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Integrable Systeme : Blatt 13

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Die Aufgaben werden in der Extraübung am 21. Juli besprochen.

Aufgabe 1. The *partition function* $p(n)$ counts the number of ways to write n as a sum of different integers. Show that the generating function for $p(n)$ satisfies

$$\sum_{n=0}^{\infty} p(n)x^n = \prod_{i=1}^{\infty}(1-x^i)^{-1}.$$

Aufgabe 2. In the Boson–Fermion correspondence, the two types of Fock–spaces are identified under the morphism

$$(*) \qquad |u> \in \mathcal{F} \quad \mapsto \quad \sum_{l\in\mathbb{Z}} z^l \langle l|e^{H(x)}|u\rangle$$

of Heisenberg algebra representations. Show that this morphism preserves energy if z^l is given energy $l^2/2$ and x_i is given energy i.

Aufgabe 3 (Combinatorics behind the Boson–Fermion correspondence)**.** The partition number $p(n)$ can be computed by either

- counting Young diagrams $(\lambda_1, ..., \lambda_l)$ with $\lambda_1 + ... + \lambda_l = n$ or
- by counting sequences $(k_1, k_2, ...)$ with $k_i \geq 0$ and

$$k_1 + 2k_2 + 3k_3 + ... = n.$$

Conclude that, for $d = \frac{l^2}{2} + n$ the dimension of $\mathcal{F}_l^{(d)}$ coincides with the dimension of the energy d–subspace of $z^l B_{l,1}$. (Identify young diagrams of the above type with the standard "excited state" basis vectors of $\mathcal{F}_l^{(d)}$ and identify sequences $(k_1, k_2, ...)$ as above with the respective basis of bosonic Fock space.)

Aufgabe 4. Show that the fermionic characteristic function

$$ch(\mathcal{F}) := \sum_{l\in\mathbb{Z};\, d\geq l^2/2} \dim(\mathcal{F}_l^{(d)}) z^l q^d$$

satisfies

$$ch(\mathcal{F}) = \prod_{j\in\mathbb{Z}+1/2;\, j>0} (1+zq^j)(1+z^{-1}q^j).$$

Aufgabe 5. Show that the bosonic characteristic function

$$ch(\mathcal{B}) := \sum_n \dim(\mathcal{B}^{(n)})q^n$$

satisfies

$$ch(\mathcal{B}) = \prod_{i\in\mathbb{N}^*}(1-q^i)^{-1}.$$

Aufgabe 6. The fact that corresponding energy subspaces of $\mathcal{F}_l$ and $z^l B_{l,1}$ have same dimensions (which implies that $(*)$ is an isomorphism) is equivalent to the Jacobi triple product identiy

$$\prod_{i\in\mathbb{N}^*}(1-zq^{i-1})(1-z^{-1}q^i)(1-q^i) = \sum_{l\in\mathbb{Z}}(-z)^l q^{\frac{l(l-1)}{2}}.$$