Prof. Dr. Christoph Bohle Tübingen SS 2019

Geometry of Manifolds II : Exercise Sheet 6

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Diese Aufgaben sind schriftlich auszuarbeiten und am 30. Mai vor der Vorlesung abzugeben. Für jede Aufgabe gibt es 4 Punkte.

Zweierabgaben sind erlaubt. Bitte bei der ersten Abgabe Matrikelnummer(n) angeben.

Aufgabe 1. Let $(M, \{.\})$ be a Poisson manifold. Show that $f \mapsto X_f$ is a Lie algebra homomorphism from $(C^\infty, \{.\})$ to $(\mathfrak{X}(M), [.])$.

Aufgabe 2. Given a *symplectic form* on a manifold M, i.e. a closed form $\omega \in \Omega^2(M)$ that is non–degenerate, then for any function $f \in C^\infty(M)$ there is a unique vector field X_f such that

$$\omega(Y, X_f) = df(Y)$$

for all $Y \in TM$. Show that

$$\{f, g\} := \omega(X_f, X_g)$$

is a Poisson bracket and that X_f defined here coincides with the definition on Poisson manifolds.

Aufgabe 3. Show that a discrete subgroup $\Gamma \subset \mathbb{R}^n$ is a lattice, i.e. there are $\mathbb{R}$–linearly independent vectors $\gamma_1,...,\gamma_k \in \Gamma$ such that

$$\Gamma = \mathrm{Span}_{\mathbb{Z}}\{\gamma_1, ..., \gamma_k\}.$$

(Hint: If $\Gamma \neq 0$, choose $\gamma_1 \in \Gamma\backslash\{0\}$ with minimal distance to 0 and prove that $\Gamma_1 := \Gamma \cap \gamma_1\mathbb{R}$ equals $\gamma_1\mathbb{Z}$. Proceed by induction and show that, given $\mathbb{R}$–linearly independent γ_1, ..., $\gamma_l \in \Gamma$ with $\Gamma_l := \Gamma \cap (\gamma_1\mathbb{R} + ... + \gamma_l\mathbb{R})$ satisfying $\Gamma_l = \mathrm{Span}_{\mathbb{Z}}\{\gamma_1, ..., \gamma_l\}$, either $\Gamma_l = \Gamma$ or there is $\gamma_{l+1} \in \Gamma\backslash\Gamma_l$ with minimal distance to Γ_l and $\Gamma_{l+1} := \Gamma \cap (\gamma_1\mathbb{R} + ... + \gamma_{l+1}\mathbb{R})$ again satisfies $\Gamma_{l+1} := \mathrm{Span}_{\mathbb{Z}}\{\gamma_1, ..., \gamma_{l+1}\}$.)

Aufgabe 4. On the tangent bundle TM of the torus of revolution given by

$$(\varphi, \theta) \mapsto \begin{pmatrix} (2+\cos(\theta))\cos(\varphi) \\ (2+\cos(\theta))\sin(\varphi) \\ \sin(\theta) \end{pmatrix}$$

we use coordinates $(v_\varphi, v_\theta, \varphi, \theta)$ to describe vectors

$$v_\varphi \frac{\partial}{\partial \varphi} + v_\theta \frac{\partial}{\partial \theta} \in T_{p=(\varphi,\theta)}M.$$

Furthermore, on TM we use the symplectic form

$$\omega = (2+\cos(\theta))^2 dv_\varphi \wedge d\varphi - 2(2+\cos(\theta))\sin(\theta) v_\varphi d\theta \wedge d\varphi + dv_\theta \wedge \theta.$$

Show that:

i) the geodesic flow on M coincides with the Hamiltonion flow of

$$H_1 = \frac{1}{2}((2+\cos(\theta))^2 v_\varphi^2 + v_\theta^2),$$

(Hint: use that a geodesic on a submanifold is a curve whose second derivative is pointwise normal to the submanifold.)

ii) the Hamiltonion flow of

$$H_2 = (2+\cos(\theta))^2 v_\varphi$$

corresponds to the 1–parameter group of rotations of M around the z–axis and conclude that

$$\{H_1, H_2\} = 0,$$

iii) the Arnold–Liouville theory applies and derive its consequences.

(One can show that the geodesic flow on any Riemannian manifold is Hamiltonian, for H equal to the squared length of tangent vectors, with respect to the symplectic form on TM obtained—via the metric—from the canonical symplectic form on T^*M. For surfaces of revolution, the geodesic flow always admits a second integral that commutes with H and whose flow corresponds to the rotational symmetry.)