPePi]. The so constructed surfaces \( f_\theta \) are automatically constrained Willmore for every \( \theta \).

**Definition 4.** [Constrained Willmore Associated Family] Let \( f \) be a constrained Willmore surface and \( q \) its conformal Hopf differential. The family surfaces \( f_\theta, \theta \in \mathbb{R} \)

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determined by the conformal Hopf differential \( q_\theta = q e^{2i\theta} \) is called the constrained Willmore associated family of \( f \).

Surfaces with the same conformal Hopf differential and the same Schwarzian derivative (which is determined by the function \( \rho \) and the real constant \( C \)) differ only by a Möbius transformation. Since both invariants depends only on one parameter for an equivariant initial surface, all surfaces in the associated family of equivariant constrained Willmore surfaces are also equivariant (and constrained Willmore), see [Hel]. In general, these surfaces are not closed, i.e., \( f_\theta : \mathbb{C} \rightarrow S^3 \) is not doubly periodic, even if the initial surface is. Moreover, since a non-isothermic surface \( f \) are already determined up to Möbius transformations by its conformal Hopf differential, only the map \( \theta \mapsto f_\theta \in C^\infty(T^2, S^3) \) is smooth, see [BuPePi].

4.1.1. The associated family of homogenous tori \( T_r \).

The homogenous tori are given by the direct product of two circles with different radii. They can be parametrized by

\[
\begin{aligned}
  f^b : \mathbb{C}/(2\pi ri \mathbb{Z} + i2\pi s \mathbb{Z}),
  f(x, y) &= \left( re^{ix}, se^{iy} \right),
  \end{aligned}
\]

for \( r, s \in \mathbb{R} \) satisfying \( r^2 + s^2 = 1 \) with rectangular conformal class given by \( b = \frac{r}{r} \).

The conformal Hopf differential (in this particular parametrization) can be computed to be

\[
q = \frac{1}{2rs} \quad \text{and} \quad \mu = \frac{r^2 - s^2}{rs}.
\]

Since homogenous tori are isothermic, hence the Lagrange multiplier \( \lambda \) in (4.2) can be chosen arbitrarily and \( q \) does not uniquely determine the surface. Thus the associated family \( f^b \) is also not uniquely determined and is only smooth in \( \theta \) for appropriate \( \rho \) and \( C \) depending on both \( \mu \) and \( \lambda \). In fact, the following choice of associated family of \( f^b \) seen as a map from \( \mathbb{C} \) to \( S^3 \) is smooth in \( \theta \) and is given by

\[
\begin{aligned}
  f^b_\theta(x, y) &= \left( r^\lambda_\theta e^{-i\frac{1}{r^\lambda_\theta} \cos(\theta)x + \frac{1}{r^\lambda_\theta} \sin(\theta)y}, s^\lambda_\theta e^{-i\frac{1}{s^\lambda_\theta} \sin(\theta)x + \frac{1}{s^\lambda_\theta} \cos(\theta)y} \right),
\end{aligned}
\]

where \( r^\lambda_\theta, s^\lambda_\theta \) are determined by the Lagrange multiplier \( \mu_\theta \) by (4.4) and \( \mu_\theta, \lambda_\theta \) satisfies

\[\mu_\theta + i\lambda_\theta = e^{-4i\theta}(\mu + i\lambda).\]

Our choice of the associated family ensures that for a non-degenerated family \( f^t \) such that the map \([0, t_0) \rightarrow C^\infty_{imm}, t \mapsto f^t\) is smooth with \( f^t \rightarrow f^0 = f^b \), also the associated family \( f^t_\theta \) has the same regularity in \( t \) for every fixed \( \theta \). In particular, the corresponding Lagrange multipliers \( \mu^t_\theta \) and \( \lambda^t_\theta \) are continuous for \( t \in [0, t_0) \) for every fixed \( \theta \). The equivariance type of the “rotated” surface \( f^b_\theta \) is given by

\[
\frac{m}{n} = \left| \frac{\cos(\theta)s^\lambda_\theta}{\sin(\theta)r^\lambda_\theta} \right| \in [1, \infty].
\]

Note that the derivative of the equivariance type by \( \theta \) is non zero at generic \( \mu, \lambda \) and \( \theta \).
4.2. Constrained Willmore Hopf cylinders.

Since tori of revolution are isothermic, they cannot cover an open set of the Teichmüller space. In contrast all conformal types can be realized as (constrained Willmore) Hopf tori (i.e., \( m = n = 1 \)), [Pin, Hel2]. The Willmore energy of the surface reduces to the (generalized) energy functional of the curve in \( S^2 \) and the conformal type of the torus translates into invariants of the curve, namely length and enclosed area. Thus a Hopf torus is constrained Willmore if and only if there exist Lagrange multipliers \( \lambda \) and \( \mu \in \mathbb{R} \) such that the geodesic curvature \( \kappa \) of its profile curve \( \gamma \) in \( S^2 \) satisfies

\[
\kappa'' + \frac{1}{2}\kappa^3 + (\mu + \frac{G}{2})\kappa + \lambda = 0, \tag{4.6}
\]

which can also be obtained from Equation 4.2 for \( 4q = \kappa + i\sqrt{G} \). We call curves (not necessarily closed) into the round \( S^2 \) with constant curvature \( G \) satisfying equation \(4.6\) constrained elastic.

Since we are interested in periodic solutions of (4.6), we can restrict ourselves to the initial values

\[
\kappa'(0) = 0 \quad \text{and} \quad \kappa(0) = \kappa_0
\]

for the Euler-Lagrange equation.

Let \( \gamma : \mathbb{R} \to S^2 \) be a curve and \( \kappa \) its geodesic curvature. We use an integrated version of the Euler Lagrange equation for constrained elastic curves obtained by multiplying (4.6) with \( \kappa' \) and integrate. The curve \( \gamma \) is therefore constrained elastic if and only if there exist real numbers \( \mu, \lambda \) and \( \nu \) such that \( \kappa \) satisfying:

\[
(\kappa')^2 + \frac{1}{4}\kappa^4 + (\mu + \frac{G}{2})\kappa^2 + \lambda \kappa + \nu = 0. \tag{4.7}
\]

Here \( \mu \) is the length constraint, \( \lambda \) is the enclosed area constraint and \( \nu \) the integration constant corresponding to the initial value \( \kappa_0 \) (which is a root of of the polynomial \( P_1 = \frac{1}{4}\kappa^4 + (\mu + \frac{G}{2})\kappa + \lambda \kappa + \nu \)).

**Remark 8.** The conformal Hopf differential of the Clifford torus in \((1, 1)\)-parametrization \( q_{(1,1)} \) is the conformal Hopf differential of the Clifford torus considered as a torus of revolution \( q_{(1,0)} \) multiplied by the imaginary unit \( i \). Thus the role of the Lagrange multipliers \( \lambda \) and \( \mu \) switch compared to Section 4.1.1.

All constrained elastic curves in \( S^2 \) can be parametrized in terms of the Weierstrass elliptic functions and limits of these as the lattice degenerates. Elliptic functions are defined on a torus \( \mathbb{C}/\Gamma \), where the lattice \( \Gamma \) is determined by its lattice invariants \( g_2 \) and \( g_3 \). For constrained elastic curves these invariants were computed in [Hel2]:

\[
g_2 = \frac{(\mu + \frac{G}{2})^2}{12} + \frac{\nu}{4} \tag{4.8}
\]

\[
g_3 = \frac{1}{216}(\mu + \frac{G}{2})^3 + \frac{1}{16}\lambda^2 - \frac{1}{24}\nu(\mu + \frac{G}{2}). \tag{4.9}
\]

The lattice \( \Gamma \) is non degenerated, i.e., has two real linear independent generators, if and only if \( D := g_2^3 - 27g_3^2 \neq 0 \). In this case we denote the generators of the lattice by \( 2\omega_1, 2\omega_2 \in \mathbb{C} \). Since by construction \( g_2, g_3 \in \mathbb{R} \), the resulting lattice \( \Gamma \) is either
rectangular or rhombic. Thus we can fix \(2\omega_1 \in \mathbb{R}\) and there exits a smallest lattice point on the imaginary axis, which we denote by \(2\omega_3\in i\mathbb{R}\). For details on elliptic functions see [?].

Now we can parametrize all solutions of (4.7) with periodic curvature \(\kappa\) such that \(D = g_2^3 - 27g_3^2 \neq 0\) according to the following theorem.

**Theorem 6 ([Hel2]).** For \(g_2, g_3\) and \(\kappa_0 \in \mathbb{R}\) with \(D = g_2^3 - 27g_3^2 \neq 0\), the curve \(\gamma = [\gamma_1 : \gamma_2] : \mathbb{R} \to \mathbb{CP}^1\) with \(\gamma_i : \mathbb{R} \to \mathbb{C}\) given by

\[
\begin{align*}
\gamma_1 &= \frac{\sigma(x + x_0 - \rho)}{\sigma(x + x_0)} e^{\zeta(x)(x + x_0)} \\
\gamma_2 &= \frac{\sigma(x + x_0 + \rho)}{\sigma(x + x_0)} e^{\zeta(-x)(x + x_0)}
\end{align*}
\]

is constrained elastic in the round \(S^2\) with curvature \(G(> 0)\). Hereby \(\sigma\) and \(\zeta\) denote respectively the Weierstrass \(\sigma\)− and \(\zeta\)−function and the parameters \(x_0, \rho \in (0, \omega_3) \subset i\mathbb{R}\) are chosen such that

\[
2\varphi(x_0) + \varphi(\rho) + \frac{1}{4}\kappa_0^2 = -\frac{1}{4}G < 0,
\]

where \(\omega_3\) is the lattice point of \(\Gamma\) with smallest length lying on the imaginary axis and the upper half plane. Moreover, all constrained elastic curves of \(S^2\) with \(D = g_3^3 - 27g_3^2 \neq 0\) and periodic curvature are obtained this way.

**Remark 9.** For given lattice invariants \(g_2\) and \(g_3\) we obtain thus a 2−parameter \((x_0\) and \(\rho)\) family of (not necessarily closed) constrained elastic curves into a 2−sphere of constant curvature \(G(> 0)\). It is shown in [Hel2] that there exist a unique \(\hat{x}_0 \in (0, \omega_3 \subset i\mathbb{R})\) such that the corresponding curve becomes elastic, i.e., \(\lambda = 0\). In this case we obtain \(\varphi(\rho) = \varphi(\omega_3) - \frac{1}{4}G\). Moreover, by [Hel2] we have \(\varphi(\rho) = \frac{1}{8}(\mu - G)\) for all \(\lambda\).

A straight forward computation, see [Hel2], shows that the curve \(\gamma\), as given by the above theorem, closes if and only if there exist \(m, n \in \mathbb{N}\) such that

\[
(4.11) \quad M(g_2, g_3, \rho) := \rho \eta_1 - \zeta(\rho)\omega_1 = \frac{i\pi n}{2m} \pi,
\]

where \(\zeta\) is the Weierstrass \(\zeta\)−function and \(\eta_1 = \zeta(\omega_1)\). \(M\) is called the monodromy of the curve.

Geometrically speaking, \(m\) is the winding number of the curve and \(n\) is the lobe number, i.e., the number of (intrinsic) periods of the curvature till the curve closes in space. Further, the closing condition is independent of \(x_0\), thus we obtain a 1−parameter family of closed constrained elastic curves in \(S^2\). Since \(\hat{x}_0\) for which the curve becomes elastic is unique, the variation of \(x_0\) near \(\hat{x}_0\) corresponds to a variation of \(\lambda\).

For the Clifford torus the profile curve is (a piece of a) geodesic in \(S^2\) and can be described using trigonometric functions. Thus we have \(D = 0\) and the curve is given as the limit curve as \(\omega_3\), the smallest lattice point lying on the imaginary axis, goes to infinity, of the family of elastic curves given by \(\omega_3\) and \(G = 1\). In this case the
corresponding limits of the Weierstraß elliptic functions and invariants are given by, see [ErMaOb]:

\[ \wp_{\infty}(z) = -a + 3a \frac{1}{\sin^2(\sqrt{3}az)} \]  
\[ \zeta_{\infty}(z) = az + \sqrt{3}a \frac{\cos(\sqrt{3}az)}{\sin(\sqrt{3}az)} \]

\[ \omega_1 = \frac{1}{\sqrt{12a}} \pi \]
\[ \eta_1 = \frac{a}{\sqrt{12a}} \pi \]

for a real number \( a \) with \( g_2^\infty = 12a^2 \) and \( g_3^\infty = 8a^3 \). Since for the Clifford torus we have \( \nu_\infty = 0 \) (and thus \( 144a^2 = 12g_2^\infty = (\mu_\infty + \frac{G}{2})^2 \)) we obtain by (4.12) that

\[ \wp_{\infty}(\rho_\infty) = \frac{1}{6}(\mu_\infty - G) = 2a - \frac{1}{4}G, \]

from which we can compute

\[ \rho_\infty = \frac{1}{\sqrt{3a}} \arcsin(\sqrt{\frac{12a}{12a-G}}). \]

By Equation (4.16) and because \( \wp(\rho) = \wp(\omega_3) - \frac{1}{4}G \) for elastic curves \( \lim_{\tau \to \infty} \wp_\tau(\omega_3) \to 2a \). Thus the invariants for family of constrained Willmore Hopf tori converging to the Clifford torus satisfy \( \omega_3 = \omega_1 \mod \Gamma \), i.e., these are wavelike solutions with \( D \to 0 \) converging from below. The closing condition (4.11) converges to

\[ \rho_\infty = a_{\sqrt{12a}} \pi - \left( a\rho_\infty + \sqrt{3a} \frac{\cos(\sqrt{3a}\rho_\infty)}{\sin(\sqrt{3a}\rho_\infty)} \right) \right) \frac{1}{\sqrt{12a}} \pi = \frac{m}{2n} \pi i. \]

For the simply wrapped Clifford torus, i.e., \( m = 1 \), the Equations (4.16), (4.17) and (4.18) yields \( a = \frac{n^2}{12} \) (and \( n > 1 \)) or equivalently \( \mu_\infty^n = n^2 - \frac{G}{2} \). Since the ratio of winding number and lobe number is rational for closed solutions, it remains constant throughout deformation induced by a continuous deformation of the parameters \( g_2, g_3 \) and \( \rho \). Suppose there is a family of embedded constrained Willmore Hopf tori with parameters \( g_2, g_3 \) and \( \rho \) converge to the parameters of the Clifford torus as computed above, then the necessary condition is thus that its Lagrange multiplier \( \mu \) converges to \( n^2 - \frac{G}{2} \), for a integer \( n \).

**Theorem 7.** For every integer \( n > 1 \) there is an embedded \( n \)-lobed 2-parameter family constrained Willmore Hopf tori \( f^n_{g_2,g_3} \), "deforming" the Clifford torus. Further, the Clifford torus is parametrized as the limit of Weierstrass elliptic functions (as in the formulas above) for \( a = \frac{n^2}{12} \). Thus the limit Lagrange multiplier of the family at the Clifford torus is \( \mu_\infty^n = n^2 - \frac{G}{2} \) (and \( \lambda = 0 \).)

**Proof.** For given \( g_2, g_3 \) with \( D \neq 0 \) we can define constrained elastic curves and their corresponding monodromy. Since for \( D \to 0 \) the Weierstrass \( \wp \) converge uniformly on every compact set, we obtain that the monodromy function \( M(g_2, g_3, \rho) \), see (4.11), remain smooth in all arguments.
We are interested in the singular case (i.e., \(D=0\)) where we have computed above that \(g_2^\infty = \frac{n^4}{12}\) and \(g_3^\infty = \frac{n^6}{216}\), and we denote by \(\rho_\infty \in i\mathbb{R}_+\) the unique solution of \(\psi_\infty(\rho_\infty) = \frac{n^2}{6} - \frac{G}{2}\) in \(i\mathbb{R}_+\). From this we obtain

\[
\frac{\partial M(n^4, n^6, \rho)}{\partial \rho} \bigg|_{\rho = \rho_\infty} = \psi_\infty(\rho_\infty) = -\frac{(n^2 + \frac{G}{2n})}{2}
\]

which is non zero for positive \(G\) and \(n\).

Thus by the implicit function theorem and Theorem 6 we obtain a 2-parameter family of curves parametrized by \(g_2\) and \(g_3\) "deforming" the Clifford torus for every \(n > 1\).

\[\square\]

**Definition 5.** In the following we will denote the 2-parameter family of constrained Willmore tori given by the above theorem by \(f_{g_2,g_3}^n\).

**Remark 10.** The surface \(f_{g_2,g_3}^n\) is homogenous if and only if \(D = g_2^3 - 27g_3^2 = 0\). Thus these surfaces can be identified within \(f_{g_2,g_3}^n\) by varying \(a \sim n^2 - \frac{G}{2}\) (and still prescribe \(M(12a^2, 8a^3, \rho_\infty(a)) = \frac{1}{n} \pi i\)). Further, for every \(g_2\) there is a unique positive \(\hat{g}_3\) such that \(f_{g_2,\hat{g}_3}^n\) is homogenous.

**Remark 11.** By the same computations it is possibly to construct a 2-parameter family \(f_{g_2,g_3}^c(D)\) of constrained Willmore Hopf cylinders with

\[M(g_2, g_3, \rho(c(D))(g_2, g_3)) = c(D)\]

for an arbitrary positive function \(c(D)\) depending on the descriminant \(D = g_2^3 - 27g_3^2\). For \(D = 0\) the values \(g_2^c\), \(g_3^c\) and \(\rho^c\) can be computed analogously to the computations above and it is possible to use implicit function theorem to solve for \(\rho^c(D)(g_2, g_3)\) for \(g_2 \sim g_2^c\) and \(g_3 \sim g_3^c\). The limit Lagrange multiplier \(\mu c\) depends only on the limit monodromy \(c(0)\).

The conformal type of a Hopf torus is given by the lattice generated by the vector \(2\pi \in \mathbb{C}\) and the vector \(\frac{1}{2}(GA + i\sqrt{G}L) \in \mathbb{C}\), where \(L\) is the length of the corresponding profile curve in the space form of curvature \(G\) and \(A\) the oriented enclosed volume, see [Pin]. These quantities can be explicitly computed, see [Hel2].

**Theorem 8.** For every lobe number \(n \geq 2\) the map \(\psi\) which assigns to \((g_2, g_3)\) the conformal class of the immersions \(f_{g_2,g_3}^n\), see Definition 5, covers an open neighborhood of the conformal class of the Clifford torus.

**Proof.** By construction we have \(\psi(g_2, g_3) = \frac{1}{2}A(g_2, g_3) + i\frac{1}{2}L(g_2, g_3)\), where \(L\) is the length and \(A\) the oriented enclosed area of the profile curve of \(f_{g_2,g_3}^n\) in the 2-sphere with constant curvature \(G = 1\).

For given \(g_2\) there exist a unique \(\hat{g}_3(g_2)\) such that the resulting surface is homogenous and moreover

\[
\partial_{g_2} A(f_{g_2,g_3}^n) \bigg|_{g_2 = \frac{n^4}{12}, g_3 = \hat{g}_3(g_2)} \neq 0.
\]

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add proof of the formula Therefore there exist a map \( g_2(g_3) \) for \( g_3 \sim \hat{g}_3 (\frac{n^4}{17}) \) such that \( A(f^n_{g_2(g_3),g_3}) = \text{const} \). It remains to show that \( L(f^n_{g_2(g_3),g_3}) \) is non constant in \( g_3 \).

Assume \( L(f^n_{g_2(g_3),g_3}) = \text{const} \), then we would have a family of constrained Willmore Hopf tori of the same conformal type. Thus the Willmore energy must be constant along the whole family \( f^n_{g_2(g_3),g_3} \). Thus \( \varphi := (\partial_{g_3} f^n(g_2(g_3), g_3)) \) \( \in \text{Ker} \delta^2 \mathcal{W} \). But close to homogenous tori this would imply that \( \varphi \) is a Möbius variation for all \( g_3 \sim \hat{g}_3 \) which leads to a contradiction, since \( f^n_{g_2(g_3),g_3} \) is not a Möbius transformation of a homogenous torus.

The so constructed family constrained Willmore Hopf tori \( f^n_{g_2,g_3} \) do not converge to the Clifford torus with the appropriate Lagrange multiplier of \( \mu = \frac{5}{2} \) (for \( G = 1 \)) for an integer \( n \) to be candidate surfaces for the constrained Willmore minimizer. However, we show that the actual constrained Willmore minimizers are associated to the constrained Willmore Hopf cylinders and inherit many properties. For the surfaces in the associated family it is harder to compute the limit Lagrange multiplier than in the Hopf case, but its normal variation at a homogenous torus turn out to be very related, as we compute in Proposition 5. For the constrained Willmore Hopf tori \( f^n_{g_2,g_3} \) we compute its infinitesimal normal variation at homogenous tori specified in the following proposition.

**Proposition 2.** The infinitesimal normal variation of the \((1-\text{parameter})\) sub family \( f^t \), smooth in \( t \sim 0 \), of \( f^n_{g_2,g_3} \) satisfying \( A(f^n_{g_2,g_3}) = \text{const} \). with \( f^0 = f^n_{\hat{g}_2, g_3} \), is given by

\[
\Phi = \sin(n \frac{y}{r_s}) \vec{n}_b,
\]

where \( f^n_{\hat{g}_2, g_3} \) is a piece of \( f^t \) parametrized as a \((1,1)\)-torus

\[
f^t : \mathbb{C}/(2\pi \mathbb{Z} + 2\pi (r^2 - ir) \mathbb{Z}) \to S^3, \quad f^t(x, y) = \left( r e^{i(x + \frac{ry}{s})}, s e^{i(-x + \frac{sy}{r})} \right)
\]

and \( y \) is the profile curve parameter and \( \vec{n}_b \) is the normal vector.

**Proof.** At \( f^n_{\hat{g}_2, g_3} \) which is \( \frac{1}{n} \)th of the Clifford torus, we obtain that \( \mu = n^2 - \frac{1}{2} = \tilde{\alpha} \) is the limit Lagrange multiplier. Further, by replacing the parameters \( g_2 \) and \( g_3 \) by \( L \) and \( A \) we obtain a family of surfaces parametrized by its conformal class and verifying the regularities assumptions needed for Lemma 1. Thus the infinitesimal normal variation lies in the kernel of \( \delta^2 \mathcal{W}_\alpha \), with \( \alpha = 4\pi^2 \tilde{\alpha} \). The kernel of \( \delta^2 \mathcal{W}_\alpha \) can be computed analogously the computations leading to Lemma 3 (the \( c = 1 \) case) and is spanned by the vectors \( \varphi_1 = \sin(2ny) \vec{n}_1 \) and \( \varphi_1 = \cos(2ny) \vec{n}_1 \), similarly to Proposition 1 both variations are equivalent up to invariance. Thus with the same arguments as in Proposition 1 we can choose with out loss of generality \( \varphi_1 \) to be the normal variation of a family \( f^t \) with \( f^0 = f^n_{n, n} \) at the Clifford torus, proving the statement in this case.

For the homogenous tori the statement holds, since we have a smooth \( 2-\text{dimensional} \) family of surfaces \( f^n_{\hat{g}_2,g_3} \) and the space of \((\alpha, \beta)\) for which \( \delta^2 \mathcal{W}_{\alpha,\beta} \) has a kernel is 1-dimensional. Thus for every \( b \sim 1 \) we have \( \beta^b \sim 0 \) there is a unique \( \alpha \sim n^2 - \frac{1}{2} \).
such that $\delta^2\mathcal{W}_{\alpha,\beta^0}$ has a kernel which is again spanned by $\varphi_1^b = \sin(n\frac{\pi}{sr})\vec{n}_b$ up to invariance, where $\vec{n}_b$ is the normal vector of the homogenous torus $f^b$. Therefore, by continuity of the family we obtain that the assertion holds for every $b$. write with spectral decomposition and formula in Section 3).

\begin{proof}
With the same computation as in the above proposition it can be shown that for a family of $n$–lobed constrained Willmore Hopf tori converge to the $k$–times covered homogenous torus with $\dot{b} = 0$ the corresponding normal variation is given by

$$\Phi = d \sin\left(n\frac{\pi}{sr}\right)\vec{n}_b + e.$$

Thus by the continuous dependence of the family on the monodromy $c(b(t))$ we obtain that any family of (not necessarily compact) constrained Willmore Hopf cylinders deforming the homogenous torus with monodromy $M = \frac{1}{c(b)}\pi i$ and $\dot{b} = 0$ has normal variation at the $a = 0$ given by

$$\Phi = \sin\left(c(b)\frac{\pi}{sr}\right)\vec{n}_b$$

for a real number $c(b) \in \mathbb{R}$.

If $\dot{b} \neq 0$ and $a(t) \neq 0$ we can split the normal variation into components:

$$\dot{f}^t = \frac{\dot{a}(t)}{\sqrt{a(t)}} \partial \sqrt{a} f^{c(b)}(a,b) + \dot{b}(t) \partial_b f^{c(b)}(a,b).$$

Because the surfaces $f^{c(b)}_{0,b}$ are homogenous, we can compute that $\partial_b f^{c(b)}(a,b)|_{a=0} = \tilde{c}$ is constant hence

$$\dot{f}^t|_{t=0} = (a_0 \sin(c(b)x) + b'(0)\tilde{c})\vec{n}_b$$

proving the statement with $d = a_0$ and $e = b'(0)\tilde{c}$. \end{proof}

4.3. (1,2)–Equivariant surfaces associated to constrained Willmore Hopf cylinders. The stability computations indicates that the candidates for constrained Willmore minimizers should have (1,2)–symmetry, see Section 3. We thus construct in the following 2–parameter families of (1,2)–equivariant tori deforming the Clifford torus whose projection into Teichmüller space cover an open neighborhood of the square conformal class. The crucial property of these candidates is that the limit Lagrange multiplier (as the surfaces converge to the Clifford torus) is $\alpha^1$, i.e., the maximum $\alpha > 0$ for which $(\delta^2\mathcal{W} - \alpha \Pi^1)(f_{\text{Cliff}}) \geq 0$.

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For our candidates we use the Ansatz that they lie in the associated family of constrained Willmore Hopf cylinders, i.e., the conformal Hopf differential of the \((1,2)\)-surface should be given by

\[4q_{(1,2)} = (\kappa + \sqrt{G})e^{2i\theta},\]

for \(G, \theta \in \mathbb{R}_+\) and \(\kappa : \mathbb{R} \to \mathbb{R}\) satisfying

\[\kappa'' + \frac{1}{2} \kappa^3 + (\mu + \frac{G}{2})\kappa + \lambda = 0\]  \hspace{1cm} (4.19)

which is the (constrained) elastic curve equation for curves on a round \(S^2\). The real constants \(\mu, \lambda \in \mathbb{R}\) are the Lagrange multipliers of the constrained Willmore Hopf surface. The corresponding Lagrange multipliers for the \((1,2)\)-equivariant surface are given by \([4.3]\) and are depending on \(\lambda, \mu\) and \(\theta\).

4.3.1. **Seifert fiber space.** We want to restrict to the \((1,2)\)-equivariant case in the following, although the constructions below can be used to construct general \((m,n)\)-tori (lying in the associated family of Hopf cylinders). We consider \(S^3 \subset \mathbb{C}^2\) with the equivalence relation\((z,w) \sim (\tilde{z}, \tilde{w}) \Leftrightarrow \exists \varphi : (\tilde{z}, \tilde{w}) = (e^{i\varphi}z, e^{2i\varphi}w).\)

We can always choose a unique representative of \([(z,w)]\) of the form \((|z|, \tilde{w}) \in S^3\), since for \(z = |z|e^{i\varphi}\) we have \((|z|e^{i\varphi}, w) \sim (|z|, e^{-2i\varphi}w).\) The orbit space \(S^3/\sim\) is a topological 2-sphere and the fibers of a point \((z, w) \in S^3 \subset \mathbb{C}^2\) is given by the curve

\[\varphi \mapsto (e^{i\varphi}z, e^{2i\varphi}w).\]

The trippel \(F = (S^3, S^3/\sim, \pi)\) defines a Seifert fiber space with two exceptional fibers over \([(1,0)]\) and \([(0,1)]\), where \(\pi : S^3 \to S^3/\sim\) is the projection map. In the following we parametrize the regular set of \(S^3/\sim\) (which is a sphere \(S^3 \setminus \{\text{two points}\}\) using polar coordinates:

\[\varphi \mapsto (e^{i\varphi}z, e^{2i\varphi}w).\]

We have always chosen a unique representative of \([(z,w)]\) of the form \((|z|, \tilde{w}) \in S^3\), since for \(z = |z|e^{i\varphi}\) we have \((|z|e^{i\varphi}, w) \sim (|z|, e^{-2i\varphi}w).\) The orbit space \(S^3/\sim\) is a topological 2-sphere and the fibers of a point \((z, w) \in S^3 \subset \mathbb{C}^2\) is given by the curve

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The trippel \(F = (S^3, S^3/\sim, \pi)\) defines a Seifert fiber space with two exceptional fibers over \([(1,0)]\) and \([(0,1)]\), where \(\pi : S^3 \to S^3/\sim\) is the projection map. In the following we parametrize the regular set of \(S^3/\sim\) (which is a sphere \(S^3 \setminus \{\text{two points}\}\) ) using polar coordinates:

\[\varphi \mapsto (e^{i\varphi}z, e^{2i\varphi}w).\]

The round metric \(g_{\text{round}}\) on \(S^3\) induces a unique metric on (regular set of) the base space, such that \(\pi\) becomes a Riemannian submersion. We denote this metric also by \(g_{\text{round}}.\) To be more specific, let \(X, Y\) be local vector fields on \(S^3/\sim\) and \(\hat{X}, \hat{Y}\) be their lifts to \(TS^3\). Then we define the metric on the base space to be

\[g_{\text{round}}(X, Y) = g_{\text{round}}((\hat{X})^\perp, (\hat{Y})^\perp),\]

where \((\cdot)^\perp\) means the projection orthogonal to the fiber direction. In terms of the coordinates \((R, \varphi)\) the metric on the base space is given by

\[g_{\text{round}} = (1 - R^2)dr^2 + \frac{R^2(1 - R^2)}{4 - 3R^2}d\varphi^2.\]

With respect to the round metric the length \(l\) of the fibers of \(F\) at \((z, w) \in S^3 \subset \mathbb{C}^2\) can be easily computed to be

\[l(z, w) = \sqrt{|z^2| + 4|w^2|} = \sqrt{4 - 3R^2}.\]
Dividing the metric $g_{\text{round}}$ point wise by $l$, which is constant along every fiber, yields new metrics $g_{(1,2)}$ on $S^3$ and on the base space $S^3/\sim$, respectively, given by

$$g_{(1,2)} = \frac{1}{4-3R^2}g_{\text{round}}.$$  

With respect to $g_{(1,2)}$ all fibers have same length $2\pi$. Let $B$ be the unit fiber direction with respect to $g_{(1,2)}$, then a bundle connection on $F$ is given by $\omega = g_{(1,2)}(B,.)$ and its curvature is computed to be $\Omega = \frac{4}{\sqrt{4-3R^2}}$.

Any closed curve $\gamma : S^1 \to S^3/\sim \setminus \{(1,0),(0,1)\}$, $y \mapsto (R(y),\varphi(y))$ gives rise to an immersed equivariant torus by

$$f(x,y) = (e^{ix}R(y), e^{2ix}\sqrt{1-R^2} e^{i\varphi(y)}).$$

The torus is embedded, if and only if the curve is. To obtain a conformal parametrization of the surface, we need the profile curve to have constant velocity $v$ with respect to $g_{(1,2)}$ and take its horizontal lift

$$\tilde{\gamma}(t) = (e^{ix}R(y), e^{2ix}(1-R(y)^2)e^{i\varphi(y)})$$

with to $S^3$ (i.e, $\tilde{\gamma}$ satisfy $\omega(\tilde{\gamma}') = 0$ and is not necessarily closed).

The conformal Hopf differential of the immersion can be computed to be

$$4q_{(1,2)} = v(\kappa_{(1,2)} + i\Omega).$$

where $\kappa_{(1,2)}$ is the geodesic curvature and $v$ the constant velocity of $\gamma$, see [Hel1].

### 4.4. Construction of candidate surfaces.

For given conformal Hopf differential $q = e^{2i\theta}(\kappa + i\sqrt{G})$ lying in the associated family of constrained Willmore Hopf solutions, we want to show how to obtain a constrained Willmore and $(1,2)$–equivariant cylinder and determine the closing conditions. Without loss of generality we always choose $G > 0$ such that the resulting $(1,2)$–surface is arc length parametrized in with respect to $g_{(1,2)}$. The curvature of constrained Willmore Hopf cylinders depends on three parameters $g_2, g_3$, and $\mu$. It can be easily computed that the derivative of the equivariance type w.r.t. $\theta$ is non zero at the Clifford torus (for $\theta = \arctan(1/2)$ and $\lambda_\theta = 0$, but $\mu_\theta \neq 0$), there exist by implicit function theorem a function $\theta(g_2, g_3, \mu)$ such that $f_{\theta(g_2, g_3, \mu)}$ is $(1,2)$–symmetric.

To construct the profile curve $\gamma$ of the $(1,2)$–equivariant surface in $S^3/\sim$, we show that $\gamma$ is already uniquely determined up to isometries by $\Omega = \text{Im}(q) = (\cos(2\theta)\kappa + \sin(2\theta)\sqrt{G})$.

The function $\Omega$ is the curvature of the connection $\omega$ which for a $(1,2)$–symmetric torus is in [Hel1] computed to be $\Omega = \frac{4}{\sqrt{4-3R^2}}$. Thus

$$R^2 = \frac{4}{3} - \frac{16}{3} \frac{1}{\Omega^2}.$$  

Further, we normalized our profile curve to be arclength parametrized. The round metric on $S^3$ induce a metric on $S^3/\sim$ given by

$$g_{\text{round}} = (1-r^2)(dR)^2 + \frac{R^2(1-R^2)}{4-3R^2} (d\varphi)^2.$$
Thus the arc length condition gives rise to a condition on $\varphi'$ for $\gamma = (R(t), \varphi(t))$:

$$(1 - R^2)(R')^2 + \frac{r^2}{4}(1 - R^2) \left( \varphi' \right)^2 = (4 - 3R^2).$$

Therefore

$$(4.20) \quad \varphi = \pm \int_0^t \frac{1}{R} \sqrt{\frac{1 - 3R^2}{1 - R^2} - (R')^2(4 - 3R^2)} dt. $$

The choice of initial value for $\varphi$ corresponds to an isometry of the ambient space $S^3/\sim$ and the choice of the sign corresponds to the orientation of the curve. Hence with choose without loss of generality we choose $\varphi(0) = 0$ and the "\(+\)" sign.

4.4.1. Step 1: Existence of a 2-parameter family of candidates. In order to get closed curves, the necessary condition is that $R$ is periodic. This holds automatically for the solutions $\Omega = \cos(2\theta) \kappa + \sin(2\theta) \sqrt{G}$ solving the elastic curve equation, where $P_4$ has simple roots, see [Hel2]. In other words non periodic solutions can only exist for limits of the $\varphi$–function. Moreover, the angle $\varphi$, defined in (4.20), has to satisfy

$$\varphi(L) - \varphi(0) = \frac{l}{k} 2\pi, \text{ for integers } l, k.$$

We want to show that there exist a two parameter family of closed curves deforming the Clifford torus using implicit function theorem. The constrained elastic curve equation (4.7) can be solved using elliptic functions. For real numbers, lattice invariants $g_2$ and $g_3$ given by

$$g_2 = \frac{1}{12}(\mu + \frac{G}{2})^2 + \frac{1}{4} \nu, \quad g_3 = \frac{1}{24}(\mu + \frac{G}{2})^3 + \frac{1}{16} \lambda^2 - \frac{1}{24}(\mu + \frac{G}{2}) \nu$$

define a Weierstrass $\varphi$–function. The function is holomorphic on to $\mathbb{C}P^1$ and doubly periodic with respect to the lattice $\Gamma \subset \mathbb{C}^2$ given by $g_2$ and $g_3$. By Lemma 2 of [Hel2] we have obtain that

$$\kappa^2 = -8Re(\varphi(x + x_0)) - \frac{2}{3}(\mu + \frac{G}{2}), $$

solves the elastic curve equation for the parameter given by $g_2$, $g_3$ and $(\mu + \frac{G}{2})$ with $x_0$ chosen such that $P_4(\kappa(0)) = 0$.

The homogenous tori as solutions to the elastic curve equation appear in this description as limits of the generic solutions where the lattice $\Gamma$ (on which the $\varphi$ function is defined) degenerates, i.e., when the discriminant given by $D = g_2^3 - 27g_3^2 \rightarrow 0$. Moreover, the limit solution is constant. For the Clifford torus we have $R' \rightarrow 0$ and $R \rightarrow \frac{1}{\sqrt{3}}$ we obtain $\varphi(L_\infty) - \varphi(0) = 5L_\infty$, where $L_\infty$ is the limit period of $R$, or equivalently the length of the curve in $g_{(1,2)}$ metric. The lattice $\Gamma$ can be normalized to be generated by $L \in \mathbb{R}$ and $L \cdot \tau \in \mathbb{C}$. Since $g_2$ and $g_3$ are real, the corresponding $\Gamma$ is either rectangular or rhombic. We have computed already that the constrained Willmore Hopf tori, that converge to a simply wrapped Clifford torus must be rhombic. Thus we construct only surfaces to rhombic $\Gamma$ here, i.e. $\text{Re}(\tau) = \frac{1}{2}$. We change the parameters slightly and use $\tau$, $L$ and $\mu_0 = \cos(4\theta) \mu + \sin(4\theta) \lambda$ instead of $\lambda$, $\mu$, $\nu$.

\[6\] As before we choose $G(\mu, \tau, L)$ such that the $(1,2)$–equivariant profile curve is arclength parametrized.
\( \varphi : \mathbb{R}^3 \to \mathbb{R}, (L, \tau, \mu_\theta) \mapsto \varphi(L, \tau, \mu_{\theta(L, \tau, \mu)}) \).

Since we start at a \((1, 2)\)–parametrized Clifford torus, associated to a \((1, 1)\)–parametrized homogenous torus (not Clifford) we have that \( \mu_\theta = \cos(4\theta)\mu + \sin(4\theta)\lambda \), with \( \lambda \neq 0 \) at the \((1, 2)\)–parametrized Clifford torus.

**Proposition 4.** There exist a function \( \mu_\theta(L, \tau) \) for \( L \sim \frac{2}{5} \pi \) and \( \tau \sim \infty \) such that 
\( \varphi(L, \tau, \mu_{\theta(L, \tau)}) = 2\pi. \)

**Remark 12.** This imply the existence of a \( 2 \)–dimensional family of \((1, 2)\)–equivariant constrained Willmore tori with extrinsic period \( 1 \).

**Proof.** We want to show that \( \frac{\partial \varphi(L, \tau, \mu_\theta)}{\partial \mu_\theta} \neq 0 \) at the simply wrapped Clifford torus. Then the assertion follows by implicit function theorem. At the Clifford torus we have \( R = \frac{1}{\sqrt{2}} \) and \( R' = 0 \), therefore

\[
\left. \frac{\partial \varphi(L, \tau, \mu_\theta)}{\partial \mu_\theta} \right|_{R = \frac{1}{\sqrt{2}}} = \int_0^L \frac{\partial R}{\partial \mu_\theta} \frac{4 - 5R^2}{R^2(1 - R^2)^{3/2}} \left| _{R = \frac{1}{\sqrt{2}}} \right. = 6\sqrt{2} \int_0^L \frac{\partial R}{\partial \mu_\theta} \, dt \tag{4.22}
\]

Further, we have

\[
\frac{\partial R}{\partial \mu_\theta} = \frac{5\sqrt{5}}{6} \frac{\partial \varnothing}{\partial \mu_\theta},
\]

Thus it remain to show that \( \frac{\partial \varnothing}{\partial \mu_\theta} \neq 0 \). Since

\[
\mu_\theta = \mu_{\theta(L, \tau, \mu)} = \cos(4\theta(L, \tau, \mu))\mu + \sin(4\theta(L, \tau, \mu))\lambda,
\]

we have

\[
\frac{\partial \varnothing}{\partial \mu} = \frac{\partial \varnothing}{\partial \mu_\theta} \frac{\partial \mu_\theta}{\partial \mu} = \frac{\partial \varnothing}{\partial \mu_\theta} \left( \cos(4\theta(L, \tau, \mu)) + 4 \frac{\partial \theta(L, \tau, \mu)}{\partial \mu}( - \sin(4\theta)\mu + \cos(4\theta)\lambda) \right).
\]

At the Clifford torus we have \( \lambda_\theta = 0 \). Therefore,

\[
\frac{\partial \varnothing}{\partial \mu} = \frac{\partial \varnothing}{\partial \mu_\theta} \cos(4\theta(L, \tau, \mu)).
\]

Now because

\[
\varnothing = \sin(2\theta)\kappa + \cos(2\theta)\sqrt{G}
\]

we obtain

\[
\frac{\partial \varnothing}{\partial \mu} = \sin(2\theta) \frac{\partial \kappa}{\partial \mu} + \cos(2\theta) \frac{\partial \sqrt{G(L, \tau, \mu)}}{\partial \mu} + \frac{\partial \theta}{\partial \mu} \kappa_{1,2}
\]

At the \((1, 2)\)–parametrized Clifford torus, we have \( \frac{\partial G(L, \tau, \mu)}{\partial \mu} = 0 \), and \( \theta = \frac{\pi}{4} - \arctan(1/2) \) Moreover, \( \lambda \neq 0 \), hence we can choose the sign of \( \kappa \), i.e., \( \kappa(0) < 0 \) and obtain

\[
\frac{\partial \kappa}{\partial \mu} = - \frac{2}{3\kappa(0)} > 0.
\]

From this we can deduce \( \frac{\partial \theta}{\partial \mu} < 0 \) and thus

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\[
\frac{\partial \Omega}{\partial \mu} = -\sin(2\theta) \frac{2}{3\kappa(0)} + \frac{\partial \kappa}{\partial \mu} > 0.
\]

4.4.2. Step 2: Candidates cover an open neighborhood of the Teichmüller space. The family of surfaces constructed in the previous section cover an open neighborhood of conform classes near the square conformal class. Firstly, the homogenous tori are contained in the family, by fixing \(\tau = \infty\) and vary \(L\).

The conformal class of a \((1,2)\)-equivariant surface is given by the length \(L\) of its profile curve \(\gamma\) w.r.t. the \(g_{(1,2)}\) metric and \(A = \int_0^L \Omega(s)ds\). For the homogenous tori we have by definition that \(L(\gamma(\infty, L))\) is non constant and

\[
\frac{\partial L}{\partial L(\gamma(\infty, L))} \neq 0.
\]

Thus there exist a map \(L(\tau)\) such that \(L(\gamma(\tau, L(\tau))) = \text{const}\). It remains to show that \(A(\gamma(\tau, L(\tau)))\) is non constant. If \(A(\gamma(\tau, L(\tau))) \equiv \text{const}\), then the family \(\gamma(\tau, L(\tau))\) would give rise to a family of constrained Willmore surfaces in the square conformal class deforming the Clifford torus. But by stability of the Clifford torus we get a family of surfaces with non constant Willmore energy. This is a contradiction to the fact that the surfaces are constrained Willmore.

Remark 13. Since the above constructed family of candidates cover an open neighborhood of \((0,1)\), we can parametrize it by its conformal type \((a,b) \sim (0,1)\) instead and denote it by \(f_{(a,b)}\) in the following.

4.4.3. Step 3: Candidates have the right limiting Lagrange multiplier. The candidate surfaces \(f_{(a,0)}\) constructed here lie in the associated family of constrained Willmore Hopf tori \(f_{(a,b)}^{\text{Hopf}}\). As the indices emphasizes, the surface \(f_{(0,b)}^{\text{Hopf}}\) is homogenous but not the Clifford torus. From Proposition 3 we have that the normal part of the variation \(\dot{f}_{(a,b)}^{\text{Hopf}} := \partial_a f_{(a,b)}^{\text{Hopf}}\) is given by \(\sin(c_{rs}x)\tilde{n}_{\text{Hopf}}\). The following Lemma relate the normal variation of \(f_{(a,0)}\) to the normal variation of \(f_{(a,b)}^{\text{Hopf}}\).

Lemma 5. For fixed let \(f_{(a,b)}^{\text{Hopf}}\) be a smooth 2-dimentional family of constrained Willmore cylinders (not necessarily compact) deforming the homogenous torus \(T_{r_0}\) and \(f_{(a,b)}\) be the associated \((1,2)\)-equivariant tori of conformal type \((a,b)\). Then we have

\[
< \partial_a f_{(a,b)}|_{a=0}, \tilde{n}^{1,2}_b > = \text{const} \cdot \sin(cx_\theta)\tilde{n}^{1,2}_b,
\]

where \(\tilde{n}^{m,n}_b\) is the normal vector of the homogenous torus with conformal class \((0,b)\) parametrized as a \((m,n)\)-equivariant surface with conformal Hopf differential \(q^{1,2} = q^{1,1}e^{2\theta}\) and \(x_\theta + iy_\theta = (x + iy)e^{\theta}\).

Remark 14. Note that \((a,b)\) is the conformal typ of the \((1,2)\)-equivariant tori and not the conformal typ of the associated Hopf cylinders. Nevertheless, for \(a = 0\) we have that both families of surfaces are homogenous.

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**Proof.** Since we are considering equivariant tori and variations of these preserving the equivariance type, we can restrict without loss of generality to the variation of the underlying profile curves. The conformal type of the surfaces are given by

\[ A_{m,n} = \int_{\gamma_{m,n}} \Omega_{m,n} ds_{m,n} \quad \text{and} \quad L_{m,n} = \int_{\gamma_{m,n}} ds_{m,n}. \]

Let \( 4q_{1,1}^{t,1} = \kappa_t + 2i \) be the conformal Hopf differential of the Hopf cylinders \( f_{t,b_0}^{Hopf} \) for a fixed \( b_0 \). Then the conformal Hopf differential of the associated \((1,2)-(1,2)\)-tori are given by

\[ 4q_{1,2} = (\cos(2\theta_t)\kappa_t - 2\sin(2\theta_t)) + i(\sin(2\theta_t)\kappa + 2\cos(\theta_t)) \]

Since \( b_0 = const \) and the degeneracy of homogenous tori we have \( \dot{A}_{1,2} = 0 = L_{1,2}. \) The first can also be computed to be

\[ \dot{A}_{m,n} = \int_{\gamma_{1,2}} \dot{\Omega} ds + \Omega \dot{ds} = \int_{\gamma_{1,2}} (\sin(2\theta_0)\dot{\kappa} + 2\dot{\theta}\kappa_0 + g_{1,2}(\dot{\gamma}_{1,2}, \gamma_{1,2})\kappa_0) ds = 0, \]

where \( \kappa_0 \) is a constant.

The second is given by

\[ \partial_a L_{1,2}(a, b_0) = \int g_{1,2}(\dot{\gamma}_{1,2}, \gamma') ds = -\int \kappa_0 g_{1,2}(\dot{\gamma}_{1,2}, N) = 0. \]

Since for \( a = 0 \) the family \( f^{(0,b)} \) are homogenous, the corresponding Hopf cylinders are homogenous too. Thus there exist a real numbers \( c \) and \( d \) such that normal variation \( \dot{f}_{Hopf}^{Hopf} := \partial_a f_{a,b}^{Hopf} \) of the Hopf cylinders \( f_{a,b}^{Hopf} \) are given by Theorem 3 to be

\[ <\dot{f}_{Hopf}^{Hopf}, \vec{n}_{1,1}^1> = d + \sin(cx). \]

From this we obtain that

\[ <\dot{\gamma}_{1,1}, \vec{n}_{1,1}^1>'' = c^2 <\dot{f}_{Hopf}^{Hopf}, \vec{n}_{1,1}^1> - d. \]

A computation shows that

\[ <\dot{\gamma}_{1,1}, \vec{n}_{1,1}^1>'' = \dot{\kappa} + (\kappa_0^2 + 1) <\dot{\gamma}_{1,1}, \vec{n}_{1,1}^1>, \]

hence

\[ \dot{\kappa} = (c^2 - \kappa_0^2 - 1) \sin(cx) - d. \]

For \( c^2 = \kappa_0^2 + 1 \) we obtain that the variation has extrinsic period 1 and the resulting surfaces (including the associated \((1,2)-(1,2)\)-equivariant ones) are all homogenous.

Together with \( \dot{A}_{1,2} = 0 \) this yields

\[ \dot{\Omega} = \sin(\theta)(c^2 - \kappa_0^2 - 1) \sin(c^2 a), \]

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where $x_\theta$ is the parameter of the $(1,2)$—equivariant profile curve. At the homogenous tori these are given by $x_\theta = \sqrt{4r^2 + s^2} (\cos(\theta)x - \sin(\theta)y)$.

By the construction of $(1,2)$—equivariant surfaces, see section 4.4, we have at homogenous tori

\[ \dot{R} = \text{const} \cdot \Omega = g_{1,2}(\hat{\gamma}_{1,2}, \vec{n}^{1,1}) \]

which proves the Lemma, since at $\vec{n}^{1,2}_{a=0} = \frac{\partial}{\partial R}$.

As a corollary we obtain

**Proposition 5.** A smooth family of $n$—lobed $(1,2)$—equivariant constrained Willmore tori deforming a Clifford torus with $\dot{A}_{1,2} = 0$ has normal variation given by

\[ \Phi = \sin(nx_\theta) . \]

In particular for $n = 1$ the limit $\alpha$—Lagrange multiplier is

\[ \lim_{f \to f_{Cliff}} \alpha \to \alpha^1 . \]

4.4.4. Step 4: Lagrange multiplier converge from below.

**Theorem 9.** Let $f(a, b)$ be the family of candidate surfaces such that $\int_0^{L(\tau)} \Omega(\tau, L, \mu_\theta) = \text{const}$ and $\lim_{a \to 0} f(0, b_1)$ and $b_1 \sim 1$. Then we have

\[
\frac{\partial}{\partial a} \frac{\partial \mu_\theta(a, b)}{\partial a} \bigg|_{a=0} = \frac{\partial}{\partial L} \frac{\partial \mu_\theta(\tau, L)}{\partial L} \bigg|_{a=0} < 0.
\]

**Proof.** By Section 4.4.2 we have a 2—parameter family of closed solutions parametrized by $\tau$ and $L$. We compute the derivatives at the Clifford torus, i.e., $\tau = \infty$. First we have

\[
\frac{\partial}{\partial \tau} \varphi(\tau, L, \mu_\theta(\tau, L)) = \frac{\partial}{\partial \tau} \varphi(\tau, L, \mu_\theta) + \frac{\partial}{\partial \mu_\theta} \varphi(\tau, L, \mu_\theta) \frac{\partial \mu_\theta}{\partial \tau} = 0.
\]

Since $\mu_\theta$ is bounded, we obtain that $\lim_{\tau \to \infty} \frac{\partial \mu_\theta}{\partial \tau} = 0$ and we can compute the second derivative at $\tau = \infty$ to be:

\[
(4.23) \quad \frac{\partial^2}{\partial \tau^2} \varphi(\tau, L, \mu_\theta(\tau, L)) = \frac{\partial^2}{\partial \tau^2} \varphi(\tau, L, \mu_\theta) + \frac{\partial}{\partial \mu_\theta} \varphi(\tau, L, \mu_\theta) \frac{\partial^2 \mu_\theta}{\partial \tau} = 0.
\]

Now we take the family of constrained Willmore tori $f_{(a, b)} = f(\tau, L(\tau), \mu_\theta(\tau, L))$ with $A = \int_0^{L(\tau)} \Omega = \text{const}$. For this family we can compute

\[
\frac{\partial A}{\partial \tau} = \int_0^{L(\tau)} \frac{\partial \Omega}{\partial \tau} + \left( \frac{\partial}{\partial \tau} \frac{\partial A}{\partial \tau} + \frac{\partial \mu_\theta}{\partial \tau} \frac{\partial L}{\partial \tau} \right) \int_0^{L(\tau)} \frac{\partial \Omega}{\partial \mu_\theta} + \left( \Omega(\tau, L(\tau)) + \int_0^{L(\tau)} \frac{\partial \Omega}{\partial L} \right) \frac{\partial L(\tau)}{\partial \tau} = 0.
\]

Again $L$ is bounded and therefore $\lim_{\tau \to \infty} \frac{\partial L}{\partial \tau} = 0$. Thud we obtain for the second derivative of the total curvature:

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\[
0 = \frac{\partial^2 A}{\partial^2 \tau} = \int_0^{L\infty} \frac{\partial^2 \Omega}{\partial^2 \tau} + \left( \frac{\partial^2 \mu_{\theta}}{\partial^2 \tau} + \frac{\partial \mu_{\theta}}{\partial L} \frac{\partial^2 L}{\partial^2 \tau} \right) \int_0^{L\infty} \frac{\partial \Omega}{\partial \mu_{\theta}} \\
+ \frac{\partial^2 L}{\partial^2 \tau} \left( \Omega(\tau, L(\tau)) + \int_0^{L\infty} \frac{\partial \Omega}{\partial L} \right).
\]

From Equation (4.23) we now get that at the Clifford torus
\[
\int_0^{L\infty} \frac{\partial^2 \Omega}{\partial^2 \tau} = \frac{1}{5\sqrt{10}} \frac{\partial^2 \varphi(\tau, L, \mu_{\theta})}{\partial^2 \tau} = -\frac{1}{5\sqrt{10}} \frac{\partial \varphi(\tau, L, \mu_{\theta})}{\partial \mu_{\theta}} \frac{\partial^2 \mu_{\theta}}{\partial^2 \tau} = -\frac{\partial^2 \mu_{\theta}}{\partial^2 \tau} \int_0^{L\infty} \frac{\partial \Omega}{\partial \mu_{\theta}}.
\]

Further we have \(\frac{\partial^2 L}{\partial^2 \tau} \neq 0\), since \(\Pi = (L, A)\) is the projection to the Teichmüller space and for \(A(L(\tau), \tau) = \text{const.}\)

Therefore,
\[
\frac{\partial \mu_{\theta}}{\partial L} \int_0^{L\infty} \frac{\partial \Omega}{\partial \mu_{\theta}} + \left( \Omega(\tau, L(\tau)) + \int_0^{L\infty} \frac{\partial \Omega}{\partial L} \right) = 0.
\]

Moreover, \(\Omega = \sin(2\theta)\kappa + \cos(2\theta) \sqrt{G}\), where \(\kappa\) is given by (4.21) in terms of the Weierstrass \(\varphi\)–function. The \(\varphi\)–function is homogenous in \(L\) of degree \(-2\) and \(\frac{\partial G}{\partial L} = 0 = \frac{\partial \theta}{\partial L}\) at homogenous tori, hence
\[
\frac{\partial \Omega}{\partial L} \big|_{L\infty} = \sin(2\theta) \frac{\partial \kappa}{\partial L} \big|_{L\infty} = -\sin(2\theta) \frac{1}{\kappa L\infty} (\kappa^2 + \frac{2}{3}(\mu + \frac{G}{2}))
\]
\[
\left( \Omega(\infty, L\infty) + \int_0^{L\infty} \frac{\partial \Omega}{\partial L} \big|_{L\infty} \right) > 0.
\]

In Section 4.4.2 we already computed
\[
\int_0^{L\infty} \frac{\partial \Omega}{\partial \mu_{\theta}} > 0.
\]

This yields \(\frac{\partial \mu_{\theta}}{\partial L} < 0\) as desired. \(\square\)

5. A classification of constrained Willmore tori

Before classifying all solutions to Euler-Lagrange equation with control on the Lagrange multiplier, we first show a technical lemma that allow us to use Bifurcation theory.

**Lemma 6.** Using the notations as before we obtain
\[
\delta^3 W_{\alpha^i, \beta^j}(\varphi^b, \varphi^b, \varphi^b)|_{f^b} = 0.
\]

Further, we obtain for the forth variation of the Willmore functional
\[
\delta^4 W_{\alpha^i, \beta^j}(\varphi^b, \cdots, \varphi^b)|_{f^b} + \delta^3 W_{\alpha^i, \beta^j}(\partial_{\alpha} \varphi|_{\alpha=0}, \varphi^b, \varphi^b)|_{f^b} \neq 0.
\]

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Proof. For fixed $b \sim 1$ the candidate surfaces $f_{(a,b)}$ let $\varphi := \partial_3 f_{(a,b)}$. Further, let
\[ \varphi^b = \lim_{a \to 0} \varphi. \] This implies $\delta \Pi^2_{f_{(a,b)}}(\varphi) = 0$. Let $\alpha(a,b)$ and $\beta(a,b)$ be the Lagrange multipliers of the candidate surfaces with $\alpha^b := \lim_{a \to 0} \alpha(a,b)$. The surface $f_{(a,b)}$ satisfy the Euler Lagrange equation tested with $\varphi$

\begin{equation}
\delta \mathcal{W}|_{f_{(a,b)}}(\varphi) = \alpha(a,b) \delta \Pi^1_{f_{(a,b)}}(\varphi) + \beta(a,b) \delta \Pi^2_{f_{(a,b)}}(\varphi) = \alpha(a,b) \delta \Pi^1_{f_{(a,b)}}(\varphi)
\end{equation}

Differentiate the equation once together with the Euler-Lagrange equation yields

\begin{equation}
\delta^2 \mathcal{W}|_{f_{(a,b)}}(\varphi, \varphi) = \alpha(a,b) \delta^2 \Pi^1(\varphi, \varphi) + \beta(a,b) \delta^2 \Pi^2(\varphi, \varphi) + \partial_3 \alpha(a,b) \delta \Pi^1(\varphi^b, \varphi^b) + \partial_3 \beta(a,b) \delta \Pi^2(\varphi^b, \varphi^b)
\end{equation}

For the third derivative we thus obtain:

\begin{equation}
\delta^3 \mathcal{W}(\varphi^b, \varphi^b, \varphi^b) = \alpha^b \delta^3 \Pi^1(\varphi^b, \varphi^b, \varphi^b) + \beta^b \delta^3 \Pi^1(\varphi^b, \varphi^b, \varphi^b) + 2 \lim_{a \to 0} \partial_3 \alpha(a,b) \delta^2 \Pi^1(\varphi^b, \varphi^b) + \lim_{a \to 0} \partial_3 \beta(a,b) \delta^2 \Pi^2(\varphi^b, \varphi^b)
\end{equation}

From the computations in Section 3 we have that $\delta^2 \Pi^1(\varphi^b, \varphi^b) > 0$ and for the candidates we have $\lim_{a \to 0} \partial_3 \alpha(a,b) = 0$. Moreover, we have that

\[ \partial_3 \beta^b|_{a=0} = \lim_{a \to 0} \partial_3 \alpha(a,b) \frac{\partial \omega(a,b)}{\partial b} = \lim_{a \to 0} \partial_3 \left( \sqrt{a} \frac{\partial \omega(a,b)}{\partial a} \right) = \sqrt{a} \partial_b \alpha^b = 0 \]

Therefore we obtain

\[ \delta^3 \mathcal{W}_{\alpha^b, \beta^b}(\varphi^b, \varphi^b, \varphi^b) = 0 \]

Differentiating the equation \[5.1\] three times and taking the limit for $\tilde{a} \to 0$ gives the following formula:

\begin{equation}
\delta^4 \mathcal{W}_{\alpha^b, \beta^b}(\varphi^b, \cdots, \varphi^b)|_{fb} = \delta^3 \mathcal{W}_{\alpha^b, \beta^b}(\partial_{\tilde{a}} \varphi|_{\tilde{a}=0}, \varphi^b, \varphi^b)|_{fb} = \lim_{\tilde{a} \to 0} \partial_{\tilde{a}} \alpha(a,b) \delta^2 \Pi^1(\varphi^b, \varphi^b) + \lim_{a \to 0} \partial_{\tilde{a}} \beta(a,b) \delta^2 \Pi^2(\varphi^b, \varphi^b).
\end{equation}

We have computed for the candidates that $\lim_{\tilde{a} \to 0} \partial_{\tilde{a}} \alpha(a,b) = \partial_b \alpha^b \leq 0$ and $\lim_{\tilde{a} \to 0} \partial_{\tilde{a}} \beta(a,b) = \partial_a \alpha(a,b) < 0$. Together with $\delta^2 \Pi^1(\varphi^b, \varphi^b) > 0$ we conclude that the second formula of the Proposition holds.

Theorem 10. For $b \sim 1$, $b \neq 1$, fixed there exist a $a^b > 0$ such that there exists 4 branches of solutions to the Euler-Lagrange equation

\begin{equation}
\delta \mathcal{W}_{\alpha, \beta}(f) = 0, \text{ with } \alpha \sim \alpha^b, \beta \sim \beta^b, f \sim f^b \text{ smoothly}
\end{equation}

and $\Pi^1(f) = a, \Pi^2(f) = b$ with $b \sim 1$ fixed and $a \in [0, a^b)$. In particular, for $\alpha = \alpha^b$ and $\Pi^2(f) = b$ the only solution is the homogenous torus.

Proof. We prove the theorem using Bifurcation Theory from Non Linear Analysis, more precisely bifurcation from simple eigenvalues, see [AmbPro].

We subdivide the proof into the following four steps:
(1) The splitting of the Euler-Lagrange equation (5.5) into an auxiliary and a bifurcation part
(2) Classification of all solutions to the auxiliary equation
(3) Classification of all solutions to the bifurcation equation
(4) Identification of the respective Teichmüller class.

For convenience we will work on $\mathbb{R}^3$ instead of $S^3$ in the following but we keep using the same notations as before, now meaning the stereographic projections of the respective maps into $\mathbb{R}^3$. We need to fix some further notations first. We will work on the following Sobolev space given by:

$$W^{4,2}(T^2_b, \mathbb{R}^3) := \{ V : T^2_b \to \mathbb{R}^3 | \text{each } V^i \in W^{4,2}(T^2_b, \mathbb{R}) \}$$

and $W^{4,2}(T^2_b, \mathbb{R})$ is the usual Sobolev space, namely

$$W^{4,2}(T^2_b, \mathbb{R}) := \{ V : T^2_b \to \mathbb{R} | V \text{ and its derivatives up to order } 4 \text{ are all } L^2 \text{ integrable with respect to } g_b = (f^b)^*(g_{\mathbb{R}^3}) \}.$$

Since tangential variations only lead to a reparametrization of the surface preserving $\mathcal{W}$ and $\Pi$ we can restrict ourselves to the space

$$W^{4,2,\perp}(T^2_b, \mathbb{R}^3) := \{ V \in W^{4,2}(T^2_b, \mathbb{R}) | V \perp T_{r_b} \text{ along } T_{r_b} \}.$$ 

Further, for an appropriate neighborhood $U(0)$ of $0 \in <\varphi^b, \tilde{\varphi}^b>_*, W^{4,2,\perp} := \{ \text{orthogonal complement of } W^{4,2,\perp} \text{ in } W^{4,2} \text{ topology} \}$ we consider the map

$$\Phi : U(0) \times \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R} \to L^{2,\perp}(T^2_b, \mathbb{R}^3) := L^2(T^2_b, \mathbb{R}^3) \cap W^{4,2,\perp}(T^2_b, \mathbb{R}^3),$$

given by

$$\Phi(V, \alpha, \beta, t, s) = \delta W_{\alpha,\beta}(f^b + V + t\varphi^b + s\tilde{\varphi}^b)$$

$$= \delta W(f^b + V + t\varphi^b + s\tilde{\varphi}^b) - \alpha \delta \Pi_{f^b + V + t\varphi^b + s\tilde{\varphi}^b} - \beta \delta \Pi_{f^b + V + t\varphi^b + s\tilde{\varphi}^b},$$

where $L^2(T^2_b, \mathbb{R}^3) := \{ f : T^2_b \to \mathbb{R}^3 | f^i \in L^2(T^2_b, \mathbb{R}) \}$ with $L^2(T^2_b, \mathbb{R})$ the usual $L^2$-Lebesgue space.

By [NdiSch1] the map $\Phi$ is smooth in $W^{4,2}$—topology and the solutions of (5.5) are exactly the zero locus of $\Phi$.

5.1. **Step (1).** We first observe that

$$\Phi(0, \alpha^b, \beta^b, 0, 0) = \delta W_{\alpha^b,\beta^b}(f^b) = 0$$

Thus we have

$$\partial_V \Phi(0, \alpha^b, \beta^b, 0, 0) \cdot V = \delta^2 W_{\alpha^b,\beta^b}(f^b)(V, .)$$

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and by \(\text{[NdiSch1]}\) \(\partial_V \Phi(0, \alpha^b, \beta^b, 0, 0)\) is a Fredholm operator of index 0. Since

\[
\delta^2 \mathcal{W}_{\alpha^b, \beta^b}(f^b)(V, V) = 0
\]

\[
\Leftrightarrow \quad V \in < \varphi^b, \bar{\varphi}^b > \oplus \text{Moeb}_{f^b} T^2_b \oplus T_{f^b} T^2_b
\]

we obtain with the same arguments in the proof of equation (3.20) of \(\text{[NdiSch1]}\) that

\[
(5.6) \quad \text{Ker}(\partial_V \Phi(0, \alpha^b, \beta^b, 0, 0)) = < \varphi^b, \bar{\varphi}^b > \oplus \text{Moeb}_{f^b} T^2_b \oplus T_{f^b} T^2_b.
\]

On the other hand, using the symmetry of \(\delta^2 \mathcal{W}_{\alpha^b, \beta^b}(f^b)\) and by the arguments of the proof of formula (3.21) of \(\text{[NdiSch1]}\) we get

\[
(5.7) \quad < \varphi^b, \bar{\varphi}^b > \oplus \text{Moeb}_{f^b} T^2_b \perp \text{Im}(\partial_V \Phi(0, \alpha^b, \beta^b, 0)) \quad \text{in} \quad L^{2,1}(T^2_b, \mathbb{R}^3).
\]

However, since \(\partial_V \Phi(0, \alpha^b, \beta^b, 0, 0)\) is Fredholm with index 0 we obtain by \(\text{(5.6)}\)

\[
(5.8) \quad \dim\left(\frac{L^{2,1}(T^2_b, \mathbb{R}^3)}{\text{Im}(\partial_V \Phi(0, \alpha^b, \beta^b, 0))}\right) = \dim(\text{Moeb}_{f^b} T^2_b < \varphi^b, \bar{\varphi}^b >)
\]

\[
= \dim\left(\frac{L^{2,1}(T^2_b, \mathbb{R}^3)}{(\text{Moeb}_{f^b} T^2_b < \varphi^b, \bar{\varphi}^b >)^{\perp} L^{2,1}(T^2_b, \mathbb{R}^3)}\right).
\]

Together with property \(\text{(5.7)}\) this yields

\[
\text{Im}(\partial_V \Phi(0, \alpha^b, \beta^b, 0, 0)) = (\text{Moeb}_{f^b} T^2_b < \varphi^b, \bar{\varphi}^b >)^{\perp} L^{2,1}(T^2_b, \mathbb{R}^3).
\]

Let

\[
Y := (\text{Moeb}_{f^b} T^2_b < \varphi^b, \bar{\varphi}^b >)^{\perp} L^{2,1}(T^2_b, \mathbb{R}^3).
\]

Since \(\text{Moeb}_{f^b} T^2_b < \varphi^b, \bar{\varphi}^b >\) is finite dimensional we obtain

\[
L^2(T^2_b, \mathbb{R}^3)^{\perp} = Y \oplus \text{Moeb}_{f^b} T^2_b < \varphi^b, \bar{\varphi}^b >,
\]

and thus

\[
L^2(T^2_b, \mathbb{R}^3) = Y \oplus \text{Moeb}_{f^b} T^2_b < \varphi^b, \bar{\varphi}^b > \oplus T_{f^b} T^2_b.
\]

The above splitting still holds (though not as orthogonal decomposition) for

\[
V \in U(0) \subset W^{4,2,1}(T^2_b, \mathbb{R}^3) \subset C^1(T^2_b, \mathbb{R}^3)
\]

small and \(t, s\) small by proposition B.3 of \(\text{[NdiSch1]}\), i.e.

\[
(5.9) \quad L^2(T^2_b, \mathbb{R}^3) = Y \oplus \text{Moeb}_{f^b + V + t\varphi^b} T^2_b < \varphi^b, \bar{\varphi}^b > \oplus T_{f^b + V + t\varphi^b} T^2_b.
\]

On the other hand, since \(\text{Moeb}_{f^b + V + t\varphi^b} T^2_b < \varphi^b, \bar{\varphi}^b >\) is finite dimensional we obtain for

\[
X := (\text{Moeb}_{f^b} T^2_b < \varphi^b, \bar{\varphi}^b >)^{\perp} W^{4,2,1}(T^2_b, \mathbb{R}^3) \subset W^{4,2,1}(T^2_b, \mathbb{R}^3)
\]

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an analogous splitting for $W^{4,2}$, i.e.,
\[ X \oplus \text{Mob}_b T_b^2 \oplus \varphi^b, \bar{\varphi}^b = W^{4,2,\perp}(T_b^2, \mathbb{R}^3). \]

To continue we first define the following projection maps:
\[
\begin{align*}
\Pi_Y : L^2(T_b^2, \mathbb{R}^3)^\perp &\to Y, \\
\Pi_{\text{Mob}_b T_b^2 \oplus \varphi^b, \bar{\varphi}^b} &\to \text{Mob}_b T_b^2 \oplus \varphi^b, \bar{\varphi}^b, \\
\Pi_X &\to W^{4,2,\perp}(T_b^2, \mathbb{R}^3) \to X.
\end{align*}
\]

(5.10)

This splitting (5.9) ensures that we can decompose the equation $\Phi = 0$ close to $(0, \alpha^b, \beta^b, 0, 0)$ into two equations which we solve successively in the following:

\[
\begin{align*}
\Pi_Y \Phi = 0 \\
\Pi_{\text{Mob}_b T_b^2 \oplus \varphi^b, \bar{\varphi}^b} \Phi = 0.
\end{align*}
\]

(5.11)

In the language of Bifurcation theory the first equation is called the auxiliary equation and the second the bifurcation equation. We deal with the auxiliary equation first.

5.2. Step (2). For
\[ \Psi := \Pi_Y \circ \Phi : U(0) \times \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R} \to Y \]
we have that
\[ \partial_V \Psi(0, \alpha^b, \beta^b, 0, 0)|_X = \Pi_Y \circ \partial_Y \Phi(0, \alpha^b, \beta^b, 0)|_X. \]
Thus the map
\[ \partial_V \Psi(0, \alpha^b, \beta^b, 0, 0) : X \to Y \]
is an isomorphism and by the implicit function theorem there exist $\epsilon_i > 0, i = 1, 2, 3,$ an open neighborhood $U_M(0) \subset \text{Mob}_b T_b^2$ and a smooth function
\[ \tilde{V} : U_M(0) \times [-\epsilon_1, \alpha^b], \alpha^b + \epsilon_1, \beta^b + \epsilon_2, \beta^b + \epsilon_2 - \epsilon_3, \beta^b + \epsilon_2 - \epsilon_3, \beta^b + \epsilon_2 - \epsilon_3] \to U(0) \cap X \subset W^{4,2,\perp}(T_b^2, \mathbb{R}^3) \]
such that $V(m, \alpha, \beta, t, s) = m + \tilde{V}(m, \alpha, \beta, t, s)$ satisfies
\[ \Psi(V(m, \alpha, \beta, t, s), \alpha, \beta, t, s) = 0 \]
for all $(m, \alpha, \beta, t, s) \in U_M(0) \times [-\epsilon_1, \alpha^b], \alpha^b + \epsilon_1, \beta^b + \epsilon_2, \beta^b + \epsilon_2 - \epsilon_3, \beta^b + \epsilon_2 - \epsilon_3, \beta^b + \epsilon_2 - \epsilon_3 - \epsilon_4, \beta^b + \epsilon_2 - \epsilon_3 - \epsilon_4].$

Further, these are the only solutions to $\Psi(V, \alpha, \beta, t, s) = 0$ with $V \in W^{4,2,\perp}(T_b^2, \mathbb{R}^3)$ close to 0 in the $W^{4,2,\perp}$-topology and $\alpha \sim \alpha^b, \beta \sim \beta^b,$ and $t, s \sim 0.$

By the definition of $\Psi$ we have classified all solutions of
\[
\Pi_Y (\delta W_{\alpha, \beta}(f^b + V + t \varphi^b + s \bar{\varphi}^b)) = \Pi_Y (\Phi(V, \alpha, \beta, t, s)) = 0
\]
with $V \in W^{4,2,\perp}(T_b^2, \mathbb{R}^3)$ close to 0, $\alpha \sim \alpha^b, \beta \sim \beta^b,$ and $t, s \sim 0.$

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5.3. Step (3). We now turn to the bifurcation equation
\[ \Pi_{\text{Mob}_{b^+} T_b^2 \oplus <\varphi^+,\varphi^b>} \Phi(V, \alpha, \beta, t, s) = 0, \]
which we split up to
\[ (5.13) \]
\[ \left\{ \begin{array}{l}
\Pi_{\text{Mob}_{b^+} T_b^2} \Phi(V, \alpha, \beta, t, s) = 0 \\
\Pi_{<\varphi^+,\varphi^b>} \Phi(V, \alpha, \beta, t, s) = 0.
\end{array} \right. \]

The first equation has already been dealt with in [NdSiSch1] (proposition B.2 and equation B.7). The Möbius invariance of \( \mathcal{W} \) and \( \Pi \) implies that every solution of \( (5.12) \) already solves the equation
\[ \Pi_{\text{Mob}_{b^+} T_b^2} \Phi(V, \alpha, \beta, t, s) = 0, \]
for \( V \in W^{4,2,1}(T_b^2, \mathbb{R}^3) \) close to 0 and \( \alpha \sim \alpha^b, \beta \sim \beta^b, t, s \sim 0. \) By Proposition 1 there exist a Möbius transformation (isometry of \( S^3 \)) \( M \) and a reparametrization \( \sigma \) of \( T_b^2 \) such that \( < M \circ \Phi(V, \alpha, \beta, t, s) \circ \sigma, M \varphi^b \circ \sigma > = 0. \) Thus we can restrict without loss of generality to the equation
\[ \Pi_{<\varphi^+,\varphi^b>} \Phi(V, \alpha, \beta, t, 0) = \Pi_{<\varphi^+>} \Phi(V, \alpha, \beta, t, 0) = 0. \]

Now the situation is very similar to the situation of bifurcation from simple eigenvalues. To abbreviate the notations let \( \Phi(V, \alpha, \beta, t) := \Phi(V, \alpha, \beta, t, 0) \) and \( V(m, \alpha, \beta, t, 0) := V(m, \alpha, \beta, t) \). We have derived that there exist a smooth function \( V \) satisfying
\[ \Pi_{\text{Mob}_{b^+} T_b^2} \Phi(V(m, \alpha, \beta, t), \alpha, \beta, t) = 0 \]
for all \( (m, \alpha, \beta, t) \in U_M(0) \times \left[ -\epsilon_1 + \alpha^b, \alpha^b + \epsilon_1 [x] - \epsilon_2 + \beta^b, \beta^b + \epsilon_2 [x] - \epsilon_3, \epsilon_3 [x] \right. \].

It remains to solve
\[ \Pi_{<\varphi^+>} \Phi(V(m, \alpha, \beta, t), \alpha, \beta, t) = 0, \]
or equivalently
\[ \Phi(V(m, \alpha, \beta, t), \alpha, \beta, t) \cdot \varphi^b = 0, \]
for \( (m, \alpha, \beta, t) \in U_M(0) \times \left[ -\epsilon_1 + \alpha^b, \alpha^b + \epsilon_1 [x] - \epsilon_2 + \beta^b, \beta^b + \epsilon_2 [x] - \epsilon_3, \epsilon_3 [x] \right. \].

We first observe that
\[
\Phi(0, \alpha^b, \beta^b, 0) \cdot \varphi^b = \delta \mathcal{W}_{\alpha^b, \beta^b}(f^b)(\varphi^b) = 0
\]
\[
\partial_t \Phi(0, \alpha^b, \beta^b, 0) \cdot \varphi^b = \delta^2 \mathcal{W}_{\alpha^b, \beta^b}(f^b)(\dot{\varphi}, \varphi^b) = 0
\]
\[
\partial_{tt} \Phi(0, \alpha^b, \beta^b, 0) \cdot \varphi^b = \delta^3 \mathcal{W}_{\alpha^b, \beta^b}(f^b)(\ddot{\varphi}, \dot{\varphi}, \varphi^b) = 0
\]
\[
\partial_{ttt} \Phi(0, \alpha^b, \beta^b, 0) \cdot \varphi^b = \delta^4 \mathcal{W}_{\alpha^b, \beta^b}(f^b)(\dddot{\varphi}, \ddot{\varphi}, \dot{\varphi}, \varphi^b) + \delta^3 \mathcal{W}_{\alpha^b, \beta^b}(f^b)(\dddot{\varphi}, \ddot{\varphi}, \dot{\varphi}, \varphi^b).
\]

Lemma 7. With the notations above we have for \( b \sim 1 \)
\[ \partial_{ttt} \Phi(0, \alpha^b, \beta^b, 0) \cdot \varphi^b \neq 0. \]
Proof. Let 
\[ f(t) := f^b + t\varphi^b + V(m, \alpha, \beta, t) \]
and \( \dot{f} := \frac{\partial f(t)}{\partial t}|_{t=0} = \varphi^b + \dot{V}(m, \alpha^b, \beta^b, 0) \). Since \( V \in X \) we have that also \( \dot{V}(m, \alpha^b, \beta^b, 0) \in X \). Further, \( f(t) \) solves the constrained Willmore equation on \( X \) from which we obtain

\[ \delta^2 W_{\alpha^b, \beta^b}(f^b)(\dot{V}, V_0) = 0 \text{ for all } V_0 \in X. \]

Because Moeb \( \oplus < \varphi^b > X^\perp \in \text{Ker } \delta^2 W_{\alpha^b, \beta^b} \) we obtain that \( \dot{V} \in \text{Ker } \delta^2 W_{\alpha^b, \beta^b} \).

Thus \( \dot{V} \in X \cap X^\perp \) and we get \( \dot{V} = 0 \) and \( \dot{f} = \varphi^b \).

For the second derivative we have that the candidates constructed in Section 4 are \( W^{4,2} \) close to the homogenous torus and thus it there are maps \( t(a, b) \) and \( V(\alpha(a, b), \beta(a, b), t(a, b)) \) such that the candidate surfaces (or rather a suitable M"obius transformation of them) have the following representation:

\[ f(a, b) = f^b + t(a, b)\varphi^b + V(m(a, b), \alpha(a, b), \beta(a, b), t(a, b)) \]

Since \( \partial_{\tilde{a}} f(a, b)|_{\tilde{a}=0} = \varphi^b \), we have that
\[ \partial_{\tilde{a}} t(a, b)|_{\tilde{a}=0} = 1 \quad \text{and} \quad \partial_{\tilde{a}} V|_{\tilde{a}=0} = (\partial_{\tilde{a}} \alpha(a, b) \partial_\alpha V + \partial_{\tilde{a}} \beta(a, b) \partial_\beta V + \partial_{\tilde{a}} t(a, b) \partial_t V)|_{\tilde{a}=0} = 0. \]

Similar arguments as for \( \dot{V} \) shows that
\[ \delta^2 W_{\alpha^1, \beta^1}(f^1)(\partial_\alpha V(\alpha^1, \beta^1, 0), \cdot) = \delta \Pi^1(\cdot) = 0 \]

from which we obtain that \( \partial_\alpha V|_{\tilde{a}=0} = 0 \). Further,
\[ \partial_{\tilde{a}}^2 \beta(a, b)|_{\tilde{a}=0} = \partial_\alpha \beta(a, b)|_{\alpha=0} = \partial_\beta \alpha^b|_{\beta=1} = 0. \]

The last equality is due to the fact that \( \alpha^b = \alpha^1 \). Moreover, we have already computed that \( \partial_{\tilde{a}} \alpha(a, b) = \partial_{\tilde{a}} \beta(a, b) = 0 \). For the second derivative we thus obtain
\[ \lim_{b \to 1} \partial_{\tilde{a}}^2 f(a, b)|_{\tilde{a}=0} = \partial_{\tilde{a}}^2 t(a, b)|_{\tilde{a}=0} \varphi^1 + (\partial_{\tilde{a}} t(a, b))^2 \dot{V}(\alpha^b, \beta^b, 0)|_{\tilde{a}=0}. \]

By Proposition 6 we thus obtain
\[ \lim_{b \to 1} \partial_{\tilde{a}}^2 (\Phi(0, \alpha^b, \beta^b, t) \cdot \varphi^b \neq 0, \]

and by continuity we get that this remains true for \( b \sim 1 \).

Now we can use classical arguments in bifurcation theory (bifurcation from simple eigenvalues), we get unique function \( t(m, \alpha, \beta) \) around \( t = 0 \) with

\[ \Phi(V(m, \alpha, \beta, t(m, \alpha, \beta)), \alpha, \beta, t(m, \alpha, \beta)) \cdot \varphi^b = 0 \]

and all solutions to

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\[ \Phi(V(m, \alpha, \beta, t), \alpha, \beta, t) \cdot \varphi^b = 0 \]

for \((m, \alpha, \beta, t) \in U_M(0) \times [-\epsilon_1 + \alpha^b, \alpha^b + \epsilon_1[\cdot] - \epsilon_2 + \beta^b, \beta^b + \epsilon_2[\cdot] - \epsilon_3, \epsilon_3] \) are of this form for sufficiently small \(\epsilon_i\) and \(U_M(0)\).

In other words,
\[ f^m_{\alpha, \beta} := f^b + V(m, \alpha, \beta, t) + t\varphi^b \]

are the only solutions to
\[ \delta W_{\alpha, \beta}(f) = 0 \quad \text{with} \quad f = f^b + W^{4,2}(T^\circ_b, \mathbb{R}^3) \]

which are \(W^{4,2}\)-close to \(f^b\) for \(< f, \varphi^b >= 0, \alpha \sim \alpha^b\) and \(\beta \sim \beta^b\). For fixed \((\alpha, \beta)\) we thus obtain two manifolds worth of solutions, each of them of dimension \(\dim(Moeb_{f^b} T^\circ_b) + 1\).

Since \(W\) and \(\Pi\) is Möbius and parametrization invariant, we get for any Möbius transformation \(M\) with
\[ M \circ (f^b + V(m, \alpha, \beta, t(m, \alpha, \beta)) + t(m, \alpha, \beta)\varphi^b) \subset \mathbb{R}^3 \]

and every
\[ \sigma \in Diff = Diff_{Tr^b} := \{ \psi : T^\circ_b \to T^\circ_b | \psi \text{ is a smooth diffeomorphism} \} \]

that the following equation holds
\[ \delta W_{\alpha, \beta}(M \circ (f^b + V(m, \alpha, \beta, t(m, \alpha, \beta)) + t(m, \alpha, \beta)\varphi^b) \circ \sigma) = 0. \]

The Möbius group \(Moeb(3)\) of \(S^3\) is a finite dimensional Lie group and for an appropriate neighborhood \(U(Id) \subset Moeb(3)\) and \((\alpha, \beta) \in [-\epsilon_1 + \alpha^b, \alpha^b + \epsilon_1[\cdot] - \epsilon_2 + \alpha^b, \alpha^b + \epsilon_2[\cdot] \) we have

\[ M \circ (f^b + V(m, \alpha, \beta, t(m, \alpha, \beta)) + t(m, \alpha, \beta)\varphi^b) \]

is \(C^1\)-close to \(f^b\) and hence we can write
\[ M \circ (f^b + V(m, \alpha, \beta, t(m, \alpha, \beta)) + t(m, \alpha, \beta)\varphi^b) \circ \sigma^\pm = f^b + W \]

for an appropriate \(W \in W^{4,2}(T^\circ_b, \mathbb{R}^3)\) and \(\sigma \in Diff\). More precisely, for the nearest point projection
\[ \Pi_{Tr^b} : U_\delta := \{ x \in \mathbb{R}^3 | dist(x, Tr^b) < \delta \} \to Tr^b \]

for an appropriate small positive \(\delta\), we have
\[ \sigma := \sigma(\phi, \alpha, \beta) = (f^b)^{-1} \circ \Pi_{Tr^b} \circ (M \circ (f^b + V(m, \alpha, \beta, t(m, \alpha, \beta)) + t(m, \alpha, \beta)\varphi^b)) \]

and

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\[ W = W(M, \alpha, \beta) = M \circ (f^b + V(m, \alpha, \beta, t(m, \alpha, \beta)) + t(m, \alpha, \beta)\varphi^b) \circ \sigma - f^b. \]

Now since \( f_{\alpha, \beta}^m \) are the only solutions to (5.14) in \( f^b + W^{4,2}\perp(T_0^2, \mathbb{R}^3) \) which are \( W^{4,2} - \)close to \( f^b \) we get

\[ W(M, \alpha, \beta) = V(m, \alpha, \beta, t(m, \alpha, \beta)) + t(m, \alpha, \beta)\varphi^b \]

for some \( m \in U_M(0) \subset \text{Moeb}_{f^b}(T_0^2) \). More precisely we have

\[ m := m(M, \alpha, \beta) = \Pi_{\text{Moeb}_{f^b}T_0^2}W(M, \alpha, \beta). \]

Since \( \tilde{V} \) is a smooth map into \( W^{4,2}(T_0^2, \mathbb{R}^3) \subset C^2(T_0^2, \mathbb{R}^3) \) we obtain that the maps

\[ (M, \alpha, \beta) \mapsto \sigma, W, m \]

are continuously differentiable into \( C^1(T_0^2, \mathbb{R}^3) \).

For \( \chi \in T_{Id, \text{Moeb}} \) we obtain

\[ \partial_\phi W(Id, \alpha^b, \beta^b) = (\chi \circ f^b) + df^b \cdot (d\psi(Id, \alpha^b, \beta^b) \cdot \chi) = P_{\chi \circ f^b}(\chi \circ f^b) \in \text{Moeb}_{f^b}T_0^2 \]

and

\[ \partial_\phi m(Id, \alpha^b, \beta^b) \cdot \chi = \Pi_{\text{Moeb}_{f^b}T_0^2}(P_{\chi \circ f^b}(\chi \circ f^b)) = P_{\chi \circ f^b}(\chi \circ f^b). \]

By definition of \( \text{Moeb}_{f^b}T_0^2 \) we thus obtain that \( \partial_\phi m(Id, \alpha^b, \beta^b) : T_{Id, \text{Moeb}} \rightarrow \text{Moeb}_{f^b}T_0^2 \) is surjective and hence by implicit function theorem and \( m(Id, \alpha^b, \beta^b) = 0 \) we have

\[ \tilde{U}_M(0) \subset m(U(0) \times \{(\alpha, \beta)\}) \]

for some open neighborhood \( \tilde{U}_M(0) \) of \( 0 \) in \( \text{Moeb}_{f^b}T_0^2 \) independent of \( (\alpha, \beta) \). Therefore we have that

\[ (5.15) \quad M \circ (f^b + V(m, \alpha, \beta, t(m, \alpha, \beta)) + t(m, \alpha, \beta)\varphi^b) \circ \sigma \]

are the only solutions to (5.14) which are \( W^{4,2} - \)close to \( f^b \).

5.4. **Step (4)**. The aim is to identify the Teichmüller class of the solutions of (5.14) given by (5.15). In particular, we show that the solutions of (5.14) induces a local diffeomorphism between the space of Lagrange multipliers (around \((\alpha^b, \beta^b)\)) to the Teichmüller space of tori around the class of the Clifford torus \((0, 1) \in \mathbb{H}^2\). Clearly, by setting

\[ V_b = V(\alpha, \beta, t(0, \alpha, \beta)) + t(0, \alpha, \beta)\varphi^b \]

we have

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\(\Pi \left( \left( M \circ (f^b + V(\alpha, \beta)) \circ \sigma \right) \right)^* g_{\mathbb{R}^3} = \Pi \left( (f^b + V(\alpha, \beta)) \right)^* g_{\mathbb{R}^3}.\)

Thus for all solutions of (5.14) we have that

\[
\begin{pmatrix}
    a(\alpha, \beta) \\
    c(\alpha, \beta)
\end{pmatrix} = \begin{pmatrix}
    \Pi^1 \left( (f^b + V(\alpha, \beta)) \right)^* g_{\mathbb{R}^3} \\
    \Pi^2 \left( (f^b + V(\alpha, \beta)) \right)^* g_{\mathbb{R}^3}
\end{pmatrix}
\]

\[
\begin{pmatrix}
    \Pi^1 \left( (f^b + V(m, \alpha, \beta)) \right)^* g_{\mathbb{R}^3} \\
    \Pi^2 \left( (f^b + V(m, \alpha, \beta)) \right)^* g_{\mathbb{R}^3}
\end{pmatrix}
\]

independently of \(m \in \tilde{U}_{M}(0)\). We first solve for \(\Pi^2\), i.e., want to solve the equation

\[c(\alpha, \beta) = c \quad \text{for} \quad c \sim b.\]

By definition we have

\[c(\alpha^b, \beta^b) = b\]

and further

\[\partial_\beta c(\alpha^b, \beta^b) = \delta \Pi^2(f^b)(\partial_\beta |_{\beta = \beta^b} V_b(\alpha^b, \beta)).\]

Then from

\[\Phi(V(\alpha, \beta), \alpha, \beta, t(0, \alpha, \beta)) = 0\]

with \(V(\alpha, \beta) := \tilde{V}(0, \alpha, \beta, t(0, \alpha, \beta))\) and

\[\partial_{V} \Phi(0, \alpha^b, \beta^b, 0) \cdot V = \delta^2 W_{\alpha^b, \beta^b}(V, .)\]

we derive that

\[\partial_{V} \Phi(0, \alpha^b, \beta^b, 0) \cdot \partial_\beta V_b(\alpha^b, \beta^b) + \partial_\beta \Phi(0, \alpha^b, \beta^b, 0) + \partial_t \Phi(0, \alpha^b, \beta^b, 0) \cdot \partial_\beta t(0, \alpha^b, \beta^b) = 0.\]

Thus we get

\[(5.16)\]

\[\delta^2 W_{\alpha^b, \beta^b}(f^b)(\partial_\beta V(\alpha^b, \beta^b), .) - \delta \Pi^2(f^b) + \delta^2 W_{\alpha^b, \beta^b}(f^b)(\varphi^b, .) \cdot (\partial_\beta t(0, \alpha^b, \beta^b) \varphi^b) = 0\]

\[\Leftrightarrow \quad \delta^2 W_{\alpha^b, \beta^b}(f^b)(\partial_\beta V(\alpha^b, \beta^b), .) = \delta \Pi^2(f^b).\]

On the other hand, there exist a \(V_0^b \in C^\infty(T^2_b, \mathbb{R}^3)\) such that \(\delta \Pi^2_{f^b}(V_0^b) \neq 0\) by proposition 3.2. of [NdiSch1]. This implies

\[\delta^2 W_{\alpha^b, \beta^b}(f^b)(\partial_\beta V(\alpha^b, \beta^b), V_0^b) \neq 0\]

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and therefore $\partial_\beta V(\alpha^b, \beta^b) \notin \text{Moeb}_{f^b} T^2_0 \oplus \langle \varphi^b \rangle$ and

$$\delta^2 \mathcal{W}_{\alpha^b, \beta^b}(f^b) (\partial_\beta V(\alpha^b, \beta^b), \partial_\beta V(\alpha^b, \beta^b)) > 0.$$ 

Hence by implicit function theorem we have for $c \sim b$ a unique $\beta(\alpha, c) \sim \beta^b$ such that

$$c(\alpha, \beta(\alpha, c)) = b \quad \text{and} \quad \beta(\alpha, b) = \beta^b$$

and the map $(\alpha, c) \to \beta(\alpha, c)$ is smooth. In particular, if $\Pi^2(f) = b$ we obtain $\beta_f = \beta^b$. It remains to determine $\Pi^1$ of the solutions of (5.14) given in (5.15). The equation is

$$a(\alpha(c, \beta), \beta) = a \quad \text{with} \quad a \sim 0.$$ 

Again we have

(5.17)

$$a(\alpha^b, \beta(\alpha^b, b)) = 0$$

$$\partial_\alpha |_{\alpha = \alpha^b} [a(\alpha, \beta(\alpha, b))] = \delta \Pi^1_{f^b} \cdot (\partial_\alpha |_{\alpha = \alpha^b} [V_b(\alpha, \beta(\alpha, b))]) = 0$$

$$\partial^2_\alpha a(\alpha, \beta(\alpha, b)) = \delta^2 \Pi^1_{f^b} (\partial_\alpha |_{\alpha = \alpha^b} [V_b(\alpha, \beta(\alpha, b))], \partial_\alpha |_{\alpha = \alpha^b} [V_b(\alpha, \beta(\alpha, b))]).$$

Now, using the fact that

$$\Phi(V(\alpha, \beta(\alpha, c), \alpha, \beta, t(0, \alpha, \beta(\alpha, c)))) = 0$$

we get

(5.18)

$$\partial_\alpha \Phi(0, \alpha^b, \beta^b, 0) + \partial_{V^b} \Phi(0, \alpha^b, \beta^b, 0) \cdot (\partial_\alpha |_{\alpha = \alpha^b} [V_b(\alpha, \beta(\alpha, b))])$$

$$+ \partial_t \Phi(0, \alpha^b, \beta^b, 0) \cdot (\partial_\alpha |_{\alpha = \alpha^b} [t(0, \alpha, \beta(\alpha, b))] \varphi^b) = 0.$$ 

Thus we obtain (using $V^\pm_b(\alpha, \beta(\alpha, b)) = V^\pm(\alpha, \beta(\alpha, b)) + t^\pm(0, \alpha, \beta(\alpha, b)) \varphi^b$)

$$-\delta \Pi^1_{f^b} + \delta^2 \mathcal{W}_{\alpha, \beta}(f^b) (\partial_\alpha |_{\alpha = \alpha^b} [V_b^\pm(\alpha, \beta(\alpha, b))], .) = 0$$

and therefore, since $\delta \Pi^1_{f^b} = 0$, we have

(5.19)

$$\delta^2 \mathcal{W}_{\alpha, \beta}(f^b) (\partial_\alpha |_{\alpha = \alpha^b} [V_b^\pm(\alpha, \beta(\alpha, b))], .) = 0,$$

which means that

$$\partial_\alpha |_{\alpha = \alpha^b} [V_b(\alpha, \beta(\alpha, b))] \in \text{Moeb}_{f^b} T^2_0 \oplus \langle \varphi^b \rangle,$$

i.e.,

$$\partial_\alpha |_{\alpha = \alpha^b} [V(\alpha, \beta(\alpha, b))] \in \text{Moeb}_{f^b} T^2.$$ 

Therefore, we get by (5.19)

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\begin{align}
\alpha^b \delta^2 \Pi^1(\partial_{\alpha}|_{\alpha=\alpha^b}[V_b(\alpha,\beta(\alpha, b))] , \partial_{\alpha}|_{\alpha=\alpha^b}[V_b(\alpha,\beta(\alpha, b))] ) \\
= \delta^2 W_{\phi^b}(f^b) (\varphi^b, \varphi^b)(\partial_{\alpha}|_{\alpha=\alpha^b}[t(0,\alpha,\beta(\alpha, b))] ) \neq 0
\end{align}

from which we can derive

$$\partial_{\alpha}^2 [a(\alpha, \beta(\alpha, b))]|_{\alpha=\alpha^b} \neq 0, \quad \text{(Recall } \alpha^b \neq 0).$$

Hence as above, using classical arguments in bifurcation theory we have that there exist two smooth branches of unique solutions \(\alpha^\pm(a, c)\) such that

$$a(\alpha^\pm(a, c), \beta(\alpha(a, c), c)) = a$$

with \(\alpha^+(0, c) = \alpha^-(0, c) = \alpha^b\). Altogether we obtain two families of smooth solutions to (up to Möbius transformations)

$$\delta W_{\alpha, \beta} = 0 \quad \alpha \sim \alpha^b \text{ and } \beta \sim \beta^b$$

such that the only solution with \(\alpha = \alpha^b\) and \(\Pi^2(f) = b\) is the homogenous torus of conformal class \((0, b)\).

\[\square\]

6. Reduction of the global situation to a local one

First, we use relaxation techniques of calculus of variations to establish Theorem 11 providing the existence of appropriate global minimizers in an open neighborhood of the square class. By appropriate global minimizer we mean those reducing our clearly global problem to a local problem, i.e., which are close to the Clifford torus in \(W^{4,2}\) with prescribed behavior of its Lagrange multipliers. Then Theorem 10 then shows that these abstract minimizers coincides with the candidate surfaces.

**Theorem 11.** For every \(b \sim 1\) there exist a \(a^b\) small with the property that for all \(a \in [0, a^b]\) the infimum of Willmore energy

\[\inf_{(a, b)} \{ W_{\alpha^b}(f) | f : T^2_b \to S^3 \text{ smooth immersion } |0 \leq |\Pi^1(f)| \leq a \text{ and } \Pi^2(f) = b \} \]

is attained by a smooth immersion \(f^{(a, b)} : T^2_b \to S^3\) of conformal type \((a, b)\) and verifying

$$\delta W_{\alpha^{(a, b)}, \beta^{(a, b)}}(f^{(a, b)}) = 0$$

with \(\alpha^{(a, b)} \to \alpha^b\) almost everywhere as \(a \to 0\) and \(\beta^{(a, b)} \to \beta^b\), where \((\alpha^b, \beta^b) \in \mathbb{R}^2\) as defined in Theorem 10.

**Proof.** By taking \(b \sim 1\) close enough, we have that (by the same arguments as in [NdiSch1], existence part) there exists \(a^b > 0\) small with the property that for all \(a \in [0, a^b]\) the minimization problem

\[\inf_{(a, b)} \{ W_{\alpha^b}(f) | f : T^2_b \to S^3 \text{ smooth immersion } |0 \leq |\Pi^1(f)| \leq a \text{ and } \Pi^2(f) = b \} \]

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is attained by a smooth immersion \( f^{(\tilde{a},b)}_a \) with conformal type \((\tilde{a}, b)\) and \( \tilde{a} \in [0, a] \) solving the Euler-Lagrange equation for (conformally) constrained Willmore tori

\[
\delta \mathcal{W}_{\alpha^{(\tilde{a},b)}, \beta^{(\tilde{a},b)}}(f^{(\tilde{a},b)}_a) = 0
\]

for some \( \alpha^{(\tilde{a},b)}, \beta^{(\tilde{a},b)} \in \mathbb{R} \).

**Step (1):** \( \tilde{a} = a \). For \( a > 0 \) the candidate surfaces \( f^{(a,b)}_a \) constructed in section 4 satisfy that \( \mathcal{W}_{\alpha^{(a,b)}, \beta^{(a,b)}}(f^{(a,b)}_a) \) is strictly decreasing for \( a \sim 0 \), since

\[
\frac{\partial \omega^{(a,b)}}{\partial a} = \alpha^{(a,b)} - \alpha^{b} < 0.
\]

This yields for all \( a > 0 \) we obtain \( \tilde{a} > 0 \).

Now, we claim that up to take \( a^b \) smaller we have \( \forall a \in [0, a^b] \) we have \( \tilde{a} = a \). Assume this is not true. Then there would exist a sequence \( a_n \to 0 \) with corresponding \( \tilde{a}_n \to 0 \) such that

\[
\alpha^{b}_n := \alpha^{a_n,b}, \ \forall n.
\]

Then arguing as in [NdiSch1] we have up invariance that

\[
f^b_n := f^{a_n,b}_{\tilde{a}_n} \to f^b \text{ smoothly.}
\]

Hence we obtain a contradiction to our classification, because \( \tilde{a}_n \neq 0 \).

**Step (2):** \( \frac{\partial}{\partial a} \omega(a) = \alpha^{(a,b)} \leq \alpha^b \text{ a.e.} \). The aim is now to show the first statement (1) of Lemma 1 with minimum regularity assumptions of the dependence of \( f^{(a,b)} \) on its conformal class, i.e., to relate \( \frac{\partial \omega(a)}{\partial a} \) with \( \alpha(a,b) \).

The minimizers \( f^{(a,b)} \) are non-degenerate up to take \( a^b \) smaller. Thus there exist a variation \( V^{(a,b)} = \Pi^b (\frac{\partial}{\partial a}) \) and \( f^{(a,b)} \) is a lift of \( (a, b) \) with respect to \( \Pi \). Hence the Euler-Lagrange equation tested with \( V \) gives

\[
(6.1) \quad \frac{\partial}{\partial a} \omega(a) = \delta \mathcal{W}_{f^{(a,b)}}(V) = \alpha(a,b) \quad \text{where } \omega(a) \text{ is differentiable.}
\]

For \( a \in [0, a^b) \) and \( b \sim 1 \) fixed let

\[
\varphi(a) := \mathcal{W}_{\alpha^{(a,b)}}(f^{(a,b)})
\]

Since the minimum is always attained at the boundary \( \tilde{a} = a \), we have that \( \varphi \) is monotone non-increasing and hence we have:

1. \( \frac{\partial}{\partial a} \varphi(a) \) exists almost everywhere in \([0, a^b)\),
2. \( \frac{\partial}{\partial a} \varphi(a) \leq 0 \) almost everywhere in \([0, a^b]\),

Moreover, let

\[
\omega(a) := \mathcal{W}(f^{(a,b)})
\]

and have

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\[ \varphi(a) = \omega(a) - \alpha^b a. \]

Thus the properties of \( \varphi \) listed above imply

1. \( \frac{\partial}{\partial a} \omega(a) \) exists almost everywhere in \( [0, a^b) \),
2. \( \frac{\partial}{\partial a} \omega(a) \leq \alpha^b \) almost everywhere in \( [0, a^b) \).

From which we obtain \( \alpha(a, b) \leq \alpha^b \) almost everywhere.

**Step (3):** \( \lim_{a \to 0} \alpha(a, b) = \alpha^b \) a.e. Since \( \delta \Pi^2_{T_b} \neq 0 \) we have that
\[
\lim_{a \to 0} \beta(a, b) = \beta^b.
\]

Thus it is only necessary to show the convergence of \( \alpha(a, b) \). Since \( \varphi(a) \) is non-increasing, it has bounded variation on \( [0, a^b] \) and we obtain
\[
\int_0^{a^b} |\alpha(s, b) - \alpha^b| ds < \infty.
\]

Hence there exist \( M \in \mathbb{R}_+ \) such that \( |\alpha(a, b)| < M \) almost everywhere. Now take a sequence \( (a_k)_{k \in \mathbb{N}} \subset [0, a^b) \) and \( a_k \to 0 \) we obtain that the corresponding sequence \( \alpha(a_k, b) \) is bounded and converge up to taking a sub sequence. By step (2) we already obtained \( \lim_{k \to \infty} \alpha(a_k, b) \leq \alpha^b \) almost everywhere, if \( \lim_{k \to \infty} \alpha(a_k, b) < \alpha^b \), we obtain by Lemma 2 that \( f^{(a_k, b)} = f^b \) for \( k \gg 1 \) contradicting the fact that \( \Pi^1(f^{(a_k, b)}) = a_k \).

Thus we can conclude that \( \lim_{a \to 0} \alpha(a, b) = \alpha^b \) almost everywhere. \( \square \)

**References**


[Bla] Blaschke, . . . ( ), no. . .


CONSTRAINED WILLMORE MINIMIZERS OF NON-RECTANGULAR CONFORMAL CLASS


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