Integrable Systeme : Blatt 4

Diese Aufgaben sind schriftlich auszuarbeiten und bis zum 12. Mai abzugeben. Für jede Aufgabe gibt es 4 Punkte.

Aufgabe 1. Show that

$$\mathcal{E}_k \circ \mathcal{E}_l \subset \mathcal{E}_{k+l}$$

and that, provided the underlying differential algebra is commutative,

$$[\mathcal{E}_k, \mathcal{E}_l] \subset \mathcal{E}_{k+l-1}.$$

Give an example for which the last inclusion doesn't hold. Explain why analogous results hold for $\mathcal{D}_k$.

Aufgabe 2. Denote by $Q = \partial + \sum_{i=1}^{\infty} q_i \partial^{-i}$ the N–th root of a family $L = Q^N$ of differential operators in $\partial = \partial_x$ with coefficients in $\mathbb{C}[[x, t_r]]$. Prove that

$$\partial_{t_r} Q = [(Q^r)_+, Q] = -[(Q^r)_-, Q] \qquad (1)$$

if and only if

$$\partial_{t_r} L = [(L^{r/N})_+, L] = -[(L^{r/N})_-, L]. \qquad (2)$$

(That (1) implies (2) is elementary. To verify the converse, use that there is $W = 1 + \sum_{i=1}^{\infty} w_i \partial^{-i}$ with coefficients in $\mathbb{C}[[x, t_r]]$ such that $Q = W \circ \partial \circ W^{-1}$ and $L = W \circ \partial^N \circ W^{-1}$. From $\partial_{t_r} Q = [\partial_{t_r} W \circ W^{-1}, Q]$ and $\partial_{t_r} L = [\partial_{t_r} W \circ W^{-1}, L]$ conclude that $\partial_{t_r} W \circ W^{-1}$ coincides with $-(Q^r)_- = -(L^{r/N})_-$ up to stuff that commutes with L and hence with Q...)

Aufgabe 3. Assume our differential algebra R has a "weight"–grading compatible with the product such that ∂ increases weights by 1. Show that if $P = \sum_{i=0}^{\infty} p_i \partial^{\alpha-i}$ and $Q = \sum_{i=0}^{\infty} q_i \partial^{\beta-i}$ such that p_i has weight i and q_i has weight i, the coefficient of $\partial^{\alpha+\beta-i}$ in $P \circ Q$ has weight i.

Assume now that R is the free differential algebra generated a scalar function u. Let $L = \partial^2 + u$ and denote by

$$Q = \partial + \sum_{i=1}^{\infty} q_i \partial^{-i}$$

the unique square root of L. Show that

$$\hat{K}_{2r+1}(u) = [(Q^{2r+1})_+, L]$$

has weight $2r+3$ if $u^{(j)}$ is assigned weight $2+j$. (Hint: use Exercise 3 on Sheet 3.)