Prof. Dr. Christoph Bohle — Tübingen SS 2014

Integrable Systeme : Blatt 2

Dr. Aaron Gerding — 17. April 2014

Diese Aufgaben sind schriftlich auszuarbeiten und bis zum 28. April abzugeben. Für jede Aufgabe gibt es 4 Punkte.

Aufgabe 1. Verify that the formula for φ_{12} given in the Bianchi–Permutability theorem for the Sine–Gordon equation indeed defines a Bäcklund transformation of φ_1 with parameter α_2 and Bäcklund transformation of φ_2 with parameter α_1. (You only have to check one of the four differential equations, because the computations are all essentially the same.)

Aufgabe 2. Compute the 2–soliton solutions of the Sine–Gordon equation. Also try 3–solitons if you feel like doing so... (Hint: use $\tan(x+y) = \frac{\tan(x)+\tan(y)}{1-\tan(x)\tan(y)}$.)

Aufgabe 3. (K–surfaces and Sine–Gordon equation) (**Doppelte Punktzahl**)

Our aim is to prove that there is a 1–1–correspondence between nowhere vanishing[1] solutions to the Sine–Gordon equation and special parametrizations of K–surfaces (up to Euclidean motion). For this we assume without proof that every surface with $K < 0$ admits an asymptotic line parametrisation, i.e., a parametrization for which the coefficients of the second fundamental form are off–diagonal.

Let $f\colon \mathcal{U} \subset \mathbb{R}^2 \to \mathbb{R}^3$ be an asymptotic line parametrisation of a K–surface and denote by N its Gauss–map. For simplicity we assume that $\mathcal{U}$ is a rectangle.

i) Explain that there is an $SO(3)$–frame $F = (X, Y, N)$ such that X and Y bisect the angles between the parameter lines. (This is essentially a fact about the null–directions of an indefinite symmetric bilinear form on a 2–dimensional Euclidean vector space and the eigenspaces of the corresponding symmetric endomorphism.)

ii) There are functions θ, ρ, $\tilde{\rho}$ with $\rho, \tilde{\rho} > 0$ such that

$$f_u = \rho(\sin(\theta)X + \cos(\theta)Y)$$

$$f_v = \tilde{\rho}(-\sin(\theta)X + \cos(\theta)Y)$$

(after possibly changing the sign of v).

[1] We exclude zeros of the Sine–Gordon solution, because they would correspond to singularities of the surface.

iii) The connection matrices U, V in the frame equations $F_u = FU$ and $F_v = FV$ then take the form

$$U = \begin{pmatrix} 0 & -\omega & -\tau\rho\cos(\theta) \\ \omega & 0 & \tau\rho\sin(\theta) \\ \tau\rho\cos(\theta) & -\tau\rho\sin(\theta) & 0 \end{pmatrix}$$

$$V = \begin{pmatrix} 0 & -\eta & -\tilde{\tau}\tilde{\rho}\cos(\theta) \\ \eta & 0 & -\tilde{\tau}\tilde{\rho}\sin(\theta) \\ \tilde{\tau}\tilde{\rho}\cos(\theta) & \tilde{\tau}\tilde{\rho}\sin(\theta) & 0 \end{pmatrix}$$

for functions τ, $\tilde{\tau}$.

iv) Show that because $K = -1$ one can assume (up to permutation of the coordinates) that $\tau = 1$ and $\tilde{\tau} = -1$. (Compute the coefficients of the Weingarten operator A given by $dN = -df \circ A$ with respect to the basis ∂_u, ∂_v and use $K = \det(A)$.)

v) Derive $\rho_v = \tilde{\rho}_u = 0$ from the Maurer–Cartan equation

$$U_v - V_u = [U, V]$$

and the compatibility condition

$$f_{uv} = f_{vu}.$$

(Maurer–Cartan yields three scalar equations and $f_{uv} = f_{vu}$ yields two more scalar equations.) The condition $\rho_v = \tilde{\rho}_u = 0$ allows to change parameters in a way that $\rho = \tilde{\rho} = 1$.

vi) Evaluate the remaining conditions obtained from the Maurer–Cartan equation for special parametrizations with $\rho = \tilde{\rho} = 1$.

vii) Explain briefly, that the above procedure can be reversed and that a solution of the Sine–Gordon equation gives rise to a K–surface with asymptotic line parametrization so that f_u and f_v have length one.