GLOBAL ASPECTS OF INTEGRABLE SURFACE GEOMETRY

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The study of surfaces in 3-space has certainly been pivotal in the development of differential geometry and geometric analysis. Real dimension two is of course special in a number of ways: the possible topological types are easy to describe, one can take advantage of complex analysis in dimension one, and visualization of the objects of study in computer experiments allows many new conjectures to be formulated and tested. It is fair to say that nowadays surface geometry serves as a terrain where one can quickly migrate between diverse areas of mathematics, such as integrable systems, moduli spaces of connections and holomorphic bundles, surface group representations, algebraic geometry of special varieties, non-linear variational problems, mathematical physics, and computational methods and visualization, bringing the ideas and techniques of one to bear on another. This article attempts to describe some of these topics and their relevance to classical problems in surface geometry in a conceptual manner. We maintain an informal style with the hope of leaving the reader with some impressions of the subject and a snapshot of some methods under current development.

There is a long tradition of physics motivating advances in surface geometry. Early on Riemann found that the limiting isotherms of the heat of a candle placed under a surface provide it with conformal coordinates. The same method can be used to construct energy minimizing (harmonic) maps from surfaces into higher dimensional (non-positively curved) target spaces. Material properties of surfaces in space, like surface tension or bending energy, give rise to equilibrium shapes and interface models such as constant mean curvature and Willmore surfaces. And from a more esoteric perspective, string theory studies surfaces in appropriate target spaces subject to variational constraints.

It is perhaps not surprising then that the notion of a quantum field theory (albeit so far only in its classical formulation) offers a unifying perspective on surface geometry. Roughly speaking, a d-dimensional QFT is described by a d-dimensional *space-time* M, a space of *fields* $\mathcal{F}$ over M, and an action functional $\mathcal{L}\colon \mathcal{F} \to \mathbb{R}$ whose critical points are the *classical* field configurations. The objective is to calculate the partition function $\int_{\mathcal{F}} e^{i\mathcal{L}(f)}$ or its mathematically more rigorous variant $\int_{\mathcal{F}} e^{-\mathcal{L}(f)}$. In either case the critical points of $\mathcal{L}$ on $\mathcal{F}$—its *classical solutions*—can be used for perturbative calculations. Zero dimensional QFTs are Gaussian integrals whereas one dimensional QFTs give rise to the path integral formulation of quantum mechanics when applied to the space of curves describing particle evolution in some target space. We are concerned with dimension two in which there are a number of geometrically interesting action functionals:

- The non-linear sigma models, where $\mathcal{F} = \mathrm{Map}(M, P)$ is a space of maps from M into some auxiliary target space P with the Dirichlet energy $\mathcal{L}(f) = \int_M |df|^2$ as the action functional. In dimension two the action is invariant under conformal changes of metrics on M and is thus defined on a Riemann surface M. The

Date: June 3, 2008.
First, third, and fourth authors supported by DFG SPP 1154 "Global Differential Geometry".

classical solutions are *harmonic maps*, which play a fundamental role in surface geometry.

- Constant mean curvature (CMC) surfaces in 3-space (see figure 1) which are the classical solutions of the area functional on the space of immersions from M into a 3-dimensional space form P when constrained by a constant enclosed volume, the "Wess-Zumino term." The Gauß normal map of such a surface is a harmonic map, making this problem a special instance of a non-linear sigma model.
- Willmore surfaces which are classical solutions to the bending or Willmore energy, the average of the mean curvature squared $\mathcal{L}(f) = \int_M H^2$, on the space of immersions. This action is invariant under the Möbius group of the target space P which can therefore be taken to be the conformal 3-sphere (and more generally the conformal n-sphere). Again, this problem is a special example of a non-linear sigma model: the conformal Gauß map, which assigns to each point of the surface the unique touching 2-sphere whose mean curvature is that of the surface at that point, is a harmonic map into de Sitter space, the space of round 2-spheres in the 3-sphere.
- More generally gauge theories in dimension four, where $\mathcal{F}$ is a space of bundles with connections over M with the total field strength $\mathcal{L}(A) = \int_M |F^A|^2$ as the action functional, have 2-dimensional reductions inducing Higgs fields which are relevant to surface geometry. For instance, non-linear sigma models into a symmetric target space $P = G/K$ are described by the Yang-Mills-Higgs equations, a coupled system for a K-connection A and a TP-valued 1-form Φ, the Higgs field. This formalism is reminiscent of one in which gauge theories describe a moduli of integrable connections (the "closed forms") whereas sigma models, under certain conditions which non-abelian Hodge theory attempts to make precise, describe the subspace of those integrable connections which serve as "harmonic representatives of a cohomology class" by splitting "harmonically" into a metric connection and a self-adjoint Higgs field.

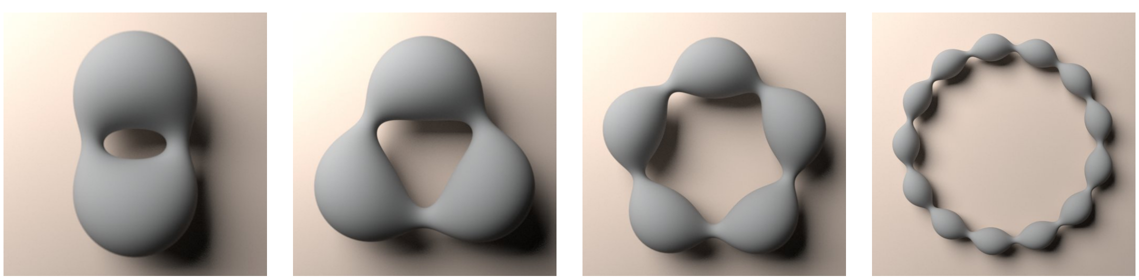

FIGURE 1. Equivariant CMC tori in $\mathbb{S}^3$ with increasing numbers of lobes. The surfaces are stereographically projected to $\mathbb{R}^3$.

One might hope that approaches used in physics to calculate the partition function of a QFT might also shed some light on solved and unsolved questions of global surface geometry. What are the classical solutions and the values of a particular action in a given topology (for instance, compact surfaces with finitely many punctures)? What are the properties of such surfaces? Which ones are embedded? And which conformal structures on M can be realized by classical solutions? Are there non-trivial examples which can be calculated in terms of special functions, such as hypergeometric functions or theta functions? Does the moduli space of solutions have some special structure (such as being a symplectic manifold or an integrable system) allowing an explicit description in terms of special functions, complementing the implicit description available from the existence results of non-linear analysis? Could such a description involve holomorphic data? The non-linear sigma model on a compact Riemann surface M with values in the 2-sphere $\mathbb{P}^1 = S^2$ is a prototypical test for intuitions about these questions. The absolute minima of the Dirichlet action (within a given homotopy class) are the meromorphic functions on M which are all described in terms of ratios of theta functions. For M of genus zero the meromorphic functions comprise all the critical points and there is a complete theory of the remaining critical points when M has genus one, where solutions are described again by theta functions [3] defined this time not on M but on an auxiliary hyperelliptic Riemann surface Σ. But we owe almost all of our current understanding of the case where M has higher genus to existence proofs and gluing constructions from non-linear analysis [25, 31, 36, 32].

The Fundamental Theorem of Surface Geometry. Connections on bundles appear initially in surface theory as they often do in the study of differential equations: as a means of using representations of the infinitesimal symmetries of an ambient space to formulate conditions for the existence of maps into the space which solve certain equations. To illustrate this, we recapitulate the fundamental theorem of surface geometry in $\mathbb{R}^3$ in an historically familiar manner that makes a first approximation to a more convenient approach we will elaborate later. A distinctive and essential feature of our setting is that $\mathbb{R}^3$ generates its own infinitesimal symmetries via the vector cross product and translation.

We can then begin by using a map $f\colon M \to \mathbb{R}^3$ to define a trivial bundle $V = f^*T\mathbb{R}^3$ over M which we view as the bundle of skew-hermitian endomorphisms of an auxiliary trivial hermitian bundle $L = \underline{\mathbb{C}}^2$, that is, $V = \mathbf{su}(L) = \underline{\mathbf{su}}(2)$. V has a flat and, in fact, trivial connection d as does L, which we also denote by d. (This notation is consistent with the fact that the connection d on L induces a connection on $\mathrm{End}(L)$ which, restricted to $\mathbf{su}(L) \subset \mathrm{End}(L)$, is given by d on V.) If f is an immersion, it splits V into a direct sum

$$V = TM \oplus \nu \tag{1}$$

of tangential and normal subbundles which is orthogonal for the standard inner product $< X, Y >= \frac{1}{2}\,\mathrm{tr}(XY^*)$ on $\mathbf{su}(2)$. And if f induces an orientation, then $\nu = N\mathbb{R}$ for a section N of V, the unit normal of f, which as an endomorphism of L has $\det N = \frac{1}{2}\,\mathrm{tr}\, N^2 = 1$. Besides spanning ν, the section N also serves as a complex structure on L (since $N^2 = -1$), and acts by conjugation on V to describe (1) as a ∓ 1-eigenbundle decomposition ($\mathrm{Ad}\, N|_{TM} = -1$ and $\mathrm{Ad}\, N|_{\nu} = 1$).

The connection d on V splits with respect to this decomposition,

$$d = \nabla^V + B \tag{2}$$

as a diagonal connection ∇^V and a skew-symmetric 1-form B. The restriction of ∇^V to TM is the Levi-Civita connection induced by f, and B acts as the second fundamental form on ν and as the Weingarten map on TM. Equivalently, ∇^V is the average of d and the gauged connection $(\mathrm{Ad}\, N)^{-1} \cdot d \cdot \mathrm{Ad}\, N$, whereas B is half their difference, from which we can compute

$$B(v) = \tfrac{1}{2}[NdN, v] \tag{3}$$

for $v \in \Gamma(V)$.

The connection d on L also has a splitting

$$d = \nabla^L + \phi \tag{4}$$

defined by requiring that ∇^L commute with N and ϕ anti-commute with N. Computing

$$\phi = \tfrac{1}{2}(d + N \cdot d \cdot N) = \tfrac{1}{2}NdN \tag{5}$$

we see that (4) corresponds to the splitting (2) insofar as ∇^V is the restriction to V of the $\mathrm{End}(L)$ connection induced by ∇^L. In the language of Clifford algebras, ∇^L is the spinor connection on the spinor bundle L corresponding to the spin structure on M which f induces as the restriction of the trivial spin structure of $\mathbb{R}^3$ to TM. Accordingly, the i eigenbundle L_+ of N is a dual spin bundle for f on M, meaning that L_+^2 with the holomorphic structure induced by $\nabla^L|_{L_+}$ is isomorphic to the dual canonical bundle K^{-1} with the holomorphic structure induced by the conformal class of $< df, df >$. We will arrive at this observation again later along a somewhat different route.

Writing $\nabla = \nabla^L$, the flatness of d on L is equivalent to

$$F^\nabla + d^\nabla\phi + \frac{1}{2}[\phi \wedge \phi] = 0.$$

Since N commutes with ∇ and anti-commutes with ϕ, we have $\mathrm{Ad}\, N(d^\nabla\phi) = -d^\nabla\phi$. Thus, the V-valued curvature 2-form of d (on L) decomposes relative to (1) into the Gauß–Codazzi equations

$$F^\nabla + \frac{1}{2}[\phi \wedge \phi] = 0 \text{ (Gauß)} \qquad \text{and} \qquad d^\nabla\phi = 0 \text{ (Codazzi)} \tag{6}$$

for the surface. $\nabla = \nabla^L$ is determined by the connection ∇^V, which in turn is determined by the Levi-Civita connection of g, the restriction of the inner product of $\underline{\mathbf{su}}(2)$ to TM. The Gauß–Codazzi equations can therefore be considered a coupled system of two non-linear partial differential equations for the pair (g, ϕ). As a TM-valued 1-form on M, ϕ decomposes into a trace part $H\,\mathrm{Id}$, where $H\colon M \to \mathbb{R}$ is the mean curvature, and a trace-free part which g makes into a bilinear form $q + \bar{q}$, where $q \in \Gamma(K^2)$ is the Hopf differential of f. The Gauß–Codazzi equations then take the form

$$K_g + H^2 - |q|_g^2 = 0 \qquad \text{and} \qquad g\partial H = \bar{\partial} q$$

regarding g as a section of $K\bar{K}$. A choice of a conformal coordinate z on M gives the local expressions $g = e^{2u}|dz|^2$ and $q = qdz^2$, from which we obtain the classical form of the Gauß–Codazzi equations

$$\triangle u + e^{2u}H^2 - e^{-2u}|q|^2 = 0 \qquad \text{and} \qquad H_z e^{2u} = q_{\bar{z}}\,. \tag{7}$$

The above construction can be reversed. We start with a Riemannian surface (M, g) together with a real rank 3 bundle $V := TM \oplus \underline{\mathbb{R}}$, the bundle metric $g \oplus dt^2$, and a self-adjoint $\beta \in \mathrm{End}(TM)$ which determines a skew-symmetric $B \in \mathrm{End}(V)$. If ∇ is the Levi-Civita connection for g, we can augment it with the trivial connection on $\underline{\mathbb{R}}$ to get a metric connection ∇^V on V. Then $d = \nabla^V + B$ is a connection which, since V splits, induces a connection with a corresponding splitting $d = \nabla^L + \phi$ as in (4) on a trivial $\mathbb{C}^2$ bundle L for which we regard $V = \mathbf{su}(L)$. We assume that ∇^L and ϕ satisfy the Gauß–Codazzi equations so that $d = \nabla^L + \phi$ is flat. Then d on V is also flat, which in turn pulls back to a flat metric connection, also called d, on the bundle π^*V over the universal cover $\widetilde{M}$ of M by the covering map π. Since π^*V is a flat bundle over a simply connected manifold, there is a global trivialization $\Psi : \pi^*V \cong \underline{\mathbb{R}}^3$ of flat metric bundles which restricts to an $\mathbb{R}^3$-valued 1-form α on the subbundle π^*TM. The projection of d onto π^*TM is the pull-back of the Levi-Civita connection ∇ of g and thus torsion-free, so that α is closed and therefore expressable as df for some isometric immersion $f\colon \widetilde{M} \to \mathbb{R}^3$.

Our interest lies, though, in immersions of M, so we view f as an equivariant function on $\widetilde{M}$ with two types of monodromy for the group of deck transformations: a translation resulting from a constant of the integration of α, and a rotation resulting from the holonomy representation $\rho\colon \pi_1(M) \to \mathbf{SO}(3,\mathbb{R})$ of the connection d. The task of adjusting our data (g, β) so that these monodromies vanish and we obtain an immersion of M, rather than its universal cover, is known as the *period closing problem* and represents an additional challenge beyond the already difficult problem of solving the Gauß–Codazzi equations.

CMC Surfaces and Loops of Flat Connections. Before presenting our approach to general surfaces in $\mathbb{R}^3$, we will focus on some techniques that have been successful for constructing CMC surfaces—at least of low genus. Using the Riemann surface structure on M induced by a given CMC immersion $f\colon M \to \mathbb{R}^3$, we write the $\mathbf{su}(2)$-valued 1-form $\phi = \Phi - \Phi^*$ with Φ now an $\mathbf{sl}(2,\mathbb{C})$-valued $(1,0)$-form on M, that is $*\Phi = i\Phi$. Since the mean curvature H is constant, the equations (6) become

$$F^{\nabla} = [\Phi \wedge \Phi^*] \qquad \text{and} \qquad \bar{\partial}^{\nabla}\Phi = 0. \tag{8}$$

These are a version of the self-duality equations over a Riemann surface [22, 23] known as the Yang-Mills-Higgs equations for the connection ∇ and the Higgs field Φ. The Hopf differential $q \in \Gamma(K^2)$, which is holomorphic for H constant, appears in this situation as $q = \frac{1}{H}\det\Phi$. The equations (8) are invariant under the S^1-action sending the pair (∇, Φ) to $(\nabla, \lambda^{-1}\Phi)$ for $\lambda \in S^1$ and, as a consequence, are equivalent to the flatness of the S^1-family of connections

$$d^{\lambda} = \nabla + \lambda^{-1}\Phi - \lambda\Phi^*\,. \tag{9}$$

In fact, d^λ extends to a $\mathbb{C}_*$-family of flat $\mathbf{SL}(2,\mathbb{C})$-connections which are unitary for $\lambda \in S^1$. Since d^λ arises from the solution of (8) given by an immersion of M, it is not only flat but also solves the period closing problem at $\lambda = 1$. The vanishing of rotational monodromy for this value just means that the family of holonomy representations $\rho^\lambda\colon \pi_1(M) \to \mathbf{SL}(2,\mathbb{C})$ of d^λ satisfies $\rho^1 = \pm 1$. The vanishing of translational monodromy has the less obvious consequence that $\frac{d}{d\lambda}|_{\lambda=1}\rho^\lambda = 0$. To see this, one can integrate (9) around an element γ of $\pi_1(M)$ to check that the zeroes of $\frac{d}{d\lambda}\rho^\lambda(\gamma)$ coincide with those of $\int_\gamma \lambda^{-2}\Phi + \Phi^*$. Evaluated at $\lambda = 1$ this expression gives the period for γ of the form $i * \phi$ which is closed by (8) and can be shown to have the same period as df. Alternatively, we can view this first-order condition as the limit at infinite radius of a zero-order condition for producing CMC surfaces in the 3-sphere which is interesting in its own right. Namely, if the connection d^λ is trivial for $\lambda = 1$ and some $\mu \in S^1$, $\mu \neq 1$, then the $\mathbf{SU}(2)$-gauge between those connections gives a CMC surface in S^3 of mean curvature $H = i\frac{1+\mu}{1-\mu}$. A similar condition exists for hyperbolic 3-space: if the connection d^λ is trivial for some $\mu \in \mathbb{C}$, $|\mu| \neq 1$, then it is also trivial at $\lambda = \bar{\mu}^{-1}$ and the gauge between those connections gives rise to a CMC surface in hyperbolic space H^3 of mean curvature $H = \frac{1+|\mu|^2}{1-|\mu|^2} > 1$.

Conversely, given a Riemann surface M, one solves the Gauß–Codazzi equations (8) (for a metric in the given conformal class) by finding a $\mathbb{C}_*$-family of flat connections d^λ of the form (9), that is, having simple poles at $\lambda = 0$ and ∞ with principal parts given by forms of type $(1,0)$ and $(0,1)$ respectively. There is a method [13], based on the Riemann-Hilbert factorization of matrix valued functions on contours, which builds such families from meromorphic $\mathbf{SL}(2,\mathbb{C})$-connections over M with unitarizable holonomy. To see the basic idea, we assume that d^λ is a flat family of the desired form and then gauge it, by a

λ-dependent gauge g, to a meromorphic family of connections. In order for the family to be meromorphic, $g : M \times \mathbb{C}_* \to \mathbf{SL}(2, \mathbb{C})$ has to solve the $\bar\partial$-problem

$$\bar\partial g = (\lambda - 1)\Phi^* g \tag{10}$$

where $\bar\partial = \frac{1}{2}(d + i * d)$ for the trivial connection d on L. Then the gauged connection has the form

$$g^{-1} \cdot d^\lambda \cdot g = d + \Psi\,, \qquad \Psi = \sum_{k \geq -1} \lambda^k \Psi_k \tag{11}$$

where, due to the flatness of d^λ, the Ψ_k are meromorphic 1-forms on M. Since the gauge g extends holomorphically as a function of λ to $\lambda = 0$, comparing λ coefficients in the usual relationship of gauge potentials under a gauge transformation shows that the Hopf differential of our surface of constant mean curvature 1 is encoded in the first order pole Ψ_{-1} in λ of Ψ via

$$q = \det \Phi = \det \Psi_{-1}\,. \tag{12}$$

Since our mean curvature is already constant, we see that Ψ characterizes d^λ up to a constant gauge. Thus Ψ serves as a kind of "Weierstraß data" which decribes the associated S^1-family of (generally non-closing) CMC surfaces up to rigid motions. Restated in the language of loop groups, we may view d^λ as a flat $\Lambda\mathbf{SU}(2)$-connection which is gauged by the $\Lambda^+\mathbf{SL}(2, \mathbb{C})$ gauge g to a meromorphic $\Lambda\mathbf{SL}(2, \mathbb{C})$ connection. Here, for a real Lie group G we denote by $\Lambda G^{\mathbb{C}}$ the group of holomorphic maps $h : \mathbb{C}_* \to G^{\mathbb{C}}$, by $\Lambda^+ G^{\mathbb{C}}$ those holomorphic maps which extend into $\lambda = 0$, and by ΛG those maps in $\Lambda G^{\mathbb{C}}$ which restrict to "loops" with values in G for $\lambda \in S^1$. Thus we arrive at the following method by which any CMC surface could, in principle, be constructed:

(i) Write down a meromorphic $\Lambda\mathbf{SL}(2, \mathbb{C})$-connection $d + \Psi$ of the form (11).
(ii) Gauge $d + \Psi$ to a $\Lambda\mathbf{SU}(2)$-connection d^λ of the form (9). The existence of an appropriate $\Lambda^+\mathbf{SL}(2, \mathbb{C})$-gauge g on the universal cover of M is guaranteed by the analog of the Iwasawa decomposition for loop groups, in our case $\Lambda\mathbf{SL}(2, \mathbb{C}) = \Lambda\mathbf{SU}(2) \cdot \Lambda^+\mathbf{SL}(2, \mathbb{C})$.
(iii) Ensure that d^λ descends to a connection on M. This is possible if the holonomy ρ_Ψ of $d + \Psi$ is unitarizable, that is, $\rho_\Psi : \pi_1(M) \to \Lambda\mathbf{SU}(2)$ after a conjugation.
(iv) Ensure that the CMC surface arising from d^λ at $\lambda = 1$ has vanishing periods. This requires that the unitary holonomy ρ^λ satisfy the closing conditions $\rho^1 = \pm 1$ and $\frac{d}{d\lambda}|_{\lambda=1}\rho^\lambda = 0$ as above (or the respective closing conditions for surfaces in S^3 and H^3).

To carry out this program one then needs to understand which meromorphic connections (11) can occur for CMC surfaces, or in other words, for what Weierstraß data (iii) and (iv) are possible. There are two instances in which this has been resolved.

The first is the case when $M = T^2$ is a torus. It is then known [34, 23, 7, 9] that the meromorphic connection $d + \Psi$ has a constant gauge potential of the form

$$\Psi = \lambda^{d-1}\eta dz \qquad \text{where} \qquad \eta = \sum_{k=-d}^{d} \lambda^k \eta_k \in \Lambda\mathbf{su}(2)$$

is a rational loop whose only poles are at $\lambda = 0, \infty$. At this point, we find an important link with integrable systems theory. We express η in the d^λ gauge by conjugating with g

to obtain the section

$$\xi = \sum_{k=-d}^{d} \lambda^k \xi_k \colon M \to \Lambda\mathbf{su}(2)\,. \tag{13}$$

As the gauge of a section parallel for $d+\Psi$, the section ξ is parallel for d^λ, giving the *Lax equation*

$$d^\lambda \xi = \nabla\xi + [\lambda^{-1}\Phi - \lambda\Phi^*, \xi] = 0\,. \tag{14}$$

Since (14) describes a flow on an adjoint orbit, the characteristic polynomial $\det(\xi - y)$ is constant on T^2. Thus the conserved quantities are recorded by the *spectral curve*—a hyperelliptic Riemann surface Σ which is the normalization of the possibly singular curve defined by

$$\det(\xi - y) = y^2 + \det\xi = 0\,. \tag{15}$$

The eigenline bundle $\mathcal{L}(p) \to \Sigma$ of ξ evolves according to (14) as p ranges over the torus T^2. In fact (by general arguments presented in [17]) the map

$$\mathcal{L}\colon T^2 \to \mathrm{Pic}(\Sigma)$$

is a linearization of the flow since $\mathcal{L}$ has constant derivative along the tangent of the Abel image of Σ at the origin. And because ξ is parallel for d^λ, the holonomy representation

$$\rho^\lambda(p)\colon \pi_1(M,p) \to \mathbf{Sl}(2,\mathbb{C})$$

of d^λ commutes with $\xi(p)$, so that $\mathcal{L}(p)$ coincides with the eigenline bundle of the family $\rho^\lambda(p)$. Thus d^λ and its associated family of (possibly non-closing) CMC tori can be reconstructed from algebro-geometric data: a hyperelliptic Riemann surface Σ and an initial holomorphic line bundle over Σ subject to certain compatibility conditions. This was first accomplished for certain data, along with the accompanying period closing problem, in [23] and [34], where it was recognized as a geometric version of traditional soliton methods of finding doubly periodic solutions (on $\mathbb{C}$) to the sinh-Gordon equation

$$\triangle u + 2\sinh(2u) = 0\,.$$

This is the Gauß equation (7) for a CMC torus since the Hopf differential can, in this case, be normalized as $q = dz^2$. In fact, [3] gives explicit formulas for all CMC tori in terms of theta functions on Σ. The simplest examples occur for Σ of genus 1. In this case, only one of the periods of the immersed torus can be closed if the ambient space is euclidean space $\mathbb{R}^3$ or hyperbolic space H^3, resulting in the classical Delaunay cylinders obtained by revolving the focal trace of a conic rolled along an axis (see figure 2). In the 3-sphere one can close both periods to obtain Delaunay tori (see figure 1), and more generally, all ([8]) CMC tori equivariant under rotations of S^3 in perpendicular planes of rationally commensurable speed. The Wente tori appear for spectral genus 2 and the shapes of CMC tori become more intricate as the spectral genus increases (see figure 3). The existence of CMC tori of arbitrary spectral genera was later confirmed in [15, 24, 10].

It should be noted that the moduli space of spectral curves for CMC tori in $\mathbb{R}^3$ is discrete. But for CMC tori in S^3, the moduli space is a one real dimensional manifold parametrized (away from singular points) by the mean curvature. This observation lies at the heart of recent developments [27] concerning the Lawson conjecture, which says that the only embedded minimal torus in S^3 is the Clifford torus—the unique minimal torus in S^3 whose spectral curve has genus zero. The basic idea is to deform (as suggested by figure 4) an embedded minimal torus via CMC tori in S^3 until one arrives at a CMC torus of spectral genus zero. With every drop in genus the spectral curve acquires double points so that

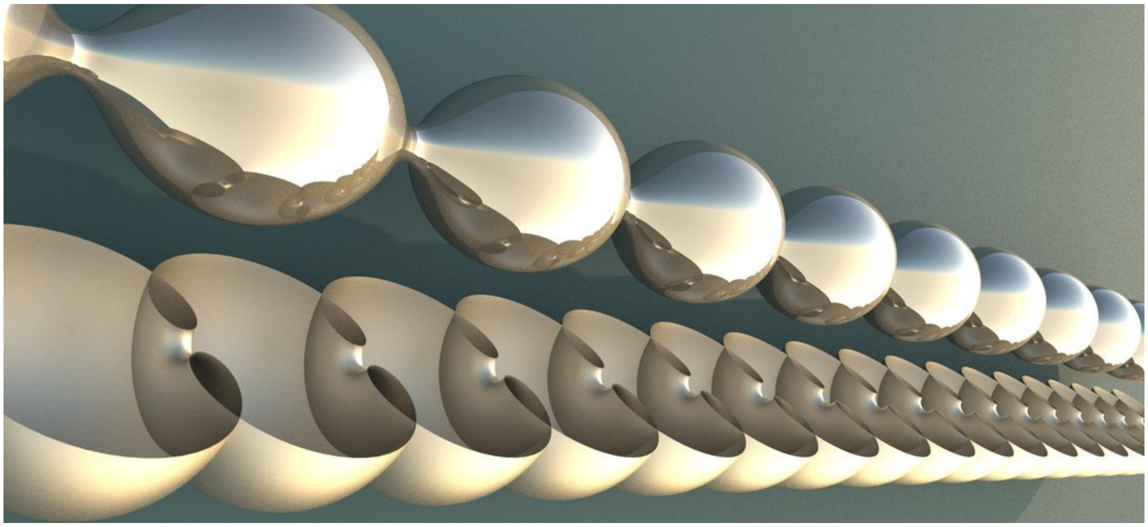

FIGURE 2. Delaunay cylinders, the embedded unduloids and immersed nodoids, have spectral genus 1.

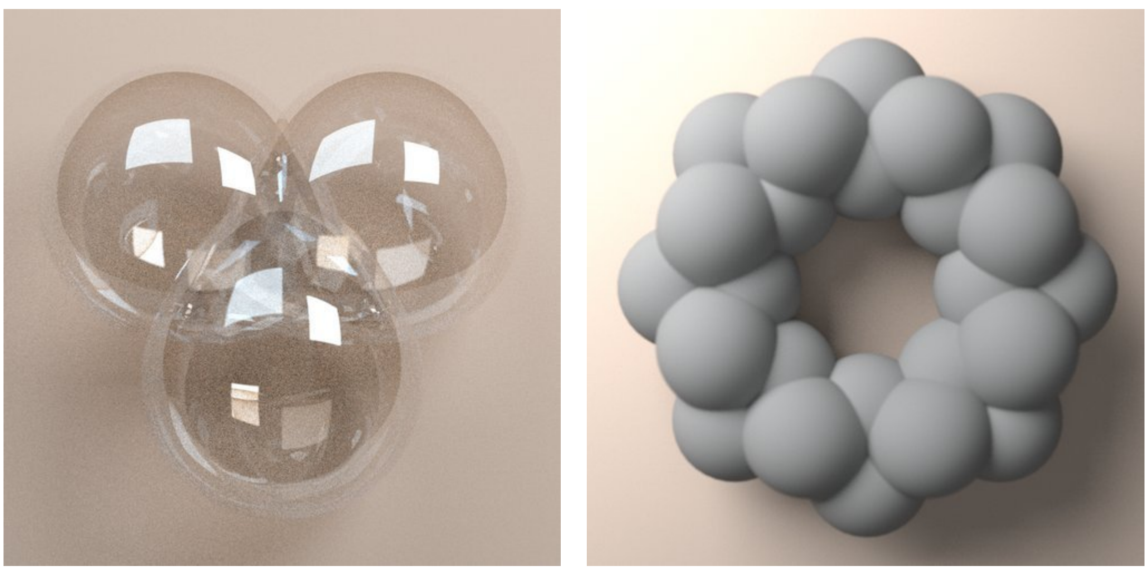

FIGURE 3. The Wente and Dobriner CMC tori have respective spectral genera 2 and 3.

if the initial spectral genus had been non-zero, one would arrive at a genus zero spectral curve with real double points, a configuration known not to give rise to an embedded CMC torus in S^3. Therefore the initial embedded minimal torus must have had spectral genus zero and thus must have been the Clifford torus. It is quite conceivable that a similar strategy might work for proving the Willmore conjecture which states that among all immersed tori, the bending energy $\int H^2$ has a global minimum of $2\pi^2$ at the Clifford torus. While the spectral curves of Willmore tori are discrete, the spectral curves of constrained Willmore tori (the classical solutions for the bending energy in a fixed conformal class) come again in families. Essentially one would try to deform the spectral curve of a

Willmore torus through spectral curves of constrained Willmore tori to the Clifford torus without increasing bending energy.

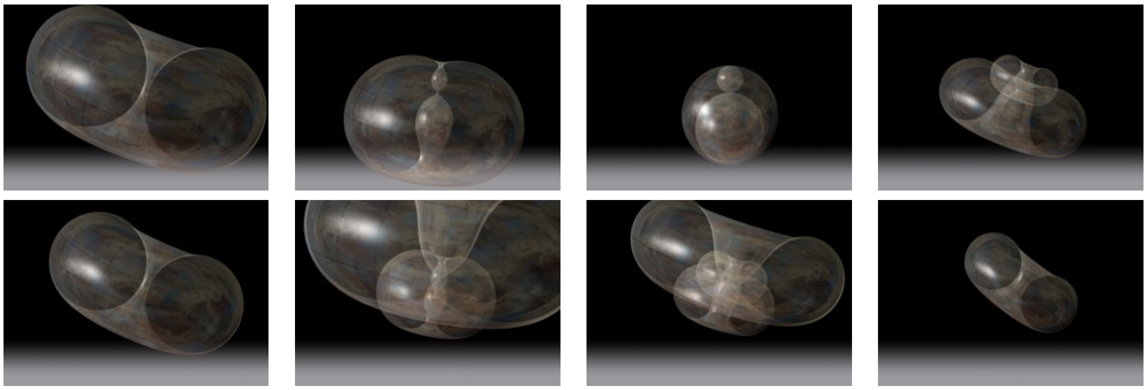

FIGURE 4. A path through the moduli space of equivariant CMC tori in $\mathbb{S}^3$. The first five images show a sequence which starts at a flat torus and passes through a chain of three spheres to arrive at a doubly-covered flat torus. The sequence goes on through a five-lobed tori terminating at a thrice-wrapped flat torus.

The second case where Weierstraß data allowing (iii) and (iv) have been determined is when $M = \mathbb{P}^1 \setminus \{p_1, \dots p_n\}$ is a punctured sphere. Here the method produces n-noids—CMC surfaces of genus zero with ends asymptotic to Delaunay cylinders. An overlap with the previous situation occurs for the Delaunay cylinders themselves which arise from the meromorphic connection

$$d + \Psi^{\mathrm{Del}}, \qquad \Psi^{\mathrm{Del}} = \begin{pmatrix} c & a\lambda^{-1} + \overline{b} \\ b + \overline{a}\lambda & -c \end{pmatrix} \frac{dz}{z} \tag{16}$$

on the twice puncture sphere with simple poles at the ends [26, 40]. The coefficients determine the neck size of the Delaunay cylinder. Because an end of an n-noid is asymptotic to a Delaunay cylinder, the holonomy for that end p_i of the associated connection $d+\Psi$ must be conjugate to that of a Delaunay cylinder. This implies [30] that the residue $\mathrm{res}_{p_i} \Psi$ must be conjugate to the residue of some Ψ^{Del} suggesting [14, 37, 40, 39] an ansatz

$$\Psi = \begin{pmatrix} 0 & \lambda^{-1} dz \\ \lambda Q/dz & 0 \end{pmatrix} \tag{17}$$

for the gauge potential with possible "apparent" singularities. The quadratic residue of the meromorphic quadratic differential Q at p_i is given by $\det \Psi^{\mathrm{Del}}$, which in turn is determined by the asymptotic neck sizes of the Delaunay ends. A simple Riemann-Roch count shows that Q, and thus the meromorphic $\Lambda\mathbf{SL}(2,\mathbb{C})$-connection $d + \Psi$, is determined up to $n-3$ analytic functions $h_k(\lambda)$ which must be holomorphic at $\lambda = 0$. Note that for 3-noids our meromorphic connection is then completely determined by the asymptotic Delaunay neck sizes and the spherical triangle inequalities on these neck sizes are the necessary and sufficient conditions for unitarizability (iii) of the holonomy [40]. This makes contact with the work of [18] in which embedded CMC 3-noids (and more generally embedded n-noids with reflectional symmetry across a plane) were classified using the conjugate cousin construction. Here a simply connected CMC surface in $\mathbb{R}^3$ is generated by a *conjugate minimal surface* in S^3 built from solutions to Plateau's problem.

For $n \geq 4$ the spherical n-gon inequalities on the asymptotic Delaunay neck sizes are only necessary conditions [2, 37] for the connection form (17) to have unitarizable holonomy. Sufficient conditions on the undetermined functions $h_k(\lambda)$ are not known. However, a recently uncovered ([12]) paper of Hilb [20] from 1908 has led to some interesting developments. In this and a sequel paper [21] Hilb, addressing a question that Felix Klein had posed in one of his courses, treats exactly the problem of determining the unitarizability of the holonomy of a Fuchsian equation with $n \geq 4$ singular points. Since Hilb works with what, in our language, is a meromorphic $\mathbf{SL}(2, \mathbb{C})$-connection, one would need to "loopify" his results. But even the loop-free version of Hilb's results has the significant surface geometric interpretation of showing the existence of n-oids of constant mean curvature 1 in hyperbolic space with non-degenerate catenoidal ends.

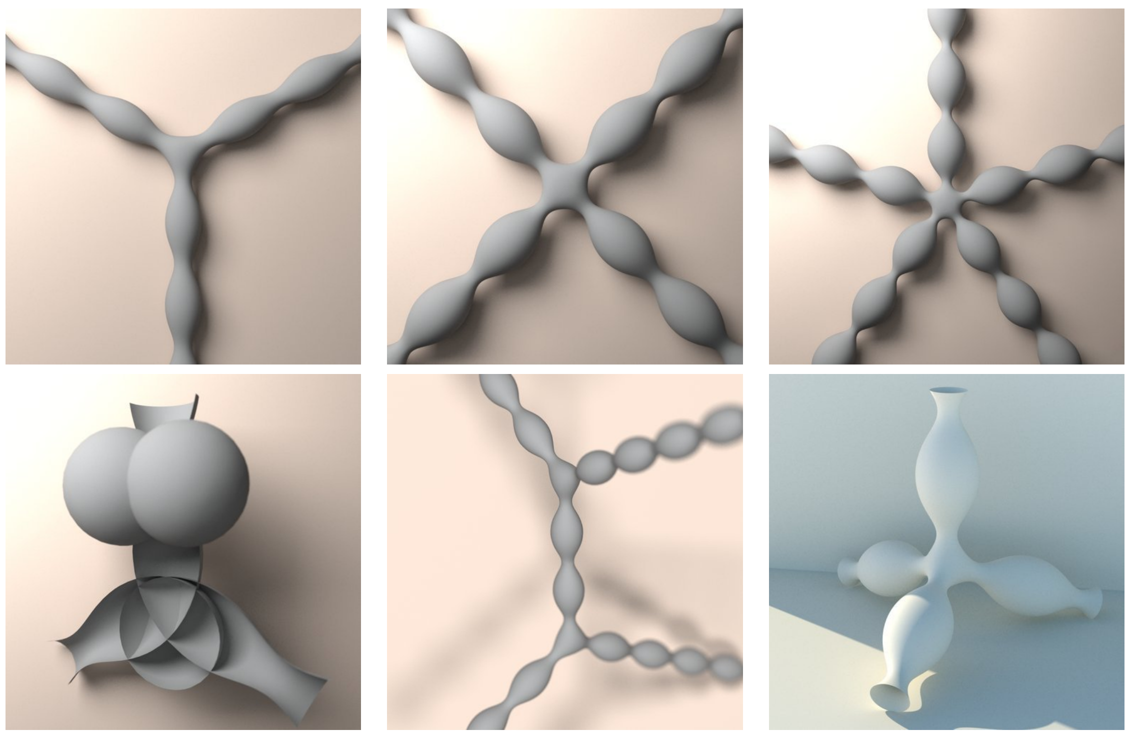

FIGURE 5. CMC n-noids. The trinoid, fournoid and fivenoid in the first row have prismatic symmetry [37]. Shown in the second row are a trinoid with a two-lobed bubbleton on one leg [28], a conjectural fournoid with central rod, and a tetranoid with tetrahedral symmetry [39].

In principle, the methods discussed so far could also be applied to more complicated variational problems in surface theory related to harmonic maps. But in most of these cases the relevant symmetry groups have higher rank or are non-compact, making it far more difficult to find explicit non-trivial examples. For instance, constructing Willmore surfaces with this approach would require loop group factorizations for the group $\mathbf{Sp}(1, 1)$.

Conformally Immersed Surfaces and Quaternionic Holomorphic Structures. A larger issue, however, seems to be how these methods might generalize to higher genus surfaces. Even though we can formulate the equations for, say, CMC surfaces $f\colon M \to \mathbb{R}^3$ in terms of holomorphic families of flat, or even meromorphic, connections on M, our ability to find solutions of those equations for a given M with non-trivial topology is limited by our inadequate understanding of special holomorphic families of holonomy representations of the fundamental group of M into $\mathbf{SL}(\mathbf{2},\mathbb{C})$. For the simplest cases of a 2-torus with its abelian holonomy, or a punctured $\mathbb{P}^1$ where holonomy is determined locally by the residues of a meromorphic connection, the approaches outlined above have been rather successful in the classification of solutions. But since a flat connection determines and is determined (at least up to gauge) by its holonomy, we cannot hope to avoid non-commutativity and non-local contributions in higher genus. Inspired by QFT, we then try to overcome the rigidity of this description by placing it within a larger context where the space of fields $\mathcal{F}$ being described is the space of all conformally immersed surfaces in 3-space. This will hopefully allow solutions to variational problems to be identified in terms of objects that are not as easily over-determined by topology and variational constraints as are connections.

The basic idea is the following: even though there is a unique holonomy for a given flat connection ∇ on a complex vector bundle, there is a range of possible *monodromies* for the holomorphic structure $\bar\partial = \nabla''$ given by this connection. Of course, requiring the monodromy to be unitary may again introduce restrictions, for example in the case of stable bundles, but in principle the monodromy of such a "half-connection" will be less rigid. To apply this idea to surfaces $f\colon M \to \mathbb{R}^3$ we will have to work with a more general object—a quaternionic holomorphic structure [33, 16] (a type of Dirac operator with potential)—but the principle remains the same.

We view a surface in $\mathbb{R}^3$ as a map into the imaginary part of an affine subset of $\mathbb{HP}^1 = S^4$, that is, $f\colon M \to \operatorname{Im}\mathbb{H} \subset \mathbb{H} \subset \mathbb{HP}^1$. Then f is the same as a quaternionic line subbundle $L \subset \underline{\mathbb{H}}^2$ of the trivial $\mathbb{H}^2$-bundle over M where $f(p) = L_p$. In other words, L is the pull-back of the tautological line bundle over $\mathbb{HP}^1$. By construction, our surface does not include the point at infinity $\infty = (1,0)\mathbb{H}$ in $\mathbb{HP}^1$ and therefore L is spanned by the section $\psi = (f,1)$. Moreover, the line bundle L is isotropic with respect to the hermitian form $< x, y >= \bar x_1 y_2 + \bar x_2 y_1$ on $\mathbb{H}^2$ since $< \psi, \psi >= 0$ if and only if f takes values in $\operatorname{Im}\mathbb{H} = \mathbb{R}^3$. Having the spanning section ψ allows us to define a flat quaternionic connection d with trivial holonomy, by requiring $d\psi = 0$, and a complex structure J, by the formula $J\psi = \psi(-N)$, where $N\colon M \to S^2 \subset \operatorname{Im}\mathbb{H}$ is the unit normal of f as a surface in $\operatorname{Im}\mathbb{H}$. Together, d and J give a quaternionic holomorphic structure

$$D = d''\colon \Gamma(L) \to \Gamma(\bar K L) \tag{18}$$

for which ψ is holomorphic, that is, $D\psi = 0$. Since $L \subset \underline{\mathbb{H}}^2$ is isotropic, we obtain the skew hermitian pairing

$$(\ ,\ )\colon L \times L \to TM^* \otimes \mathbb{H} \qquad \text{given by} \qquad (\varphi, \tilde\varphi) =< \varphi, d\tilde\varphi > \tag{19}$$

which, on the diagonal in $L \times L$, gives imaginary valued forms $(\varphi, \varphi) \in TM^* \otimes \mathbb{R}^3$. If f is also conformal, the pairing satisfies $*(\varphi, \tilde\varphi) = (\varphi, J\tilde\varphi) = (J\varphi, \tilde\varphi)$. The quaternionic holomorphic structure D is compatible with the pairing (19) as it obeys the product rule

$$d(\varphi, \tilde\varphi) = (D\varphi, \tilde\varphi) + (\varphi, D\tilde\varphi)\,. \tag{20}$$

Therefore, the pairing of any holomorphic section $\varphi \in H^0(L)$ with itself is a closed $\mathbb{R}^3$-valued 1-form (φ, φ) whose primitive $\int(\varphi, \varphi)\colon \widetilde{M} \to \mathbb{R}^3$ gives a conformal immersion (branched at the isolated zeros of φ) of the universal covering with translational periods. In particular, as (19) indicates, the holomorphic section $\psi = (f, 1)$ pairs to $(\psi, \psi) = df$.

To unravel the geometry of our setup and compare it to our previous descriptions of surface geometry, we first split the quaternionic line bundle $L = L_+ \oplus L_-$ into the two complex line subbundles given by the $\pm i$-eigenbundles of the complex structure J. Since $L_- = L_+ j$ and J is quaternionic linear these complex line bundles are in fact isomorphic and we may view the rank 2 complex vector bundle $(L, J) = E \oplus E$ as the double of the complex line bundle $E = L_\pm$. The flat connection d also decomposes as

$$d = \nabla + \phi\,, \qquad \phi = \frac{1}{2} J dJ \tag{21}$$

where $\nabla J = 0$, making ∇ a direct sum connection on $E \oplus E$ of two copies of a complex connection on E, and ϕ, as the complex anti-linear part of d, is an $\mathrm{End}_-(L)$-valued 1-form on M. Similarly, the holomorphic structure $D = d''$ decomposes into

$$D = \bar\partial + \phi'' \tag{22}$$

with $\bar\partial$ the double of a complex holomorphic structure on E and ϕ'' an $\mathrm{End}_-(L)$-valued $(0,1)$-form with respect to the complex structure J. Thus by restricting (20) to sections of E we see that the canonical holomorphic structure d of K corresponds to $\bar\partial$ on E via the Leibniz rule, making $(\,,) : E^2 \to K$ a holomorphic isomorphism and E a spin bundle. ∇ is then the spin connection induced by the Levi-Civita connection on M. One might have already suspected this to be the case since the relation between J and $-N$ allows us to use the Gauß-Bonnet theorem to calculate $\deg E = g - 1$ for compact M with genus g.

In order to understand the geometric content ϕ carries in our setting, we note that $\mathrm{End}_-(L) = \mathrm{Hom}(\bar E, E)$ so that $K \,\mathrm{End}_-(L) = K^2(\bar E E)^{-1}$ and $\bar K \,\mathrm{End}_-(L) = \bar E E$. Then ϕ' is given by the Möbius invariant Hopf differential $q/|df|$ and ϕ'' is given by the mean curvature half-density $H|df|$. Here H is the mean curvature function on M and q is the Hopf differential. Therefore ϕ is in essence the second fundamental form of the surface f.

It is also helpful to express our data in the more traditional "matrix" notation of, for example, [23]. Since complex anti-linear maps cannot be written as complex matrices, we will need to modify the splitting $(L, J) = E \oplus E$. We decompose instead with respect to the d-constant complex structure I which acts as multiplication by (the quaternion) i on L, giving $(L, I) = E \oplus \bar E$. A quaternionic linear endomorphism of L is linear, in particular, with respect to the complex structure I, and can therefore be written as a matrix. Since $J = \mathrm{diag}(i, -i)$ in this decomposition, we have

$$\phi' = \begin{pmatrix} 0 & q/|df| \\ -\bar q/|df| & 0 \end{pmatrix} \qquad \text{and} \qquad \phi'' = \begin{pmatrix} 0 & H|df| \\ -H|df| & 0 \end{pmatrix}. \tag{23}$$

The holomorphic structure D is therefore given by the Dirac operator with potential

$$D = \begin{pmatrix} \bar\partial & U \\ -U & \partial \end{pmatrix}, \qquad U = H|df| \tag{24}$$

on $\Gamma(E \oplus \bar E)$. The relationship of our data to that of [23] is brought into view by decomposing the second fundamental form

$$\phi = \Phi - \Phi^*\,, \qquad \Phi = \phi^{(1,0)}$$

into types with respect to the constant complex structure I, rather than J, resulting in the more traditional expression

$$\Phi = \begin{pmatrix} 0 & q/|df| \\ -H|df| & 0 \end{pmatrix} \tag{25}$$

for the $(1,0)$-part of the second fundamental form. In particular, we recover the forms (8) and (9) of the Gauß–Codazzi equations for CMC surfaces written in terms of the spin connection ∇ and the holomorphic Higgs field Φ.

This formulation makes contact with the generalized Weierstraß representation for conformally immersed surfaces in $\mathbb{R}^3$ as presented in [33, 41, 38, 42]. Here one chooses a complex holomorphic spin bundle E over a Riemann surface M along with a half density $U \in \Gamma(\bar{E}E)$ and solves the equation $D\varphi = 0$ on the quaternionic spin bundle $L = E \oplus \bar{E}$ with complex structure $J = \operatorname{diag}(i, -i)$. Then $f = \int(\varphi, \varphi)$ is a conformal immersion (branched at the isolated zeros of φ) into $\mathbb{R}^3$ with normal N, where $J\varphi = \varphi(-N)$. The Willmore energy of f is given by $\int U^2$. Notice that for $U = 0$ the formula for f becomes the classical Weierstraß representation for minimal surfaces in terms of complex holomorphic spinors [29].

We remark that there is also a construction (see [16]) of conformal immersions of M which (in our context) makes use of *anti-holomorphic* structures on L. For a given immersion $f : M \to \mathbb{H}$ the section $\psi = (f, 1) \in \Gamma(L)$ is in the kernel of d'. Since f is conformal, it scales ψ to another element of the kernel of d', so that f appears as the ratio of the two anti-holomorphic sections ψ and ψf (which are of course holomorphic sections if we view d' as a holomorphic structure for $-J$). Starting with an arbitrary anti-holomorphic structure $d' = \partial + \phi'$, where ϕ' is determined as above by a choice of Möbius invariant Hopf differential, we then produce conformal immersions of M without monodromy by taking ratios of sections in the kernel of d'. It is important to notice, though, that while the difference between anti-holomophic and holomorphic structures is only a matter of the sign of J, this latter construction is not geometrically equivalent to the Weierstraß representation. The ratio construction has the Möbius group as its symmetry group since it depends on the Möbius invariant potential ϕ'. The Weierstraß representation, on the other hand, uses the stretch-rotationally invariant potential ϕ'' and therefore has the euclidean similarities as its symmetry group.

Finally, note that both of these constructions define only conformal immersions, rather than the isometric immersions guaranteed by the classical fundamental theorem of surface geometry. For the latter we must choose a representative (M, g) of the conformal class of the Riemann surface M. Then E is a Riemannian spin bundle, that is, E has a hermitian structure inducing g on K and $\bar{E} = E^{-1}$. The density bundle is thus a trivial $\mathbb{R}$-bundle and ϕ is a putative second fundamental form determined by H and q. The Gauß–Codazzi equations for our data, (6) and (7), express that d is a flat connection. From here we can reconstruct an isometrically immersed surface up to rotational and translational periods.

The Spectral Variety of a Conformally Immersed Surface. After having introduced our approach to conformal surface theory in 3-space, we outline a construction of spectral data corresponding to the operator D defined by (18) for a general conformally immersed surface in $\mathbb{R}^3$ of arbitrary genus. Our motivation, as alluded to earlier, is a description of a surface $f\colon M \to \mathbb{R}^3$ of genus $g > 1$ solving some variational problem (CMC, Willmore, etc.) in terms of algebro-geometric data akin to the spectral curve and line bundle flow defined previously for tori. But in higher genus the holonomy spectral curve of the family (9) of flat connections d^λ of a CMC surface $f\colon M \to \mathbb{R}^3$ does not provide much more than a holomorphic mapping of $\mathbb{C}_*$ into the *representation variety* of $\pi_1(M)$ in $\mathbf{SL}(2, \mathbb{C})^{2g}$ (a hypersurface defined by the commutator relation on $\pi_1(M)$). We find that the non-commutativity of $\pi_1(M)$ leaves this picture largely impractical, so we take advantage of the range of monodromies offered by the "half-connection" D to define data that refer only to the abelian subset of this range. This also has the benefit that our theory is extended to include all conformally immersed surfaces in $\mathbb{R}^3$ beyond just the "classical solutions" such as CMC or Willmore surfaces.

We then define the *spectral variety* Σ of D to be the set of abelian representations $h\colon \pi_1(M) \to \mathbb{H}_*$ realized by monodromies of non-trivial holomorphic sections for D. In other words, the points $h \in \Sigma$ are the abelian representations for which there are non-trivial sections $\varphi \in \Gamma(\pi^* L)$ on the universal cover $\pi\colon \widetilde{M} \to M$ solving $D\varphi = 0$ and $\gamma^*\varphi = \varphi h_\gamma$ for $\gamma \in \pi_1(M)$ acting by deck transformations on $\widetilde{M}$. Note that when φ is scaled by a quaternion, the representation h gets conjugated. Due to the commutativity we may therefore assume that $h\colon H_1(M,\mathbb{Z}) \to \mathbb{C}_*$ is a representation into $\mathbb{C}_*$ of the abelianization of the fundamental group, the homology group of the surface M. Also, since conjugation by j preserves $\mathbb{C}_* \subset \mathbb{H}_*$ and L is spin, $h \in \Sigma$ implies that $\bar{h}$ and h^{-1} are also contained in Σ giving rise to a real structure and a holomorphic involution on Σ.

The spectral variety Σ has the surface-geometric interpretation as a natural family of conformal immersions $f^h\colon \widetilde{M} \to \mathbb{R}^3$ of the universal cover of M, containing the original surface f, whose periods commute, meaning that the rotational periods all have the same axis. This is provided by the generalized Weierstraß representation which for $h \in \Sigma$ gives the (branched) conformally immersed surface $f^h\colon \widetilde{M} \to \mathbb{R}^3$ as the integral of the 1-form $df^h = (\varphi, \varphi)$. Since h takes values in $\mathbb{C}_*$ every surface f^h then has stretch-rotational periods given by $\gamma^* df^h = \bar{h} df^h h$ around the i-axis in $\mathbb{R}^3 = \operatorname{Im}\mathbb{H}$.

As the name "spectral variety" suggests, Σ is indeed an analytic variety and has a rather special structure. If M is compact of genus $g \geq 1$ any representation $h\colon H_1(M,\mathbb{Z}) \to \mathbb{C}_*$ is given by the periods of harmonic forms $\omega \in \operatorname{Harm}(M,\mathbb{C})$ on M. Thus $h = \exp(\int \omega)$ and $\Sigma \subset \operatorname{Harm}(M,\mathbb{C})/\Gamma$ where Γ is the rank $2g$ lattice of integer period harmonic forms $\eta = \eta' - \overline{\eta'}$ satisfying $\int_\gamma \eta \in 2\pi i \mathbb{Z}$. Equivalently, $\Sigma \subset \operatorname{Harm}(M,\mathbb{C})$ can be viewed a Γ-periodic subset of harmonic forms on M. In this interpretation a harmonic form ω lies in the spectral variety Σ if and only if the lift of D to the universal cover $\widetilde{M}$ is gauged by $\exp(\int_* \omega)$ to an operator

$$D_\omega = \exp(-\int_* \omega) \cdot D \cdot \exp(\int_* \omega)$$

with non-trivial kernel. Notice that even though $\exp(\int_* \omega)$ is a function on $\widetilde{M}$, the gauged operator D_ω again descends to M. This can be seen from (24) which gives the explicit expression

$$D_\omega = \begin{pmatrix} \bar{\partial} + \omega'' & U \\ -U & \partial + \omega' \end{pmatrix}$$

showing also that D_ω is a holomorphic family of Dirac type operators over $\operatorname{Harm}(M,\mathbb{C})$. Since E is a spin bundle its degree is $g-1$ and the Riemann-Roch theorem gives zero for the index of $\bar{\partial}$, which in turn gives the index of the family D_ω. The techniques of [35, 1] then provide a holomorphic determinant line bundle as will be shown in [19]. Therefore Σ is the zero locus of the holomorphic determinant $\det D_\omega$ on $\operatorname{Harm}(M,\mathbb{C})$ and as such is an analytic variety. These same techniques yield evidence that Σ is asymptotic for large $\ln|h|$ to the spectral variety $\Sigma_0 \subset \operatorname{Harm}(M,\mathbb{C})$ of the "vacuum" operator

$$\bar{\partial}_\omega = \begin{pmatrix} \bar{\partial} + \omega'' & 0 \\ 0 & \partial + \omega' \end{pmatrix}.$$

But $\bar{\partial} + \omega''$ ranges over the Picard group $\operatorname{Pic}_{g-1}(M)$ of holomorphic structures of line bundles of degree $g-1$ and has non-trivial kernel along the theta divisor $\Theta \subset H^0(K)$. The vacuum spectrum Σ_0 thus consists of Γ-translates of the hypersurface

$$\Theta \times H^0(K) \cup \overline{H^0(K)} \times \bar{\Theta}.$$

Using this, a deformation argument then shows that Σ is indeed an analytic *hypersurface* in the complex cylinder $\mathrm{Harm}(M, \mathbb{C})/\Gamma$ asymptotic to Σ_0 for large $\ln |h|$.

The kernel of D_ω is generically 1-dimensional for $\omega \in \Sigma$ and we obtain the kernel line bundle $\mathcal{L} \to \Sigma$ whose generic fiber $\mathcal{L}_\omega = \varphi^\omega \mathbb{C}$ is spanned by a solution $D_\omega \varphi^\omega = 0$. Fixing $p \in M$ we can evaluate $\mathcal{L}_\omega(p) \subset L_p$ to obtain a line subbundle $\mathcal{L}(p) \subset \underline{L_p}$ of the trivial L_p-bundle over Σ.

There is some expectation that the spectral variety can be compactified if the original surface is a "classical solution" of an elliptic variational problem. Algebro-geometric techniques could then be applied to show that Σ and the map

$$M \to \mathrm{Pic}(\Sigma) \quad \text{assigning} \quad p \mapsto \mathcal{L}(p) \tag{26}$$

into the Picard group of holomorphic line bundles over Σ provide sufficient data for the reconstruction of the surface in $\mathbb{R}^3$. Ideally, (26) would be identified as the restriction to the Abel image of M in $\mathrm{Jac}(M)$ of a linear embedding of $\mathrm{Jac}(M)$ into $\mathrm{Pic}(\Sigma)$. The only evidence so far that this might be the case comes from the case when $M = T^2$ has genus $g = 1$. Then Σ is one dimensional [6, 5], and can be compactified to a finite genus curve for CMC [11] and Willmore tori [4, 38], both solutions to an elliptic variational problem. Here the map (26) is a (real) linear embedding of $T^2 \cong \mathrm{Jac}(T^2)$ into the Picard group of Σ which is in fact tangent to the (real part of the) Abel image of Σ. Moreover, this case is consistent insofar as Σ is indeed the spectral curve (15) defined by the holonomy of the family of flat connections (9) and $\mathcal{L}(p)$ has the same flow with respect to $p \in M$ as does the eigenline bundle of the holonomy. At present it is still unclear though how to detect spectral varieties arising from higher genus CMC surfaces.

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