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PhD dissertation summary[1]

The Plateau problem, Fuchsian equations and the Riemann–Hilbert problem

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1 Context and results

The starting point of my work is René Garnier's resolution of the Plateau problem for minimal surfaces. His proof relies on a resolution of the Riemann–Hilbert problem, and on isomonodromic deformations of Fuchsian equations.

Minimal surfaces are surfaces whose mean curvature is identically zero. They are the critical points of the area functional for the variations fixing the boundary. The Plateau problem is the following: *for any given closed Jordan curve Γ of the three-dimensional Euclidean space, to prove that there exists a regular minimal surface with the disk topology, which is bounded by the curve Γ.* The first known resolutions of the Plateau problem are independently obtained in the begining of the 1930's by Douglas [4] and Radó [8] by the variational method. But their proofs are not entirely complete, since they failled to exclude the possibility of isolated branch points on the minimal disks. A complete resolution of the Plateau problem is finally obtained in the 1970's.

In 1928, though, Garnier published a paper [6] in which he claimed he solved the Plateau problem in the three-dimensional Euclidean space. The most significant part of his proof is the case of polygonal boundary curves. Published two years before the resolutions of Douglas and Radó, that paper seems today to be totally forgotten, if not ignored at the time it was published. Even if the existence of such a resolution is known by some specialists, when I began my PhD thesis, nobody seemed to be able to tell how this proof was working, nor even whether it was correct or not. Garnier's method to solve the Plateau problem is very different from the variational one. His paper is longer and more technical than Douglas's one. However, unlike Douglas, he obtained everywhere regular minimal disks (without branch points). Moreover, Garnier's approach is more geometrical, inspired by previous works of Weierstrass [12], Riemann [9], Schwarz [11] and Darboux [1]. It is also much more constructive than the variational method.

However, Garnier's paper is sometimes really complicated, and even obscure or incomplete. The first step of my work was to understand his proof. In my thesis dissertation and in the paper [2], I then wrote an improved version of Garnier's proof, by giving some alternative easier proofs, and by filling some gaps. This work mainly relies on a systematic use of Fuchsian systems (instead of equations) and on a reality condition satisfied by

[1] My PhD thesis is available at `http://tel.archives-ouvertes.fr/tel-00452508/fr`

these Fuchsian systems. This reality condition plays a central role in Garnier's method, even if himself never explicitly mentioned it. To complete his proof, I established that this condition can be deduced from the monodromy. This clarification of the fondations of Garnier's proof has already enabled me to extend his result to maximal disks in the three-dimensional Minkowski space (Theorem 2), and lets me hope to be able to generalize it to other situations.

My framework is somewhat broader than Garnier's one, since I consider possibly unclosed polygonal boundary curves. We denote by $n+3$ ($n \in \mathbb{N}^*$) the number of edges (of possibly infinite length), and we restrict ourselves to polygonal curves whose $(n+3)$-tuple of oriented directions of edges $D = (D_1, \dots, D_{n+3})$ belongs to a set $\mathcal{D}^n$ with generic restrictions. We obtain the following result:

Theorem 1 (The Plateau problem for polygonal boundary curves)**.** *For every (possibly unclosed) polygonal curve P with $n+3$ edges of the Euclidean space $\mathbb{R}^3$, whose oriented directions of edges belong to the set $\mathcal{D}^n$, there exists a regular minimal disk in $\mathbb{R}^3$ bounded by the curve P. Moreover, if P is unclosed, the minimal disk has a helicoidal end.*

The improvements I made on Garnier's method enable me to apply it when the ambient space is the three-dimensional Minkowski space $\mathbb{R}^{2,1}$, namely $\mathbb{R}^3$ endowed with the Lorentzian metric

$$\langle\ ,\ \rangle = \mathrm{d}X_1^2 + \mathrm{d}X_2^2 - \mathrm{d}X_3^2.$$

A conformal immersion of a Riemann surface into $\mathbb{R}^{2,1}$ is said to be *maximal* if its mean curvature vanishes everywhere, and if it is spacelike (*i.e.* its induced metric is Riemannian). To solve the Plateau problem for polygonal boundary curves, we have to consider only polygons whose oriented directions of edges are spacelike, *i.e.* whose director vectors u satisfy $\langle u, u\rangle > 0$. However, this assumption is not enough for us to exclude the possibility of singularities – more precisely, of lightlike lines. I established the following result.

Theorem 2. *For every (possibly unclosed) polygonal curve P with $n+3$ edges, whose oriented directions of edges belong to the set $\mathcal{D}^n$ and are spacelike, there exists a maximal disk in $\mathbb{R}^{2,1}$ (which may have singularities) bounded by the curve P. Moreover, if P is unclosed, the maximal disk has a helicoidal end.*

This theorem is a new result, since the only known result for the Plateau problem for maximal surfaces is due to Quien [7] and deals with boundary curves of regularity $C^{3,\alpha}$.

2 The framework of Fuchsian systems

Garnier's method relies on a one-to-one correspondence between a class of Fuchsian equations and the space of minimal disks with a polygonal boundary curve. To solve the Plateau problem, the idea is instead of looking for a minimal disk with a given boundary, to look rather for its associated equation. The main part of the proof of Theorem 1 thus belongs to the domain of Fuchsian equations and Fuchsian systems.

Let us consider a second-order Fuchsian equation[2] on the Riemann sphere $\mathbb{P}^1$

$$y'' + p(x)y' + q(x)y = 0 \qquad (E_0)$$

and let S denote its singular set, which is finite. The solutions of Equation (E_0) are holomorphic functions on the universal covering space of $\mathbb{P}^1 \setminus S$. The *monodromy* of

[2] a linear ODE is said to be Fuchsian if all its singularities are Fuchsian, that is to say if they are simple poles of the coefficient $p(x)$, and simple or double poles of the coefficient $q(x)$.

Equation (E_0) is an equivalent class of representations of the fondamental group of $\mathbb{P}^1 \setminus S$, that measures the lack of uniformity of solutions around the singularities.

The *Riemann–Hilbert problem* is to prove that there always exists a Fuchsian equation with a given monodromy and a given singular set. If the given monodromy is irreductible, then we get a positive answer, provided that we authorize additional parameters: the apparent singularities. The apparent singularities are the singularities at which every solution is uniform.

We are also interested in *isomonodromic deformations* of Fuchsian equations: if we suppose that the position of the singularities depends on a variable parameter, how shall we describe the set of Fuchsian equations with a given common monodromy? For the Fuchsian equations we are interested in, isomonodromic deformations are described by the Garnier system, a completely integrable Hamiltonian system generalizing the sixth Painlevé equation.

Unlike Garnier, I mostly exclusively work with first-order 2×2 Fuchsian systems (instead of equations). When the position of singularities is varying, isomonodromic deformations of Fuchsian systems are described by the Schlesinger system, which is an integrable system. The Schlesinger system has the Painlevé property, whereas the Garnier system does not: it is free of movable branch point and essential singularity.

3 Outline of the dissertation

After a first introductory chapter, Chapters 2, 3 and 4 are devoted to the proof of Theorem 1. In Chapter 2, we draw up the relation established by Garnier's method between minimal surfaces with polygonal boundary curves and Fuchsian equations: we build a kind of dictionary. Then, in Chapter 3, for any given oriented directions $D = (D_1, \ldots, D_{n+3}) \in \mathcal{D}^n$, we explicitly construct by means of isomonodromic deformations the family of all minimal disks with a polygonal boundary curve of directions D. We finally prove by studying the length ratios, in Chapter 4, that every polygonal curve of directions D is the boundary of an element of that family. Chapter 5 is devoted to the proof of Theorem 2, which is not merely obtained by fitting the previous results to the Minkowski space, but requires some additional works.[3]

Chapter 1. Minimal surfaces and Fuchsian equations

General aspects of both minimal surfaces and Fuchsian equations domains are presented. The key-point for minimal surfaces is the spinorial Weierstrass representation: every minimal disk can be conformally represented on the upper half-plane

$$\mathbb{C}_+ = \{x \in \mathbb{C}\,,\ \Im(x) > 0\}$$

by an immersion $\mathcal{X} : \mathbb{C}_+ \to \mathbb{R}^3$ of the form

$$\mathcal{X}(x) = \Re\left(\int_{x_0}^{x} \left(i\left(G^2 - H^2\right), G^2 + H^2, 2iGH\right) \mathrm{d}x\right),$$

where the functions G and H, called the Weierstrass data, are holomorphic on $\mathbb{C}_+$, without commun zero. Conversely, every such functions G and H define a minimal conformal

[3] Paper [2] is made up of Chapters 1, 2, 3 and 4. Chapter 5 will be the subject of a later paper.

immersion. The induced metric and the Hopf differential of immersion $\mathcal{X}$ are then given by

$$\mathrm{d}s^2 = \left(|G|^2 + |H|^2\right)^2 |\mathrm{d}x|^2, \qquad Q = i\left(GH' - HG'\right)\mathrm{d}x^2.$$

We also provide a detailed introduction to Fuchsian equations and Fuchsian systems.

Chapter 2. The equation associated with a minimal disk with a polygonal boundary curve

This chapter is not directly devoted to the resolution of the Plateau problem. We rather show how one can associate a Fuchsian equation with a given minimal disk with a polygonal boundary curve, and we study the properties of this equation. The aim is to obtain a characterization of the equations that come from a minimal disk with a polygonal boundary curve.

Let us consider a minimal conformal immersion $\mathcal{X} : \mathbb{C}_+ \to \mathbb{R}^3$ of the upper half-plane into $\mathbb{R}^3$, and let (G, H) denote its Weierstrass data. Since they are linearly independent, the functions G and H are solutions of a unique second-order linear differential equation

$$y'' + p(x)y' + q(x)y = 0. \tag{E}$$

This is the equation associated with the minimal conformal immersion $\mathcal{X}$. When immersion $\mathcal{X}$ represents a minimal disk with a polygonal boundary curve, it appears that there is a nice correspondence between the geometry of $\mathcal{X}$ and analytical properties of its associated equation (E). The singularities of Equation (E) are then of two types: the pre-images of the verticies of the polygonal curve, which are real:

$$t_1 < \cdots < t_n < t_{n+1} = 0,\ t_{n+2} = 1,\ t_{n+3} = \infty,$$

and the umbilics, which are apparent singularities. By applying the Schwarz reflection principle, we prove that Equation (E) can be extended on the whole Riemann sphere $\mathbb{P}^1$, on which it is a real Fuchsian equation, and we determine how the Weierstrass data are transformed around the verticies t_i: we prove that the monodromy of Equation (E) is entirely determined by the oriented directions $D_1, \ldots, D_{n+3}$ of the polygonal boundary curve. Notice that there is no natural translation of the edge lengths of the polygonal boundary curve in terms of property of Equation (E).

For any given $(n + 3)$-tuple of oriented directions $D \in \mathcal{D}^n$, we define the space $\underline{\mathcal{E}}_D^n$ consisting in second-order linear differential equation equations that come from a minimal disk with a polygonal boundary curve of direction D. A three-points characterization of this space is deduced from the previous description: an equation (E) belongs to $\underline{\mathcal{E}}_D^n$ if and only if it satisfies the three following conditions:

(i) Equation (E) is Fuchsian on the Riemann sphere $\mathbb{P}^1$. Its singularities are the pre-images of the verticies $t_1, \ldots, t_{n+3}$, and the umbilics, which are apparent singularities. The exponents at the singularity t_i are $(1 - \alpha_i)/2$ and $-(1 - \alpha_i)/2$, where α_i is the measure of the angle between the oriented directions D_{i-1} and D_i such that $0 < \alpha_i < 1$.

(ii) A system of generators of the monodromy of Equation (E) is contained in the group $\mathrm{SU}(2)$, and it is expressed by cyclic products of the rotations of angle π and of direction D_i (half-turns).

(iii) Equation (E) is real, and the n-tuple of singular points $t = (t_1, \ldots, t_n)$ belongs to the simplex
$$\pi^n = \left\{(t_1, \ldots, t_n) \in \mathbb{R}^n \;\middle|\; t_1 < \cdots < t_n < 0\right\}.$$

In particular, the space $\underline{\mathcal{E}}_D^n$ is isomonodromic. From every equation in $\underline{\mathcal{E}}_D^n$, we can construct a polygonal curve of oriented direction D that bounds at least one minimal disk (by choosing an appropriate fundamental solution (G, H) of the equation). This polygonal curve is uniquely determined by the equation up to translations and to homotheties. We obtain by this way the family $\mathcal{P}_D$ of all (equivalent classes of) polygonal curves of direction D that bound at least one minimal disk.

For every oriented direction $D \in \mathcal{D}^n$, we need a more explicit description of the space $\underline{\mathcal{E}}_D^n$, in order to be able to study the family $\mathcal{P}_D$. For this purpose, we have to solve a Riemann–Hilbert problem (condition (ii)), to simplify the reality condition (iii) and to use isomonodromic deformations of Fuchsian systems. It is the subject of the following chapter.

Chapter 3. Isomonodromic deformations

For any given oriented direction $D \in \mathcal{D}^n$, we aim to get an explicit description of the family $\mathcal{P}_D$ of polygonal curves for which the Plateau problem has a positive answer. Actually, it is more convenient to define this family from first-order 2×2 Fuchsian systems, than from Fuchsian equations. Since we make the correspondence between Fuchsian systems and Fuchsian equations explicit, we can translate the three conditions (i), (ii) and (iii) relative to Fuchsian equations into three new conditions (a), (b) and (c) relative to Fuchsian systems (conditions (ii) and (b) are the same). The space $\underline{\mathcal{A}}_D^n$ of Fuchsian systems satisfying conditions (a), (b) and (c) is “bigger” than the space $\underline{\mathcal{E}}_D^n$, since a one-parameter family of systems corresponds to each single equation. Garnier uses Fuchsian systems like a punctual tool in the study of Fuchsian equations of the space $\underline{\mathcal{E}}_D^n$. On the contrary, we directly link the family $\mathcal{P}_D$ with the space $\underline{\mathcal{A}}_D^n$, in order to work exclusively with systems. This approach considerably simplifies the resolution of the Plateau problem.

To get a description of the space $\underline{\mathcal{A}}_D^n$, we first establish a necessary and sufficient condition **C1** for a Fuchsian system to satisfy the reality condition (c). Condition **C1** only depends on the monodromy of the system. Since we prove that monodromies satisfying condition (b) automatically satisfy condition **C1**, the space $\underline{\mathcal{A}}_D^n$ is then described by conditions (a) and (b) only. Garnier never mentions such a result, nor even the fact that we need it: he never checks that the equations he constructs are real. Let us notice that we have an analogous characterization of real Fuchsian equations from their monodromy, and that this result provides a charaterization of the real solutions of the Garnier system in terms of their monodromy data.

Since the Riemann–Hilbert problem always has a positive answer for first-order 2×2 Fuchsian systems, the space $\underline{\mathcal{A}}_D^n$ is never empty. We then prove that it contains an isomonodromic family of Fuchsian systems $(A_D(t), t \in \pi^n)$, described by the Schlesinger system and parametrized by the position of singularities $t = (t_1, \ldots, t_n)$, which is homeomorphic to the family $\mathcal{P}_D$. Precisely, for each value of t in the simplex π^n, there is a 2×2 invertible matrix $\mathbf{Y}(x, t)$ of solutions of the Fuchsian system $A_D(t)$, whose first line $(G(x, t), H(x, t))$ constitutes the Weierstrass data of a minimal disk with a polygonal boundary curve in $\mathcal{P}_D$. We denote this polygonal boundary curve by $P_D(t)$, and we thus obtain that the family $\mathcal{P}_D$ is described by the Schlesinger system:
$$\mathcal{P}_D = (P_D(t), t \in \pi^n).$$

Since the Schlesinger system has the Painlevé property – whereas the Garnier system does not –, we can easily deduce, using the reality condition (c), that the polygonal curve $P_D(t)$ analytically depends on t varying inside the simplex π^n.

Chapter 4. Length ratios

The goal of this chapter, that ends the proof of Theorem 1, is to show that every polygonal curve of oriented direction D belongs to the family $\mathcal{P}_D$. Since we work up to translations and to homotheties, a coordinate system on the space of equivalent classes of polygonal curves of oriented direction D is given by n ratios of edge lengths, between the $n+1$ finite lengths. To solve the Plateau problem, we thus have to prove that the n-tuple of length ratios of the polygonal curves belonging to the family $\mathcal{P}_D$ take all the values in the set $]0,+\infty[^n$.

Using the description of the family $\mathcal{P}_D$ by the Schlesinger system, we get the following expression of the length ratios of the polygonal boundary curve $P_D(t)$:

$$r_i(t) = \frac{\displaystyle\int_{t_i}^{t_{i+1}} \left(|G(x,t)|^2 + |H(x,t)|^2\right) \mathrm{d}x}{\displaystyle\int_0^1 \left(|G(x,t)|^2 + |H(x,t)|^2\right) \mathrm{d}x}$$

($i = 1, \ldots, n$). We then define the length-ratio function F_D associated with the oriented direction $D \in \mathcal{D}^n$

$$F_D : \pi^n \to]0,+\infty[^n, \qquad F_D(t) = (r_1(t), \ldots, r_n(t)).$$

We show that the length-ratio function F_D can be holomorphically extended in a simply-connected neighbourhood of the simplex π^n contained in the space

$$\mathcal{B}^n = \{(t_1, \ldots, t_n) \in (\mathbb{C}^* \smallsetminus \{1\})^n \quad | \quad \forall i \neq j \quad t_i \neq t_j\}.$$

To end the proof of Theorem 1, we have to establish the following theorem.

Theorem 3. *For any given oriented direction $D \in \mathcal{D}^n$, the length-ratio function F_D : $\pi^n \to]0,+\infty[^n$ is surjective.*

The proof of Theorem 3 mainly relies on the behavior of the function F_D at the boundary of the simplex π^n. The boundary of π^n is formed of faces that are lower-dimensional simplexes, characterized by equalities of the type $t_i = t_{i+1}$, that is to say by the fact that some singularities t_i are "missing". Let us consider a face P^k of dimension k of the boundary of the simplex π^n ($1 \leq k \leq n-1$). It is homeomorphic to the simplex π^k, and it is characterized by $n-k$ "missing" singularities t_i. We define the $(k+3)$-tuple of oriented directions $D' \in \mathcal{D}^k$, obtained from $D = (D_1, \ldots, D_{n+3})$ by "removing" the directions D_i corresponding to the singularities t_i that are missing. We establish the following proposition.

Proposition 1. *The function F_D can be continuously extended on the face P^k. Moreover, its restriction to P^k coincides, up to homeomorphisms, with the lower-dimensional length-ratio function*

$$F_{D'} : \pi^k \to]0,+\infty[^k.$$

Even if the geometrical meaning of this result is natural, its proof constitutes the most difficult step of the proof of Theorem 3. It is based on the behavior of the solutions of the Schlesinger system at its fixed singularities, that is to say at the points t such that $t_i = t_j$, $i \neq j$. This is a known part of Garnier's work [5], that has been further developed and generalized by Sato, Miwa et Jimbo [10]. By fitting this results to our situation, we estabish Proposition 1.

An induction on the number $n+3$ of verticies and a degree argument then enable us to conclude the proof of Theorem 3, and thus of Theorem 1 as well.

Chapter 5. The Plateau problem in Minkowski space

This chapter is devoted to the proof of Theorem 2. We apply the method we have developed in the previous chapters to the case where the ambient space is the three-dimensional Minkowski space $\mathbb{R}^{2,1}$.

We also have a spinorial Weierstrass representation for maximal conformal immersions $\mathcal{X} : \mathbb{C}_+ \to \mathbb{R}^{2,1}$ in terms of two holomorphic functions G and H without common zero. The induced metric and the Hopf differential of immersion $\mathcal{X}$ are then given by its Weierstrass data:

$$\mathrm{d}s^2 = \left(|G|^2 - |H|^2\right)^2 |\mathrm{d}x|^2, \qquad Q = i\left(GH' - HG'\right)\mathrm{d}x^2.$$

Immersion $\mathcal{X}$ would thus be singular if the moduli of the functions G and H coincide. Unfortunately, we do not have any controle on this condition by the equation associated with immersion $\mathcal{X}$. That's why we have to consider the class of *generalized* maximal disks with a polygonal boundary curve.

For any given $(n+3)$-tuple of spacelike oriented directions $D \in \mathcal{D}^n$, we define the space $\underline{\mathcal{E}}^n_D(2,1)$ of second-order linear differential equations associated with a generalized maximal disk with a polygonal boundary curve. It is very similar to the space $\underline{\mathcal{E}}^n_D$ (conditions (i) and (ii) above are the same). The only difference comes from the expression of the monodromy, since it depends on the isometries of the ambient space. However, this monodromy, which is also given by the direction D, satisfies condition **C1**. We also define the corresponding space $\underline{\mathcal{A}}^n_D(2,1)$ of Fuchsian systems, and we obtain by the same way the description by the Schlesinger system of the family $\mathcal{P}_D(2,1)$ of polygonal curves of oriented direction D that bound at least one generalized maximal disk.

The other difference with the Euclidean case is the expression of the length ratios, since it is also related to the ambient metric. On the boundary, the singularities of the generalized maximal disks we construct are isolated. Since the sign of the quantity $|G(x,t)| - |H(x,t)|$ changes at this singular points, the length ratios are given by:

$$r_i(t) = \frac{\displaystyle\int_{t_i}^{t_{i+1}} \left||G(x,t)|^2 - |H(x,t)|^2\right| \mathrm{d}x}{\displaystyle\int_0^1 \left||G(x,t)|^2 - |H(x,t)|^2\right| \mathrm{d}x}.$$

It is thus more difficult to derive from this expression the complex-analyticity of the length-ratio function F_D, and to study its behavior at the points $t_i = t_{i+1}$. The implicit function theorem enables us to obtain a simpler expression of the length ratios (without moduli), that can be used to study the function F_D.

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