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Contents

Introduction vii

Chapter I. The Hopf Fibration 1

1. The Hopf Map 1
2. The Topology of the Hopf Fibration 3
3. The Geometry of the Hopf Fibration 8

Chapter II. Conformal Fibrations and Complex Geometry 11

4. CR Manifolds 11
5. Conformal Submersions 19
6. Conformal Fibrations by Circles 22

Chapter III. Riemannian Submersions on 3−Manifolds 33

7. Connections and Curvature 33
8. Riemannian Submersions and Curvature 37
9. Curvature on Fibered 3−manifolds 39

Chapter IV. Conformally Flat Circle Bundles over Surfaces 45

10. Circle Bundles 45
11. Conformally Flat Circle Bundles 48
12. Classification over compact Surfaces 53

Chapter V. Harmonic Morphisms on 3−Manifolds 57

13. The Chern-Simons Invariant of 3−manifolds 57
14. Harmonic Morphisms with one-dimensional Fibers 61
15. Harmonic Morphisms on Conformally Flat 3−Spheres 64

Index of Symbols 73

Bibliography 75

Danksagung 77

Introduction

The Hopf fibration was introduced by Heinz Hopf in 1931 in order to show that the third homotopy group of the 2−dimensional sphere is non-trivial. It is taken as an important example not only in topology but in geometry, too. For example, it is well-known that the Hopf fibration is the only submersion of S^3 to S^2 up to diffeomorphisms, and it is also a non-trivial example of a Riemannian submersion. Later on, it was shown that the Hopf map is the only Harmonic morphism up to isometries of the round 3−sphere and up to Möbius transformations on the 2−sphere.

The aim of this work is to show the uniqueness of the Hopf fibration under weaker geometrical conditions. It is assumed to be the only conformal submersion of S^3. We are going to prove this conjecture under the restriction that the fibers are circles. Moreover, we are going to show that it is the only submersive harmonic morphism of a conformally flat 3−sphere if the curvature of its horizontal distribution is nowhere vanishing. A further result is the classification of conformally flat circle bundles over compact oriented surfaces. In case of S^3 as total space it gives us another uniqueness theorem for the Hopf fibration.

We start this work with a discussion of the Hopf fibration and give some well-known geometrical and topological properties with proofs.

In the second chapter we investigate conformal fibrations whose fibers are circles. We give a new description of the space of possibly degenerated, oriented circles as a complex quadric. It is used to show that the map, which assigns every base point to its fiber in the space of circles, is a holomorphic curve. As a corollary the fibration on hand is the Hopf map.

The third chapter is devoted to the theory of Riemannian submersions. We collect formulas for the Levi-Civita connection and for the Riemannian curvature tensor. We decompose the curvature tensor in order to give obstructions for the 3−space being conformally flat. We apply this to the case of Riemannian fibrations of 3−dimensional manifolds.

The topic of the fourth chapter are conformally flat circle bundles over compact, oriented surfaces. We use the results of the previous chapter to obtain partial differential equations, which must be satisfied on the surface. Under the conditions on hand these equations are reduced to an ordinary differential equation. Using topological invariants we solve it and classify conformally flat circle bundles.

In the last chapter we discuss the Chern-Simons functional for conformal 3-spaces, which is a global invariant. We recall basic facts about harmonic morphisms. We apply Chern-Simons theory to the case of a conformally flat 3−sphere admitting a harmonic morphism to compute certain integrals of curvature functions. With these integrals we can prove that the horizontal gradient of the curvature function of the horizontal distribution vanishes. It is the key step to the classification of submersive harmonic morphisms on conformally flat 3−spheres with a nowhere vanishing curvature of its horizontal distribution.

CHAPTER I

The Hopf Fibration

We start with some topological and geometrical properties of the Hopf fibration.

1. The Hopf Map

The Hopf fibration is a map from the 3−dimensional sphere onto the 2−sphere. It was introduced by Hopf [**H**] in order to show that the third homotopy group of the 2−sphere is non-trivial.

Definition. The Hopf fibration $\pi\colon S^3 \to S^2$ is the restriction of the projection $\mathbb{C}^2 \setminus \{0\} \to \mathbb{C}\mathrm{P}^1$ to the sphere $S^3 = \{(z,w) \in \mathbb{C}^2 \mid\mid z\mid^2 + \mid w \mid^2 = 1\}$:

$$(z,w) \in S^3 \subset \mathbb{C}^2 \mapsto [z:w] \in \mathbb{C}\mathrm{P}^1 \cong S^2.$$

We identify the complex projective space $\mathbb{C}\mathrm{P}^1$ with the 2−sphere in the standard way, i.e.

$$[z:w] \in \mathbb{C}\mathrm{P}^1 \mapsto \frac{1}{\mid z \mid^2 + \mid w \mid^2}(\mid w \mid^2 - \mid z \mid^2, 2z\bar{w}) \in S^2 \subset \mathbb{R} \times \mathbb{C}.$$

Obviously, the orientation of the complex space $\mathbb{C}\mathrm{P}^1$ is consistent with the one of the sphere induced by the outer normal. A formula for the Hopf fibration, considered as a map to $S^2 \subset \mathbb{R} \times \mathbb{C}$, is given by

$$\pi(z,w) = (\mid w \mid^2 - \mid z \mid^2, 2z\bar{w}).$$

It follows from the first definition that the fibers of the Hopf map are the circles given by

$$\pi^{-1}([z,w]) = \{e^{i\varphi}(z,w) \mid e^{i\varphi} \in S^1\}.$$

As the kernel of $\pi\colon \mathbb{C}^2 \setminus \{0\} \to \mathbb{C}\mathrm{P}^1$ at $p = (z,w)$ is given by

$$\ker \mathrm{d}_p \pi = \{\lambda(z,w) \mid \lambda \in \mathbb{C}\} \subset \mathbb{C}^2 \cong T_p\mathbb{C}^2,$$

the kernel of $\pi\colon S^3 \to S^2$ at $p = (z, w) \in S^3$ is given by

$$\ker \mathrm{d}_p \pi = \{\lambda i(z, w) \mid \lambda \in \mathbb{R}\} \subset (z, w)^\perp \cong T_p S^3.$$

Therefore, a result of Ehresman implies that π is a fiber bundle as a submersion of a compact space. One can see it also more directly: Consider the action

$$S^3 \times S^1 \to S^3; (z, w) \cdot e^{i\varphi} \mapsto (ze^{i\varphi}, we^{i\varphi}).$$

The orbits of this action are the fibers of the Hopf fibration, thus the Hopf fibration becomes a principal S^1-bundle, see 7.2 for a definition. Trivializations of this bundle over the affine sets $\mathbb{C}\mathrm{P}^1 \setminus \{[1 : 0]\}$ and $\mathbb{C}\mathrm{P}^1 \setminus \{[0 : 1]\}$ are given by

$$([z : 1], e^{i\varphi}) \mapsto \frac{e^{i\varphi}}{\sqrt{1+ \mid z \mid^2}}(z, 1)$$

and

$$([1 : w], e^{i\varphi}) \mapsto \frac{e^{i\varphi}}{\sqrt{1+ \mid w \mid^2}}(1, w).$$

The $3-$dimensional sphere is a Lie group, as one might see as follows: Consider $S^3 \subset \mathbb{H}$ as the subset of quaternions of length 1. The Euclidean scalar product on $\mathbb{H}$ is given by $< x, y >= \mathrm{Re}\,(x\bar{y}) = \mathrm{Re}\,(\bar{x}y)$, where $\bar{x}$ is the conjugate quaternion of $x \in \mathbb{H}$. Since $\mid xy \mid=\mid x \mid\mid y \mid$ the sphere $S^3 \subset \mathbb{H}$ is closed under multiplication in $\mathbb{H}$. The identity is 1, and the inverse of x is given by $x^{-1} = \frac{1}{|x|^2}\bar{x}$. The tangent space at 1 can be identified with $\mathrm{Im}\,\mathbb{H}$. We trivialize the tangent bundle of the Lie group S^3 via left translation, i.e.

$$\mu \in \mathrm{Im}\,\mathbb{H} \mapsto x\mu \in T_x S^3 = x^\perp \subset \mathbb{H}.$$

Then the integral curves of the vector field $\bar{I} \in \Gamma(TS^3) = \Gamma(S^3, \mathrm{Im}\,\mathbb{H})$ given by $\bar{I}_x = x^{-1}\mathbb{i}x \in \mathrm{Im}\,\mathbb{H}$ are exactly the fibers of the Hopf fibration, because of $xx^{-1}\mathbb{i}x = \mathbb{i}x$ and the description of the kernel of the Hopf fibration above. Obviously, this vector field is right invariant.

On the other hand, by considering integral curves of the left invariant vector field I with $I_x = x\mathbb{i}$, one obtains the conjugate Hopf fibration given by $\tilde{\pi}(z_0, z_1) = [z_0 : \bar{z}_1]$. Both maps, the Hopf and the conjugate Hopf fibration, differ only by the orientation reversing map

$$\bar{\ }\colon\ \mathbb{C}\mathrm{P}^1 \to \overline{\mathbb{C}\mathrm{P}}^1; [z_0 : z_1] \mapsto [z_0 : \bar{z}_1]$$

of the target spaces. We will show in the next section that each fibration of S^3 is homotopic to one of these two maps.

We give another description of the Hopf fibration. Consider the map $\rho\colon S^3 \to \mathrm{SO}(3)$ which is defined by its action

$$\rho(x)(\mu) = x\mu x^{-1}$$

on $\mu \in \mathrm{Im}(\mathbb{H}) \cong \mathbb{R}^3$. It is well-known that this is the universal covering of SO(3), therefore S^3 is the spin group Spin(3), see [**Fr**] for definitions. If one takes $\mu = \mathbb{i}$, one obtains a map

$$f\colon S^3 \to S^2 \subset \mathrm{Im}\,\mathbb{H}; x \mapsto x\mathbb{i}x^{-1}.$$

The differential of f at $x \in S^3$ is given by

$$\mathrm{d}_x\, f(v) = xv\mathbb{i}\bar{x} + x\mathbb{i}\bar{v}\bar{x},$$

so the left invariant vector field I spans the kernel $\ker \mathrm{d}_p\, f$ at any point $p \in S^3$. Now it is easy to compute that

$$f \circ\ ^{-1}\colon S^3 \to S^2; x \mapsto x^{-1}\mathbb{i}x$$

differs from the Hopf fibration only by a constant transformation of the 2−sphere. For the same reasons f is the conjugate Hopf fibration up to an element of SO(3) acting on S^2. As their geometric properties are equal we regard them as the same and we call $f \circ\ ^{-1}$ and f the Hopf and the conjugate Hopf fibration, too.

2. The Topology of the Hopf Fibration

In this section we describe how to classify submersions $\pi\colon S^3 \to M$ from S^3 onto a manifold M of strictly lower dimension. Of course, M must be compact and connected.

Proposition. *There is no submersion from the* 3−*sphere onto a manifold M with dimension* 1.

Proof. Since M is compact and connected we have $M \cong S^1$. Further, one can lift every map $f\colon S^3 \to S^1$ to a function $\hat{f}\colon S^3 \to \mathbb{R}$

with $e^{i\hat{f}} = f$, because the 3−dimensional sphere is simply connected. Therefore f has critical values and cannot be a submersion. □

We now consider surfaces as target space for submersions. First we show that M is diffeomorphic to a quotient of the sphere S^2.

2.1. Lemma. *If $\pi\colon S^3 \to M$ is a submersion to a connected 2−manifold then M is diffeomorphic to S^2 or $\mathbb{R}P^2$. The later case occurs if and only if the fibers are not connected.*

Proof. The fibers of π are closed 1−dimensional submanifolds. Let $\tilde{M}$ be the space of the connected components of all fibers. It is evident that $\tilde{M}$ has an unique structure as a manifold such that the map $c\colon \tilde{M} \to M$, which assigns each component of a fiber to the fiber, is a smooth covering map, and there exists an unique submersion $\tilde{\pi}\colon S^3 \to \tilde{M}$ with $c \circ \tilde{\pi} = \pi$.

If S^2 covers a surface M, then M has a finite fundamental group. The only two compact surfaces with a finite fundamental group are S^2 and $\mathbb{R}P^2$.

It remains to be proven that in the case that π has connected fibers, the base is the 2−sphere. Let $p \in S^3$ with $\pi(p) = q \in M$ and $F = \pi^{-1}(q)$ the fiber through p. Then, by taking p as base point in F and S^3, and q as base point in M, one gets the following exact sequence for the Homotopy groups

$$\begin{aligned} &... \longrightarrow \pi_n(F) \longrightarrow \pi_n(S^3) \longrightarrow \pi_n(M) \longrightarrow \pi_{n-1}(F) ... \\ &... \longrightarrow \pi_1(M) \longrightarrow \pi_0(F) \longrightarrow \pi_0(S^3) \longrightarrow \pi_0(M). \end{aligned}$$

A proof can be found in [**Br**]. With $F = S^1$ we deduce from $\pi_1(S^3) = 0$ and $\pi_0(S^1) = 0$ that $\pi_1(M) = 0$, but the only compact surface with trivial fundamental group is the 2−sphere. □

In the following we restrict ourselves to fibrations of S^3 with connected fibers. We will show that they are homotopic to the Hopf or the conjugate Hopf fibration.

Let $\pi\colon S^3 \to S^2$ be a submersion, and equip S^3 with a Riemannian metric such that the length of each fiber is 2π. Note that the fibers are oriented, so there exists an unique vector field T in positive fiber direction of length 1. We consider the following action of S^1 on S^3 : Let $\Phi\colon \mathbb{R} \times S^3 \to S^3$ be the flow of T, then

$$S^3 \times S^1 \to S^3;\ (q, e^{i\varphi}) \mapsto \Phi_\varphi(q)$$

is a well defined group action. The quotient map is π and $S^1 \to S^3 \to S^2$ becomes a principal S^1-bundle.

We consider the 1−dimensional complex representation $\rho\colon S^1 \to \mathrm{GL}(\mathbb{C}^1)$ and the associated complex line bundle $L = S^3 \times_\rho \mathbb{C}$, see 7.2 for a definition. The complex line bundles over compact oriented surfaces are characterized by their degree which is given by

$$\frac{i}{2\pi}\int_M \Omega,$$

where $[\Omega]$ is the first Chern character of L or more elementary, Ω is the $(\mathrm{Im}(\mathbb{C})-$valued) curvature of any metric connection on L. A proof can be found in [**Pe**].

It is well-known that the line bundle associated to the Hopf fibration has degree -1. In fact it is the tautological bundle over $\mathbb{C}\mathrm{P}^1$, as we will show below. The bundle associated to the conjugate Hopf fibration has degree 1.

2.2. Theorem. *Let $L \to M$ be a complex hermitian line bundle over a compact oriented surface and denote by $S(L) \to M$ the associated unitary frame bundle of L. Then the total space $S(L)$ is diffeomorphic to S^3 if and only if $M = S^2$ and the degree of L is ± 1.*

Proof. Clearly, the total space $S(L)$ is a compact oriented manifold of dimension 3. If it is diffeomorphic to S^3 we obtain from 2.1 that $M = S^2$, because $S(L) \to M$ is a fibration with connected fibers. We can choose two points $N \neq S \in S^2$, and trivializations

$$A = S(L)_{|\ S^2\setminus\{N\}} \cong_\Psi \mathbb{C} \times S^1$$

and

$$B = S(L)_{|S^2\setminus\{S\}} \cong_\Phi \mathbb{C} \times S^1$$

of $S(L)$ such that the transition function is like

$$\Phi \circ \Psi^{-1} \colon \mathbb{C} \setminus \{0\} \times S^1 \to \mathbb{C} \setminus \{0\} \times S^1; (z, e^{i\varphi}) \mapsto (1/z, (z/\mid z \mid)^{-n} e^{i\varphi}),$$

where n is the degree of the associated line bundle L. We compute the first fundamental group via the Seifert-Van Kampen Theorem, see [**Br**]: We have $\pi_1(A) = \pi_1(B) = \mathbb{Z}$ and $\pi_1(A \cap B) = \mathbb{Z}^2$. Let $p_i \colon \mathbb{Z}^2 \to \mathbb{Z}$ denote the projections onto the factors. For the embeddings $\imath \colon A \cap B \to A$ and $j \colon A \cap B \to B$ and the induced maps on the fundamental groups one obtains:

$$\imath_* = p_2 \colon \pi_1(A \cap B) = \mathbb{Z}^2 \to \pi_1(A) = \mathbb{Z}$$

and

$$j_* = -np_1 + p_2 \colon \pi_1(A \cap B) = \mathbb{Z}^2 \to \pi_1(A) = \mathbb{Z}.$$

The free product of $\pi_1(A)$ and $\pi_1(B)$ with amalgation $\pi_1(A \cap B)$ is

$$\mathbb{Z} *_{\mathbb{Z}^2} \mathbb{Z} = \mathbb{Z} \setminus (\deg(L)\mathbb{Z}),$$

which is by Seifert-Van Kampen the first fundamental group $\pi_1(S(L)$. Therefore $S(L)$ is simply connected, and consequently it is diffeomorphic to the 3−sphere, if and only if the degree is $\deg(L) = \pm 1$. □

2.3. Corollary. *The space of fibrations $S^3 \to S^2$ has exactly two components. Any two fibrations π_0, π_1 in the same component are homotopically equivalent in the following sense: There is a smooth path $\pi_t \colon S^3 \times [0; 1] \to S^2$, which is a fibration for all $t \in [0; 1]$ connecting π_0 and π_1.*

Proof. For the proof note that any two hermitian metrics on a line bundle over a surface can be deformed smoothly into each other, and that any two line bundles of the same degree are isomorphic. □

2.4. The linking number of the Fibers. We are going to investigate whether the fibers are linked with each other.

Definition. Let $\alpha, \beta \colon S^1 \to \mathbb{R}^3$ be oriented, embedded, closed and disjoint curves in the Euclidean 3−space. The linking number $\{\alpha, \beta\}$ of α and β is defined as the mapping degree of

$$f_{\alpha,\beta} \colon S^1 \times S^1 \mapsto S^2; (s, t) \mapsto \frac{\alpha(s) - \beta(t)}{\mid \alpha(s) - \beta(t) \mid}.$$

By the definition of the mapping degree, we can compute the linking number as

$$\{\alpha,\beta\} = \frac{1}{4\pi}\int_{S^1\times S^1} f^*_{\alpha,\beta}\,\mathrm{vol}_{S^2} = \frac{1}{4\pi}\int_{S^1\times S^1} \frac{1}{\mid \alpha-\beta \mid^3}\det(\alpha-\beta,\alpha',\dot{\beta}),$$

where $'$ denotes the derivative with respect to s and $\dot{}$ the derivative with respect to t. This formula was already known to Gauss. Note that the mapping degree does not depend on the order of the curves, i.e. $\{\alpha,\beta\} = \{\beta,\alpha\}$.

There is another way to describe the linking number for more general ambient spaces: Again, let $\alpha,\beta\colon S^1 \to M$ be oriented, embedded, closed and disjoint curves in a simply connected $3-$space. The linking number $\{\alpha,\beta\}$ of α and β is defined as follows. Choose an embedding of the unit disc

$$f\colon \{z\in\mathbb{C} \mid\mid z \mid\leq 1\} \to M$$

such that the oriented curve α is given by the boundary map $f_{|S^1} = \alpha$ and such that β intersects f transversely. The intersection number of f and β only depend on $f_{S^1} = \alpha$, and not on the embedding of f.

We only sketch why both definitions coincide: Choose a local orientation preserving diffeomorphism $x\colon U\subset M \to \mathbb{R}^3$ such that α and β lie in U. We assume that α and f are the circle and the disc of radius 1 around 0 in the $x-y$ plane. Whenever β intersects f at t_i, there is an unique point s_i on the circle such that $f_{\alpha,\beta}(s_i,t_i) = e_1 \in S^2$, and the intersection is positive orientated if and only if the differential $\mathrm{d}\, f_{\alpha,\beta}(s_i,t_i)$ is orientation preserving. Therefore, the assertion easily follows from the description of the mapping degree by counting the signed preimages of a regular point.

Obviously, the linking number of any two fibers of the Hopf map is the same. By taking the fibers $\pi^{-1}([1:0])$ and $\pi^{-1}([1:1])$, and the stereographic projection $S^3\setminus\{(0,0,0,1)\} \to \mathbb{R}^3$, the linking numbers are determined to be 1.

3. The Geometry of the Hopf Fibration

We briefly discuss the geometry of the Hopf fibration. We are going to introduce many pretty properties of the Hopf fibration that makes it so special.

The first observation is that the fibers of the Hopf map are great circles, i.e. they are intersections of S^3 with $2-$dimensional linear subspaces. In fact, the first definition of the Hopf fibration shows that the fiber through $(z, w) \in S^3 \subset \mathbb{C}^2$ is exactly the intersection of S^3 with the complex linear subspace spanned by (z, w). Evidently, this shows that the fibers are geodesics in the round sphere.

Next, we consider the map

$$f\colon S^3 \to S^2; x \mapsto x\mathbb{i}x^{-1}$$

which differs from the conjugate Hopf fibration only by a isometry of S^2. We have $\mathrm{d}_x f(v) = x(v\mathbb{i} - \mathbb{i}v)x^{-1}$ and it follows that

$$\frac{1}{2}\,\mathrm{d}_x f_{|\mathbb{i}^\perp} : \mathbb{i}^\perp \subset T_x S^3 \to T_{f(x)} S^2$$

is an isometry for each $x \in S^3$. Therefore, if S^2 is equipped with the metric of constant sectional curvature 4, the conjugate Hopf fibration becomes a Riemannian submersion, see III for a definition. The same is valid for the Hopf fibration.

Instead of investigating the fibers of the Hopf fibration one can study the preimages of closed curves in S^2. Topologically they are tori, but their geometry is much more interesting. The preimages of great circles in S^2 are Clifford tori, which are believed to be the absolute minimizers of the Willmore functional for tori. Moreover, Pinkall [**Pi**] used this method to show that every compact Riemannian surface of genus one can be conformally embedded in S^3, such that after a stereographic projection, the embedding is an algebraic surface in $\mathbb{R}^3$ of degree eight at most.

Another nice feature of the Hopf fibration is obtained by considering a locally defined harmonic functions f on S^2. Its pull-back $f \circ \pi$ on S^3

is harmonic, too. That means that the Hopf fibration is a so called harmonic morphism. It follows from a result in V, or more directly, from the formulas in chapter III.

3.1. The associated bundle. We are going to prove that the line bundle associated to the Hopf fibration, i.e. the tautological bundle, is a spin bundle of the Riemannian surface $\mathbb{C}\mathrm{P}^1$. This will be done in a different manner than usual in Riemannian geometry, see [**Fr**].

We recall the notation for holomorphic structures very briefly: The canonical bundle of a Riemannian surface (M, J) consists of complex valued and complex linear 1−forms

$$K_p = \{\omega\colon T_pM \to \mathbb{C} \mid i\omega = \omega \circ J\},$$

and similarly for the anti-canonical bundle

$$\bar{K}_p = \{\omega\colon T_pM \to \mathbb{C} \mid i\omega = -\omega \circ J\}.$$

A complex linear first order differential operator

$$\bar{\partial}\colon \Gamma(L) \to \Gamma(\bar{K} \otimes L)$$

is called holomorphic structure on L if it satisfies the Leibniz rule

$$\bar{\partial}(fs) = \bar{\partial}(f) \otimes s + f \otimes \bar{\partial}s$$

for all functions $f\colon M \to \mathbb{C}$ and sections $s \in \Gamma(L)$. Here

$$\bar{\partial}(f) = \frac{1}{2}(\mathrm{d}\, f + i\, \mathrm{d}\, f \circ J)$$

denotes the complex anti-linear part of $\mathrm{d}\, f$. In fact, $\bar{\partial}\colon \Gamma(\mathbb{C}) \to \Gamma(\bar{K})$ defines the standard holomorphic structure on the trivial bundle $\mathbb{C} \to M$. The use of holomorphic structures is due to the fact that the kernel of $\bar{\partial}$ can be considered as the space of holomorphic sections in L. Therefore we call $(L, \bar{\partial})$ a holomorphic line bundle.

One of the most important examples of a holomorphic bundle is the canonical bundle $K \to M$ equipped with the exterior differential operator d as holomorphic structure: It is easy to show that

$$\mathrm{d}\colon \Gamma(K) \to \Omega^2(M; \mathbb{C}) \cong \Gamma(\bar{K} \otimes K)$$

satisfies the Leibniz rule, and that the holomorphic sections are exactly the $1-$forms ω which can be locally written as $\omega = f\,\mathrm{d}\,z$, where f is holomorphic, and z is a holomorphic coordinate.

Definition. A holomorphic line bundle $(L, \bar{\partial})$ over a Riemannian surface M is called spin bundle if the square $(L \otimes L, \bar{\partial} \otimes \bar{\partial})$ is isomorphic to the canonical bundle (K, d) as a holomorphic bundle.

Proposition. *The complex line bundle $L = S^3 \times_\rho \mathbb{C}$ associated to the Hopf fibration, where ρ is the standard representation $\rho\colon S^1 \to \mathrm{GL}(\mathbb{C}^1)$ of S^1, is isomorphic to the tautological bundle. Equipped with its standard holomorphic structure, it is a spin bundle.*

PROOF. The tautological bundle Σ is given by

$$\{\lambda(z, w) \in \mathbb{C}^2 \mid \lambda \in \mathbb{C};\ [z : w] \in \mathbb{C}\mathrm{P}^1\} \to \mathbb{C}\mathrm{P}^1,$$

so the fiber of the tautological bundle over $[z : w] \in \mathbb{C}\mathrm{P}^1$ is equivalent to the complex line $[z : w] \subset \mathbb{C}^2$. The (locally defined) holomorphic sections of the tautological bundle are given by the (locally defined) holomorphic maps $f\colon \mathbb{C}\mathrm{P}^1 \to \mathbb{C}^2$ with $[f] = \mathrm{Id}\colon \mathbb{C}\mathrm{P}^1 \to \mathbb{C}\mathrm{P}^1$. We define

$$\Phi\colon \Sigma \otimes \Sigma \to K$$

by $\Phi((z, 1) \otimes (z, 1)) = \mathrm{d}\,z$ and $\Phi((1, w) \otimes (1, w)) = -\,\mathrm{d}\,w$. Clearly, it is well-defined. The holomorphic sections are mapped to holomorphic $1-$forms.

It remains to show that $L \cong \Sigma$. A linear isomorphism is given by

$$L = S^3 \times_\rho \mathbb{C} \to \Sigma; [(z, w), \lambda] \mapsto \lambda(z, w). \tag{3.1}$$

□

Remark. We will see in the next chapter (4.3), that there exist a holomorphic structure on L induced by the canonical CR holomorphic structure on S^3. It will be proved that this holomorphic structure is equivalent to the one induced via 3.1.

CHAPTER II

Conformal Fibrations and Complex Geometry

In this chapter we show how to study conformal fibrations on manifolds of dimension 3 by using complex methods.

4. CR Manifolds

We start with a short introduction of CR (Cauchy-Riemann) manifolds. Many ideas in the theory of CR manifolds, and in fact the subject of CR manifolds, are influenced by complex manifolds. We recall some facts.

4.1. Complex Manifolds. A complex manifold $(M, \mathcal{A})$ is given by a even dimensional real manifold M with an sub-atlas $\mathcal{A}$, such that all transition functions $V \subset \mathbb{R}^{2n} \cong \mathbb{C}^n \to U \subset \mathbb{R}^{2n} \cong \mathbb{C}^n$ of $\mathcal{A}$ are holomorphic. The most important example of a complex manifold is the complex projective space $\mathbb{C}\mathrm{P}^n$. The affine coordinates provide an atlas of holomorphic charts.

On complex manifolds one has the notation of holomorphic functions, complex tangent vectors and so on. The definitions of the first is clear. The definition of the second is technically the same as in the real case, too. A complex linear derivative on the space of germs of holomorphic functions is called a complex tangent vector. For each point $p \in M$ this is a complex $n-$dimensional vector space which will be denoted by $T_p^{(1,0)}M$. The relationship to the tangent space of the underlying real manifold M can be described as follows: The complexified tangent space $T_pM \otimes \mathbb{C}$ is the set of complex linear derivatives on the space of germs of complex-valued functions. Clearly, the complexified tangent space also acts on germs of holomorphic functions. We denote the kernel of this action by $T_p^{(0,1)}M$. We do the same with the space of anti-holomorphic functions, i.e. functions $f\colon U \subset M \to \mathbb{C}$ such that $\bar{f}$ is holomorphic, and we obtain that the kernel of the action of $T_pM \otimes \mathbb{C}$

on the space of anti-holomorphic functions can be identified with the complex tangent space $T_p^{(1,0)}M$. Moreover we get the decomposition

$$T_pM \otimes \mathbb{C} = T_p^{(1,0)}M \oplus T_p^{(0,1)}M,$$

and there is an unique operator

$$\mathcal{J} \colon TM \otimes \mathbb{C} \to TM \otimes \mathbb{C}$$

such that the complexified tangent space splits into the $\pm i$ eigenspaces of $\mathcal{J}$. In fact, $\mathcal{J}$ is even defined on TM, and with any holomorphic chart $\psi = (x_k + iy_k)_{k=1,..n}$ it is given by $\mathcal{J}(\frac{\partial}{\partial x_k}) = i\frac{\partial}{\partial y_k}$. Note that $(TM, \mathcal{J})$ and $(T^{(1,0)}M, i)$ are complex isomorphic.

This leads us directly to the following definition: An almost complex structure $\mathcal{J}$ is an endomorphism $\mathcal{J} \in \mathrm{End}(TM)$ with $\mathcal{J}^2 = -\mathrm{Id}$. A complex function is said to be holomorphic (with respect to $\mathcal{J}$) if $\mathrm{d}\, f \circ \mathcal{J} = i \,\mathrm{d}\, f$. A deep result of Newlaender and Nirenberg is that on a manifold with an almost complex structure there is an atlas consisting of holomorphic functions if and only if the Nijenhuis tensor N given by

$$N(X,Y) = [X,Y] + \mathcal{J}[\mathcal{J}X, Y] + \mathcal{J}[X, \mathcal{J}Y] - [\mathcal{J}X, \mathcal{J}Y]$$

vanishes. Thus a complex manifold is obtained as the transition functions are obviously holomorphic.

It would be tedious to discuss the geometry of complex manifolds here. Whenever its methods or results are needed below, we will review them shortly. As an introduction we refer to [**GriHa**].

4.2. CR manifolds. Before defining an abstract CR manifold we shall consider an important example. Let $M \hookrightarrow E$ be a real hypersurface of a $n-$dimensional complex manifold E. We define the Levi distribution H_p of M at p as

$$H_p := T_pM \cap \mathcal{J}(T_pM),$$

where $\mathcal{J}$ is the complex structure of E. The restriction of $\mathcal{J}$ is a complex structure on that space. We also consider the complex $n-1$ dimensional spaces

$$T^{(1,0)}M = TM \otimes \mathbb{C} \cap T^{(1,0)}E$$

and

$$T^{(0,1)}M = TM \otimes \mathbb{C} \cap T^{(0,1)}E.$$

Then there is the complex linear isomorphism

$$H \to T^{(1,0)}M; X \mapsto \frac{1}{2}(X - i\mathcal{J}(X))$$

as in the case of complex manifolds. Note that $T^{(1,0)}M$ is an integrable sub-bundle of $TM \otimes \mathbb{C}$.

Negating the question whether any two real hypersurfaces of $\mathbb{C}^2$ are (at least locally) biholomorphically equivalent was probably the first result in CR geometry. Cartan [**Ca1**] found invariants of CR manifolds, which determine whether CR submanifolds are equivalent via a holomorphic diffeomorphism. We only consider an invariant which is easy to handle and plays a major role in the whole subject of CR manifolds. The Levi form $\mathcal{L}$ is given by the curvature of the Levi distribution H :

$$\mathcal{L}\colon H \times H \to TM/H; (X,Y) \mapsto [X,Y] \mod H.$$

This is a skew-symmetric complex bilinear form, and, in the case of oriented M, its index is an invariant of the CR structure. A CR structure is called strictly pseudoconvex if the index is extremal, i.e. $\pm(n-1)$.

We give the definition of an abstract CR manifold.

Definition. A CR structure (of hypersurface type) on a manifold M of dimension $2n-1$ is a complex sub-bundle

$$T^{(1,0)} \subset TM \otimes \mathbb{C}$$

of complex dimension $n-1$ which is (formally) integrable and satisfies

$$T^{(1,0)} \cap \overline{T^{(1,0)}M} = \{0\}.$$

Here we used

$$T^{(0,1)} := \overline{T^{(1,0)}M} = \{\bar{v} \mid v \in T^{(1,0)}M\},$$

where $\overline{x \otimes z} := x \otimes \bar{z}$ for all $x \otimes z \in TM \otimes \mathbb{C}$. Note that CR hypersurfaces satisfy the integrability conditions automatically.

As in the case of complex manifolds, there is another way to define a CR manifold. Consider the $2n-2$ dimensional bundle

$$H := \mathrm{Re}\{T^{(1,0)}M \oplus T^{(0,1)}M\} \subset TM$$

called the Levi distribution with the complex structure

$$\mathcal{J}\colon H \to H; \mathcal{J}(v+\bar{v}) = i(v-\bar{v}).$$

Then the integrability of $T^{(1,0)}M$ can be written as

$$[\mathcal{J}X, Y] + [X, \mathcal{J}Y] \in \Gamma(H)$$

and

$$[X,Y] - [\mathcal{J}X, \mathcal{J}Y] + \mathcal{J}[\mathcal{J}X, Y] + \mathcal{J}[X, \mathcal{J}Y] = 0.$$

The Levi form is defined as in the case of CR hypersurfaces. A function $f\colon M \to \mathbb{C}$ is called CR function or CR holomorphic if

$$\mathrm{d}\, f(\mathcal{J}X) = i\,\mathrm{d}\, f(X)$$

for all $X \in H$. Similarly, a map $\psi\colon (M,H) \to (N,\tilde{H})$ between CR manifolds is called a CR map or CR holomorphic if

$$\mathrm{d}\,\psi(\mathcal{J}X) = \tilde{\mathcal{J}}\,\mathrm{d}\,\psi(X).$$

This is equivalent to

$$\mathrm{d}\,\psi(T^{(1,0)}M) \subset T^{(1,0)}N.$$

The question whether every abstract CR manifold of hypersurface type can be embedded as a hypersurface into a complex space can only be answered in the affirmative in case of analytic CR structures, or in case of strictly pseudo-convex CR structures with dimensions ≥ 9. A counterexample was given by Nirenberg in dimension 3. For more details and references see [**Bo**] or [**DT**].

4.3. Example. The round 3−sphere $S^3 \subset \mathbb{C}^2$ is obviously a CR hypersurface of $\mathbb{C}^2$. We consider functions $f\colon S^3 \to \mathbb{C}$ with the property that $f(pe^{i\varphi}) = e^{-i\varphi} f(p)$ for all $p \in S^3$ and $e^{i\varphi} \in S^1 \subset \mathbb{C}$. Such functions are exactly the sections of the associated bundle $L = S^3 \times_\rho \mathbb{C}$ which gives us the tautological bundle of $\mathbb{C}\mathrm{P}^1$, compare 3.1. Every such function can be extended to a function $\hat{f}\colon \mathbb{C}^2 \setminus \{0\} \to \mathbb{C}$ via

$\hat{f}(zp) = z^{-1}f(p)$ for $p \in S^3$ and $0 \neq z \in \mathbb{C}$. Obviously $\hat{f}$ is holomorphic if and only if f is CR holomorphic. One can obtain the associated section s_f of f of the tautological bundle $\sum \to \mathbb{C}\mathrm{P}^1$ of $\mathbb{C}\mathrm{P}^1$ via

$$s_f([z:w]) = \hat{f}(z,w)(z,w).$$

That shows that s_f is holomorphic if and only if f is CR holomorphic.

4.4. The Tangent Sphere Bundle. We end this section by illustrating the canonical CR structure of the tangent sphere bundle of an oriented conformal 3−space $(M,[g])$. Note that the CR structure here differs from the natural CR structure on the tangent sphere bundle of a Riemannian manifold of arbitrary dimension.

For a metric $g \in [g]$ in the conformal class of M, the unit sphere bundle of (M,g) is

$$S_gM := \{v \in TM \mid g(v,v) = 1\}.$$

We identify the unit sphere bundles of two metric $S_gM \sim S_{e^{2\lambda}g}M$ via

$$v \in S_gM \sim e^{-\lambda}v \in S_{e^{2\lambda}g}M.$$

This gives us the tangent sphere bundle $p\colon SM \to M$ of the conformal space $(M,[g])$. As an alternative it could have been defined as the bundle of oriented lines in the tangent spaces. Note that it can be considered as a (non-linear) sub-bundle of TM if and only if we fix a metric $g \in [g]$. We denote the elements of the tangent sphere bundle by $[v] \in SM$. Any two representatives v and $\tilde{v}$ of $[v]$ are positive multiples of each other.

The Levi distribution of the tangent sphere bundle is defined by

$$H_{[v]} := \{X \in T_{[v]}SM \mid \mathrm{d}_{[v]}\, p(X) \perp v\}.$$

At $[v] \in SM$, the complex structure $\mathcal{J}$ on H is given as follows: Let $g \in [g]$ be a metric in the conformal class and let $v \in T_pM$ be a vector of length 1 with respect to g. Consider the subspace $V := v^\perp \subset T_pM$. The tangent space of T_pM can be canonically identified with T_pM. By using g one obtains an exact sequence

$$0 \to V \to H \subset T_{[v]}SM \to_{p_*} V \to 0.$$

On V there is the canonical complex structure given by the cross-product $\mathcal{J}(w) = v \times w$ with respect to the orientation and g. Due to the Levi-Civita connection ∇^g there is an unique decomposition $T_vTM = \mathcal{V} \oplus \mathcal{H}$, where the parts are isomorphic to $T_{p(v)}M$, such that the covariant derivative ∇s for each section $s \in \Gamma(TM)$ is given by the vertical part $\pi^{\mathcal{V}}\,\mathrm{d}\,s$ of the derivative $\mathrm{d}\,s\colon TM \to TTM$, see 7.2 or [**SW**] for details. Of course, this splitting induces a decomposition of the sequence above, so there exists an unique complex structure $\mathcal{J}$ on H compatible with $\mathcal{J}$ on V and the sequence above. Note that the vertical part of H and the complex structure restricted to it do not depend on the choice of the splitting. To see that $\mathcal{J}$ is well-defined we use the formula

$$\tilde{\nabla}_X(e^{-\lambda}Y) = e^{-\lambda}(\nabla_X Y + Y \cdot \lambda X)$$

for the Levi Civita connection $\tilde{\nabla}$ of $\tilde{g} = e^{2\lambda}g$ and $X \perp Y \in \Gamma(TM)$. Therefore, the horizontal part $H_v \cap \mathcal{H}_v^{\tilde{g}}$ of the Levi distribution H_v for the new metric $\tilde{g}$ can be identified with

$$\{v \cdot e^{-\lambda}p_*X \oplus X \mid X \in H_v \cap \mathcal{H}_v^g\},$$

where $p_*X \in \mathcal{V}_v \cong T_p(v)M$. The complex structure defined on H via $\tilde{g}$ is compatible to the one on H defined by g, as

$$\tilde{\mathcal{J}}(v \cdot e^{-\lambda}p_*X \oplus X) = v \cdot e^{-\lambda}\mathcal{J}(p_*X) \oplus \mathcal{J}(X).$$

4.5. The Tangent Sphere Bundle of S^3. Instead of proving that the CR structure $\mathcal{J}$ satisfies the integrability conditions on every tangent sphere bundle, we consider the special case of the conformally flat S^3, and verify that its tangent sphere bundle is in fact a CR hypersurface.

We need to recall some facts of the twistor projection $\pi\colon \mathbb{C}\mathrm{P}^3 \to S^4$, first. In general the twistor space P of a four dimensional Riemannian manifold M is the bundle of almost complex structures compatible with the metric. The total space of the twistor projection inherits a canonical complex structure as follows: The fibers are round two spheres, as for a fixed vector N of length 1 any almost complex hermitian structure

$\mathcal{J}$ is determined by the unit vector $\mathcal{J}(N)$ of length 1 in $N^\perp$. Although the identification with a 2-sphere is not canonical but rather depends on the choice of N, the induced complex structure on the vertical space $\mathcal{V}$ is well-defined. Using the embedding of the space of complex hermitian structures P into $\operatorname{End}(TM)$ and the connection induced by the Levi-Civita connection on $\operatorname{End}(TM)$, one obtains the horizontal bundle as the tangents to parallel curves of complex structures, see 7 for more detailed [**SW**]. This horizontal space projects isomorphic onto the tangent space of M. The complex structure $\mathcal{J}$ lifts to the horizontal space $\mathcal{H}_\mathcal{J}$ at $\mathcal{J}$. Combined with the complex structure on $\mathcal{V}$ it defines the canonical almost complex structure on the twistor space.

There is a lovely description in case of the round S^4. For details of the quaternionic part consult [**BFLPP02**]. Consider $\mathbb{H}^2$ as a right vector space. Via the complex structure given by right multiplication with $\mathbb{i}$ we can identify $\mathbb{H}^2 \cong \mathbb{C}^4$. The twistor projection is given by

$$\pi\colon\ \mathbb{C}\mathrm{P}^3 \to S^4 \cong \mathbb{H}\mathrm{P}^1; e \in \mathbb{C}\mathrm{P}^3 \mapsto l = e\mathbb{H} \in \mathbb{H}\mathrm{P}^1,$$

which maps complex lines e to quaternionic lines $e\mathbb{H} = e \oplus e\mathbb{j}$. It matches with the definition of the twistor fibration, as we describe now. Like in the real or complex case, it is $T_l\,\mathbb{H}\mathrm{P}^1 \cong \operatorname{Hom}_\mathbb{H}(l, \mathbb{H}^2/l)$ via $\mathrm{d}_x\,p(v) \cong (x \mapsto v + x\mathbb{H})$, where $p\colon \mathbb{H}^2 \setminus \{0\} \to \mathbb{H}\mathrm{P}^1$ is the canonical projection. The round metric g on $\mathbb{H}\mathrm{P}^1$ is given by

$$g(\mathrm{d}_x\,p(v), \mathrm{d}_x\,p(w)) = \frac{2}{\langle\langle x, x\rangle\rangle}\operatorname{Re}\langle\langle v, w\rangle\rangle,$$

where $\langle\langle, \rangle\rangle$ is the standard quaternionic hermitian inner product on $\mathbb{H}^2$. Thus it is easy to see that every hermitian complex structure on $T_l\,\mathbb{H}\mathrm{P}^1 \cong \operatorname{Hom}_\mathbb{H}(l, \mathbb{H}^2/l)$ is given by post-composition with a quaternionic linear complex structure $\tilde{\mathcal{J}} \in \operatorname{End}_\mathbb{H}(\mathbb{H}^2/l)$ or, as l and $\mathbb{H}^2/l$ are both quaternionic $1-$dimensional, by pre-composition with a quaternionic linear complex structure $\mathcal{J} \in \operatorname{End}_\mathbb{H}(l)$. Hence the almost complex hermitian structures on $T_l\,\mathbb{H}\mathrm{P}^1$ are exactly the quaternionic linear complex structures on l. We can identify the complex line $e \in \mathbb{C}\mathrm{P}^3$ with hermitian structure on $T_{e\mathbb{H}}\,\mathbb{H}\mathrm{P}^1$ given by pre-composition with the unique complex structure $\mathcal{J}\colon l \to l$ such that

$$e = \{v \in l \mid \mathcal{J}(v) = v\mathbb{i}\}$$

is the $\mathbb{i}$ eigenspace of $\mathcal{J}$.

The round $S^3 \subset \mathbb{H}\mathrm{P}^1$ is the space of isotropic lines of the indefinite hermitian metric $\langle , \rangle$ given by

$$\langle v, w \rangle = \bar{v}_1 w_1 - \bar{v}_2 w_2.$$

Then the real quadric $Q := \pi^{-1}(S^3) \subset \mathbb{C}\mathrm{P}^3$ is the set of isotropic lines of

$$(,) \colon \mathbb{C}^4 \times \mathbb{C}^4 \to \mathbb{C}; (z, w) = \bar{z}_1 w_1 + \bar{z}_2 w_2 - \bar{z}_3 w_3 - \bar{z}_4 w_4$$

in $\mathbb{C}^4$. Evidently that is a CR hypersurface of $\mathbb{C}\mathrm{P}^3$. Let N be the oriented unit normal vector field of $S^3 \hookrightarrow S^4 \cong \mathbb{H}\mathrm{P}^1$. There is an one-to-one correspondence between almost complex hermitian structures and elements of the unit sphere bundle via

$$\mathcal{J} \in \mathrm{End}(T_l \, \mathbb{H}\mathrm{P}^1) \mapsto [\mathcal{J}(N)] \in S_l S^3. \tag{4.1}$$

We need a better understanding of the map above. Fix the embedding $x \in S^3 \subset \mathbb{H} \mapsto [x : 1] \in \mathbb{H}\mathrm{P}^1$. Using $\langle\langle , \rangle\rangle$ we can identify

$$T_{[x:1]} \, \mathbb{H}\mathrm{P}^1 \cong \mathrm{Hom}_{\mathbb{H}}([x : 1]; \mathbb{H}^2/[x : 1]) = \mathrm{Hom}([x : 1], [x : 1]^\perp)$$

and obtain for the oriented unit normal vector N_x at $x \in S^3 \hookrightarrow \mathbb{H}\mathrm{P}^1$ the quaternionic homomorphism which maps $(x, 1)$ to $\frac{1}{2}(x, -1)$. The tangent bundle of S^3 is given by $TS^3 \cong S^3 \times \mathrm{Im}\,\mathbb{H}$ via left translation. The tangent vector $\mu \in \mathrm{Im}\,\mathbb{H}$ at $x \in S^3$ corresponds to quaternionic linear mapping which assigns $\frac{1}{2}(x, -1)\mu$ to $(x, 1)$. Therefore, the almost complex hermitian structure of $T_{[x:1]} \, \mathbb{H}\mathrm{P}^1 \cong \mathrm{Hom}([x : 1], [x : 1]^\perp)$, which maps N to the vector given by $\mu \in \mathrm{Im}\,\mathbb{H}$, is the complex structure $\mathcal{J} \in \mathrm{End}([x : 1])$ with $\mathcal{J}(x, 1) = (x, 1)\mu$ via pre-composition. The complex line e corresponding to the tangential vector μ at x is determined to be

$$e = \{(x, 1)\lambda \mid \mathcal{J}(x, 1)\lambda = (x, 1)\lambda\mathbb{i}\} = \{(x, 1)\lambda \mid \mu\lambda = \lambda\mathbb{i}\}. \tag{4.2}$$

It remains to show that the mapping in 4.1 is CR holomorphic. Note that the vertical spaces of both bundles are contained in the respective Levi distributions, and the complex structures restricted to them are clearly the same by the description of the complex structure on the vertical space of the twistor space. The intersection of the Levi distribution and the horizontal space of Q is given by $\mathcal{J}(\hat{N})^\perp$, where $\hat{N}$

is the (horizontal lift of the) oriented unit normal field of $S^3 \hookrightarrow \mathbb{HP}^1$. The complex structure on it is defined by $\mathcal{J}(v) = \mathcal{J}(N) \times v$. Therefore it remains to show that the horizontal bundles correspond, where the horizontal bundle of SS^3 is given by the Levi-Civita connection of the round metric of S^3. It is easily deduced by the definition together with $\nabla_X(\mathcal{J}(N)) = (\nabla_X \mathcal{J})(N) + \mathcal{J}(\nabla_X N)$ and the fact that the unit normal field N of S^3 in S^4 is parallel.

5. Conformal Submersions

Conformal submersions are generalizations of Riemannian submersions as conformal immersions to isometric ones:

Definition. A submersion $\pi\colon P \to M$ between Riemannian manifolds (P, g) and (M, h) is called conformal if for each $p \in P$ the restriction of the differential to the complement of its kernel

$$\mathrm{d}_p\,\pi\colon\ \ker \mathrm{d}_p\,\pi^\perp \to T_{\pi(p)}M$$

is conformal.

Of course, the definition does not depend on the representative of a conformal class. The Hopf fibration is a basic example. We call $\mathcal{V} = \ker \mathrm{d}\,\pi$ the vertical space and its orthogonal complement $\mathcal{H} = \ker \mathrm{d}\,\pi^\perp$ the horizontal space.

5.1. Proposition. *A submersion between Riemannian manifolds is conformal if and only if the differential at a point, restricted to the horizontal space, is conformal and the Lie derivative of any vertical field on the metric restricted to the horizontal space is conformal, that means for all $V \in \Gamma(\ker \mathrm{d}\,\pi)$ and $X, Y \in \ker \mathrm{d}\,\pi^\perp$*

$$\mathcal{L}_V g(X, Y) = v(V) g(X, Y),$$

where $v(V)$ does only depend on V.

Proof. We only show that the condition is satisfied for the Lie derivative, the other direction follows by backtracking the proof below. Let $\hat{X}, \hat{Y} \in \Gamma(\mathcal{H})$ be the horizontal lifts of vector fields $X, Y \in \Gamma(TM)$. Then $[\hat{X}, V] \in \mathcal{V}$ and $[\hat{Y}, V] \in \mathcal{V}$ for any vertical field V, and we compute

$$\begin{aligned} v(V)g(\hat{X},\hat{Y}) &= V\cdot(g(\hat{X},\hat{Y})) \\ &= V\cdot(g(\hat{X},\hat{Y})) - g([V,\hat{X}],\hat{Y}) - g(\hat{X},[V,\hat{Y}]) \\ &= \mathcal{L}_V g(\hat{X},\hat{Y}). \end{aligned}$$

We used that $g(\hat{X},\hat{Y}) = \Lambda h(X,Y)$ and $v(V) = V\cdot\Lambda$ for the first equality. □

5.2. The CR structure of a fibered conformal $3-$manifold. Let $\pi\colon P\to M$ a submersion of a conformal oriented $3-$space to an oriented surface. We define a CR structure on P as follows: We set $H := \mathcal{H} = \ker \mathrm{d}\,\pi^{\perp}$ and define $\mathcal{J}\colon H\to H$ to be the rotation by $\frac{\pi}{2}$ in positive direction. We extend $\mathcal{J}$ to TP by setting $\mathcal{J}(T) = 0$ for $T\in\ker\mathrm{d}\,\pi$ and call $\mathcal{J}\in\mathrm{End}(TP)$ the CR structure on P. Note that $\mathcal{J}$ can be computed as follows: Fix a metric in the conformal class and the unit length vector field T in positive fiber direction, then $\mathcal{J}(X) = T\times X$.

Since conformal and complex structures correspond on oriented surfaces, one obtains:

Proposition. *There exist local defined, non-constant holomorphic functions on (P,H) which are constant along the fibers if and only if π is a conformal submersion for a suitable conformal structure on M.*

We come to a characterization of conformal submersions due to Pinkall, which gives the opportunity to study them by using methods of complex geometry. Another approach is done by [**BaWo2**].

Proposition. *A submersion $\pi\colon P\to M$ between an oriented conformal $3-$space to an oriented surface is conformal for a suitable complex structure on M if and only if the map Ψ, through which each point in P is assigned to the oriented fiber direction $[T]\in SP$, is CR holomorphic.*

Proof. We fix a Riemannian metric $g\in[g]$ on P. By definition of the CR structure of SP, $\mathrm{d}\,\Psi$ maps the Levi distribution of M into that of SP. Moreover it is obvious that the composition of the projection onto the horizontal part of the Levi distribution, which depends on the choice of g, and $\mathrm{d}\,\Psi$ commute with the $\mathcal{J}'$s. But the vertical part of $\mathrm{d}\,\Psi$, which again is depending on $g \in [g]$, is given by ∇T, where T is the unique vector field in positive fiber direction of length 1. It remains to show that π is conformal if and only if

$$\mathcal{J}\nabla_X T = T \times \nabla_X T = \nabla_{T\times X} T = \nabla_{\mathcal{J}X} T$$

for all $X \perp T$. But this equation is clearly equivalent to the characterization of 5.1. □

In the case of $P = S^3$ we have seen that SS^3 is a CR hypersurface of $Q \hookrightarrow \mathbb{C}\mathrm{P}^3$, hence Ψ is a CR immersion of S^3 with the induced CR structure into complex projective space. One might conjecture that the image of Ψ is at least locally the intersection of Q with a suitable complex surface . We were not able to prove this in general. A counterexample may be found on subsets of S^3.

Proposition. *Let $\pi\colon S^3 \to S^2$ be an analytic conformal submersion. Then the image of the oriented fiber tangent map Ψ is the intersection of a complex surface $A \subset \mathbb{C}\mathrm{P}^3$ and $SS^3 \cong Q \subset \mathbb{C}\mathrm{P}^3$.*

Proof. In the case of an analytic submersion the CR structure and the map Ψ are analytic, too. According to the real analytic embedding theorem, see [**Bo**], S^3 can be locally CR embedded into $\mathbb{C}^2$. Using the analyticity again, the map Ψ can be extended uniquely to a holomorphic map on a neighborhood of $S^3 \subset \mathbb{C}^2$. It remains to prove that the images of the extensions do match to a complex surface. This follows easily by the uniqueness of the extensions. □

Conversely, the intersection of a surface $A \subset \mathbb{C}\mathrm{P}^3$ with Q defines an oriented $1-$dimensional distribution (locally) which gives rise to conformal foliations or, if one restricts oneself to small open subsets, to conformal submersions.

Example. The easiest examples of surface in $\mathbb{C}\mathrm{P}^3$ are complex planes. Consider the one given by $A := \{[z] \in \mathbb{C}\mathrm{P}^3 \mid z_4 = 0\}$. As A intersects each tangent sphere $Q_p \cong \mathbb{C}\mathrm{P}^1$ over a point $p \in S^3$ exactly once, it defines a distribution on the whole S^3. In fact the integral curves of this distribution are the fibers of the conjugate Hopf fibration.

To prove this note that

$$A \cap Q = \{[z_1 : z_2 : 1 : 0] \mid \mid z_1 \mid^2 + \mid z_2 \mid^2 = 1\}.$$

That set corresponds to the complex structure which maps $1 = N \in TS^{3\perp}$ to $\mathbb{i} \in \operatorname{Im}\mathbb{H} = TS^3$, compare with 4.2.

6. Conformal Fibrations by Circles

We are going to prove that each conformal fibration of S^3 is, up to conformal transformations, the Hopf fibration. First we shortly describe the group of conformal transformations of spheres, for more details and proofs one should consult [**KP**].

6.1. Conformal transformations of spheres. The easiest way to obtain the group of conformal transformations is to use the light-cone model: Consider the Minkowski space $(\mathbb{R}^{n+2}, (,))$ for $n \geq 3$ with symmetric bilinear form

$$(,) = -\,\mathrm{d}\,x_0 \otimes \mathrm{d}\,x_0 + \sum_{i=1..n+1} \mathrm{d}\,x_i \otimes \mathrm{d}\,x_i.$$

The $n-$dimensional null-quadric $\{l \in \mathbb{R}P^{n+1} \mid q(x) = 0 \ \forall \ x \in l\}$ can be identified with the sphere S^n in the following way: Obviously, each null-line l intersects the affine hyperplane

$$A := \{x \in \mathbb{R}^{n+2} \mid (x, e_0) = -1\}$$

in exactly one point, but the set of these points is given by

$$\{x = (1, v) \mid v \in \mathbb{R}^{n+1}, \ \parallel v \parallel = 1\} \cong S^n.$$

The conformal structure is given via this identification from the standard round sphere. Hyperspheres are given as the intersection of hyperplanes with the space of null-lines.

We denote by $O((,))$ the group of linear isometries of $(\mathbb{R}^{n+2}, (,))$, and by $M(n)$ the subgroup which preserves the half-space $\{x \in \mathbb{R}^{n+2} \mid x_0 \geq 0\}$. Clearly, $M(n)$ acts on S^n, and it maps hyperspheres into hyperspheres. By the theorem of Liouville $M(n)$ must be a subgroup of the space of conformal transformations of S^n. It can be shown that this is already the whole group, see [**KP**]. We call the elements Möbius transformations.

Like one embeds $\mathbb{R}^{n+1} \hookrightarrow \mathbb{R}^{n+2}$ isometrically in the canonical way, one easily gets that any Möbius transformation on $S^{n-1} \hookrightarrow S^n$ can be extended to a Möbius transformation on S^n. This, together with the fact that the Möbius group of $S^4 \cong \mathbb{HP}^1$ is given by $\mathrm{GL}(2, \mathbb{H})/\{r \,\mathrm{Id} \mid r \in \mathbb{R}\}$ verifies that the Möbius group of S^3, which is given as the set of isotropic lines in $\mathbb{H}^2$ with respect to q with $q(v) = v_1\bar{v}_1 - v_2\bar{v}_2$, can be considered as the subgroup of $\mathrm{GL}(2, \mathbb{H})$, which leave q invariant, modulo its kernel, see [**BFLPP02**] for details.

6.2. The space of circles in S^3. We are interested in submersions of S^3 where the fibers are circles. Thus we need a good description of the space of circles in S^3 first. The following goes back to Laguerre, see also [**Ba**] for more details.

Definition. A circle in $S^3 \subset \mathbb{R}^4$ is the (nonempty) intersection of S^3 with an affine 2−plane A. It is oriented if the 2−plane A is oriented.

To get a better understanding of the space of circles we consider S^3 again as the space of null lines in $\mathbb{R}P^4$ with respect to the bilinear form $(,) = -\,\mathrm{d}\,x_0 \otimes \mathrm{d}\,x_0 + \sum_{i=1..4} \mathrm{d}\,x_i \otimes \mathrm{d}\,x_i$ on $\mathbb{R}^5$:

$$S^3 = \{l \in \mathbb{R}P^4 \mid (x, x) = 0 \ \ \forall x \in l\}.$$

Then every affine (oriented) 2−plane $A \subset \{x \in \mathbb{R}^5 \mid (x, e_0) = -1\}$ gives rise to an (oriented) 3−dimensional subspace $V_A \subset \mathbb{R}^5$. If A is given by $A = \{e_0 + \lambda v + \mu w \mid \lambda, \mu \in \mathbb{R}\}$, where $v \wedge w$ is positive oriented, then the orientation of $V_A = \mathrm{span}\{e_0, v, w\}$ is given by $e_0 \wedge v \wedge w$. We observe

Lemma. *Let $V \subset \mathbb{R}^5$ be a 3−dimensional subspace. Then the intersection $PV \cap S^3 \neq \emptyset$ if and only if there exist $v \in V \setminus \{0\}$ such that $(v, v) = 0$.*

Consider the subset $\mathcal{G}_3^- : = \{V \in \mathcal{G}_3 \mid \exists v \in V : \ (v, v) < 0\}$ of the Grasmannians $\mathcal{G}_3$ of oriented 3−dimensional subspaces, and for $V \in \mathcal{G}_3^-$ the space $V^\perp \subset \mathbb{R}^5$. In consequence of Sylvester's law of inertia the bilinear form $(,)_{|V^\perp}$, restricted to $V^\perp$, is positive definite. Let X, Y be an positive oriented orthonormal basis of $V^\perp$. Then $[X - iY] \in \mathbb{C}P^4$ is independent of the choice of X and Y.

We consider the complexification $(\mathbb{C}^5, (,))$ of $(\mathbb{R}^5, (,))$ with its bilinear form $(,) = -dz_0^2 + dz_1^2 + .. + dz_4^2$, the 3−quadric

$$Q^3 = \{l \in \mathbb{C}P^4 \mid (x,x) = 0 \quad \forall x \in l\}$$

and the induced embedding $S^3 \hookrightarrow Q^3$, i.e. S^3 is the set of real points in Q^3.

Lemma. *There is a natural bijection between* G_3^- *and* $Q^3 \backslash S^3$ *given by*

$$V \in G_3^- \mapsto [X - iY] \in Q^3.$$

Proof. For any $V \in G_3^-$ and any orthonormal basis X, Y of $V^\perp$ we get

$$(X - iY, X - iY) = (X,X) - (Y,Y) - 2i(X,Y) = 0,$$

therefore $[X - iY] \in Q^3$. Since $X, Y \neq 0$ we have $[X - iY] \notin S^3 \subset Q^3$.

Conversely let $v \neq 0$ with $[v] \in Q^3 \setminus S^3$. We write $v = X - iY$ where $X, Y \in \mathbb{R}^5$. Since $[v] \notin \mathbb{R}P^4$, we know that X and Y are linearly independent. Then

$$(X - iY, X - iY) = (X,X) - (Y,Y) - 2i(X,Y) = 0$$

implies that $(X,Y) = 0$ and $(X,X) = (Y,Y)$. We have to show that $(X,X) = (Y,Y) > 0$. Assume $(X,X) = (Y,Y) \leq 0$. After some normalization X and Y satisfy $X = e_0 + v,\ Y = \lambda e_0 + w$, where $v, w \in e_0^\perp,\ \lambda > 0$. Then we obtain the following system

$$\begin{aligned} 0 &= (X,Y) = (e_0 + v, \lambda e_0 + w) = -\lambda + (v,w) \\ 0 &\geq (X,X) = (e_0 + v, e_0 + v) = -1 + (v,v) \\ 0 &\geq (Y,Y) = (\lambda e_0 + v, \lambda e_0 + v) = -\lambda^2 + (w,w). \end{aligned}$$

A solution can only be found if $w = \lambda v$, contradicting our assumption that X and Y are linearly independent. Therefore $(X,X) = (Y,Y) > 0$ and $V = \text{span}(X,Y)^\perp$ contains a timelike direction. □

If we regard the points of S^3 as degenerated circles of radius 0 we obtain

Proposition. *The space of (possibly degenerated) oriented circles in S^3 is given by Q^3.*

In order to illustrate the geometric meaning of such a description of the space of circles see the lemma below

6.3. Lemma. *The set of points of a (oriented) circle $k = [v] \in Q^3$ is given by*

$$\{[X] \in S^3 \subset \mathbb{R}P^4 \subset \mathbb{C}\mathrm{P}^4 \mid (X, v) = 0\}.$$

Proof. If $[v] \in S^3$, i.e. k is degenerated, we can write $v = e_0 + w$ with $w \in e_0^\perp \subset \mathbb{R}^5$, and one easily gets that

$$\{[X] \in S^3 \subset \mathbb{R}P^4 \mid (X, e_0 + w) = 0\} = \{[v]\}.$$

If k is non-degenerated, we have $k = [X - iY]$ and the set of points in k is given by $S^3 \cap V$, where $V = \mathrm{span}(X, Y)^\perp$. □

We fix the metric of curvature 1 on S^3, which is equivalent as regarding S^3 as the unit sphere in the affine hyperplane $\{e_0 + v \mid v \in e_0^\perp\} \subset \mathbb{R}^5$ instead of the space of null lines, then the geodesics are exactly the great circles in S^3. In our notion they are equivalent to

6.4. Lemma. *The space of great circles is given as $Q^3 \cap Pe_0^\perp$.*

Proof. A great circle $[X] \in Q^3$ is characterized by the fact that for every point $[e_0 + v] \in [X] \subset S^3$ we get $[e_0 - v] \in [X] \subset S^3$. Therefore 6.3 implies that $(X, e_0) = 0$. □

There is a way to parametrize the space of (non-degenerated) oriented circles, which do not contain a given point $N \in S^3$, by $\mathbb{C}^3 \setminus \mathbb{R}^3$ holomorphically, such that the real part is the center of the circle in $\mathbb{R}^3$ and the imaginary part is the oriented radius, where one identifies $\mathbb{R}^3$ with $S^3 \setminus \{N\}$ via a suitable stereographic projection. Baird [**Ba**] did it in order to show that conformal fibrations of subsets of S^3 by circles are given by holomorphic curves. Another way to obtain this result is to embed the space of oriented circles holomorphically into the infinite dimensional almost complex manifold of oriented, closed and non-intersecting curves. Every conformal fibration of S^3 gives rise to a holomorphic curve of the complex space of fibers into the space of curves, i.e. into the space of all possible fibers.

We will state a new proof of this, which is based on another description of the space of circles. First we introduce a 2−form Ω. Let $Q \subset \mathbb{C}\mathrm{P}^3$ be the tangent sphere bundle of $S^3 \subset \mathbb{H}\mathrm{P}^1$, and consider the indefinite metric $\langle , \rangle$ on $\mathbb{H}^2$ as in 4.5. We decompose $\langle , \rangle$ into $(1, \mathbb{i})-$ and $(\mathbb{j}, \mathbb{k})-$ parts

$$\langle , \rangle = (,) + \Omega \mathbb{j},$$

where we regard $(,)$ as a hermitian form on $\mathbb{C}^4$ and Ω as a complex 2−form on $\mathbb{C}^4$. Note that Ω is non-degenerated and $\Omega \wedge \Omega \neq 0$ in the complex 1−dimensional space $\Lambda^4 \mathbb{C}^4$.

Lemma. *Let Ω be as described above, and $[T], [S] \in Q \subset \mathbb{C}\mathrm{P}^3$ lying over different points in S^3. Then there exists an oriented circle such that $[T]$ and $[S]$ are tangent to it in positive direction if and only if $\Omega(T, S) = 0$.*

Proof. We are only going to prove that $\Omega(T, S) = 0$ is a necessary condition. The reverse direction follows in analogous manner. As we have seen above, every conformal transformation of S^3 is given by a quaternionic linear map $\Phi \in \mathrm{GL}(2, \mathbb{H})$ with $\Phi^* \langle , \rangle = \langle , \rangle$. Then Φ acts on $Q \subset \mathbb{C}\mathrm{P}^3$, too. Clearly, this can be regarded as the differential of the conformal transformation Φ acting on the tangent sphere bundle. Therefore we can assume that $[T], [S] \in Q$ are tangent in positive direction to the circle given by $\{[z : 1] \in \mathbb{H}\mathrm{P}^1 \mid z \in S^1 \subset \mathrm{span}(1, \mathbb{i})\}$ with positive oriented tangents given by the set

$$\{[z : 0 : 1 : 0] \in \mathbb{C}\mathrm{P}^3 \mid z \in S^1 \subset \mathbb{C}\},$$

but this is the intersection of Q with the projectivation of the null-plane $\mathrm{span}(e_0, e_3)$ with respect to Ω. □

6.5. Remark. The proof of the lemma above shows that every circle is given by the intersection of Q with a contact line in $\mathbb{C}\mathrm{P}^3$, i.e. the projectivation of a 2−plane on which Ω vanishes. This intersection already determines the contact line. Conversely, the intersection of a contact line with Q is the set of tangents of a circle or, in the case of a plane spanned by $v, v\mathbb{j}$, the tangent sphere over a point $v\mathbb{H} \in S^3 \subset \mathbb{H}\mathrm{P}^1$.

The space of contact lines in $\mathbb{C}\mathrm{P}^3$ is given by the complex 3−quadric Q^3 via Klein correspondence, see also [**Bry**] for more details: Consider the 5−dimensional subspace W of $\Lambda^2(\mathbb{C}^4)$ defined as

$$W := \{\omega \in \Lambda^2(\mathbb{C}^4) \mid \Omega(\omega) = 0\}.$$

It is easy to verify that $\frac{1}{2}\Omega \wedge \Omega$ is non-degenerated on W, thus the space of null-lines with respect to $\frac{1}{2}\Omega \wedge \Omega$ is projectively equivalent to the 3−dimensional quadric Q^3. It is an easy computation that every null-line in W has the shape $[v \wedge w]$, hence it corresponds to the contact line $P\,\mathrm{span}(v, w)$ in $\mathbb{C}\mathrm{P}^3$. Conversely, the contact line $P\,\mathrm{span}(v, w)$ in $\mathbb{C}\mathrm{P}^3$ determines the null-line $[v \wedge w]$ as an element of $Q^3 \subset W$.

Instead of showing that the equivalence of the 3−quadrics in $\mathbb{C}\mathrm{P}^4$ and PW can be chosen such that corresponding points give the same circles, we prove an analogon of 6.3. To do so we introduce the real structure

$$\sigma\colon \Lambda^2(\mathbb{C}^4) \to \Lambda^2(\mathbb{C}^4); v \wedge w = (v\mathbb{j}) \wedge (w\mathbb{j})$$

on $\Lambda^2(\mathbb{C}^4)$. This is a real linear map with $\sigma^2 = \mathrm{Id}$. In fact, σ can be restricted to the space of Ω−null 2−vectors: $\sigma\colon W \to W$. Moreover, the real points $[\omega] \in Q^3 \cap PW$, i.e. $\sigma(\omega) = \omega$, have exactly the shape $\omega = v \wedge v\mathbb{j}$, which means that they can be regarded as quaternionic lines $v\mathbb{H}$ or points in $S^3 \subset \mathbb{H}\mathrm{P}^1$, compare 6.5.

6.6. Proposition. *A point $p = v\mathbb{H} \in S^3 \subset \mathbb{H}\mathrm{P}^1$ represented by the null-line $[\omega] = [v \wedge v\mathbb{j}] \in Q^3 \subset PW$ lies on an oriented circle k represented by $[\eta] \in Q^3 \subset PW$ if and only if*

$$\frac{1}{2}\Omega \wedge \Omega(\omega \wedge \eta) = 0.$$

PROOF. If p lies on the circle k we could choose $[v], [w] \in \mathbb{C}\mathrm{P}^3$ tangent to k, i.e. k is given by $[\eta] = [v \wedge w] \in Q^3$, with p is given by $[\omega] = [v \wedge v\mathbb{j}] \in Q^3$. Then

$$\frac{1}{2}\Omega \wedge \Omega(\omega \wedge \eta) = \frac{1}{2}\Omega \wedge \Omega(v \wedge v\mathbb{j} \wedge v \wedge w) = 0.$$

Conversely, let $\omega = v \wedge v\mathbb{j}$ and $\eta = w \wedge \tilde{w}$ representing p and k in Q^3 with $\frac{1}{2}\Omega\wedge\Omega(\omega\wedge\eta) = 0$. As $\frac{1}{2}\Omega\wedge\Omega \neq 0 \in \Lambda^4(\mathbb{C}^4)$ we deduce that $v, v\mathbb{j}, w$ and $\tilde{w}$ are linear dependent. If $\eta = w \wedge \tilde{w}$ would represent a degenerate circle, we could assume $\tilde{w} = w\mathbb{j}$, and $v, v\mathbb{j}, w, w\mathbb{j}$ are (complex) linear dependent if and only if they lie on the same quaternionic line. That means $v\mathbb{H} = w\mathbb{H} \in S^3 \subset \mathbb{H}\mathrm{P}^1$ and $p = k$. Otherwise, the complex plane $\mathrm{span}(w, \tilde{w})$ would intersect the complex plane $\mathrm{span}(v, v\mathbb{j})$. For an element $\hat{w} \in \mathrm{span}(w, \tilde{w}) \cap \mathrm{span}(v, v\mathbb{j})$, $[\hat{w}] \in Q \subset \mathbb{C}\mathrm{P}^3$ is tangent to the circle k at p. □

We will only work with the second model below. If $\pi\colon S^3 \to \mathbb{C}\mathrm{P}^1$ is a conformal fibration, the fibers have following induced orientation: Let T be tangent to the fiber and $A, B \perp T$ such that $\pi_* A \wedge \pi_* B > 0$ represents the orientation of the Riemannian surface $\mathbb{C}\mathrm{P}^1$. We say T is in positive direction to the fiber if $T \wedge A \wedge B > 0$, i.e. (T, A, B) is a positive orientated basis of S^3.

6.7. Theorem. *Let $\pi\colon S^3 \to \mathbb{C}\mathrm{P}^1$ be a conformal fibration such that all fibers are circles with the induced orientation. Then the curve $\gamma\colon \mathbb{C}\mathrm{P}^1 \to Q^3$ which maps each $p \in \mathbb{C}\mathrm{P}^1$ to the oriented circle $\pi^{-1}(p)$, given by an element in $Q^3 \subset PW$, is holomorphic.*

Proof. We fix a metric on S^3 and let T be the unit length tangent vector in positive fiber direction. The fibers of π are the closed integral curves of the flow Φ of T. There exists an $s > 0$ such that for all $\phi := \Phi_s\colon S^3 \to S^3$ is a fixpoint free diffeomorphism. As an example, one could take any s with $s < \mathit{length}(\pi^{-1}(p))$ for all $p \in S^2$. Let A, B be the horizontal lifts of vector fields defined on the base space such that (T, A, B) is an oriented orthonormal frame field. Because of $\pi_*[A, T] = 0 = \pi_*[B, T]$ there exists functions λ, μ such that $\phi_* A = A + \lambda T$ and $\phi_* B = B + \mu T$.

We have already seen that the tangent map $\Psi\colon S^3 \to Q \subset \mathbb{C}\mathrm{P}^3$ is CR-holomorphic. Let ψ be a local CR holomorphic lift to $\mathbb{C}^4$. We define $\chi = \psi \circ \phi$. Using the second model for the space of circles we see that γ is given by

$$\gamma(\pi(p)) = [\psi(p) \wedge \chi(p)] = [\psi(p) \wedge T \cdot \psi(p)] \in Q^3 \subset PW.$$

We set $\hat{\gamma} = \psi \wedge \chi\colon U \subset S^3 \to W \subset \Lambda^2(\mathbb{C}^4)$ and compute

$$\begin{aligned} A \cdot \hat{\gamma} &= A \cdot \psi \wedge \chi + \psi \wedge A \cdot \chi \\ &= A \cdot \psi \wedge \chi + \psi \wedge (A \cdot \psi) \circ \phi + \psi \wedge (\lambda T \cdot \psi) \circ \phi \\ B \cdot \hat{\gamma} &= B \cdot \psi \wedge \chi + \psi \wedge B \cdot \chi \\ &= B \cdot \psi \wedge \chi + \psi \wedge (B \cdot \psi) \circ \phi + \psi \wedge (\mu T \cdot \psi) \circ \phi. \end{aligned}$$

With $\mathcal{J}A = B$ and from the fact that Ψ is CR holomorphic we deduce that

$$iA \cdot \psi = B \cdot \psi \mod \Psi.$$

As $[\hat{\gamma}] = [\psi \wedge T \cdot \psi]$ we have

$$(\mathcal{J}A) \cdot (\gamma \circ \pi) = i(A \cdot (\gamma \circ \pi)),$$

which means that $\gamma \circ \pi$ is CR holomorphic. As $\mathrm{d}\,\pi$ maps the Levi distribution of S^3 CR holomorphically onto the tangent space of $\mathbb{C}\mathrm{P}^1$ and $\gamma \circ \pi$ is CR holomorphic, it is evident that γ must be holomorphic. $\square$

Obviously, all results above can be rephrased for conformal submersions defined on subsets of S^3. If one considers a globally defined conformal fibration of S^3 one can use the fact that every point in S^3 lies in exactly one fiber to obtain global information of the curve γ :

6.8. Lemma. *Let $\pi\colon S^3 \to S^2$ be a conformal fibration with circles as fibers. Then the holomorphic curve $\gamma\colon \mathbb{C}\mathrm{P}^1 \to Q^3$ given by 6.7 has degree* 1.

Proof. Let n be the degree of $\gamma\colon \mathbb{C}\mathrm{P}^1 \to Q^3 \subset PW$. We take a basis $(e_0, .., e_4)$ of W consisting of real points, i.e. $\sigma(e_i) = e_i$, such that

$$(,) := \frac{1}{2}\Omega \wedge \Omega = -e_0^* \otimes e_0^* + e_1^* \otimes e_1^* + .. + e_4^* \otimes e_4^*,$$

and such that we have $\gamma(0) = [e_3 + ie_4]$ for a suitable holomorphic coordinate z on $\mathbb{C}\mathrm{P}^1$. By changing the holomorphic coordinate z we can assume that the point $[e_0 + e_3] \in S^3 \subset Q^3$ lies on the circle $\gamma(\infty)$. We take a holomorphic lift

$$\hat{\gamma}(z) = v_0 + zv_1 + .. + z^n v_n$$

of γ to W with $v_0 = e_3 + ie_4$. Since every point $p \in S^3$ lies on exactly one circle, namely $\gamma(\pi(p))$, it results from 6.6 that γ intersects each hyperplane $Pv^\perp \cap Q^3$ in exactly one point, where $[v] \in S^3 \subset \mathbb{R}P^4 \subset PW$. But every hyperplane must intersected $n-$times by γ counted with multiplicities, hence we know that γ intersects each such hyperplane $Pv^\perp$ in exactly one point with order n. Therefore

$$Pv_0^\perp \cap S^3 = \cup_{\varphi \in [0,2\pi]} [e_0 + \cos\varphi e_1 + \sin\varphi e_2]$$

proves

$$v_1, v_2, .., v_{n-1} \in \cap_{\varphi \in [0,2\pi]} (e_0 + \cos\varphi e_1 + \sin\varphi e_2)^\perp = \operatorname{span}\{e_3, e_4\}.$$

Using $(\hat{\gamma}(z), \hat{\gamma}(z)) = 0$ this implies inductively

$$v_1 = \mu_1 v_0, .., v_{n-1} = \mu_{n-1} v_0$$

for suitable $\mu_i \in \mathbb{C}$, and

$$(v_0, v_n) = (v_n, v_n) = 0.$$

As $(v_n, e_0 + e_3) = 0$ we see

$$v_n = ae_0 + be_1 + ce_2 + a(e_3 + ie_4)$$

with $a^2 = b^2 + c^2$ and $bc \neq 0$. For the holomorphic coordinate $\omega = 1/z$ we obtain another holomorphic lift

$$\tilde{\gamma}(\omega) = (ae_0 + be_1 + ce_2 + a(e_3 + ie_4)) + (\omega\mu_{n-1} + .. + \omega^n)v_0$$

of γ, thus γ intersects the hyperplane $P(e_0 + e_3)^\perp$ in $z = \infty$ $n-$times if and only if $\mu_1 = .. = \mu_{n-1} = 0$.

First we consider the case $a = 0$, i.e. $v_n = be_1 + ce_2$ with $b^2 + c^2 = 0$. After a change of the coordinate z on $\mathbb{C}\mathrm{P}^1$ we obtain the form

$$\hat{\gamma}(z) = e_3 + ie_4 + z^n(e_1 + ce_2)$$

where $c^2 = -1$, so the circle $\gamma(1)$ contains the point $[e_0 + \frac{1}{\sqrt{2}}(e_1 - e_3)] \in S^3$. The intersection of γ and the hyperplane $P(e_0 + \frac{1}{\sqrt{2}}(e_1 - e_3))^\perp$ is at $z = 1$ only of order 1 because

$$(\hat{\gamma}'(1), e_0 + \frac{1}{\sqrt{2}}(e_1 - e_3)) = \frac{n}{\sqrt{2}} \neq 0.$$

Since γ must intersect $(e_0 + \frac{1}{\sqrt{2}}(e_1 - e_3))^\perp$ at $z = 1$ with order n we see that $n = 1$ in case of $a = 0$.

If $a \neq 0$ we can change the coordinate z by a factor such that $[e_0 - e_3]$ lies on $\gamma(1)$. This implies $0 = (\hat{\gamma}(1), e_0 - e_3) = -a - (a + 1)$. The curve γ intersects $P(e_0 - e_3)^\perp$ at $z = 1$ with multiplicity n, which implies for $n > 1$ that $0 = (\hat{\gamma}'(1), e_0 - e_3) = -a - (a + n)$ contradicting $0 = -2a + 1$. Thus $n = 1$. □

With this result it is easy to prove the following theorem of the uniqueness of conformal fibrations with circles as fibers:

6.9. Theorem. *Up to conformal transformations of S^2 and S^3, every conformal fibration of S^3 by circles is the Hopf fibration.*

PROOF. We will show that for each curve $\lambda\colon \mathbb{C}\mathrm{P}^1 \to Q^3$ which is given by a conformal submersion via 6.7 there is a projective isomorphism $Q^3 \to Q^3$ resulting from a conformal transformation of S^3, such that λ is mapped to the curve $\gamma\colon \mathbb{C}\mathrm{P}^1 \to Q^3$ given by

$$\gamma([z:w]) = [z(e_1 + ie_2) + w(e_3 + ie_4)],$$

where $e_0, .., e_4$ is a basis of W as in the proof of 6.8. Using this basis, $S^3 \subset Q^3$ and the sphere in the light-cone model can be identified, and we see that the real orthogonal transformations $\Phi\colon W \to W$, i.e. $\sigma \circ \Phi = \Phi \circ \sigma$, and $\Phi^*(,) = (,)$, with $(\Phi(e_0), e_0) < 0$ are exactly the conformal transformations of S^3.

As in the proof of 6.8 we can take a holomorphic coordinate z on $\mathbb{C}\mathrm{P}^1$ such that, after a conformal transformation of S^3 and the induced transformation of W we have $\lambda(0) = [e_3 + ie_4]$, and $[e_0 + e_3]$ lies on $\lambda(\infty)$. Thus $\hat{\lambda}(z) = e_3 + ie_4 + zb(e_1 \pm ie_2)$, where $b \in \mathbb{C}$. The curves λ and γ coincide after applying the conformal transformation $\tilde{z} = 1/bz$ on S^2 and the conformal transformation on S^3 given by the linear isometry of W with

$$e_2 \mapsto \pm e_2;\ e_i \mapsto e_i,\ i \neq 2.$$

□

CHAPTER III

Riemannian Submersions on 3−Manifolds

In this chapter we develop the general theory of Riemannian submersions from 3−manifolds onto surfaces. It will be shown that the geometry is determined by the geometry of the surface, the (mean) curvature of the fibers, and by the curvature of the horizontal distribution, which can be identified with a function. We compute the curvature of the 3−manifold in these quantities explicitly. All the spaces are assumed to be oriented.

7. Connections and Curvature

First we will discuss connections on submersions in general. Afterwards we will have a glance at principal connections on principal bundles. We will see that the curvature of the (non-linear) connection canonical associated to a Riemannian submersion, or more general to a conformal submersion, play an important role for the geometry of the submersion.

7.1. Connections on submersions. Let $\pi\colon P \to M$ be a submersion. The vertical bundle is given in a natural way, i.e. $\mathcal{V} = \ker \mathrm{d}\,\pi$, in contrast to the complementary bundles. Sometimes there are canonical complements. For example if there is a Riemannian metric P on g one can take the orthonormal complement. But in general, the choice of a horizontal bundle is a geometric structure on its own.

Definition. A section of a submersion $\pi\colon P \to M$ is a map $s\colon M \to P$ such that $\pi \circ s = \mathrm{Id}$.
A connection ∇ on a submersion $\pi\colon P \to M$ is a decomposition $TP = \mathcal{V} \oplus \mathcal{H}$ of the tangent bundle into vertical and horizontal parts.
A connection ∇ on a submersion $\pi\colon P \to M$ defines the vertical derivative $\nabla s = \pi^{\mathcal{V}} \circ \mathrm{d}\,s$ of a section, which is the projection on the vertical part of the derivative of s.

Given a connection ∇, the fibers can be identified in the following manner: If $\gamma\colon I \to M$ is a curve one defines the horizontal lift $\hat{\gamma}\colon I \to P$ through a point $p \in \pi^{-1}(\gamma(I))$ as a curve through p such that $\pi \circ \hat{\gamma} = \gamma$ and $\hat{\gamma}' \in \mathcal{H}$. It is clear that such a curve always exists, at least locally, and is unique. The horizontal curves define the parallel transport of the connection: One maps the points of the fiber over $p \in M$ to the points in the fiber over $q \in M$ via the horizontal lifts of a fixed curve in M from p to q.

The question on hand is whether every map $f\colon N \to M$ provides a horizontal lift. For example in case of $\mathrm{Id}\colon M \to M$ a horizontal lift is a section s such that $\nabla s = 0$. The answer is given by the Frobenius theorem.

Definition. The curvature of a connection ∇ on a submersion is given by the vertical valued two form

$$\Omega \in \Omega^2(P;\mathcal{V}),\ \Omega(X,Y) = -\pi^{\mathcal{V}}([\pi^{\mathcal{H}}\hat{X}, \pi^{\mathcal{H}}\hat{Y}]),$$

where $\hat{X}, \hat{Y}$ are arbitrary continuations of X and Y.

Theorem (Frobenius)**.** *Let $\pi\colon P \to M$ be a submersion with connection ∇. A necessary and sufficient condition for the existence of horizontal sections through every $p \in P$ is that the curvature vanishes, i.e.* $\Omega = 0$.

There are many ways to generalize the theorem. For example, for horizontal lifts of submanifolds $\imath\colon N \to M$ into a bundle $P \to M$. To do so, one has to consider the so-called pullback connection $\imath^*\nabla$ on the pullback bundle $\imath^*P \to N$ given by

$$\imath^*P = \{(n,p) \mid \imath(n) = \pi(p)\} \to N,$$

see [**SW**] for details.

Remark. One can restrict oneself to submersions with extra structure, as to principal or vector bundles. It is common that only connections which respect the additional structure are considered. In the case of principal bundles one requires that the decomposition is invariant under the group action, and in the case of vector bundles the parallel transport should be linear for each curve.

7.2. Principal Bundles. For the general theory of principal bundles one can consult [**KN**], or more briefly [**Fr**]. A principal $G-$bundle $\pi\colon P \to M$ with structure group G is given by a right action of $P \times G \to P$ such that there exist local trivializations $P_{|U} \to U \times G$, which commute with the action of G. A principal connection on $\pi\colon P \to M$ is a $G-$invariant decomposition $TP = \mathcal{V}P \oplus \mathcal{H}P$. The vertical space $\mathcal{V}P = \ker \mathrm{d}\,\pi$ can be identified with $M \times \mathfrak{g}$, where $\mathfrak{g}$ is the Lie algebra of G : Let $\xi \in \mathfrak{g}$. A vertical $G-$invariant vector field $\hat{\xi}$ with the values $\hat{\xi}_p$ can be defined by differentiating the curve $t \mapsto p \cdot \exp(t\xi)$ at $t = 0$. A global trivialization is

$$(p, \xi) \in M \times (g) \mapsto \hat{\xi}_p \in \mathcal{V}_p.$$

Note that any connection is given by its projection on the vertical bundle. In case of a principal bundle, this projection can be considered as a $\mathfrak{g}-$valued $1-$form $\omega \in \Omega^1(P; \mathfrak{g})$.

Definition. A principal connection is a $1-$form $\omega \in \Omega^1(P; \mathfrak{g})$ with the properties:

- for all $\xi \in \mathfrak{g} : \omega(\hat{\xi}) = \xi$;
- for all $g \in G$ and $X \in TP$ we have $R_g^* \omega(X) = \mathrm{Ad}_{g^{-1}}(\omega(X))$, where Ad is the Adjoint action of G on $\mathfrak{g}$.

The first property says that ω is a projection, and the second that the decomposition is $G-$invariant.

By definition, the curvature of a connection is given by the vertical valued horizontal $2-$form Ω with $\Omega(X, Y) = -\pi^{\mathcal{V}}([\pi^{\mathcal{H}} X, \pi^{\mathcal{H}} Y])$, which can identified with a $\mathfrak{g}-$valued $2-$form.

7.3. Proposition. *The curvature $\Omega \in \Gamma^2(P; \mathfrak{g})$ of a connection ∇ is a $\mathfrak{g}-$valued Ad$-$equivariant horizontal $2-$form, that is*

$$R_g^* \Omega(X, Y) = Ad_{g^{-1}} \Omega(X, Y).$$

It is

$$\Omega = \mathrm{d}\,\omega + \frac{1}{2}[\omega \wedge \omega],$$

where $[\omega \wedge \eta]$ is defined by $[\omega \wedge \eta](X, Y) := [\omega(X) \wedge \eta(Y)] - [\omega(Y) \wedge \eta(X)]$.

PROOF. Since $[R_{g_*}\pi^{\mathcal{H}}X, R_{g_*}\pi^{\mathcal{H}}Y] = R_{g_*}[\pi^{\mathcal{H}}X, \pi^{\mathcal{H}}Y]$ is valid for all $g \in G$ one obtains

$$\begin{aligned} R_g^*\Omega(X,Y) &= -\,\omega([R_{g_*}\pi^{\mathcal{H}}X, R_{g_*}\pi^{\mathcal{H}}Y]) \\ &= -\,\omega(R_{g_*}[\pi^{\mathcal{H}}X, \pi^{\mathcal{H}}Y]) = Ad_{g^{-1}}(\omega([\pi^{\mathcal{H}}X, \pi^{\mathcal{H}}Y]) \\ &= Ad_{g^{-1}}\Omega(X,Y). \end{aligned}$$

We state $\Omega = \mathrm{d}\,\omega + \frac{1}{2}[\omega \wedge \omega]$. To prove it take the horizontal lifts $\hat{X}, \hat{Y}$ of X, Y and Killing fields $\hat{\xi}, \hat{\chi}$, and compute

$$(\mathrm{d}\,\omega + \frac{1}{2}[\omega \wedge \omega])(\hat{X}, \hat{Y}) = -\omega([\hat{X}, \hat{Y}]) = \Omega(\hat{X}, \hat{Y}),$$

$$(\mathrm{d}\,\omega + \frac{1}{2}[\omega \wedge \omega])(\hat{X}, \hat{\xi}) = \hat{X} \cdot \xi - \omega([\hat{X}, \hat{\xi}]) = 0 = \Omega(\hat{X}, \hat{\xi}),$$

$$(\mathrm{d}\,\omega + \frac{1}{2}[\omega \wedge \omega])(\hat{\xi}, \hat{\chi}) = -\omega([\hat{\xi}, \hat{\chi}]) + [\xi, \chi] = 0 = \Omega(\hat{\xi}, \hat{\chi}).$$

□

Principal bundles and vector bundles and their connections are related as follows. Consider a representation $\rho\colon G \to \mathrm{GL}(V)$ of G on a vector space V. Let $\pi\colon P \to M$ be a principal bundle, then there exists a vector bundle with typical fiber V :

$$P \times_\rho V := P \times V/\sim$$

with $(p, v) \sim (\tilde{p}, \tilde{v})$, if and only if there exists a $g \in G$ such that $\tilde{p} = p \cdot g$ and $\tilde{v} = \rho(g^{-1})(v)$. This vector bundle is called the associated vector bundle to (P, ρ). Vice versa, given a vector bundle $V \to M$ with typical fiber F, one can define the principal $GL(F)-$bundle $P \to M$ as

$$P = \{(m, f) \mid m \in M, f\colon F \to V_m \textit{ linear isomorphism}\}.$$

Elements of P are called frames. If one has a geometric structure on V, which is invariant under some group G, one can define a principal $G-$bundle as the set of frames which respect this structure.

There is a relationship between connections on principal and its associated vector bundles, too. Every linear connection defines the parallel transport between the fibers which are connected by curves. The parallel transport is a linear isomorphism between the fibers respecting the geometric structure, which is defined on the vector bundle. Thus there is a natural way to obtain the frames of one fiber from the frames of another fiber via parallel transport. It can be shown that this parallelism is due to a principal connection on the frame bundle. This procedure could be done the other way round, hence a covariant derivative on the associated bundles is defined by a principal connection.

8. Riemannian Submersions and Curvature

In this section we collect the basics of Riemannian submersions. For definitions and proofs see [**B**].

Let $\pi\colon (P,g) \to (M,h)$ be a Riemannian submersion between oriented Riemannian manifolds, that means that the differential $\mathrm{d}\,\pi_{|\ker \mathrm{d}\,\pi^{\perp}}$, restricted to the orthogonal complement of its kernel, is an isometry at every point. We consider the vertical distribution $\mathcal{V} = \ker(d\pi)$ and the horizontal distribution which is defined as its orthogonal complement $\mathcal{H} = \mathcal{V}^{\perp}$. This decomposition defines a connection in the sense of 7.

We denote vertical vector fields by $U,\ V,$ .. and vector fields on the base by $A,\ B,$ ... . Their horizontal lifts are denoted by $\hat{A}, \hat{B}, \ldots$. If one studies Riemannian submersions one has to deal with three different Levi-Civita Connections: Denote by ∇^M, ∇ and D the connection on the base, on the total space and on the fibers.

The fibers of a Riemannian submersion $\pi\colon P \to M$ are closed submanifolds, thus it is natural to consider their second fundamental forms:

8.1. Proposition. *The second fundamental forms of the fibers gives rise to a tensor field $\mathcal{S} \in T^{2,1}(P)$:*

$$\mathcal{S}(X,Y) = \pi^{\mathcal{H}}\nabla_{\pi^{\mathcal{V}}X}\pi^{\mathcal{V}}Y + \pi^{\mathcal{V}}\nabla_{\pi^{\mathcal{V}}X}\pi^{\mathcal{H}}Y.$$

It satisfies

$$\begin{aligned}
\mathcal{S}(U,V) &= \nabla_U V - D_U V,\\
\mathcal{S}(U,V) &= \mathcal{S}(V,U),\\
\mathcal{S}(A,B) &= \mathcal{S}(A,U) = 0,\\
g(\mathcal{S}(U,V),A) &= -g(U,\mathcal{S}(V,A)).
\end{aligned}$$

The fibers of the submersion can be identified with each other via parallel transport with respect to the connection $TP = \mathcal{V} \oplus \mathcal{V}^{\perp}$. Then the Lie derivative of $g_{|\mathcal{V}}$ is determined as

$$(L_{\hat{A}} g_{\mathcal{V}})(U,V) = g(\nabla_U \hat{A}, V) + g(\nabla_V \hat{A}, U) = 2g(\hat{A}, \mathcal{S}(U,V)).$$

This shows that the fibers are isometric to each other via parallel transport if and only if $\mathcal{S} = 0$, which is proved by a theorem of R. Hermann, see [**B**]. Moreover, the derivative of the volume V of the fibers is given by

$$\mathrm{d} \log V_p(A) = \int_{\pi^{-1}(p)} \operatorname{tr} g(\mathcal{S}, \hat{A}) \,\mathrm{dvol}, \tag{8.1}$$

see [**GLP**]. This is not surprising since $\frac{1}{\dim \pi^{-1}(p)} \operatorname{tr} \mathcal{S}$ is the mean curvature vector of the fiber which is 0 if and only if the fibers are minimal.

There is a equivalent definition for the Tensor field introduced in 7.1.

8.2. Proposition. *The tensor field $\mathcal{T}$ given by*

$$\mathcal{T}(X,Y) = \pi^{\mathcal{H}} \nabla_{\pi^{\mathcal{H}} X} \pi^{\mathcal{V}} Y + \pi^{\mathcal{V}} \nabla_{\pi^{\mathcal{H}} X} \pi^{\mathcal{H}} Y$$

satisfies

$$\begin{aligned} \mathcal{T}(U,A) &= \mathcal{T}(U,V) = 0, \\ \mathcal{T}(A,B) &= -\mathcal{T}(B,A) = -\frac{1}{2}\Omega(A,B), \\ g(\mathcal{T}(A,U),B) &= -g(\mathcal{T}(A,B),U). \end{aligned}$$

Proof. We only do prove that $\mathcal{T}(A,B) = \frac{1}{2}\Omega(A,B)$. We state that $\pi^{\mathcal{V}} \nabla_{\hat{A}} \hat{A} = 0$, for every vertical field U, as

$$g(U, \nabla_{\hat{A}} \hat{A}) = -g(\nabla_{\hat{A}} U, \hat{A}) = -g(\nabla_U \hat{A}, \hat{A}) = -\frac{1}{2} U \cdot g(\hat{A}, \hat{A}) = 0.$$

We used that $[U, \hat{A}]$ is vertical for horizontally lifted fields $\hat{A}$. The formula follows from

$$\pi^{\mathcal{V}}[\hat{A}, \hat{B}] = \pi^{\mathcal{V}}(\nabla_{\hat{A}} \hat{B} - \nabla_{\hat{B}} \hat{A}).$$

□

The tensor field $\mathcal{T}$ vanishes if and only if the horizontal distribution is integrable. Notice that in this case the second fundamental form of the leaves of the horizontal distribution also vanishes.

We are now able to give formulas for the Levi-Civita connection of (P, g) explicitly.

8.3. Proposition. *The Levi-Civita connection of the total space of a Riemannian submersion is computed as follows:*

$$\nabla_U V = D_U V + \mathcal{S}(U, V)$$
$$\nabla_U X = \mathcal{S}(U, X) + \pi^{\mathcal{H}} \nabla_U X$$
$$\nabla_X U = \pi^{\mathcal{V}} \nabla_X U + \mathcal{T}(X, U)$$
$$\nabla_X Y = \mathcal{T}(X, Y) + \pi^{\mathcal{H}} \nabla_X Y,$$

where X, Y are horizontal and U, V are vertical vector fields. In case of basic vector fields $\hat{A}, \hat{B}$ one obtains

$$\nabla_{\hat{A}} \hat{B} = \mathcal{T}(\hat{A}, \hat{B}) + \hat{\nabla}^M_A B.$$

These formulas can be used to compute the Riemannian curvature. As an example, the sectional curvature of a horizontal plane spanned by orthonormal $\hat{A}$, $\hat{B}$ is given by

$$K(\hat{A} \wedge \hat{B}) = K(A \wedge B) - \frac{3}{4} \Omega(\hat{A} \wedge \hat{B})^2, \tag{8.2}$$

where $K(A \wedge B)$ is the sectional curvature of corresponding plane $A \wedge B$ on the base.

9. Curvature on Fibered 3−manifolds

We first collect some basic facts of curvature tensors on Riemannian 3−manifolds and show how they are to be transformed after conformal changes of the metric.
Afterwards we specify the results of the previous section for the case of 3−spaces fibered by curves.

9.1. Curvature on 3−manifolds. We will not present a detailed study of the decomposition of the Riemannian curvature tensor, since we are only interested in 3−manifolds. We will not distinguish between endomorphisms and bilinear forms, but for better reading we denote the first by capital and the latter by small letters.

On manifolds of dimension 1 there is no Riemannian curvature. On surfaces it is determined by the scalar curvature. With 3 dimensions it is determined by the Ricci or the Schouten curvature, respectively:

Proposition. *The Riemannian curvature tensor* $r \in \Gamma(T^{(4,0)}M)$ *on* $3-$*manifolds with Riemannian metric* g *is given by*

$$r = -s \cdot g,$$

where the Schouten tensor is $s = \mathrm{ric} - \frac{1}{4}\,\mathrm{scal}\, g$ *and the Kulkarni-Nomizu product is given by*

$$\begin{aligned} h \cdot k(x,y,z,t) =& h(x,z)k(y,t) + h(y,t)k(x,z) \\ & - h(x,t)k(y,z) - h(y,z)k(x,t). \end{aligned}$$

Proof. Let x, y, z be a orthonormal basis of $T_pM,\ p \in M$. Using the basic symmetries of the Riemannian curvature it is obvious that

$$r(x,z,z,y) = \mathrm{ric}(x,y) = s(x,y) = -s \cdot g(x,z,z,y)$$

and

$$\begin{aligned} 2r(x,y,y,x) &= \mathrm{ric}(x,x) + \mathrm{ric}(y,y) - \mathrm{ric}(z,z) \\ &= s(x,x) + s(y,y) - s(z,z) + \frac{1}{4}\,\mathrm{scal} \\ &= 2s(x,x) + 2s(y,y) = -2s \cdot g(x,z,z,x). \end{aligned}$$

Other cases are followed by permuting x, y and z. □

We consider a conformal change of the metric $\tilde{g} = e^{2\lambda} g$. The Levi-Civita connection of $\tilde{g}$ is

$$\tilde{\nabla}_X Y = \nabla_X Y + X \cdot \lambda Y + Y \cdot \lambda X - g(X,Y)\,\mathrm{grad}\,\lambda.$$

Proposition. *The Schouten tensor* $\tilde{S}$ *of the conformally changed metric* $\tilde{g} = e^{2\lambda} g$ *is*

$$\tilde{S} = S - B_\lambda, \tag{9.1}$$

where S *is the Schouten tensor of* g *and*

$$B_\lambda = \mathrm{Hess}\,\lambda - \mathrm{d}\,\lambda \otimes \mathrm{grad}\,\lambda + \frac{1}{2}\,\|\,\mathrm{grad}\,\lambda\,\|^2\,\mathrm{Id}\,.$$

Proof. We choose orthonormal vector fields X, Y, Z such that their commutators at $p \in M$ vanish. We compute

$$\begin{aligned}\tilde{s}(X,Y) =& \tilde{r}(X, e^{-\lambda}Z, e^{-\lambda}Z, Y) = g(\tilde{R}(X,Z)Z,Y)\\ =& g(\tilde{\nabla}_X\tilde{\nabla}_Z Z - \tilde{\nabla}_Z\tilde{\nabla}_X Z, Y)\\ =& g(\tilde{\nabla}_X(\nabla_Z Z + 2Z\cdot\lambda Z - \operatorname{grad}\lambda)\\ & - \tilde{\nabla}_Z(\nabla_X Z + X\cdot\lambda Z + Z\cdot\lambda X), Y)\\ =& r(X,Z,Z,Y) + X\cdot\lambda Y\cdot\lambda - \operatorname{hess}\lambda(X,Y)\\ =& s(X,Y) - b_\lambda(X,Y),\end{aligned}$$

and similarly for the other cases. □

In contrast to surfaces, there exist 3−spaces with metrics, where, even locally, no conformal transformation to a flat metric can be found. Hence the question on hand is what the obstructions of being conformally flat are. The answer was given by Schouten, see [**HJ**].

9.2. Theorem. *A Riemannian* 3−*manifold is conformally flat if and only if the covariant derivative of the Schouten tensor is totally symmetric.*

Proof. A locally defined function λ is to be found, such that $S = B_\lambda$. With $v = \operatorname{grad}\lambda$, this is equivalent to

$$\nabla v = S + g(v,.)v - \frac{1}{2}g(v,v)\operatorname{Id}.$$

Note that any vector field which solves the equation above is the gradient of a function automatically, because its covariant derivative is symmetric. We define the vector valued horizontal 1−form $\eta \in \Omega^1(TM; p^*TM)$ by

$$\eta_{|v}(X) := S(p_*X) + g(v, p_*X)v - \frac{1}{2}g(v,v)p_*X,$$

where $p\colon TM \to M$ is the projection. Using $r = -s\cdot g$, one easily computes for horizontal vectors $X, Y \in \mathcal{H}TM \subset TTM$ and for the pullback connection on p^*TM :

$$\begin{aligned}d^\nabla\eta(X,Y) &= \nabla_X(\eta(Y)) - \nabla_Y(\eta(X)) - \eta([X,Y])\\ &= (\nabla_X S)(Y) - (\nabla_Y S)(X)s + R(X,Y)v.\end{aligned}$$

Therefore, the theorem is a corollary of the following Lemma. □

Lemma. *Let* ∇ *be a connection on a vector bundle* $\pi\colon V \to M$, *and* $\Omega \in \Omega^2(\mathrm{End}(V))$ *its curvature. The equation*

$$\nabla v = \omega_v$$

for $v \in \Gamma(V)$ *and* $\omega \in \Omega^1(V;\pi^*V)$ *has a local solution around each point if and only if*

$$d^\nabla \omega(X,Y) = \Omega(\pi_* X, \pi_* Y)v$$

for horizontal vectors $X, Y \in \mathcal{H}V \subset TV$.

Proof. In order to furnish the proof the Frobenius theorem is applied. On TV, the connection ∇ induces a decomposition $TV = \mathcal{V}V \oplus \mathcal{H}V$. We define the distribution

$$\mathcal{D}_v := \{\omega_{|v}(\pi_* X) \oplus X \mid X \in \mathcal{H}V\} \subset TV.$$

An integral manifold of $\mathcal{D}$ gives rise to a locally defined section $v \in \Gamma(V)$ with $\nabla v = \omega_v$. Thus it remains to show that $\mathcal{D}$ is integrable. Let $\hat{X}, \hat{Y}$ be lifted horizontal fields in $\mathcal{H}V$ of $X, Y \in \Gamma(TM)$. We compute

$$\begin{aligned}[\omega_{|v}(X) \oplus \hat{X}, \omega_{|v}(Y) \oplus \hat{Y}] =&[\omega_{|v}(X), \omega_{|v}(Y)] + [\omega_{|v}(X), \hat{Y}] + [\hat{X}, \omega_{|v}(Y)] \\ &+ [\hat{X}, \hat{Y}] \\ =&\nabla_X(\omega_{|v}(Y)) - \nabla_Y(\omega_{|v}(X)) \\ &+ (-\Omega(X,Y)(v) \oplus \widehat{[X,Y]}) \\ =&\omega_{|v}([X,Y]) \oplus \widehat{[X,Y]} \in \mathcal{D},\end{aligned}$$

see [**SW**] for more details. □

9.3. Remark. Regarding S as a $TM-$valued $1-$form such that

$$\begin{aligned} d^\nabla S(X,Y) &= \nabla_X(S(Y)) - \nabla_Y(S(X)) - S([X,Y]) \\ &= \nabla_X(S(Y)) - S(\nabla_X Y) - \nabla_Y(S(X)) + S(\nabla_Y X) \\ &= (\nabla_X S)(Y) - (\nabla_Y S)(X),\end{aligned}$$

the condition that the covariant derivative of the Schouten tensor is totally symmetric is equivalent to the fact that its absolute exterior derivative vanishes.

Remark. The decomposition of the Riemannian curvature tensor can be generalized to higher dimensions as follows: Defining the Schouten tensor for $n \geq 3$ as $S := \frac{1}{n-2}(\mathrm{Ric} - \frac{\mathrm{scal}}{2n-2}\,\mathrm{Id})$, we get $r =$

$s \cdot g + w$, where w is the Weyl tensor. It was proven that w vanishes for dimension 3. In general, the Schouten tensor changes under conformal transformation as shown above, and the Weyl tensor (as a bilinear form) is invariant under these transformations of the metric. Thus the condition that $w = 0$ is necessary for being conformally flat. For $n \geq 4$ $w = 0$ implies $d^\nabla S = 0$, and therefore this condition is also sufficient.

9.4. Levi-Civita Connection on Fibered 3−manifolds. To obtain the formulas below, we will use orthonormal vector fields A, B defined on an open subset $U \subset S^2$ such that $e^\lambda(A+iB)$ is a holomorphic vector field on the surface for some function $\lambda\colon U \to \mathbb{R}$. This condition is equivalent to $[e^\lambda A, e^\lambda B] = 0$. Note that the Gaussian curvature is given by $K = \Delta\lambda$. We denote by $\hat{A}, \hat{B} \in \Gamma(\mathcal{H})$ the horizontal lifts of A and B.

There exists an unique positive oriented vertical vector field T of length 1. Of course, we have $D_T T = 0$, and the second fundamental form is equal to the mean curvature, thus we have $\nabla_T T = \mathcal{S}(T, T)$. The curvature Ω of the horizontal distribution can be identified with a real valued 2−form $\Omega := g(\Omega, T)$. Using the metric and the orientation, we define the curvature function of the horizontal distribution by

$$H := \Omega(\hat{X} \wedge \hat{Y}) = -g(T, [\hat{X}, \hat{Y}]), \tag{9.2}$$

where $\hat{X}, \hat{Y}$ are locally defined positive oriented basis fields of $\mathcal{H}$. Let $\mathcal{J} \in \mathrm{End}(TP)$ be the CR structure of the Riemannian submersion π, i.e. $\mathcal{J}(\hat{A}) = \hat{B}$, $\mathcal{J}(\hat{B}) = -\hat{A}$ and $\mathcal{J}(T) = 0$. Using 8.3 one obtains

9.5. Proposition. *The Levi-Civita connection on a fibered* 3−*space is given by*

$$\begin{aligned}
\nabla T =& \frac{1}{2} H\mathcal{J} + \nabla_T T \otimes g(., T) \\
\nabla X =& \pi^{\mathcal{H}} \nabla_{\pi^{\mathcal{H}}(.)} X + \mathcal{T}(., X) + \mathcal{S}(., X) \\
=& \pi^{\mathcal{H}} \nabla_{\pi^{\mathcal{H}}(.)} X + \frac{1}{2} HT \otimes g(., \mathcal{J}X) \\
&+ (\frac{1}{2} H\mathcal{J}X - g(\nabla_T T, X)T) \otimes g(., T),
\end{aligned}$$

where X is horizontal and of length 1. *For the lifted vector fields $\hat{A}, \hat{B}$ one obtains*

$$\nabla \hat{A} = \hat{B} \otimes g(., \frac{1}{2}HT + \mathcal{J}\,\mathrm{grad}\,\lambda) + T \otimes g(., \frac{1}{2}H\hat{B} - g(\nabla_T T, \hat{A})T)$$

$$\nabla \hat{B} = -\hat{A} \otimes g(., \frac{1}{2}HT + \mathcal{J}\,\mathrm{grad}\,\lambda) - T \otimes g(., \frac{1}{2}H\hat{A} + g(\nabla_T T, \hat{B})T).$$

We only cite the formulas for the Schouten tensor, which can be computed easily.

9.6. Proposition. *The Schouten tensor on a fibered* 3−*manifold is given by*

$$\begin{aligned} s(T,T) =& (-\frac{1}{2}K + \frac{5}{8}H^2 + \frac{1}{2}\,\mathrm{div}\,\nabla_T T)T, \\ s(T,X) =& g(\mathcal{J}(H\nabla_T T - \frac{1}{2}\,\mathrm{grad}^h\, H), X), \\ s(X,Y) =& (-\frac{3}{8}H^2 + \frac{1}{2}K)g(X,Y) + \frac{1}{2}g(\nabla_X \mathcal{J}\nabla_T T, \mathcal{J}Y) \\ & + \frac{1}{2}g(\nabla_{\mathcal{J}X}\mathcal{J}\nabla_T T, Y) + \frac{1}{2}g(\nabla_T T, \mathcal{J}X)g(\nabla_T T, \mathcal{J}Y) \\ & - \frac{1}{2}g(\nabla_T T, X)g(\nabla_T T, Y), \end{aligned}$$

where X, Y are horizontal vectors.

By 9.2, the obstruction of being conformally flat is that the absolute exterior derivative of the Schouten tensor vanishes. This tensor is very difficult to understand but for a nice component:

9.7. Proposition. *A necessary condition that a fibered* 3−*manifold is conformally flat is*

(9.3)

$$0 = \Delta^h H + g(\nabla_T T, \mathrm{grad}\, H) + 2H(H^2 - K + \| \nabla_T T \|^2) + 3H\,\mathrm{div}\,\nabla_T T,$$

where $\Delta^h = -\,\mathrm{div}\,\mathrm{grad}^h$ is the horizontal Laplacian.

Proof. One easily computes that the right hand side of 9.3 is equal to the component $2g(d^\nabla S(A \wedge B), T)$ of the absolute exterior derivative of the Schouten tensor, which vanishes on conformally flat 3−spaces. □

CHAPTER IV

Conformally Flat Circle Bundles over Surfaces

In this chapter we study Riemannian submersions of conformally flat, oriented $3-$spaces onto oriented surfaces with connected, compact and minimal fibers. We will show that they are determined by a solution of an ODE locally and give an example with non-constant curvature. Afterwards we are going to classify all such Riemannian submersions over compact, oriented surfaces.

10. Circle Bundles

First, we describe in which situations Riemannian submersions with minimal fibers occur. Then we specify the formulas of the previous section by including the condition for the total space of being conformally flat.

10.1. Proposition. *A compact, oriented Riemannian $3-$manifold P admitting the existence of a Riemannian submersion $\pi\colon P \to M$ to an oriented surface M with connected and minimal fibers is given by a compact, oriented Riemannian $3-$space P with a free circle action $P \times S^1 \to P$ leaving the metric invariant.*

Proof. In consequence of 8.1, the fibers of a Riemannian submersion, if all being compact, connected and minimal, are diffeomorphic to S^1 and with constant length. We assume the length to be 2π. Let T be the vector field of length 1 in positive direction of the fibers and consider its flow Φ. Clearly, every integral curve of T is closed with period 2π, hence the flow induces a free and proper circle action. By using the same arguments as in the proof of 5.1, the metric is invariant under this action, as π is a Riemannian submersion and the fibers are minimal.

Conversely, the space of orbits has an unique structure as a Riemannian manifold such that the canonical projection is a Riemannian submersion. As the metric is invariant under the action the fibers are minimal. □

The proposition above shows that $\pi\colon P \to M$ is a principal bundle with structure group S^1, see 7.2 for a definition. We will call it a circle bundle and the metric g on P a bundle metric.

We are going to compute the Levi-Civita connection and the curvature of the total space. Let T be the vector field in positive fiber direction of length 1. And let A, B be the horizontal lifts of orthonormal fields of the base M, such that $e^{\lambda}(\pi_* A + i\pi_* B)$ is a holomorphic field on the surface for an appropriate function λ. This condition on λ is equivalent to $[A, B] = B \cdot \lambda A - A \cdot \lambda B$. Since the fibers are minimal, the tensor field $\mathcal{S}$ defined in 8.1 vanishes. Moreover, from

$$0 = g(\mathcal{S}(T,T), A) = g(\nabla_T T, A) = -g(T, \nabla_T A) = -g(T, [T, A])$$

one obtains that the horizontal distribution given by $\mathcal{H} = \ker \mathrm{d}\,\pi^{\perp}$ gives rise to a principal connection. The curvature Ω of this connection is real valued and invariant under S^1, since S^1 is abelian. Therefore, the function H defined in 9.2, i.e. $\Omega = H\pi^* \operatorname{vol}_M$, is constant along the fibers, and it generates a function on the surface, which will be denoted by H, too. Remind the definitions of the CR structure $\mathcal{J} \in \operatorname{End}(TP)$ of a Riemannian submersion and the tensor field $\mathcal{T}$ given in the previous chapter. Then 9.5 turns into

Proposition. *The Levi-Civita connection of the total space of a circle bundle $P \to M$ with circle metric g and with T as above and X horizontal of length 1 is given by the following:*

$$\begin{aligned}
\nabla T &= \frac{1}{2} H \mathcal{J} \\
\nabla X &= \pi^{\mathcal{H}} \nabla_{\pi^{\mathcal{H}}(.)} X + \mathcal{T}(., X) \\
&= \pi^{\mathcal{H}} \nabla_{\pi^{\mathcal{H}}(.)} X + \frac{1}{2} H T \otimes g(., \mathcal{J}X) + \frac{1}{2} H \mathcal{J} X \otimes g(., T).
\end{aligned}$$

We obtain in case of horizontal lifted fields A and B as above

$$\begin{aligned}
\nabla A &= B \otimes g(., \frac{1}{2} H T + \mathcal{J} \operatorname{grad}(\lambda \circ \pi)) + T \otimes g(., \frac{1}{2} H B) \\
\nabla B &= -A \otimes g(., \frac{1}{2} H T + \mathcal{J} \operatorname{grad}(\lambda \circ \pi)) - T \otimes g(., \frac{1}{2} H A).
\end{aligned}$$

The Schouten tensor, see 9.6, becomes in case of geodesic fibers very simple, too:

Proposition. *The Schouten tensor of the total space of a circle bundle $P \to M$ with circle metric g is given by*

$$\begin{aligned} s(T,T) =& (-\frac{1}{2}K + \frac{5}{8}H^2)T \\ s(T,X) =& g(-\frac{1}{2}\mathcal{J}\,\mathrm{grad}\,H, X) \\ s(X,Y) =& (-\frac{3}{8}H^2 + \frac{1}{2}K)g(X,Y) \end{aligned} \tag{10.1}$$

where X, Y are arbitrary horizontal vectors and T is defined as above.

We have already proven that a 3−space is conformally flat if and only if the integrability condition $d^\nabla S = 0$ is satisfied in the last chapter. Using the formulas for the Levi-Civita connection and the Schouten tensor one easily computes

$$\begin{aligned} &g(d^\nabla \mathcal{S}(\hat{A} \wedge \hat{B}), T) = \frac{1}{2}(-\Delta H + 2H(K - H^2)), \\ &g(d^\nabla \mathcal{S}(A \wedge T), B) = \frac{1}{2}(\mathrm{Hess}\,H(A,A) - H(H^2 - K)), \\ &g(d^\nabla \mathcal{S}(B \wedge T), A) = \frac{1}{2}(\mathrm{Hess}\,H(B,B) - H(H^2 - K)), \\ &g(d^\nabla \mathcal{S}(\hat{B} \wedge T), \hat{B}) = -g(d^\nabla \mathcal{S}(A \wedge T), A) = \frac{1}{2}\,\mathrm{Hess}\,H(A,B). \\ &g(d^\nabla \mathcal{S}(A \wedge B), B) = -g(d^\nabla \mathcal{S}(A \wedge T), T) = A \cdot (\frac{1}{2}K - \frac{3}{4}H^2) \end{aligned}$$

and

$$g(d^\nabla \mathcal{S}(B \wedge A), A) = -g(d^\nabla \mathcal{S}(B \wedge T), T) = B \cdot (\frac{1}{2}K - \frac{3}{4}H^2).$$

The terms on the right hand side are conditions on the total space. But as the fibers are minimal and H is constant along every fiber, these terms are actually defined on the base. We obtain

10.2. Theorem. *A circle metric g of a circle bundle $\pi\colon P \to M$ over a surface M with Gaussian curvature K and curvature function H of the horizontal distribution is conformally flat if and only if*

$$\mathrm{Hess}\,H = \nabla\,\mathrm{grad}\,H = H(H^2 - K)\,\mathrm{Id}$$

and

$$2K - 3H^2 = \alpha$$

for some constant α.

11. Conformally Flat Circle Bundles

We are going to illustrate how the conditions of being conformally flat reduces to an ODE. First we investigate the equation $\operatorname{Hess} H = f \operatorname{Id}$ on a surface M with Riemannian metric. In the following we set $U := \{x \in M \mid \operatorname{grad} H \neq 0\}$.

11.1. Lemma. *Assume* $\operatorname{Hess} H = f \operatorname{Id}$ *is satisfied on a surface. Then, for each point* p *with* $\operatorname{grad}_p H \neq 0$, *there exist an open set* $V \subset U$ *and a conformal chart* $(x, y)\colon V \to \mathbb{R}^2$ *such that the metric* g *on* $V \subset M$ *and the function* H *do only depend on* x.

Proof. Define on U the vector field $X = \frac{1}{\|\operatorname{grad} H\|} \operatorname{grad} H$ and $Y = \mathcal{J} X$, where $\mathcal{J}$ is the complex structure induced by g and by the orientation, i.e. X, Y should form a positive oriented orthonormal basis. We set $h = \mathrm{d}\, H(X)$ which is equivalent to $\operatorname{grad} H = hX$. Using $\operatorname{Hess} H = f \operatorname{Id}$ and $\| X \| = 1$ we get

$$\begin{aligned} f &= \mathrm{d}\, h(X), \\ 0 &= \mathrm{d}\, h(Y), \\ \nabla X &= \frac{f}{h} Y \otimes g(., Y), \end{aligned} \tag{11.1}$$

and

$$\nabla Y = -\frac{f}{h} X \otimes g(., Y). \tag{11.2}$$

We prove the existence of a locally defined function $l\colon V' \subset U \to \mathbb{R}$ such that $[lX, lY] = 0$. Then lX, lY are the Gaussian basis fields of a conformal chart $(x, y)\colon V \subset V' \to \mathbb{R}^2$. With

$$[X, Y] = \nabla_X Y - \nabla_Y X = -\frac{f}{h} Y$$

we get

$$Y \cdot f = Y \cdot X \cdot h = X \cdot Y \cdot h - [X, Y] \cdot h = 0$$

and

$$\begin{aligned}[lX, lY] &= l^2[X, Y] - (Y \cdot l)lX + (X \cdot l)lY \\ &= (-\frac{f}{h}l + (X \cdot l))lY - (Y \cdot l)lX.\end{aligned}$$

We denote the dual basis of X and Y by $\omega_1 = g(., X), \omega_2 = g(., Y)$. Then $[lX, lY] = 0$ is equivalent to

$$\mathrm{d}\, l = l\frac{f}{h}\omega_1.$$

As $\mathrm{d}\,\frac{f}{h}\omega_1 = 0$ we get a solution of $\mathrm{d}\, q = \frac{f}{h}\omega_1$ on each simply connected open set $V' \subset U$. Then $l = e^q$ is a nowhere vanishing solution of $\mathrm{d}\, l = l\frac{f}{h}\omega_1$. We obtain a conformal chart $(x, y) \colon V \subset V' \to \mathbb{R}^2$ with Gaussian basis fields lX, lY. The metric is given by

$$g = l^2(dx \otimes dx + dy \otimes dy),$$

which only depends on x, since $\frac{\partial l}{\partial y} = l(Y \cdot l) = 0$. □

11.2. Remark. From the proof of this lemma we also get that there always exist a chart $(x, y) \colon V \to \mathbb{R}^2$ with Gaussian basis fields X, lY, with l satisfying $\mathrm{d}\, l = \frac{f}{h}l\omega_1$, such that the metric g and the function H only depend on x.

We restrict our attention to the case where the function f is given by Theorem 10.2.

11.3. Lemma. *Let H be a non-constant solution of*

$$\operatorname{Hess} H = H(H^2 - K) = -\frac{1}{2}(H^3 + \alpha H) \tag{11.3}$$

for some constant α. Then every critical point p of H, for which an integral curve γ of $\operatorname{grad} H$ *exists with $p = \lim_{t \to \pm\infty} \gamma(t)$, is a regular critical point.*

PROOF. By assumption, there is a point $q \in U$ near p with $H(q) \neq H(p)$, such that the integral curve of $\operatorname{grad} H$ (if $H(q) < H(p)$) or of $-\operatorname{grad} H$ (if $H(q) > H(p)$) through q is going to p. Consequently, the integral curve of X or $-X$ is a geodesic going to p, too. Let $\gamma \colon [0; b] \to M$ be the geodesic with $\gamma(0) = q, \gamma'(0) = X(q)$ and $\gamma(b) = p$. Consider the function $H(x) := H \circ \gamma(x)$. As a cause of 11.1 and 11.3 it satisfies the ODE

$$2H''(x) + H^3(x) + \alpha H(x) = 0 \tag{11.4}$$

with final value $H'(b) = 0$. If p would be a singular critical point of H, we would have $H^3(b) + \alpha H(b) = 0$. By Picard-Lindelöf, its only solution would be constant contradicting $H(p) \neq H(q)$. □

The geodesic polar coordinates around $p \in M$ are defined to be the composition of the inverse of the exponential map at p with the Euclidean polar coordinates of T_pM. In the case of a surface, we denote these coordinates by (r, φ), or by $(r, e^{i\varphi})$ if the image of the polar coordinates is $\mathbb{R}^{>0} \times S^1$. Of course, the polar coordinates depend on the choice of the direction which is mapped to the direction induced by $1 \in S^1 \subset \mathbb{R}^2$.

11.4. Proposition. *Let $(r, e^{i\varphi})\colon V \to]0; R[\times S^1$ be geodesic polar coordinates around a regular critical point p of a function H which solves the equations in 10.2. Then the metric is locally given by*

$$g = dr^2 + (\frac{L}{2\pi})^2 d\varphi^2. \tag{11.5}$$

Moreover, L and H are only depending on r with $L(r) = cH'(r)$ for some constant $c \neq 0$.

Proof. First we show that the integral curves of $X = \frac{\operatorname{grad} H}{\|\operatorname{grad} H\|}$ and $Y = \mathcal{J}X$ near critical points coincide with the coordinate lines of a geodesic polar coordinate system. We have proven that H has only regular critical points. Thus we can assume $p \in M$ to be a non-degenerate local minimum. There exists a neighborhood $V \subset U \subset M$ of p such that every integral curve of $-X = -\frac{1}{\|\operatorname{grad} H\|} \operatorname{grad} H$ starting at a point $q \in V \setminus \{p\}$ goes to p in finite time. As the integral curves of $-X$ are geodesics by 11.1, there is a normal neighborhood V of p, i.e. a set which is diffeomorphic to an open set in T_pM via exponential map. Thus, every geodesic emanating from p is an integral curve of X for small $t > 0$.

We are going to show that the chart given by 11.2 is the same as the geodesic polar coordinate system. The proof of 11.3 yields that the value $H(q)$ for $q \in V$ depends only on the length of the integral curve

$-X$ from q to p. And since these integral curves are geodesics its length is equivalent to the distance $d(p,q)$ between $p,q \in M$. Altogether we have that the integral curves of X and Y are the coordinate lines of the geodesic coordinate system and that $X = \frac{\partial}{\partial r}$.

Let $r\colon V \setminus \{p\} \to \mathbb{R}$ be the distance function of p. Consider the length function $L(r) = \int_{\gamma_r} g(.,Y)$ of circles with radius r around p. By using notations and results of 11.1 we get,

$$\mathrm{d}\, g(.,Y) = \mathrm{d}\,\omega_2 = \frac{f}{h}\,\mathrm{vol}_M\,.$$

Applying Stokes theorem and the fact that f and h are constant along circles around p we obtain

$$L' = \frac{f}{h}L. \tag{11.6}$$

This shows that the integrability factor l with $[X, lY] = 0$ and the length function L are the same up to a constant. We may fix this constant as 2π, i.e., $2\pi l = L$. By 11.2, lY is a Killing field and therefore a Jacobi field along every integral curve of X. Fix a geodesic γ emanating from p. Then $lY(r) := (lY)\circ\gamma(r)$ has the same initial values as the Jacobi field $\frac{\partial}{\partial\varphi}$. With $2\pi l = L$ and $\| Y \| = 1$ we have $lY(0) = 0$. Moreover, the equation $lY' = \nabla_X lY = \frac{f}{h}lY$ and the well known fact that $L'(0) = 2\pi$ show that $lY'(0) = \mathcal{J}\gamma'(0)$. This yields $\frac{\partial}{\partial\varphi} = lY$.

It remains to show that there is a constant $c \neq 0$ such that $l(r) = cH'(r)$. We have that f and h are locally given by $h(r) = H'(r)$ and $f(r) = h'(r) = H''(r)$. Thus all non-vanishing solutions of

$$l' = \frac{f}{h} = H''/H'l$$

are of shape cH' for a constant c. □

11.5. Corollary. *Every non-constant solution H of equation 11.3 on a compact surface has regular critical points only.*

Proof. First we will show the existence of a regular critical point. Note that by equation 11.1 a geodesic γ through a point $q \in U$ with initial value $\gamma'(0) = X_q$ is an integral curve of X (in U). Since M is complete, γ is defined for all t. But by definition of X the integral curve of X is obviously not defined for all $t > 0$. Thus γ does not stay in U

for all time and there exists $t_0 > 0$ with $t_0 := \inf\{t > 0 \mid \gamma(t) \notin U\}$. Evidently, $p := \gamma(t_0)$ is a regular critical point by 11.3.

Let $q \in M$ be another critical point of H, and $\gamma\colon [0;b] \to M$ be a geodesic from p to q. In 11.4 γ is proven to be an integral curve of X on the interval $]0;a_1[$, where $a_1 := \inf\{t \in 0;b] \mid \operatorname{grad}_{\gamma(t)} H = 0\}$. By using 11.3 again, $\gamma(a)$ is a regular critical point. Thus γ is again an integral curve of X but on the interval $]a_1;a_2[$, where a_2 is defined in the same manner as a_1. As the number of regular critical points is finite, q is to be reached after a finite number of steps by an integral curve of X. Thus by using 11.3 a last time, q is regular. □

11.6. Example. We end this section by giving an example of a conformally flat circle bundle over a non-compact surface with non-constant curvature. Let $M :=]1-\epsilon, 1+\epsilon[\times S^1$ for some $1 > \epsilon > 0$, and let $P = M \times S^1$ be the circle bundle with the projection $\pi\colon P \to M$ on the first factor. On M there are globally defined and commuting basis fields $\frac{\partial}{\partial r}$ and $\frac{\partial}{\partial \varphi}$ with the dual basis $\mathrm{d}\, r$ and $\mathrm{d}\,\varphi$. We define a Riemannian metric on M by

$$g = \mathrm{d}\, r \otimes \mathrm{d}\, r + l^2(r)\, \mathrm{d}\,\varphi \otimes \mathrm{d}\,\varphi$$

where $l\colon M \to \mathbb{R}$ is a nowhere vanishing function which only depends on r. For any function $f\colon M \to \mathbb{R}$ we set $f' := \frac{\partial f}{\partial r}$. We determine the Levi-Civita connection of g, with $X := \frac{\partial}{\partial r}$ and $Y := \frac{1}{l}\frac{\partial}{\partial \varphi}$, as $\nabla X = l'Y \otimes \mathrm{d}\,\varphi$ and $\nabla Y = -l'X \otimes \mathrm{d}\,\varphi$. The Gaussian curvature of M is given by

$$K = -\frac{l''}{l}.$$

Let $H\colon M \to \mathbb{R}$ be a function only depending on r and satisfying $\operatorname{Hess} H = H(H^2 - K)\operatorname{Id}$ and $2K - 3H^2 = \alpha$ for some constant α. Using 11.1 one easily obtains that there exists a constant c such that $l = cH'$. The condition of the total space being conformally flat turns into

$$H'' = -\frac{1}{2}H^3 - \frac{\alpha}{2}H. \tag{11.7}$$

It is not difficult to verify that any function H satisfying 11.7 satisfies the condition above, too, with a constant c such that $l = cH'$. Therefore there exists a family of solutions to these equations. They are depending on the choice of α, $H(1)$ and $H'(1)$. For each solution H the metric g is determined up to a constant c via $l = cH'$.

We proved the existence of a surface with Riemannian metric g and Gaussian curvature K together with a function H satisfying the equations above. It remains to show that for every solution H there is a circle metric on $\pi\colon P \to M$ such that the curvature function of the horizontal distribution is given by H. To do so, we define T to be the infinitesimal generator of the circle action on P, i.e. if one uses the product coordinates (r, φ, t) on $P =]1-\epsilon, 1+\epsilon[\times S^1 \times S^1$ then $T = \frac{\partial}{\partial t}$. Define the connection $1-$form of this circle bundle by

$$\omega = \mathrm{d}\, t + \frac{c}{2} H^2 \,\mathrm{d}\, \varphi.$$

Evidently, it defines a principal connection on P. Moreover, the symmetric bilinear form

$$\tilde{g} = \pi^* g + \omega \otimes \omega$$

is strictly positive definite and invariant under circle action. Therefore $\tilde{g}$ is a circle metric. In 9.2 and 7.3 it was proven that the curvature function $\tilde{H}$ of the horizontal distribution is given by $\mathrm{d}\,\omega = \tilde{H}\pi^* \mathrm{vol}_M$. We have

$$\mathrm{d}\,\omega = HcH' \,\mathrm{d}\, r \wedge \mathrm{d}\,\varphi = H\pi^* \mathrm{vol}_M,$$

thus $(P, \tilde{g})$ is in fact a conformally flat circle bundle over an oriented surface. By 8.2 the sectional curvature of the horizontal distribution is given by

$$K - \frac{3}{4}H^2 = \frac{\alpha}{2} + \frac{3}{4}H^2$$

which is clearly non-constant unless H is constant.

12. Classification over compact Surfaces

We are going to classify conformally flat circle bundles over compact oriented surfaces. Let M be a compact surface. With the same notations as in 11.1 we state the following lemma:

12.1. Lemma. *The integral curves of Y are complete in U. Moreover they are closed.*

PROOF. First we show completeness. Let $\gamma\colon [a,b[\to U$ be an integral curve of Y. Since M is compact there exists a sequence $t_n \to b$ such that $\gamma(t_n) \to p \in M$. But γ' is of constant length 1, thus $\gamma(t) \to p$ for $t \to b$. Assume that $p \notin U$, i.e. $\operatorname{grad}_p H = 0$. In 11.1 it was proven that $\| \operatorname{grad} H(\gamma(t)) \|$ is constant for any integral curve of Y, which is obviously not zero contradicting $\operatorname{grad}_p H = 0$.

It remains to show that $\gamma\colon \mathbb{R} \to U \subset M$ is closed. If not, γ would be injective. As M is compact, there would be a sequence $t_n \to \infty$ such that $\gamma(t_n) \to p$ for $n \to \infty$. With the same arguments as above we get $p \in U$. Since $\operatorname{grad} H \neq 0$ on U there is a neighborhood V of p such that $H(q) = H(p)$ if and only if p and q lie on the same integral curve of Y. Because H is constant along γ we have $H(\gamma(t_n)) = H(p)$. Therefore, $\gamma(t_n)$ and p lie on the same integral curve of Y for any n large enough. By using $\| Y \| = 1$ we have that γ would pass p infinitely often, which contradicts γ to be injective. $\square$

12.2. Proposition. *Let $H\colon M \to \mathbb{R}$ be a solution of*

$$\operatorname{Hess} H = H(H^2 - K) = -\frac{1}{2}(H^3 + \alpha H).$$

We have that either H is constant or $M = S^2$ and there are exactly two critical points $N, S \in S^2$ of H. In the second case $S^2 \setminus \{N, S\} \cong I \times S^1$ for some interval I such that the induced metric g on $I \times S^1$ is given by

$$g = dr^2 + (\frac{L(r)}{2\pi})^2 d\varphi^2,$$

where $(r, e^{i\varphi})$ are the product coordinates on $I \times S^1$, and $L\colon I \to \mathbb{R}$ is the length function of circles in S^2 around S. Moreover, H does only depend on r satisfying $L(r) = cH'(r)$ for some constant $c \neq 0$.

PROOF. Note that $M = S^2$ if H is not constant is a consequence of the Morse theory and the fact that the Hessian of H at every critical point is strictly definite by $\operatorname{Hess} H = H(H^2 - K)\operatorname{Id}$.

Let S denote the absolute minimum of H and let $N \in M$ be the absolute maximum of H. Fix a geodesic $\gamma\colon [0, R] \to M$ from S to N and denote its parameter by r. We have the functions $H(r) := H \circ \gamma(r)$ and $h(r) := H'(r)$. They are related to the length $L(r)$ of the circles with radius r around S by $L(r) = ch(r)$ for some constant c as shown in the proof of 11.4. We have that $\exp_S$ is a diffeomorphism if it is restricted to $B_R := \{v \in T_S M \mid \| v \| < R\}$. The rest of the proof is obvious. $\square$

Theorem. *Let g be a conformally flat circle metric on a circle bundle $\pi\colon P \to M$ over a compact oriented surface M. Then M is of constant curvature K. Moreover, we have that $\pi\colon P \to M$ is either the trivial bundle or $M = S^2$ and π is the Hopf or conjugate Hopf fibration.*

Proof. If H is constant we have $0 = \operatorname{Hess} H = H(H^2 - K)$. By Gauss-Bonnet, $H^2 - K \geq 0$ for surfaces M_g of genus $g \geq 1$. The equality is valid if and only if $g = 1$, $K = H = 0$. Thus $H = 0$, and $\pi\colon P \to M_g$ must be the trivial bundle $M \times S^1$ with the product metric.

Assume that H is non-constant, and let S and N be the absolute minimum and maximum of H, respectively. Let $\gamma\colon [0; R] \to S^2; t \mapsto \gamma(t)$ be a geodesic from S to N. We already showed, that the equations $\operatorname{Hess} H = H(H^2 - K)$ and $2K = 3H^2 + \alpha$ turn into

$$2H''(t) + H^3(t) + \alpha H(t) = 0 \tag{12.1}$$

with $H'(0) = H'(R) = 0$, where we use $H(t) = H \circ \gamma(t)$ for short.

We denote $H(0) = A$ and $H(R) = B$. Then every solution of 12.1 satisfies

$$\begin{aligned} 4(H'(t))^2 &= -\,H^4(t) - 2\alpha H^2(t) + 2\alpha A^2 + A^4 \\ &= -\,(H(t) - A)(H(t) + A)(H^2(t) + 2\alpha + A^2) \end{aligned} \tag{12.2}$$

by the law of conservation. As the functions H and K are the curvatures of complex line bundles with degree d and 2, respectively, the integrals of H and K are given by

(12.3)

$$\begin{aligned} -2\pi d &= \int_{S^2} H dA = c \int_0^R H(t)H'(t)dt = \frac{c}{2}(B^2 - A^2) \\ 4\pi &= \int_{S^2} K dA = \frac{c}{2} \int_0^R (3H^2(t) + \alpha)H'(t)dt = \frac{c}{2}(B^3 + cB - A^3 - cA). \end{aligned}$$

Since $H'(R) = 0$, 12.2 and 12.3 imply

$$0 = B^2 + 2\alpha + A^2. \tag{12.4}$$

With $L'(0) = 2\pi$ and 12.1 we obtain

$$\begin{aligned} 0 &= \frac{4\pi}{c} + A^3 + \alpha A, \\ 0 &= \frac{4\pi}{c} + B^3 + \alpha B. \end{aligned} \tag{12.5}$$

An easy algebraic computation reveals that the equations 12.3, 12.4 and 12.5 have no common solution. Thus, H must be constant and therefore K must be constant, too. That implies that g must be the round metric of S^2. Using 12.3 and $H(H^2-K) = 0$, we get $H = \pm 2$ and $K = 4$, or $H = 0$. In the latter case, π is the trivial bundle $S^2 \times S^1$ with product metric. If $H = \pm 2$ and $K = 4$, we have that S^3 is equipped with the round metric, by 10.1. As the fibers are geodesics, they have to be circles, thus 6.9 yields the theorem. □

CHAPTER V

Harmonic Morphisms on $3-$Manifolds

In the last chapter we study harmonic morphisms from Riemannian $3-$manifolds to surfaces. These are maps which pull harmonic functions back to harmonic functions. It is well-known that being a harmonic morphism is a very strong condition, see 14.3 or [**BaWo**]. We will show, under a natural assumption, that the only conformally flat metric on S^3 which posses a submersive harmonic morphism onto a surface is the round metric, and that the map must be the Hopf fibration up to isometries.

13. The Chern-Simons Invariant of $3-$manifolds

Our proof of the uniqueness theorem 15.6 uses a global invariant of conformal $3-$manifolds given by the Chern-Simons functional. In their fundamental work, [**CS**], Chern and Simons introduced a geometric invariant of connections. Their theory plays an important role in the topology and knot theory of three-manifolds, since Witten had shown its connection to the Jones polynomial. We will only discuss the Chern-Simons functional for Levi-Civita connections, which is based on [**Ch**].

13.1. The Frame Bundle. We need to introduce basic concepts of Riemannian geometry via frame bundles, see 7.2 for the theory of connections on principal bundles. Let $F(M) \to M$ be the bundle of linear frames on a $n-$dimensional Riemannian manifold (M, g), and $O(M) \to M$ its reduction to orthonormal frames. One of these two bundles is meant, if we write $\pi\colon P \to M$. Recall that the elements of the frame bundle $F(M)$ or $O(M)$ at a point $p \in M$ are linear isomorphism or linear isometries $f\colon \mathbb{R}^n \to T_pM$. The canonical form $\theta \in \Gamma(P; \mathbb{R}^n)$ on P is defined by

$$\theta_f(X) = f^{-1}(\mathrm{d}\,\pi(X)).$$

Proposition. *There exists an unique principal connection* $\omega \in \Gamma(O(M);\mathfrak{so}(n))$ *on* $O(M)$ *such that*

$$\mathrm{d}\,\theta(X,Y) = -\omega(X)(\theta(Y)) + \omega(Y)(\theta(X)),$$

where $\mathfrak{so}(n)$ *act on* $\mathbb{R}^n$ *in the standard way. The induced linear connection on* TM *is the Levi-Civita connection. And there is exactly one connection on* $F(M)$*, denoted by* ω*, too, which reduces to* ω *on* $O(M)$*.*

Proof. The proof is based on Cartan's moving frame method applied to all frames, i.e. to the frame bundle. For details, one should consult [**KN**]. □

13.2. Remark. Consider a locally defined section $s \in \Gamma(U; O(M))$. The vector fields $X_i = s(e_i)$ define an orthonormal basis of T_pM for all $p \in U \subset M$. Consider the dual 1−forms $\theta_i = g(.,X_i)$. It is clear that their collection (θ_i) is the canonical form θ_s along $s \in \Gamma(O(M))$. Thus the proposition above yields

$$\mathrm{d}\,\theta_j = -\sum_i \omega_{ij} \wedge \theta_i.$$

Here, the connection form ω is given by a skew symmetric matrix of 1−forms on M via $s^*\omega = (\omega_{ij}) \in \mathfrak{so}(n)$. The ω_{ij} can be obtained from the covariant derivatives of the X_j via

$$\nabla X_j = \sum_i \omega_{ij} \otimes X_i.$$

The $\mathfrak{so}(n)$−valued curvature 2−form can be computed from $s^*\Omega = (\Omega_{ij})$ with

$$\Omega_{ij} = \mathrm{d}\,\omega_{ij} + \sum_k \omega_{ik} \wedge \omega_{kj} = \frac{1}{2}\sum_{k,l} r(X_k, X_l, X_i, X_j)\theta^k \wedge \theta^l, \tag{13.1}$$

where r is the Riemannian curvature tensor.

13.3. Chern-Simons Functional. Let M always be a compact oriented 3−manifold with trivial tangent bundle. We are going to use the bundle $\mathrm{SO}(M) \to M$ of oriented orthonormal frames instead of $O(M) \to M$.

Proposition. *Let* g *be a metric on* M *and* $\omega \in \Omega^1(\mathrm{SO}(M), \mathfrak{so}(3))$ *be the Levi-Civita connection. For any section* $s\colon M \to \mathrm{SO}(M)$ *we* *define*

$$\begin{aligned}CS(M,g,s) :=& \frac{1}{8\pi^2}\int_M \operatorname{tr}(-\frac{1}{2}s^*\omega \wedge s^*\,\mathrm{d}\,\omega - \frac{1}{3}s^*\omega\wedge s^*\omega\wedge s^*\omega)\\ =& \frac{1}{8\pi^2}\int_M \operatorname{tr}(-\frac{1}{2}s^*\omega\wedge s^*\Omega + \frac{1}{6}s^*\omega\wedge s^*\omega\wedge s^*\omega).\end{aligned}$$

The functional only depends on the conformal class of g and on the homotopy type of s. Consequently, the Chern-Simons functional

$$CS((M,[g])) := CS(M,g,s) \mod \mathbb{Z} \in \mathbb{R}/Z$$

is a conformal invariant.

Proof. Consider two sections $s, \tilde{s} \in \Gamma(\mathrm{SO}(M))$. Then there is an unique function $f\colon M \to \mathrm{SO}(M)$ such that $\tilde{s} = sf$. In such a case we have

$$\tilde{s}^*\omega = \mathrm{d}\,ff^{-1} + fs^*\omega f^{-1}$$

and

$$\tilde{s}^*\Omega = fs^*\Omega f^{-1}.$$

Here, the multiplication is meant to be the matrix multiplication of the matrix Lie group $\mathrm{SO}(M)$. Note that $\mathrm{d}\,ff^{-1}$ is the pullback $f^*\omega_{mc}$ of the Maurer-Cartan form ω_{mc} of the group $\mathrm{SO}(M)$. Using $\operatorname{tr}(\alpha\wedge\beta) = (-1)^{\deg\alpha\deg\beta}\operatorname{tr}(\beta\wedge\alpha)$ and $d\operatorname{tr}(\alpha) = \operatorname{tr}(\mathrm{d}\,\alpha)$, we have

$$\begin{aligned}CS(M,g,\tilde{s}) =& CS(M,g,s) + \frac{1}{8\pi^2}\int_M \frac{1}{6}\operatorname{tr}(f^*\omega_{mc}\wedge f^*\omega_{mc}\wedge f^*\omega_{mc})\\ &+ \frac{1}{16\pi^2}\int_M \mathrm{d}\operatorname{tr}(\mathrm{d}\,ff^{-1}\wedge \mathrm{d}\,s^*\omega)\\ =& CS(M,g,s) + \deg(f),\end{aligned}$$

where the last equation follows from

$$\frac{1}{6}\operatorname{tr}(f^*\omega_{mc}\wedge f^*\omega_{mc}\wedge f^*\omega_{mc}) = f^*\operatorname{vol}_{\mathrm{SO}(3)}.$$

Here, $\mathrm{SO}(3)$ is equipped with the metric of constant sectional curvature $\frac{1}{4}$ which has a volume of $8\pi^2$.

It remains to prove that the functional is independent of the metric in a conformal class. We denote

$$f^*\omega = \begin{pmatrix} 0 & \omega^1 & \omega^2 \\ -\omega^1 & 0 & \omega^3 \\ -\omega^2 & -\omega^3 & 0 \end{pmatrix},$$

for short, and

$$\begin{aligned} f^*\Omega &= \begin{pmatrix} 0 & \Omega^1 & \Omega^2 \\ -\Omega^1 & 0 & \Omega^3 \\ -\Omega^2 & -\Omega^3 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & \mathrm{d}\,\omega^1 - \omega^2 \wedge \omega^3 & \mathrm{d}\,\omega^2 + \omega^1 \wedge \omega^3 \\ -\mathrm{d}\,\omega^1 + \omega^2 \wedge \omega^3 & 0 & \mathrm{d}\,\omega^3 - \omega^1 \wedge \omega^2 \\ -\mathrm{d}\,\omega^2 - \omega^1 \wedge \omega^3 & -\mathrm{d}\,\omega^3 + \omega^1 \wedge \omega^2 & 0 \end{pmatrix}. \end{aligned}$$

By using this we get

(13.2)

$$CS(M,g,f) = \frac{1}{8\pi^2}\int_M \omega^1 \wedge \mathrm{d}\,\omega^1 + \omega^2 \wedge \mathrm{d}\,\omega^2 + \omega^3 \wedge \mathrm{d}\,\omega^3 - 2\omega^1 \wedge \omega^2 \wedge \omega^3.$$

Now, we consider for an arbitrary function $\lambda\colon M \to \mathbb{R}$ the curve $g_t = e^{2t\lambda}g$ of metrics. Using 13.2 and the well known formula

$$\tilde{\nabla}_A B = \nabla_A B + A \cdot \lambda B + B \cdot \lambda A - g(A,B)\,\mathrm{grad}\,\lambda$$

for the Levi-Civita connection of the metric $\tilde{g} = e^{2\lambda}g$, one obtains the following connection forms ${\omega^i}_t$ for the frames $f_t = e^{-t\lambda}f \in \Gamma(\mathrm{SO}(M,g_t))$:

$$\begin{aligned} {\omega^1}_t =& \omega^1 - tX_1 \cdot \lambda\theta_2 + tX_2 \cdot \lambda\theta_1 \\ {\omega^2}_t =& \omega^2 - tX_1 \cdot \lambda\theta_3 + tX_3 \cdot \lambda\theta_1 \\ {\omega^3}_t =& \omega^3 - tX_2 \cdot \lambda\theta_3 + tX_3 \cdot \lambda\theta_2, \end{aligned}$$

and we compute

$$\begin{aligned}
8\pi^2 \frac{\partial CS(M, g_t, f_t)}{\partial t}_{|t=0} =& \int_M - \mathrm{d}(\sum_i \omega^i \wedge \frac{\partial \omega^i}{\partial t}) + 2 \sum_i \frac{\partial \omega^i}{\partial t} \wedge \Omega^i \\
=& 2 \int_M \sum_i \frac{\partial \omega^i}{\partial t} \wedge \Omega^i \\
=& 2 \int_M X_2 \cdot \lambda(\theta^1 \wedge \Omega^1 - \theta^3 \wedge \Omega^3) \\
& - 2 \int_M X_1 \cdot \lambda(\theta^2 \wedge \Omega^1 + \theta^3 \wedge \Omega^2) \\
& + 2 \int_M X_3 \cdot \lambda(\theta^1 \wedge \Omega^2 + \theta^2 \wedge \Omega^3) = 0.
\end{aligned}$$

The last equation is a consequence of $\sum_j \Omega_{ij} \wedge \theta^j = 0$, which follows immediately from 13.1 and the symmetries of the curvature tensor. This computation is valid for any metric g and any infinitesimal conformal change, thus the Chern-Simons functional only depends on the conformal structure. □

We end this section by stating two theorems illustrating the geometric significance of the Chern-Simons functional. The proofs can be found in [**CS**].

Theorem. *The critical values $CS(M, [g])$ of the Chern-Simons functional are exactly the conformally flat spaces $(M, [g])$.*

Theorem. *A necessary condition that a conformal $3-$space $(M, [g])$ admits a conformal immersion into $\mathbb{R}^4$ is given by $CS(M, [g]) = 0$.*

14. Harmonic Morphisms with one-dimensional Fibers

We give a short introduction of harmonic morphisms. We refer the reader to the monograph [**BaWo**] for a detailed study.

14.1. Harmonic Mappings. Let $f\colon (P, g) \to (M, h)$ be a smooth map between Riemannian manifolds. The energy functional of f is given by

$$E(f) := \frac{1}{2} \int_P \| \mathrm{d} f \|^2 \mathrm{vol}_P .$$

Here $\| \, . \, \|$ is the Hilbert-Schmidt norm of operators, i.e. $\| \mathrm{d} f \|^2 = \mathrm{tr}_g f^* h$. It is a generalization of the energy of real-valued functions and one defines harmonic maps as the critical values of this functional.

Both, the functional and the Euler-Lagrange equation have the same shape as for functions. Consider the differential of f as a section

$$\mathrm{d}\, f \in \Gamma(P; T^*P \otimes f^*TM),$$

and equip the bundle with the product connection of the Levi-Civita connection on T^*P and the pullback $f^*\nabla$ of the Levi-Civita connection on TM. Note that in case of functions, i.e. $(M,h) = (\mathbb{R}, <,>)$, the product connection on $T^*P \otimes T\mathbb{R} \cong T^*P$ is equal to the Levi-Civita connection on T^*P.

Proposition. *A map $f\colon (P,g) \to (M,h)$ between Riemannian manifolds is a harmonic map if and only if the tension field*

$$\tau(f) := \mathrm{tr}\, \nabla\, \mathrm{d}\, f \in \Gamma(f^*TM)$$

vanishes.

A proof can be found in [**J**], where a whole chapter is devoted to the study of harmonic maps. In the case of functions, the tension field is given by the negative of the Laplacian and both definitions coincide.

14.2. Harmonic Morphisms. A harmonic morphism is a map

$$f\colon (P,g) \to (M,h)$$

between Riemannian manifolds, such that for any locally defined harmonic function $u\colon U \subset M \to \mathbb{R}$ the composition $u \circ f$ is harmonic on $\pi^{-1}(U) \subset P$.

Of course, constant maps are harmonic morphisms. Further holomorphic maps between Riemannian surfaces or isometries are harmonic morphisms, too. The composition of two harmonic morphisms is again a harmonic morphism. Therefore, in the case of a surface as target space, a map is a harmonic morphism for a metric on the surface if and only if it is one for any other metric in the same conformal class. This allows us to speak about harmonic morphisms into Riemannian surfaces.

There is a useful characterization of harmonic morphisms given by Fuglede ([**Fu**]) and Ishihara ([**Is**]). We specify in the case of submersions:

14.3. Theorem. *A submersion between Riemannian manifolds is a harmonic morphism if and only if it is harmonic and conformal.*

Proof. We will not give a proof of this theorem in general because we only consider harmonic morphisms from $3-$dimensional spaces to surfaces, so we restrict ourselves to this case. The general proof is very similar but it has more technical difficulties.

Let $p \in M$ be an arbitrary point on M, and $(E_i)_{i=1,2,3}$ be an orthonormal frame around $\pi^{-1}(p)$ on (P, g) such that $\mathrm{d}\,\pi(E_3) = 0$. Let $h \in [h]$ be a metric in the conformal class of the surface. For any smooth function $f \colon U \subset M \to \mathbb{R}$, which is locally defined around p, we compute

$$\begin{aligned}
\Delta(f \circ \pi) &= -\sum_i (\nabla_{E_i}(\mathrm{d}\, f \circ \mathrm{d}\,\pi), E_i) \\
&= -\sum_i (\nabla_{E_i} h(\operatorname{grad} f \circ \pi, \mathrm{d}\,\pi), E_i) \\
&= -\sum_i (h(\pi^* \nabla_{E_i} \operatorname{grad} f \circ \pi, \mathrm{d}\,\pi), E_i) \\
&\quad - \sum_i (h(\operatorname{grad} f \circ \pi, \nabla_{E_i} \mathrm{d}\,\pi), E_i) \\
&= -\sum_i h(\nabla_{\mathrm{d}\,\pi(E_i)} \operatorname{grad} f, \mathrm{d}\,\pi(E_i)) - \mathrm{d}\, f(\tau(\pi)).
\end{aligned}$$

If π is harmonic and conformal with square dilation Λ defined by $\pi^* h = \Lambda g_{|\mathcal{V} \times \mathcal{V}}$, we have

$$\Delta(f \circ \pi) = -\sum_i h(\nabla_{\mathrm{d}\,\pi(E_i)} \operatorname{grad} f, \mathrm{d}\,\pi(E_i)) = \Lambda(\Delta f) \circ \pi,$$

and we obtain that π is a harmonic morphism.

Conversely, let π be a submersive harmonic morphism. As π remains to be a harmonic morphism in spite of a conformal change of the metric on the surface, we may assume that h is flat around p. Let $z = x + iy$ be an isometric chart. Consider the locally defined functions $f(z) = x^2 - y^2$ and $g(z) = xy$. It is clear that they are harmonic on a small set around p, and $\mathrm{d}_p\, f = \mathrm{d}_p\, g = 0$. The term

$0 = \Delta(f \circ \pi) = \Delta(g \circ \pi)$ and a short algebraic computation yield

$$\mathrm{d}\,\pi(E_1) \perp \mathrm{d}\,\pi(E_2)$$

and

$$\| \mathrm{d}\,\pi(E_1) \| = \| \mathrm{d}\,\pi(E_2) \| .$$

And as p is arbitrary we get that π is conformal. Therefore, for any harmonic function f on M, we have $0 = \Delta(f \circ \pi) = \mathrm{d}\, f(\tau(\pi))$, thus π is also a harmonic map. □

We will use another characterization of harmonic morphisms, which is due to Baird and Eells [**BE**]:

14.4. Theorem. *A submersion π from a Riemannian $3-$space (P, g) to a Riemannian surface $(M, [h])$ is a harmonic morphism if and only if it is conformal and has minimal fibers.*

Proof. It remains to be shown that a conformal submersion of a $3-$dimensional manifold to a surface is harmonic if and only if the fibers are geodesics. Using the formulas of 9, and the notations as in the proof of 14.3, it is easy to determine the tension field as $\tau(\pi) = -\,\mathrm{d}\,\pi(\nabla_{E_3} E_3)$. □

15. Harmonic Morphisms on Conformally Flat $3-$Spheres

We will now study harmonic morphisms on the conformally flat $3-$sphere under two assumptions: We only consider submersive harmonic morphisms, and we restrict ourselves to the case where the curvature of the horizontal distribution nowhere vanishes. The latter is exactly the case if the induced CR structure on S^3 is strictly pseudoconvex. By changing one of the orientations either on the $3-$space or on the surface the curvature function H of the horizontal distribution changes sign. So we will assume in the following that $H > 0$, and we say that the horizontal distribution is of positive curvature.

15.1. Lemma. *Let $\pi\colon (S^3, \tilde{g}) \to (S^2, [h])$ be a submersive harmonic morphism such that its horizontal distribution has positive curvature. By a conformal change g of the metric on S^3 and with a metric $h \in [h]$ on S^2, π is a Riemannian submersion, such that the mean curvature of the fibers is given by $\nabla_T T = -\,\mathrm{grad}^h \log H$ with respect to the* *new metric. For this metric, the function*

$$p \in S^2 \mapsto \int_{\pi^{-1}(p)} \frac{1}{H} g(.,T)$$

is constant on the two sphere. We fix that constant to be π*, then* g *and* h *are unique.*

Proof. Let g and h be metrics in the given conformal classes such that π is a Riemannian submersion. Note that these metrics are unique up to the multiplication by the same function defined on S^2. As the metric changes by the factor $e^{2\lambda}$, the curvature function H changes by the factor $e^{-\lambda}$. Hence there is a unique choice of the metrics g and h such that $\int_{\pi^{-1}} \frac{1}{H} = \pi$.

It remains to show that for this choice of g and h the (mean) curvature of the fibers is given by $\nabla_T T = -\operatorname{grad}^h \log H$. In 14.4 we have showed that by changing the metric g in a suitable way the mean curvature of the fibers vanishes. Using the formula for the Levi-Civita connection of a conformally changed metric we have that this is equivalent to the existence of a function $\lambda\colon S^3 \to \mathbb{R}$ such that $\nabla_T T = \operatorname{grad} \lambda$.

Let $\hat{A}, \hat{B}$ be the horizontal lift of positive orthonormal basis fields A, B on the surface. As $\hat{A}$ is a horizontal lift, the commutator $[\hat{A}, T]$ is vertical and we get

$$[\hat{A}, T] = \nabla_{\hat{A}} T - \nabla_T \hat{A} = -g(\nabla_T \hat{A}, T)T = g(\hat{A}, \nabla_T T) = \hat{A} \cdot \lambda T,$$

and similarly $[\hat{B}, T] = \hat{B} \cdot fT$. By definition, we have $H = -g(T, [\hat{A}, \hat{B}])$. The Jacobi identity for commutators of the vector fields $\hat{A}, \hat{B}, T$ yields

$$T \cdot \lambda = -T \cdot \log H.$$

Hence, $\lambda = -\log H + f \circ \pi$, where f is some function defined on the surface. If we change the metric g by the factor $e^{2\lambda}$, the fibers become geodesics, thus all of them have the same length. We obtain

$$\begin{aligned} const = \int_{\pi^{-1}} \tilde{g}(., \tilde{T}) &= \int_{\pi^{-1}} e^{\lambda} g(., T) \\ &= \int_{\pi^{-1}} \frac{e^f}{H} g(., T) = e^f \pi. \end{aligned}$$

Thus f must be constant and consequently $\nabla_T T = -\operatorname{grad}^h \log H$. □

For the rest of this section we are going to use the metrics given by the lemma above.

15.2. Corollary. *Let the metrics on S^3 and S^2 be given as in 15.1. Then we have $\int_{S^2} \operatorname{vol}_{S^2} = \pi$, and consequently*

$$\int_{S^3} \frac{1}{H} \operatorname{vol}_{S^3} = \pi^2.$$

Proof. The equation $\nabla_T T = -\operatorname{grad}^h \log H$ induces

$$\mathrm{d}\,\frac{1}{H}\theta_3 = \pi^* \operatorname{vol}_{S^2}.$$

Since $\int_{\pi^{-1}(p)} \frac{2}{H} = 2\pi$, we get that $\frac{2}{H}\theta_3$ is a principal connection 1−form of the principal S^1−bundle $\pi\colon S^3 \to S^2$ with principal action given by the flow of $\frac{H}{2}T$. The curvature form of this connection is given by $2\operatorname{vol}_{S^2}$. But the degree of this bundle is -1, and can be determined as

$$-2\pi \deg(S^3 \to S^2) = \int_{S^2} \mathrm{d}(\frac{2}{H}\theta_3) = \int_{S^2} 2\operatorname{vol}_{S^2}.$$

Therefore $\int_{S^2} \operatorname{vol}_{S^2} = \pi$, and $\int_{S^3} \frac{1}{H} \operatorname{vol}_{S^3} = \int_{S^2} \pi \operatorname{vol}_{S^2} = \pi^2$ by Fubini. □

We are going to compute the Chern-Simons functional in the geometric quantities H, K and $\nabla_T T = -\operatorname{grad}^h \log H$. We need:

15.3. Lemma. *Let $\pi\colon S^3 \to S^2$ be a Riemannian submersion and $X \in \Gamma(S^3, \mathcal{H})$ be a horizontal vector field of length 1. Then the mapping degree of*

$$p \in \pi^{-1}(q) \mapsto \mathrm{d}\,\pi_p(X) \in S^1 \subset T_q S^2$$

is ± 2 for each fiber, where the sign is given by the sign of the degree of the bundle π.

Conversely, let A be a non-vanishing vector field of length 1 defined on $U \subset S^2$, and $e^{i\varphi}\colon \pi^{-1}(U) \to S^1$ be a map, where $S^2 \setminus U$ is simply connected with nonempty interior. Then $\cos\varphi \hat{A} + \sin\varphi \mathcal{J}\hat{A}$ can be extended to a globally defined, non-vanishing, horizontal vector field of S^3 if and only if the mapping degree of $e^{i\varphi}$ is ± 2 for each fiber, with the same sign as above.

Proof. Every submersion of S^3 is homotopic to the Hopf fibration, which has degree -1, or to the conjugate Hopf fibration with degree 1, thus the proof of the lemma reduces to these cases. The fibers of the Hopf fibration (or the conjugate Hopf fibration) are given by the oriented integral curves of the right invariant vector field $\bar{I}$ (or the left invariant vector field I), and for the right invariant horizontal field $\bar{J}$, the mapping degree of the projection is -2 (or 2 for the left invariant J), compare with section 1. Every other horizontal vector field is given by $X = e^{i\varphi}J$ for some well-defined $e^{i\varphi}\colon S^3 \to S^1$. Further we have that S^3 is simply connected, hence the logarithm $\varphi\colon S^3 \to \mathbb{R}$ is well-defined, too, and consequently X also has degree -2 (or 2). The proof of the inverse direction is done in a similar way. □

15.4. Proposition. *Let g be the conformally flat metric in 15.1. Let T be the unique vertical vector field of length 1 in positive direction and let X, Y be horizontal. Then the Chern-Simons functional with respect to a positive oriented orthonormal frame $s = (X, Y, T)$, is*

$$8\pi^2 CS(S^3, g, s) = \int_{S^3} 4\frac{K}{H} - \frac{1}{2}H^3 \,\mathrm{vol},$$

where K is the Gaussian curvature of the surface and H is the curvature function of the horizontal distribution.

Proof. Let A, B be a positive oriented orthonormal frame on $V := S^2 \setminus \{p\}$ for some $p \in S^2$, and let $\lambda\colon V \to \mathbb{R}$ be a function such that $e^{\lambda}(A - iB)$ is holomorphic. We denote the horizontal lifts by A and B, too. For any frame $s = (X, Y, T)$, there is a function $e^{i\varphi}\colon \pi^{-1}(V) \to \mathbb{R}$ with $X + iY = e^{i\varphi}(A + iB)$. By using 9.5, we obtain the following connection forms for the frame $s = (X, Y, T)$:

$$\begin{aligned} \omega^1 &= \mathcal{J}^* \,\mathrm{d}\,\lambda - \frac{1}{2}H\theta_3 - \mathrm{d}\,\varphi, \\ \omega^2 &= -\frac{1}{2}H\theta_2 + g(\nabla_T T, X)\theta_3, \\ \omega^3 &= \frac{1}{2}H\theta_1 + g(\nabla_T T, Y)\theta_3, \end{aligned} \tag{15.1}$$

where $\omega_1, \omega_2, \omega_3$ is the dual frame of A, B, T, and $\theta_1 = \cos\varphi\omega_1 + \sin\varphi\omega_2, \theta_2 = -\sin\varphi\omega_1 + \cos\varphi\omega_2, \theta_3 = \omega_3$ is the dual frame of X, Y, T. With 13.2 and $\nabla_T T = -\operatorname{grad}^h \log H$, we get the integrand μ of the Chern-Simons functional on $\pi^{-1}(V)$:

$$\begin{aligned} \mu =& \Big(\frac{1}{2}H(H^2 - K) + T\cdot\varphi\big(\frac{1}{2}H^2 - K\big) + \frac{1}{2}\Delta^h H \\ &+ g(\operatorname{grad} H, \mathcal{J}\operatorname{grad}\varphi - \operatorname{grad}\lambda + 2\operatorname{grad}^h \log H\Big) \,\mathrm{vol}\,. \end{aligned}$$

The condition 9.3 for g to be conformally flat turns in case of $\nabla_T T = -\operatorname{grad}^h \log H$ into:

$$
(15.2)\qquad
\begin{aligned}
0 =& 2H\Delta^h \log H + H(H^2 - K)\\
=& 2\Delta^h H + 2g(\operatorname{grad}^h H, \operatorname{grad}^h \log H) + H(H^2 - K),
\end{aligned}
$$

thus we obtain

$$
\begin{aligned}
\mu =& (-\frac{1}{2}H(H^2 - K) + T \cdot \varphi(\frac{1}{2}H^2 - K)\\
& - \frac{3}{2}\Delta^h H + g(\operatorname{grad} H, \mathcal{J} \operatorname{grad}\varphi - \operatorname{grad}\lambda) \operatorname{vol}.
\end{aligned}
$$

For $n \in \mathbb{N}$ we set $U_n := \pi^{-1}(B_n)$, where the $B_n \subset V = S^2 \setminus \{p\}$ are open connected and simply connected subsets with piecewise smooth boundary, such that $\lim_{n\to\infty} \int_{S^2\setminus B_n} \operatorname{vol}_{S^2} = 0$ and $\bar{B}_n \neq S^2$. We will specify these sets later on.

Let $\eta \in \Omega^1(V)$ be a 1−form on the surface such that $\mathrm{d}\,\eta = 4 \operatorname{vol}_{S^2}$. Consider a map $e^{i\varphi}\colon \pi^{-1}(V) \to S^1$ with $\mathrm{d}\,\varphi = \eta - \frac{4}{H}\theta_3$. To prove the existence of $e^{i\varphi}$, we state $\mathrm{d}(\eta - \frac{4}{H}\theta_3) = 0$, then we use that

$$
\int_{\pi^{-1}(q)} \eta - \frac{4}{H}\theta_3 = -4\pi \in \ker(t \mapsto e^{it})
$$

for a generator $\pi^{-1}(q)$, $q \neq p$, of the first homology group of $\pi^{-1}(V)$. We have shown in 15.3 that there exists for each U_n a globally defined orthonormal frame $s = (X, Y, T)$ with $X = \cos\varphi A + \sin\varphi B$ on U_n. Thus

(15.3)

$$
\begin{aligned}
8\pi^2 CS(S^3, g, s) =& \int_{S^3} \mu = lim_{n\to\infty} \int_{U_n} \mu\\
=& \lim_{n\to\infty} \int_{U_n} -\frac{1}{2}H(H^2 - K) - \frac{4}{H}(\frac{1}{2}H^2 - K) + \frac{1}{2}H(4 - K) \operatorname{vol}\\
& + \lim_{n\to\infty} \frac{1}{2} \int_{\partial U_n} H \star (-\mathcal{J}^*\eta - \mathrm{d}\,\lambda)\\
=& \lim_{n\to\infty} \int_{U_n} (\frac{4K}{H} - \frac{1}{2}H^3) \operatorname{vol} + \lim_{n\to\infty} \frac{1}{2} \int_{\partial U_n} H \star (-\mathcal{J}^*\eta - \mathrm{d}\,\lambda)\\
=& \int_{S^3} (\frac{4K}{H} - \frac{1}{2}H^3) \operatorname{vol} + \lim_{n\to\infty} \frac{1}{2} \int_{\partial U_n} H \star (-\mathcal{J}^*\eta - \mathrm{d}\,\lambda),
\end{aligned}
$$

where the third equality follows from

$$\delta(\mathcal{J}^*\eta + \mathrm{d}\,\lambda) = K - 4 - g(\mathrm{d}\log H, \mathcal{J}^*\eta + \mathrm{d}\,\lambda)$$

and the following application of Stokes for each $n \in \mathbb{N}$:

$$\begin{aligned} &\int_{U_n} g(\operatorname{grad} H, \mathcal{J}\operatorname{grad}\varphi - \operatorname{grad}\lambda)\,\mathrm{vol} \\ &= \int_{U_n} H\delta(-\mathcal{J}^*\eta - \mathrm{d}\,\lambda)\,\mathrm{vol} \\ &+ \int_{\partial U_n} H \star (-\mathcal{J}^*\eta - \mathrm{d}\,\lambda) \\ &= \int_{U_n} H(4-K) + g(\operatorname{grad} H, \operatorname{grad}\lambda - \mathcal{J}\operatorname{grad}\varphi)\,\mathrm{vol}\,. \\ &+ \int_{\partial U_n} H \star (-\mathcal{J}^*\eta - \mathrm{d}\,\lambda). \end{aligned} \tag{15.4}$$

It remains to show that $\lim_{n\to 0}\int_{\partial U_n} H \star (-\mathcal{J}^*\eta - \mathrm{d}\,\lambda) = 0$. Define the function $h\colon S^2 \to \mathbb{R}$ by

$$q \in S^2 \mapsto \int_{\pi^{-1}(q)} H\theta_3.$$

Note that for any form $\alpha \in \Omega^1(S^2)$ we have $\star\pi^*\alpha = \pi^* \star \alpha \wedge \theta_3$. We apply Fubini to obtain

$$\begin{aligned} \int_{\partial U_n} H \star (-\mathcal{J}^*\eta - \mathrm{d}\,\lambda) &= \int_{\pi^{-1}(\partial B_n)} H \star (-\mathcal{J}^*\eta - \mathrm{d}\,\lambda) \\ &= -\int_{\partial B_n} h(\eta + \star\,\mathrm{d}\,\lambda), \end{aligned} \tag{15.5}$$

where the Hodge star $\star$ in the last line is the one on the surface. On $V \subset S^2$ it is

$$\mathrm{d}(\eta + \star\,\mathrm{d}\,\lambda) = (4-K)\,\mathrm{vol}\,.$$

By using 15.2 and the Hodge theorem respectively we have

$$\int_{S^2}(4-K)\,\mathrm{vol} = 0,$$

thus $(4-K)\,\mathrm{vol}$ is an exact differential form on S^2, i.e. there exist $\alpha \in \Omega^1(S^2)$ with $\mathrm{d}\,\alpha = (4-K)\,\mathrm{vol}$. Therefore we have $\eta + \star\,\mathrm{d}\,\lambda = \alpha + \beta$ on V, where β is a closed 1−form on V. Since V is simply connected $\beta = \mathrm{d}\,f$ for some $f\colon V \to \mathbb{R}$. Because of Stokes theorem, and because

α and h are defined on the whole S^2, we obtain

$$\lim_{n\to\infty}\int_{\partial B_n} h\alpha = -\lim_{n\to\infty}\int_{S^2\setminus B_n} \mathrm{d}(h\alpha) = 0. \tag{15.6}$$

The only term left to investigate is $\lim_{n\to\infty}\int_{\partial B_n} h\,\mathrm{d}\,f$. We consider two cases: In the first the function h has an isolated critical point or is constant on an open set. Then, there are closed embedded curves $\gamma_n\colon S^1 \to S^2$ and a point $p \in S^2$, such that h is constant along each γ_n, and such that $\lim_{n\to\infty}\mathrm{vol}(B_n) = \mathrm{vol}_{S^2}$, with B_n be the component of $S^2 \setminus \mathrm{Im}\,\gamma_n$ and $p \notin B_n$. It follows immediately that

$$\lim_{n\to\infty}\int_{\partial B_n} h\,\mathrm{d}\,f = \pm\lim_{n\to\infty}\int_{\gamma_n} h\,\mathrm{d}\,f = 0.$$

In the other case, there would be $p \in S^2$ with $\mathrm{grad}_p\, h \neq 0$. Furthermore, there would be a small neighborhood $\tilde{V}$ around p such that f is bounded on $S^2 \setminus \tilde{V}$ and for each $c \in \mathbb{R}$ the set $h^{-1}(c) \cap \tilde{V}$ is empty or consist of the image of a smooth, connected and open curve. In fact, one could choose $\tilde{V}$ with a diffeomorphism

$$x\colon V \to I\times]h(p)-\epsilon; h(p)+\epsilon[$$

on an open interval $I =]a;b[$, such that $h \circ x^{-1}(I \times \{c\}) = \{c\}$. We may assume that x can be extend to a diffeomorphism of an open neighborhood of the closure of $\tilde{V}$. Let

$$B_n := S^2 \setminus x^{-1}([a,b] \times [h(p) - \frac{1}{n}; h(p) + \frac{1}{n}]).$$

Then $\lim_{n\to\infty}\mathrm{vol}(B_n) = \mathrm{vol}(S^2)$ and its boundary γ_n is a piecewise smooth, oriented and closed curve which consists of the points

$$\mathrm{Im}\,\gamma_n = \partial x^{-1}([a,b] \times [h(p) - \frac{1}{n}; h(p) + \frac{1}{n}]).$$

With $c_n := h(p) - \frac{1}{n}$ and $d_n := h(p) + \frac{1}{n}$ we get

$$\begin{aligned}\lim_{n\to\infty}\int_{\partial B_n} h\,\mathrm{d}f &= -\lim_{n\to\infty}\int_{\gamma_n} h\,\mathrm{d}f\\ &= -\lim_{n\to\infty}(c_n(f(x^{-1}(b,c_n)) - f(x^{-1}(a,c_n)))\\ &\quad + d_n(f(x^{-1}(b,d_n)) - f(x^{-1}(a,d_n))))\\ &\quad - \lim_{n\to\infty}\int_{x^{-1}(\{b\}\times[c_n;d_n])} h\,\mathrm{d}f\\ &\quad + \lim_{n\to\infty}\int_{x^{-1}(\{a\}\times[c_n;d_n])} h\,\mathrm{d}f = 0.\end{aligned}$$

□

15.5. Corollary. *Let g be the conformally flat metric on S^3 given by 15.1. Then*

$$\int_{S^3} H^3 \,\mathrm{vol} = 16\pi^2.$$

PROOF. As $H > 0$, π is homotopic to the Hopf fibration, the induced frames are homotopic with the same Chern-Simons functional. It is easy to compute the Chern-Simons functional for the induced section $\tilde{s} = (\bar{J}, \bar{K}, \bar{I}) \in \mathrm{SO}(S^3)$:

$$CS(S^3, g_{round}, \tilde{s}) = 1.$$

With 15.4 and

$$\int_{S^3} \frac{K}{H}\,\mathrm{vol} = \int_{S^2} \pi K \,\mathrm{vol}_{S^2} = 4\pi^2,$$

one obtains

$$\int_{S^3} H^3 \,\mathrm{vol} = 16\pi^2.$$

□

We are now able to classify submersive harmonic morphisms on conformally flat S^3 with nowhere vanishing curvature of the horizontal distribution.

15.6. Theorem. *Let $\pi\colon (S^3, g) \to (M^2, h)$ be a submersive harmonic morphism of a conformally flat (S^3, g) to a surface M. Assume that the curvature of the horizontal distribution is nowhere vanishing. Then $M = S^2$, g is the round metric and π is, up to isometries of S^3, the Hopf fibration.*

Proof. If we change the orientation of the surface, the curvature function H of the horizontal distribution changes sign. Therefore, we can assume that $H > 0$.

As the metric g on S^3 is conformally flat, 15.2 can be used to obtain

$$0 = 2H\Delta^h \log H + H(H^2 - K).$$

By dividing this equation by H^2 and then integrating it, we obtain

$$\begin{aligned} 0 &= \int_{S^3} \frac{1}{H^2}(2H\Delta^h \log H + H(H^2 - K)) \operatorname{vol} \\ &= \int_{S^3} (\frac{2}{H}\Delta^h \log H + H - \frac{K}{H}) \operatorname{vol} \\ &= - \int_{S^3} \frac{2}{H^3} \parallel \operatorname{grad}^h H \parallel^2 \operatorname{vol} + \int_{S^3} H \operatorname{vol} - 4\pi^2. \end{aligned}$$

This shows that

$$\int_{S^3} H \operatorname{vol} \geq 4\pi^2$$

and with equality if and only if $\operatorname{grad}^h H = 0$.

Consider the measure $\mu = \frac{1}{H} \operatorname{vol}$ on S^3. Then the Cauchy-Schwartz inequality, 15.2 and 15.5 give us

$$\begin{aligned} \int_{S^3} H \operatorname{vol} &= \int_{S^3} H^2\mu \leq (\int_{S^3} H^4\mu)^{\frac{1}{2}}(\int_{S^3} 1^2\mu)^{\frac{1}{2}} \\ &=(\int_{S^3} H^3 \operatorname{vol})^{\frac{1}{2}}(\int_{S^3} \frac{1}{H} \operatorname{vol})^{\frac{1}{2}} = \sqrt{16\pi^2}\sqrt{\pi^2} = 4\pi^2. \end{aligned}$$

Therefore, $\int_{S^3} H = 4\pi^2$ and $\operatorname{grad}^h H = 0$. If $\operatorname{grad}_p H \neq 0$ for a point $p \in S^3$, the level sets of H near p would be integral curves of the horizontal distribution $\mathcal{H}$ which is a contradiction to the non-integrability of the horizontal distribution $\mathcal{H}$ measured by $H \neq 0$. Thus, $\nabla_T T = 0$, and we are in the case of a circle bundle. The results of IV prove the theorem.

□

Index of Symbols

$CS(M,[g])$	Chern-Simons functional of a conformal $3-$ space $(M,[g])$
$\mathcal{J}$	complex or CR structure
Δ	laplacian, $\Delta^h = \mathrm{d}\,\delta + \delta\,\mathrm{d}$
Δ^h	horizontal laplacian, $\Delta^h = -\operatorname{div}\operatorname{grad}^h$
grad^h	horizontal gradient, $\operatorname{grad}^h = \pi^{\mathcal{H}}\operatorname{grad}$
$\star$	Hodge star operator
H	curvature of the horizontal distribution
$\pi^{\mathcal{H}}$	orthogonal projection onto the horizontal distribution
$\pi^{\mathcal{V}}$	orthogonal projection onto the vertical distribution
Q^3	space of oriented, possibly degenerated circles in S^3
Q	tangent sphere bundle of S^3

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Erklärung

Die selbständige und eigenhändige Anfertigung der vorliegenden Dissertation versichere ich an Eides statt. Ich habe nur die angegebenen Hilfsmittel benutzt und die Arbeit ohne unerlaubte Hilfe angefertigt.

Ich erkläre hiermit, dass ich die Arbeit erstmalig und nur an der Humboldt-Universität zu Berlin eingereicht habe. Ich habe weder ein Promotionsverfahren andernorts eingeleitet noch erfolglos beendet. In dem angestrebten Fach besitze ich nicht den Grad eines Doktors der Naturwissenschaften.

Der Inhalt der dem Verfahren zugrunde liegenden Promotionsordnung (Amtliches Mitteilungsblatt der HU, Nr. 34/2006) ist mir bekannt.

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