Minimal surfaces with polygonal boundary and Fuchsian equations

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Abstract

We present a forgotten approach due to René Garnier [3] to the Plateau problem in the case of a polygonal boundary in the three-dimensional Euclidean space. We associate to each minimal surface with polygonal boundary a second-order Fuchsian equation, which is strongly related to the geometry of the surface. The monodromy of the equation is described by the polygonal boundary of the surface. Our aim is to use this equation to solve the Plateau problem: the unknown is not a minimal surface anymore, but its associated equation instead. We see that the Plateau problem is transformed into a Riemann-Hilbert problem and its solution then involves monodromy preserving deformations.

Introduction

We present a method to solve the Plateau problem in the case of polygonal boundary: we are looking for surfaces with zero mean curvature and polygonal boundary, with the topology of a disk, and everywhere regular, except possibly at the vertices of the polygon (this is natural to ask for the surface to be regular at the edges of the polygon, in order to be able to apply the Schwarz symmetry principle). To this end, we will associate to such a surface a Fuchsian equation. This equation was first mentioned in a short paper of K. Weierstrass [8] published on 20 December 1866, and few days later in a presentation of Riemann's works by Hattendorf on 6 January 1867 at the Royal Society of Göttingen [6]. G. Darboux [1] studied this equation to solve the Plateau problem, but it was R. Garnier [3] who finally succeed in 1928, two years before the proofs of Rado and Douglas. His paper seems today to be totally forgotten; we do not know how it has been received at the time, neither whether his proof is correct or not. Even if his work looks less powerful than the variational approach, it is more constructive and it uses more geometrical tools. We intend here to introduce his paper and to rewrite it in a more modern manner, thanks in particular to the recent interest on Garnier's work on Fuchsian systems and isomonodromic deformations. But it is still a work in progress.

The key point of the method is the conformal representation of Weierstrass for real minimal surfaces: every conformal minimal immersion defined on the upper half-plane $\mathbb{C}_+ = \{z \in \mathbb{C}\,,\ \Im(z) > 0\}$ is of the form:

$$X(z) = \Re \int \begin{pmatrix} i(G^2 - H^2) \\ G^2 + H^2 \\ 2iGH \end{pmatrix} \tag{1}$$

where the functions G and H are called the *Weierstrass data* of the surface. They are holomorphic at the points where the surface is regular. The boundary of the surface is represented on the real axis. The stereographic projection from the North Pole of the Gauss map $N(z)$ at a point $z \in \mathbb{C}_+$ is $-G(z)/H(z)$, and the Hopf differential is given by:

$$Q(z)dz^2 = i(GH' - HG')dz^2.$$

The first and second quadratic fundamental forms are:

$$I = (G\overline{G} + H\overline{H})^2 dz d\bar{z},$$
$$II = Q(z)dz^2 + \overline{Q(z)}d\bar{z}^2.$$

For a minimal surface with a polygonal boundary with $n+3$ vertices, we define:

$$t_1, \dots, t_{n+3} \in \mathbb{R} \cup \{\infty\},$$

the pre-image of the vertices of the polygon. The functions G and H are unique up to the sign and to the composition of a conformal representation of the upper half-plane into itself, that is to say a Möbius transformation. This enable us to choose:

$$t_{n+1} = 0, \quad t_{n+2} = 1, \quad t_{n+3} = \infty.$$

To obtain the minimal surfaces we are looking for, the Weierstrass data G and H shall satisfy the two following properties:

- G and H are holomorphic on the upper-half plane $\mathbb{C}_+$,
- G and H are continuous on each (t_i, t_{i+1}).

Under these assumptions, the Gauss map admits a limit at each vertex of the polygon (which is orthogonal to the adjacent edges at the vertex). We call $N(t_i)$ the limit normal vector at t_i.

The functions G and H are linearly independent (except in the uninteresting case where the minimal surface lies in a plane), thus (G, H) is a fundamental system of solutions of a second-order linear differential equation: this will be the equation associated to the minimal surface. In Section 1, we study the differential equation associated to a minimal surface which has a polygonal boundary, and we show that it is Fuchsian. The position and the nature of its singularities and the values of their exponents are given by the surface. The monodromy of the equation is completely determined by the directions of the edges of the polygonal boundary of the surface.

Section 2 is directly concerned with the Plateau problem: for a given polygon, we want to prove the existence of a minimal surface bounded by the polygon. The idea is rather to prove the existence of its associated second-order Fuchsian equation. The unknown equation has prescribed monodromy: we are led to solve a Riemann-Hilbert problem. But the points $t_1, \dots, t_n$ are now variables; actually, we will obtain a family of equations, whose monodromy-preserving deformations are described by the Garnier system, a completely integrable Hamiltonian system which gives a generalization of the sixth Painlevé equation to an arbitrary number of singularities.

1 The Fuchsian equation

In this section, we consider a conformal minimal immersion X whose image $(\mathcal{M})$ has a polygonal boundary $(\mathcal{P})$ of vertices $A_1, \dots, A_{n+3}$. We call $V_1, \dots, V_{n+3}$ the angles between two adjacent edges of $(\mathcal{P})$. We suppose $(\mathcal{P})$ is not degenerated and $(\mathcal{M})$ does not overlap itself, that is:

$$0 < V_i < \pi \quad \text{or} \quad \pi < V_i < 2\pi.$$

1.1 Quaternionic description

By direct computations, we can see that, when we make a rotation of the surface of angle φ and of axis $\delta = (\delta_1, \delta_2, \delta_3)$, the Weierstrass data (G_1, H_1) corresponding to the new position $(\mathcal{M}_1)$ of the surface are linear combinations of G and H:

$$\begin{pmatrix} G_1 & H_1 \end{pmatrix} = \begin{pmatrix} G & H \end{pmatrix} A$$

where $A \in SU(2)$ and, more precisely, we can write:

$$A = \cos\left(\frac{\varphi}{2}\right) I_2 - i \sin\left(\frac{\varphi}{2}\right) \sum_{j=1}^{3} \delta_j \sigma_j. \tag{2}$$

The σ_j denote the Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We obtain the usual identification of the three-dimensional Euclidean space with the space of imaginary quaternions $\Im \mathbb{H}$:

$$X = (X_1, X_2, X_3) \mapsto \widetilde{X} = -i \sum_{j=1}^{3} X_j \, \sigma_j$$

Indeed, if X is the immersion associated to (G, H) and Y the immersion associated to (G_1, H_1), we have:

$$\widetilde{Y} = \overline{A}^t \widetilde{X} A.$$

This result leads us to introduce the second-order differential equation which has (G, H) as fundamental system of solutions:

$$D^2 y + p^*(z) D y + q^*(z) y = 0, \tag{3}$$

where D denotes the differentiation with respect to z. The functions p^* and q^* defined on $\mathbb{C}_+$ are given by:

$$p^* = -\frac{GH'' - HG''}{GH' - HG'} \qquad q^* = \frac{G'H'' - H'G''}{GH' - HG'},$$

and we can already see that p^* and q^* have two kind of singularities: the points where the Weierstrass data G and H are singular (that is to say, the t_i's), and the points where they are holomorphic, but at which the Wronskian $GH' - HG'$ vanishes (the umbilics of the conformal immersion X).

Different minimal conformal immersions can define the same equation. To study equation (3) we can remplace the Weierstrass data (G, H) by any linear combinations of them. In particular, we can change the position of the orthonormal basis (e_1, e_2, e_3) of $\mathbb{R}^3$. We still did not use the fact that the boundary of $(\mathcal{M})$ is polygonal: actually, equation (3) is defined for any minimal surface (except for plane surfaces). But, in the case of polygonal boundary, it appears that there is a nice traduction of the geometry of $(\mathcal{M})$ in terms of analytical properties of equation (3), as we shall see in the following.

1.2 The equation is defined on the entire complex plane

We know that the coefficients p^* and q^* are meromophic on $\mathbb{C}_+$. In order to show that they can be meromorphically continued on the entire complex plane, we need the following lemma.

Lemma 1. *The coefficients p^* and q^* are real on each (t_i, t_{i+1}).*

Proof. It is sufficient to find for each $i = 1, \dots, n+3$ two independent solutions which are both real or purely imaginary on (t_i, t_{i+1}). We move the surface $(\mathcal{M})$ such that the edge (A_i, A_{i+1}) becomes parallel to the second coordinate vector $e_2 = (0, 1, 0)$. As we have seen, the equation does not change. We still call $(\mathcal{M})$ the surface in the new position, and G and H its Weierstrass data. If we express for example that on (t_i, t_{i+1}), the third component X_3 of the immersion is constant, and that $< N, e_2 >= 0$, we get on (t_i, t_{i+1}):

$$\begin{cases} -G/H \in \mathbb{R} \\ GH \in \mathbb{R} \end{cases}, \qquad \text{i.e.:} \quad \begin{cases} G\overline{H} = \overline{G}H \\ GH = \overline{GH}. \end{cases}$$

Since X is an immersion, G and H cannot vanish simultaneously, and it follows that on (t_i, t_{i+1}) G^2, H^2 and GH are real: G and H are both real on (t_i, t_{i+1}), or purely imaginary. □

Now, since the coefficients p^* and q^* are meromorphic on $\mathbb{C}_+$ and real on the real axis (except in a finite number of points $t_1, \dots, t_{n+3}$, whose nature we do not know), they can be analytically continued on the lower half-plane $\mathbb{C}_- = \{z \in \mathbb{C}\,,\ \Im(z) < 0\}$ by setting:

$$\forall\, z \in \mathbb{C}_- \qquad \begin{array}{l} p^*(z) = \overline{p^*}(\bar{z}), \\ q^*(z) = \overline{q^*}(\bar{z}), \end{array}$$

and thus the points t_i are not branch points for p^* and q^*.

Remark 1. We recover the Schwarz symmetry principle. In this position of $(\mathcal{M})$, the Weierstrass data G and H can be analytically continued on $\mathbb{C}_-$ through the segment (t_i, t_{i+1}) (by the same way than p^* and q^*), and so does the conformal immersion X. We obtain $n + 3$ continuations of (G, H) and X. Let X^i be the continuation of X through (t_i, t_{i+1}). When the ith edge is parallel to the second coordinate vector, we have:

$$\forall\, z \in \mathbb{C}_- \qquad X^i(z) = \begin{pmatrix} -X_1(\bar{z}) \\ X_2(\bar{z}) \\ -X_3(\bar{z}) \end{pmatrix}.$$

It appears that the minimal surface $(\mathcal{M}_i)$ defined by the conformal immersion X^i on the lower half-plane is the symmetric of $(\mathcal{M})$ with respect to the vector e_2, i.e. to (A_i, A_{i+1}): the edges of $(\mathcal{P})$ are symmetry axis for $(\mathcal{M})$. Moreover, the symmetric points on $(\mathcal{M})$ and $(\mathcal{M}_i)$ correspond to conjugate values of z.

1.3 Generalities about complex differential equations

Before giving the properties of equation (3) associated to $(\mathcal{M})$, we remind basic facts on linear differential equations defined on the complex plane (see [4] for more complete explanations). We restrict ourselves to the case of second-order equations. We consider:

$$D^2y + p(z)Dy + q(z)y = 0, \tag{4}$$

where the functions p and q are meromorphic on the Riemann sphere. The singular points of equation (4) are the poles of p and q. A singular point is said *regular* if it is a pole of order 1 for p and 2 for q. A regular singular point z_0 is said *non-logarithmic* if there is a fundamental system of solutions (called *canonical* at z_0) of the form:

$$g(z) = (z - z_0)^{\alpha}\varphi(z), \quad h(z) = (z - z_0)^{\beta}\psi(z),$$

where φ and ψ are holomorphic, non-vanishing functions at z_0, and α and β are complex numbers called the *exponents* at z_0. The form of the solutions at a logarithmic singularity is more complicated, but we will not use it. Moreover, the second-order polynomial equation whose solutions are α and β is called the *characteristic equation* at z_0. It is given by:

$$x^2 + (a - 1)x + b = 0,$$

where:

$$a = \lim_{z \to z_0} (z - z_0)p(z), \quad b = \lim_{z \to z_0} (z - z_0)^2 q(z).$$

To determine the nature of the point $z = \infty$, we describe the equation and its solutions in terms of the local parameter $w = 1/z$ at the point $w = 0$. By a direct computation, we find that infinity is a regular singularity if and only if the functions:

$$\tfrac{1}{w}p\left(\tfrac{1}{w}\right), \quad \tfrac{1}{w^2}q\left(\tfrac{1}{w}\right)$$

are holomorphic at $w = 0$. If so, the characteristic equation at infinity is:

$$x^2 + (1 - a_\infty)x + b_\infty = 0,$$

where:

$$a_\infty = \lim_{w\to 0} \tfrac{1}{w} p\left(\tfrac{1}{w}\right), \quad b_\infty = \lim_{w\to 0} \tfrac{1}{w^2} q\left(\tfrac{1}{w}\right).$$

If all the singularities of equation (4) (including infinity) are regular, equation (4) is said *Fuchsian.* The exponents of a Fuchsian equation are related together by the *Fuchs relation*: if equation (4) has r regular singular points $z_1, \ldots, z_r = \infty$ which have respectively α_i and β_i as exponents, then we have:

$$\sum_{i=1}^{r} (\alpha_i + \beta_i) = r - 2. \tag{5}$$

This comes from the fact that the sum of the residues of p is equal to zero. Since p is a meromorphic function on the Riemann sphere, with at most poles of order one at the z_i's, $i = 1, \ldots, r-1$, and with a zero at infinity, it is of the form:

$$p(z) = \sum_{i=1}^{r-1} \frac{a_i}{z - z_i},$$

where a_i is the residue of p at z_i: $a_i = 1 - \alpha_i - \beta_i$. From the characteristic equation at infinity, we get:

$$a_r - 1 = \alpha_r + \beta_r,$$

with

$$a_r = \lim_{z\to\infty} z p(z) = \sum_{i=1}^{r-1} a_i,$$

which gives us the Fuchs relation (5).

1.4 Nature of the t_i's

We have seen (remark 1) that for each $i = 1, \ldots, n+3$, the fundamental system (G, H) has not the same continuations through (t_{i-1}, t_i) and through (t_i, t_{i+1}): the functions G and H are not uniform at the point t_i. By geometrical considerations, we can express how the functions G and H are transformed around t_i, that is to say, we can determine the monodromy of equation (3).

Lemma 2. *The circuit matrix S_i of the fundamental system (G, H) around t_i ($i = 1, \ldots, n+3$) is given by:*

$$S_i = -\cos(V_i)\, I_2 - i \sin(V_i) \sum_{j=1}^{3} \langle N(t_i), e_j \rangle \sigma_j, \tag{6}$$

where $N(t_i)$ is the Gauss map of $(\mathcal{M})$ at the vertex A_i.

Proof. Let (G_i, H_i) be the analytic continuation of (G, H) along a loop ℓ_i with a base point in $\mathbb{C}_+$ which encircles t_i once counterclockwise, and leaves the other singularities outside. It is still a fundamental system of equation (3). Then S_i is the unique matrix in $GL_2(\mathbb{C})$ such that:

$$\begin{pmatrix} G_i & H_i \end{pmatrix} = \begin{pmatrix} G & H \end{pmatrix} S_i.$$

The functions G_i and H_i are the Weierstrass data of a new minimal surface $(\widetilde{\mathcal{M}}_i)$. We compute S_i by determining the transformation which maps $(\mathcal{M})$ on $(\widetilde{\mathcal{M}}_i)$. When we follow ℓ_i, we first cross (t_{i-1}, t_i): the continuation of X through (t_{i-1}, t_i) parameterizes the symmetric of $(\mathcal{M})$ with respect to the edge (A_i, A_{i+1}) (remark 1). By the same way, when we go back to the base point of ℓ_i, we cross (t_i, t_{i+1}), and we finely get that $(\widetilde{\mathcal{M}}_i)$ is obtained from $(\mathcal{M})$ by two successive symmetries: first with respect to (A_{i-1}, A_i), then with respect to (A_i, A_{i+1}). The product of the two successive symmetries is the rotation of axis $(A_i, N(t_i))$ and of angle $2(\pi - V_i)$. We finally get the expressions of S_i from (2). □

The free group generated by the matrices S_i is a representation of the monodromy of equation (3). Notice that the monodromy is completely determined by the constants V_i and the vectors $N(t_i)$ $(i = 1, \ldots, n+3)$, i.e. by the directions of the edges of the polygon $(\mathcal{P})$.

Corollary 1. *The singular points $t_1, \ldots, t_{n+3}$ are regular and non-logarithmic. For $i = 1, \ldots, n+2$, the exponents at t_i are:*

$$\alpha_i = -\frac{1}{2} + \frac{V_i}{2\pi}, \quad \beta_i = \frac{1}{2} - \frac{V_i}{2\pi} + r_i,$$

where r_i is a non-negative integer. If $\pi < V_i < 2\pi$, then r_i is positive.

Proof. We are now looking for local properties of the solutions of equation (3): for $i = 1, \ldots, n+3$, we can choose the position of the orthonormal basis such that the third coordinate vector $e_3 = (0, 0, 1)$ and the Gauss map $N(t_i)$ are parallel. The Weierstrass data (G, H) corresponding to this position have for circuit matrix at t_i:

$$S_i = \begin{pmatrix} -e^{iV_i} & 0 \\ 0 & -e^{-iV_i} \end{pmatrix},$$

and thus are of the form:

$$G(z) = (z - t_i)^{-\frac{1}{2} + \frac{V_i}{2\pi}} \varphi(z),$$
$$H(z) = (z - t_i)^{\frac{1}{2} - \frac{V_i}{2\pi}} \psi(z),$$

where φ and ψ are uniform functions in a neighbourhood of t_i. Since the primitives:

$$\int G^2, \int H^2, \int GH,$$

which compose the expression of the immersion X, must take finite values at the point t_i, we deduce that t_i is not an essential singularity for φ and ψ. It forces t_i to be a regular singularity. Moreover, it is non-logarithmic and its exponents are of the form:

$$-\frac{1}{2} + \frac{V_i}{2\pi} + r_i', \qquad \frac{1}{2} - \frac{V_i}{2\pi} + r_i'',$$

where r_i' and r_i'' are integers. This is all the monodromy can tell us about the exponents.

To be more precise, we have to study the local behavior of the fundamental system (G, H) in the neighbourhood of a singularity t_i, using the expression of the conformal immersion X at the vertices of the polygon $(\mathcal{P})$. We do not get the same values whether t_i is at infinity or not: we suppose now the point t_i is not at infinity. We call α_i the smallest exponent at t_i and β_i the greatest. In any case, $r_i := \alpha_i + \beta_i$ is an integer. Up to a rotation of angle π and of horizontal axis, we can suppose that H is a canonical solution at t_i for the exponent β_i[1], i.e. we have in a neighbourhood of t_i:

$$G(z) \sim a(z - t_i)^{\alpha_i}$$
$$H(z) \sim b(z - t_i)^{\beta_i},$$

where a and b are non-zero constants. It follows that :

$$X(z) - X(t_i) \sim \Re \begin{pmatrix} \frac{ia^2}{2\alpha_i+1}(z - t_i)^{2\alpha_i+1} \\ \\ \frac{a^2}{2\alpha_i+1}(z - t_i)^{2\alpha_i+1} \\ \\ \frac{2iab}{r_i+1}(z - t_i)^{r_i+1} \end{pmatrix}$$

By letting $z - t_i$ take infinitely small real values, first positive, then negative, we can see that:

$$V_i = (2\alpha_i + 1)\pi,$$

[1] As we have seen in the first part of the proof, if e_3 and $N(t_i)$ are parallel, then the system (G, H) is canonical at $z = t_i$. Actually, since the stereographic projection of N is $-G/H$, if $N(t_i) = e_3$, H has the greatest exponent, and if $N(t_i) = -e_3$, G does.

that is to say: $r'_i = 0$ and $r''_i = r_i$. Since the vertex $X(t_i) = A_i$ has finite components, $2\alpha_i + 1$ is non-negative, and since $r_i > 2\alpha_i$, we get $r_i \geq 0$. If $\pi < V_i < 2\pi$, $\alpha_i < \beta_i$ leads to $r_i \geq 1$. □

Corollary 2. *The exponents at $t_{n+3} = \infty$ are:*

$$\alpha_{n+3} = \frac{1}{2} + \frac{V_{n+3}}{2\pi}, \quad \beta_{n+3} = -\frac{1}{2} - \frac{V_{n+3}}{2\pi} + r_{n+3},$$

where r_{n+3} is an integer greater or equal than 2. If $\pi < V_i < 2\pi$, then $r_{n+3} \geq 3$.

Proof. After changing the variable z into $w = 1/z$, the immersion X becomes:

$$X\left(\frac{1}{w}\right) = -\Re \int \begin{pmatrix} i\left(G^2\left(\frac{1}{w}\right) - H^2\left(\frac{1}{w}\right)\right) \\ G^2\left(\frac{1}{w}\right) + H^2\left(\frac{1}{w}\right) \\ 2iG\left(\frac{1}{w}\right) H\left(\frac{1}{w}\right) \end{pmatrix} \frac{dw}{w^2}.$$

As before, we choose the function H as a canonical solution at $w = 0$ for the greatest exponent β_{n+3}:

$$G\left(\frac{1}{w}\right) \sim a w^{\alpha_{n+3}},$$
$$H\left(\frac{1}{w}\right) \sim b w^{\beta_{n+3}}.$$

Thus, we obtain: $V_{n+3} = (2\alpha_{n+3} - 1)\pi$, and $r_{n+3} \geq 2$. □

We have shown that the points t_i (the first kind of singularities) are regular, and we have seen that the functions G and H are holomorphic in $\mathbb{C} \setminus \{t_1, \ldots, t_n, 0, 1\}$. Therefore, the solutions at the other singularities (the umbilics) will all be holomorphic. One can easily check that it forces this other kind of singularities to be regular. Equation (3) has thus only regular singularities, i.e.:

Lemma 3. *Equation* (3) *is a Fuchsian equation.*

1.5 The apparent singularities

We are now looking for the position, nature and values of exponents of the singularities which are different from the t_i's. We have seen that a point where the functions G and H are holomorphic is a pole for the coefficients p^* or q^* if and only if the Wronskian $GH' - HG'$ vanishes: these singularities of equation (3) are the umbilics of the immersion X (note that, for a minimal surface, the principal curvatures at an umbilic point vanish). We can be more precise about the exponents and the number of these singularities.

Lemma 4. *The singularities of equation* (3) *different from the* t_i*'s are exactly the umbilic points of the conformal minimal immersion* X. *They are regular and of apparent type. The exponents at a singularity* $\lambda \in \mathbb{C} \setminus \{t_1, \dots, t_n, 0, 1\}$ *are* 0 *and an integer* $m \geq 2$, *such that* $m+1$ *is the order of contact between* $(\mathcal{M})$ *and its tangent plane at* λ.

An *apparent singularity* is a regular non-logarithmic singularity whose exponents are integers. It is characterized by the fact that the solutions are all uniform in its neighbourhood: the first part of the lemma is obvious. We just have to compute the exponents.

Proof. We consider a point $\lambda \in \mathbb{C} \setminus \{t_1, \dots, t_n, 0, 1\}$, i.e. a regular point of the surface $(\mathcal{M})$. We choose again the directions of the coordinate axis such that e_3 and the Gauss map $N(\lambda)$ of $(\mathcal{M})$ at λ are parallel. In this position, we can write:

$$X(z) - X(\lambda) = \Re \begin{pmatrix} (z-\lambda)\ \varphi_1(z) \\ (z-\lambda)\ \varphi_2(z) \\ (z-\lambda)^{m+1}\ \varphi_3(z) \end{pmatrix}, \tag{7}$$

where the functions φ_i are holomorphic at λ, and φ_1 or φ_2 does not vanish at λ, neither does φ_3. The integer $m+1$ is the order of contact between $(\mathcal{M})$ and its tangent plane at λ $(m \geq 1)$. The generic case corresponds to the value $m = 1$. By definition:

$$Q = < \frac{d^2 X}{dz^2}, N >,$$

thus $Q(\lambda) = 0$ if and only if $m \geq 2$. Therefore, if $m = 1$, the point λ is not an umbilic and it is an ordinary point of equation (3). We suppose now m greater or equal than 2. It follows from (7) that the functions G and H satisfy:

- $\int G^2$ or $\int H^2$ is of the form $(z-\lambda)\varphi$,
- $\int GH = (z-\lambda)^{m+1}\varphi$,

where φ denotes any holomorphic non-vanishing function at λ. We suppose for example that:

$$\int G^2 = (z-\lambda)\varphi,$$

and thus the point λ is not a zero of G. By the second assumption, we have:

$$H = (z-\lambda)^m \varphi.$$

Since λ is not a zero of G and is a zero of order m of H, it is a singularity of equation (3) of exponents 0 and m. □

On $\mathbb{C}_-$ the singular points are the conjugates of the singular points on $\mathbb{C}_+$ (they correspond to symmetric points on the surface) and thus are also regular and of apparent type. We have the same result for their exponents. Since the residue of the meromorphic function p^* at an apparent singularity is $1 - m \neq 0$, we have a finite number, say N, of apparent singularities: $\lambda_1, \ldots, \lambda_N$. Moreover:

Lemma 5. *Equation* (3) *has at most* $n - 1$ *singular points of apparent type.*

Proof. This is a direct consequence of the Fuchs relation (5). Let m_k be the non-zero exponent of the singularity λ_k, $k = 1, \ldots, N$. Since equation (3) has $n + 3 + N$ singular points, including infinity, the Fuchs relation becomes:

$$\sum_{i=1}^{n+3} r_i + \sum_{k=1}^{N} m_k = n + 1 + N.$$

From the minorations given in corollary 1, corollary 2 and lemma 4, we deduce:

$$N \leq n - 1.$$

□

If the number of apparent singularities is maximal: $N = n - 1$, the Fuchs relation forces the constants to be as lower as possible:

$$r_i = 0 \quad (i = 1, \ldots, n+2), \qquad r_{n+3} = 2,$$
$$m_k = 2 \quad (k = 1, \ldots, n-1).$$

We can see the case $N < n-1$ like deriving from the precedent case by merging of $n-1-N$ apparent singularities with t_i's or with other apparent singularities.

1.6 Conclusion

We have associated to each minimal surface with a polygonal boundary $(\mathcal{P})$ a second-order Fuchsian equation (3) such that:

- the coefficients p^* and q^* are real on the real axis,
- the singularities $t_1 < \ldots < t_n < t_{n+1} = 0$, $t_{n+2} = 1$, $t_{n+3} = \infty$ and $\lambda_1, \ldots, \lambda_N$ $(N \leq n - 1)$, are all distinct and non-logarithmic, and the Riemann scheme is:

$$\begin{pmatrix} z = t_i & z = \lambda_k \\ \alpha_i & 0 \\ r_i - \alpha_i & m_k \end{pmatrix}$$
$$i = 1, \ldots, n+3, \quad k = 1, \ldots, N,$$

where the α_i are given by the polygon $(\mathcal{P})$ and the integers $r_i \geq 0$, $r_{n+3} \geq 2$ and $m_k \geq 2$ can be seen on the surface $(\mathcal{M})$,

- the singular points λ_k are apparent, they are real or conjugate by pairs,
- the monodromy is determined by the directions of the edges of the polygon $(\mathcal{P})$.

Remark 2. Conversely, given a polygon $(\mathcal{P})$, we can associate to each second-order Fuchsian equation satisfying the properties enumerated above, a minimal surface which has a polygonal boundary $(\mathcal{P}_1)$. This boundary is not necessarily the initial polygon $(\mathcal{P})$, but $(\mathcal{P})$ and $(\mathcal{P}_1)$ have the same number of vertices, and the directions of their edges are the same[2].

2 Monodromy-preserving deformation

We now turn to the Plateau problem. We consider a $(n+3)$-gon $(\mathcal{P})$ in $\mathbb{R}^3$ and we want to prove the existence of a second-order Fuchsian equation whose associated minimal surface (in the sense of remark 2) is bounded by $(\mathcal{P})$. The constants V_i and the vectors $N(t_i)$ $(i = 1, \ldots, n+3)$ associated to $(\mathcal{P})$ are fixed; the constants t_i and λ_k are now unknown. We are looking for an equation satisfying the properties given in subsection 1.6. More precisely, we want to express all the equations satisfying these conditions, in order to be able to show that one of them defines a minimal surface whose polygonal boundary has edges of the same lengths than the initial boundary $(\mathcal{P})$.

In particular, for each authorized value of the constants r_i and m_k (satisfying the Fuchs relation), we want to show that we can choose the unknowns t_i and λ_k such that the equation has the given Riemann scheme and monodromy: the Plateau problem becomes a Riemann-Hilbert problem. It is known that, to get a positive answer to a Riemann-Hilbert problem with $n+3$ effective singularities, we need n apparent singularities, although we want an equation with at most $n-1$ apparent singularities. Even so, we will first look for the equations having n apparent singularities. There is another reason to add this singularity: in the quadrilateral case $(n = 1)$, the unknown equation should

[2]Indeed, the fundamental systems of the equation having the matrices S_i given by (6) for circuit matrices at the t_i's form a one-dimensional space (defining a conjugate family of minimal surfaces). It is possible to choose a system (G, H) in this space which is real or purely imaginary on (t_1, t_2). Then, one can show that the minimal surface defined by (G, H) has a polygonal boundary $(\mathcal{P}_1)$, whose vertices are the images of $t_1, \ldots, t_{n+3}$. The directions of the edges of $(\mathcal{P}_1)$ are determined by the matrices S_i: they are those of the polygon $(\mathcal{P})$. All possible choices for the system (G, H) define homothetic minimal surfaces.

have no apparent singularity. However, if it had one, its isomonodromic deformations would be known and described by the sixth Painlevé equation P_{VI}. Actually, we have a similar result for an arbitrary number of vertices: the isomonodromic deformations of a Fuchsian equation with $n+3$ effective non-logarithmic singularities and n apparent singularities are given by the so-called *n-dimensional Garnier system* (subsection 2.2). At the end, in order to obtain an equation with the right number of apparent singularities, the strategy will be to let the unwanted λ_k collapse with a real singularity t_i. With n apparent singularities, the Fuchs relation (5) becomes:

$$\sum_{i=1}^{n+3} r_i + \sum_{k=1}^{n} m_k = 2n+1,$$

which naturally contradicts the minorations on the r_i's and on the m_k's. We set all the m_k's equal to 2. The r_i's then satisfy:

$$\sum_{i=1}^{n+3} r_i = 1,$$

that is to say, one of the r_i's, $i = 1, \ldots, n+2$, is equal to -1, or $r_{n+3} = 1$. We choose for example $r_{n+1} = -1$. Geometrically, it means that the vertex A_{n+1} is at infinity. To obtain the equation we want, one λ_k should tend to $t_{n+1} = 0$.

In order to simplify the form of the equation we are looking for, we make the following transformation:

$$y^* = \Phi(z)y,$$
$$\Phi(z) = \prod_{i=1}^{n+2} (z - t_i)^{\alpha_i}.$$

If y^* is a solution of the unknown equation, then y is a solution of a new Fuchsian equation:

$$D^2 y + p(z) D y + q(z) y = 0, \tag{8}$$

which satisfies three types of conditions:

(i) Its singularities $t_1, \ldots, t_n$, $t_{n+1} = 0$, $t_{n+2} = 1$, $t_{n+3} = \infty$ and $\lambda_1, \ldots, \lambda_n$ are all distinct and non-logarithmic, and its Riemann scheme is:

$$\begin{pmatrix} z = t_i & z = \infty & z = \lambda_k \\ 0 & \alpha & 0 \\ \theta_i & \alpha + \theta_{n+3} & 2 \end{pmatrix}$$
$$i = 1, \ldots, n+2, \qquad k = 1, \ldots, n,$$

where:

$$\theta_i := \beta_i - \alpha_i \quad \text{and} \quad \alpha := \sum_{i=1}^{n+3} \alpha_i \quad \text{are known.}$$

Moreover, the singular points λ_k are apparent.

(ii) The monodromy is given.

(iii) The coefficients p and q are real on the real axis, the points t_i are real, $t_1 < \ldots < t_n < 0$, and the points λ_k are real or conjugate by pairs.

We first give, in subsection 2.1, the general form of the equations satisfying (i): they depend on $3n$ complex parameters. In subsection 2.2, we consider these parameters as variables and we characterize which ones define equations with the same monodromy: to a given monodromy correspond a n-parameters family of equations. We only consider the family corresponding to the monodromy defined by $(\mathcal{P})$. We will then have to prove that there are equations of this family satisfying the "condition of reality" (iii).

In this section, we present more classical results about Fuchsian equations and the Garnier system. For details, see Iwasaki, Kimura, Shimomura and Yoshida [4].

2.1 General form of the unknown equation

We describe the second-order Fuchsian equations (8) satisfying the condition (i). Since the singular points are regular, the coefficients p and q are of the form:

$$\begin{aligned} p(z) &= \sum_{i=1}^{n+2} \frac{a_i}{z-t_i} + \sum_{k=1}^{n} \frac{c_k}{z-\lambda_k}, \\ q(z) &= \sum_{i=1}^{n+2} \frac{b_i}{(z-t_i)^2} + \sum_{k=1}^{n} \frac{d_k}{(z-\lambda_k)^2} \\ &\quad - \sum_{i=1}^{n+2} \frac{K_i}{z-t_i} + \sum_{k=1}^{n} \frac{\mu_k}{z-\lambda_k}. \end{aligned}$$

The characteristic equation at $z = t_i$ is:

$$x^2 + (a_i - 1)x + b_i = 0,$$

and its roots are: 0 and θ_i. Thus we have:

$$a_i = 1 - \theta_i, \quad b_i = 0.$$

Similarly, for an apparent singularity λ_k, we get:

$$c_k = -1, \quad d_k = 0.$$

Furthermore, if we express that $z = \infty$ is regular singular, and that α and $\alpha + \theta_{n+3}$ are solutions of the characteristic equation at $z = \infty$, we obtain:

$$\begin{aligned} K_{n+1} &= -\sum(\lambda_k + 1)\mu_k + \sum(t_i + 1)K_i + \alpha(\alpha + \theta_\infty), \\ K_{n+2} &= \sum \lambda_k \mu_k - \sum t_i K_i - \alpha(\alpha + \theta_\infty), \end{aligned}$$

and thus the coefficients p and q of the Fuchsian equation (8) are given by:

$$\begin{cases} p(z) = \displaystyle\sum_{i=1}^{n+2} \frac{1-\theta_i}{z - t_i} - \sum_{k=1}^{n} \frac{1}{z - \lambda_k}, \\ q(z) = \displaystyle\frac{\alpha(\alpha+\theta_\infty)}{z(z-1)} - \sum_{i=1}^{n} \frac{t_i(t_i - 1)K_i}{z(z-1)(z-t_i)} \\ \qquad\quad + \displaystyle\sum_{k=1}^{n} \frac{\lambda_k(\lambda_k - 1)\mu_k}{z(z-1)(z-\lambda_k)}. \end{cases} \tag{9}$$

Therefore, equation (8) depends on $4n$ unknown parameters:

$$t_1, \ldots, t_n, \ \lambda_1, \ldots, \lambda_n, \ \mu_1, \ldots, \mu_n, \ K_1, \ldots, K_n.$$

However, all these choices do not define an equation which has only apparent singularities at the λ_k's. The following proposition, which is a direct application of Frobenius' method, gives a criterion of non-logarithmic singularity of equation (8).

Proposition 1. *The points $\lambda_1, \ldots, \lambda_n$ are non-logarithmic singular points of equation* (8) *with coefficients p and q defined by* (9) *if and only if*

$$K_i = M_i \sum_{k=1}^{n} M^{k,i} \left(\mu_k^2 - \sum_{j=1}^{n+2} \frac{\theta_j - \delta_{ij}}{\lambda_k - t_j} \mu_k + \frac{\alpha(\alpha+\theta_\infty)}{\lambda_k(\lambda_k - 1)} \right),$$

where M_i and $M^{k,i}$ are defined by:

$$M_i = -\frac{\Lambda(t_i)}{T'(t_i)} \quad \textit{and} \quad M^{k,i} = \frac{T(\lambda_k)}{(\lambda_k - t_i)\Lambda'(\lambda_k)},$$

with:

$$\Lambda(z)=\prod_{k=1}^{n}(z-\lambda_k)\quad \textit{and}\quad T(z)=\prod_{i=1}^{n+2}(z-t_i).$$

The residues K_i can thus be seen as rational functions in (λ,μ,t), and the equations of the form (8) satisfying (i) thus only depend on (λ,μ,t): they are now denoted by $E(\lambda,\mu,t)$.

2.2 The Garnier system

We now consider (λ,μ,t) as variables. We set:

$$\mathcal{B}=\{(t_1,\ldots,t_n)\in(\mathbb{C}\setminus\{0,1\})^n/\forall i\neq j\quad t_i\neq t_j\},$$

and we characterize the submanifolds of $\{(\lambda,\mu,t)\in\mathbb{C}^n\times\mathbb{C}^n\times\mathcal{B}\}$ on which the equations $E(\lambda,\mu,t)$ have constant monodromy. Garnier [2] found the system of partial differential equations that describes isomonodromic deformations of $E(\lambda,\mu,t)$. Okamoto [5] showed the Hamiltonian structure of this system:

Definition 1. *The n-dimensional Garnier system $(\mathcal{G}_n)$ is the Hamiltonian system:*

$$\begin{cases}\dfrac{\partial\lambda_i}{\partial t_j} &= \dfrac{\partial K_j}{\partial\mu_i}\\[2mm] \dfrac{\partial\mu_i}{\partial t_j} &= -\dfrac{\partial K_j}{\partial\lambda_i}\end{cases}$$

$(i,j=1,\ldots,n)$. The Hamiltonians $K_i=K_i(\lambda,\mu,t)$ are given in proposition 1.

Theorem 1. *(i) $(\mathcal{G}_n)$ is completely integrable.*

(ii) Let M be a manifold in (λ,μ,t)-space. Then the family $E(\lambda,\mu,t)$ $((\lambda,\mu,t)\in M)$ is monodromy-preserving if and only if M is a submanifold of an integral manifold of $(\mathcal{G}_n)$.

We obtain the family of equations $(E(t),t\in\mathcal{B})$ which have the monodromy defined by the polygon $(\mathcal{P})$. If we show that for $t\in\mathcal{B}\cap\mathbb{R}^n$ the singularities λ_k and the residues μ_k are real or conjugate by pairs, then the coefficients p and q would be real on the real axis, and the equations $(E(t),t\in\mathcal{B}\cap\mathbb{R}^n)$ would satisfy the condition (iii). Therefore, for every $t\in\mathcal{B}\cap\mathbb{R}^n$, we could define a fundamental system of solutions $(G(\cdot,t),H(\cdot,t))$ of equation $E(t)$ such that its corresponding minimal surface is bounded by a polygon $(\mathcal{P}_t)$ whose directions of edges are those of the initial polygon $(\mathcal{P})$ (remark 2). But these minimal surfaces have their $(n+1)$th vertex at infinity. We will still have to find a point

t which makes vanish an apparent singularity λ_k and matches $n-1$ lengths of the polygon $(\mathcal{P}_t)$ with the corresponding lengths of $(\mathcal{P})$ (this is sufficient for $(\mathcal{P}_t)$ and $(\mathcal{P})$ to have the same form). The length of the ith edge of the polygon $(\mathcal{P}_t)$ is given by:

$$l_i(t) = \int_{t_i}^{t_{i+1}} \left(|G(z,t)|^2 + |H(z,t)|^2\right)^2 dz.$$

We want thus to prove that there is at least one $t \in \mathcal{B} \cap \mathbb{R}^n$ satisfying the following system of equations:

$$\begin{cases} \prod_{k=1}^{n} \lambda_k(t) = 0, \\ l_i(t) = c_i, \quad i = 1, \ldots, n-1, \end{cases}$$

where c_i is the length of the ith edge of $(\mathcal{P})$. One of the difficulties comes from the fact that the Garnier system does not enjoy the Painlevé property: its solutions (λ, μ) may have movable branch points or essential singularities.

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