The n-dimensional Analogue of the Catenary: Prescribed Area

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Abstract. We study "heavy" n–dimensional surfaces which are suspended from some given boundary data φ and have prescribed surface area A. Using a fixed point argument we show existence of a solution provided A is close to the area of the corresponding minimal surface spanned by φ.

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The equilibrium condition for a *heavy, inextensible* and *flexible* surface M of constant mass density which is exposed to a vertical gravitational field has been derived by several authors, see Lagrange [L, pp 158–162], Cisa de Gresy [GG, pp 274–276], Jellett [J, pp 349–354] and Poisson [P, pp 173–187]. It turns out that there are several model problems available, which are due to different notions of flexibility and inextensibility, and, which are all worth to be investigated. Quite generally Poisson [P] considers (flexible–inextensible) surfaces in $\mathbb{R}^{\nvDash}$, which are exposed to an arbitrary force field $F = (X, Y, Z)$, and, – using direct arguments from mechanics – he deduces a system of partial differential equations which, in addition to the unknown function u, also involves two independent "tensions" T and T' which describe the forces inside the surface. Of particular interest is the case where the tension coincide, i.e. $T = T'$. Then the system of p.d.e.'s reduces to the single equation

$$Z - pX - qY + \frac{T}{k^2}\left[\,(1+q^2)u_{xx} - 2pq\,u_{xy} + (1+p^2)u_{yy}\,\right] = 0, \tag{1}$$

where we have set $p = u_x = \frac{\partial u}{\partial x}$, $q = u_y = \frac{\partial u}{\partial y}$, $k^2 = 1 + p^2 + q^2$ and T satisfies

$$X\,dx + Y\,dy + Z\,dz + dT = 0, \tag{2}$$

that is the external force F must have a potential U and $T = U + c$. ¿From (1) and (2) Poisson [P] deduces:

(A) The *minimal surface equation* by taking
$X = Y = Z = 0, \quad T = \text{const.}$;

(B) The equation for a *capillary surface* by taking
$X = Y = 0, \qquad Z = \frac{a+bz}{k}$
as the equilibrium condition of a flexible surface which is covered by a heavy fluid;

(C) The equation of a *heavy surface in a gravitational field* by taking $X = Y = 0$, $Z = g\varepsilon$, where g denotes the gravitational constant and ε is the density of the surface. The tension T is then given by $T = -\lambda - g\varepsilon z$, $\lambda \in \mathbb{R}$, and hence (1) implies the condition

$$g\varepsilon - \frac{\lambda + g\varepsilon z}{k^2}\left\{(1+q^2)u_{xx} - 2pqu_{xy} + (1+p^2)u_{yy}\right\} = 0. \tag{3}$$

Assuming $g\varepsilon = 1$ we are thus led to the equation

$$\sqrt{1+|Du|^2} \quad \operatorname{div} \frac{Du}{\sqrt{1+|Du|^2}} = \frac{1}{(u+\lambda)} \qquad \text{in } \Omega \subset \mathbb{R}^{\nvDash}, \lambda \in \mathbb{R}, \tag{4}$$

as a model equation for the equilibrium condition of an inextensible, flexible, heavy surface of constant mass density in a vertical gravitational field.

A further application in architecture lends special interest to equation (4) , cp. [BHT] and [O]. In fact turning a hanging solution u of (4) upside down gives the optimal shape of a cupola.

Here we are concerned with the Dirichlet problem in $\mathbb{R}^{\ltimes}$ connected with equation (4) and, in addition, we require a solution u to have prescribed area A, i.e.

$$\int_\Omega \sqrt{1+|Du|^2}\, dx = A.$$

In other words we consider the following problem

(P). *Let $\Omega \subset \mathbb{R}^{\ltimes}$ be a bounded domain of class $C^{2,\alpha}$ and suppose $\varphi \in C^{2,\alpha}(\bar\Omega)$ is given. For some prescribed value $A \in \mathbb{R}^+$ one has to find a function $u \in C^{2,\alpha}(\bar\Omega)$ and some $\lambda \in \mathbb{R}$ such that*

$$\sqrt{1+|Du|^2}\, \mathit{div} \frac{Du}{\sqrt{1+|Du|^2}} = \frac{1}{(u+\lambda)} \qquad \text{in } \Omega \tag{5}$$

$$u = \quad \varphi \qquad \text{on } \partial\Omega, \text{ and}$$

$$A(u) = \int_\Omega \sqrt{1+|Du|^2}\, dx = A. \tag{6}$$

Observe that problem (P) can also be considered as the n-dimensional mathematical analogue of the (one-dimensional) catenary problem: *To find a surface M = graph u of prescribed area A and boundary φ with lowest possible center of gravity.* Indeed, since the x_{n+1} -coordinate of the center of gravity is given by the quotient

$$\left(\int_\Omega \sqrt{1+|Du|^2}\,dx\right)^{-1} \int_\Omega u\,\sqrt{1+|Du|^2}\,dx$$

this amounts to the minimization of the integral

$$\int_\Omega u\sqrt{1+|Du|^2}\,dx$$

subject to the constraints

$$\int_\Omega \sqrt{1+|Du|^2}\,dx = A \quad \text{and} \quad u = \varphi \text{ on } \partial\Omega.$$

Now, introducing a Lagrange multiplier λ one obtains (5) and (6) as the equilibrium condition for this problem.

Nitsche [N, p 146] has shown by way of example that the above variational problem has no solution whatever the value of A might be. Thus one has to use more refined techniques from the calculus of variations in order to construct merely relative minima, say. In this paper, however, we tackle equation (5) directly and prove suitable a priori estimates which enable us to apply some fixed point argument.

Clearly, there is an obvious necessary condition on the number A, namely that $A \geq A_0$, A_0 denoting the infimum of area of all graphs bounded by φ.
But, surprisingly, and in contrast to the one-dimensional situation, there is a further necessary condition namely that $A \leq a_1(\varphi)$, $a_1(\varphi)$ denoting some specific number depending on the boundary values φ. In fact, it was pointed out by Nitsche [N] that the Euler equation (5) in the corresponding rotationally symmetric case has no solution, provided $A > a_1(\varphi)$.

In light of the above remarks the following existence result is natural (and, in a sense, optimal).

Theorem. *Let $\Omega \subset \mathbb{R}^{\ltimes}$ be a bounded, mean-convex domain of class $C^{2,\alpha}$ and suppose $\varphi \in C^{2,\alpha}(\bar{\Omega})$. Then there exists some number $A_1 > A_0$ depending only on $n, \Omega, |\varphi|_{2,\alpha}$, such that for all numbers $A \in (A_0, A_1]$ there is some $\lambda \in \mathbb{R}$ and a function $u = u_\lambda \in C^{2,\alpha}(\bar{\Omega})$ which solves problem (P) i.e.*

$$\sqrt{1+|Du|^2}\ div\ \frac{Du}{\sqrt{1+|Du|^2}} = \frac{1}{(u+\lambda)} \qquad in\ \Omega \tag{5}$$

$$u = \varphi \qquad \text{on } \partial\Omega, \qquad \text{and}$$

$$A(u) = \int_\Omega \sqrt{1+|Du|^2}\, dx = A. \tag{6}$$

Observe that equation (5) is an equation of mean curvature type with (variable) mean curvature $H = H(u, Du) = (u+\lambda)^{-1}(1+|Du|^2)^{-1/2}$ and that $H_u \le 0$, i.e. H is monotone with the "wrong" monotonicity behaviour.

For the proof of the Theorem we use Schauder's fixed point theorem in combination with suitable a priori and monotonicity estimates. We first consider solutions
$u = u_{f,\lambda} \in C^{2,\alpha}(\bar{\Omega})$ of the related problem

$$\sqrt{1+|Du|^2} \quad \text{div}\ \frac{Du}{\sqrt{1+|Du|^2}} = (f+\lambda)^{-1} \quad \text{in } \Omega, \tag{7}$$

$$u = \varphi \qquad \text{on } \partial\Omega,\ \text{where}$$

$f \in C^{1,\alpha}(\bar{\Omega})$ is some positive function and $\lambda \in \mathbb{R}$ denotes some positive number.
Let $c(n) = n^{-1}\omega_n^{-1/n}$ stand for the isoperimetric constant, $\omega_n = |B_1^n(0)|$ the measure of

the unit ball, and put

$$h := \sup_{\partial\Omega} \varphi, \quad k_0 := \inf_{\partial\Omega} \varphi \quad \text{and} \quad \lambda_0 := (1+\sqrt{2^{n+1}})\, c(n)|\Omega|^{1/n}.$$

As a first step we establish a priori bounds for $\sup_\Omega u$ and $\inf_\Omega u$.

Lemma 1. *Let $u = u_{f,\lambda} \in C^{2,\alpha}(\bar\Omega)$ denote a solution to the Dirichletproblem (7). If λ, λ_0 and k_0 satisfy*

$$\lambda \geq \lambda_0 = (1+\sqrt{2^{n+1}})c(n)|\Omega|^{1/n} \qquad \textit{and}$$

$$k_0 \geq (1+\sqrt{2^{n+1}})^2 c(n)|\Omega|^{1/n} = (1+\sqrt{2^{n+1}})\lambda_0$$

then we have the inequality

$$h \geq u_{f,\lambda} \geq \lambda_0.$$

The proof follows the argument given in [DH]. For completeness we sketch it here. Since f, λ are positive the first inequality follows from the maximum principle. To prove the second relation we choose $\delta \geq -k_0$ and put
$w := \min(u+\delta, 0),\ A(\delta) := \{x \in \Omega \ : \ u < -\delta\}$. Multiplying (7) with w , integrating by parts and using $w_{|\partial\Omega} = 0$, we obtain

$$\int_\Omega \frac{|Dw|^2}{\sqrt{1+|Dw|^2}} = \int_{A(\delta)} \frac{|w|}{(f+\lambda)\sqrt{1+|Du|^2}}, \qquad \text{whence}$$

$$\int_\Omega |Dw| \leq |A(\delta)| + \frac{1}{\lambda_0} \int_{A(\delta)} |w|.$$

We use Sobolev's inequality on the left and Hölder's inequality on the right hand side and obtain with $c(n) = n^{-1}\omega_n^{-1/n}$ the relation

$$|w|_{n/n-1}\ \{c^{-1}(n) - \lambda_0^{-1}|\Omega|^{1/n}\} \leq |A(\delta)|,$$

where $|w|_{n/n-1}$ stands for the $L_{n/n-1}$-norm of w. Another application of Hölder's inequality yields

$$(\delta_1 - \delta_2)|A(\delta_1)| \leq \left\{ \frac{c(n)\lambda_0}{\lambda_0 - c(n)|\Omega|^{1/n}} \right\}\ |A(\delta_2)|^{1/n\,+1}$$

for all $\delta_1 \geq \delta_2 \geq -k_0$. In view of a well known Lemma due to Stampacchia, [St, Lemma 4.1] this is easily seen to imply

$$\left|A(-k_0 + 2^{n+1}c_1|A(-k_0)|^{1/n}\right| = 0, \qquad \text{where}$$

$$c_1 = \frac{c(n)\lambda_0}{\lambda_0 - c(n)|\Omega|^{1/n}}. \qquad \text{By definition this means that}$$

$$u \geq k_0 \; - \; \frac{2^{n+1}\lambda_0\; c(n)\; |\Omega|^{1/n}}{\lambda_0 - c(n)\; |\Omega|^{1/n}}.$$

Since $k_0 \geq (1+\sqrt{2^{n+1}})\lambda_0$ and $\lambda_0 = (1+\sqrt{2^{n+1}})c(n)|\Omega|^{1/n}$ we finally obtain $u \geq \lambda_0$ as desired. $\square$

To derive a gradient estimate at the boundary we rewrite (7) into

$$(1+|Du|^2)\Delta u - D_iuD_juD_{ij}u = (f+\lambda)^{-1}(1+|Du|^2). \tag{8}$$

We can then apply the results of Serrin [Se 1], see also [GT, chap. 14.3]. Equation (8) satisfies the structure condition (14.41) and the r.h.s. is of order $0(|Du|^2)$. So we obtain a gradient estimate on the boundary which is independent of $|Df|$:

$$\sup_{\partial\Omega} |Du_{f,\lambda}| \leq c_2 = c_2(n,\Omega,h,|\varphi|_{2,\Omega}),$$

provided only that $\partial\Omega$ has non-negative (inward) mean curvature.
It is not possible to derive interior gradient estimates independent of $|Df|$, but we can prove

$$\sup_{\Omega} |Du_{f,\lambda}| \leq \max\{2, 1/4\ \sup|Df|, 2e^{4(h\lambda_0^{-1}-1)}\ \sup_{\partial\Omega}|Du_{f,\lambda}|\} \tag{9}$$

Estimate (9) can be obtained from a careful analysis of the structure condition in [GT, chap 15]. For a selfcontained proof, which uses the geometric nature of equation (7) we refer to [DH].
Having proved the C^1estimates we now infer from general theory and Schauder-estimates the inequality

$$|u_{f,\lambda}|_{2,\alpha,\Omega} \leq C\,(n,\Omega,\lambda_0,h,|\varphi|_{2,\alpha,\Omega},|f|_{0,\alpha,\Omega}) \tag{10}$$

for any solution $u_{f,\lambda}$ of (7) provided

$$\lambda \geq \lambda_0 = (1 + \sqrt{2^n + 1})\ c(n)|\Omega|^{1/n},$$

$k_0 = \inf_{\partial\Omega} \varphi \geq (1+\sqrt{2^{n+1}})\lambda_0$ and $f \geq 0$. Here the constant C only depends on the quantities indicated.

Note that by Arzela-Ascoli this already implies that $u_{f,\lambda} \to u_0$ in $C^2(\Omega)$ as $\lambda \to \infty$, where u_0 denotes the unique (area minimizing) minimal surface spanned by φ. In particular we have $A(u_{f,\lambda}) \to A(u_0)$ as $\lambda \to \infty$ where $A(u) = \int_\Omega \sqrt{1 + |Du|^2}dx$ denotes the area of the graph of u.

Later on it will be important to investigate this convergence somewhat more carefully.

For f fixed we now consider the behaviour of solutions $u_{f,\lambda}$ of equation (7) as λ varies. In particular we show that $u_{f,\lambda}$ increases with increasing values of λ. More precisely we have

Lemma 2. *Let u_{f,λ_1} and u_{f,λ_2} denote two solutions of the Dirichletproblem (7) with r.h.s. $(f + \lambda_1)^{-1}$ and $(f + \lambda_2)^{-1}$ respectively, where $\lambda_1 \geq \lambda_2 \geq \lambda_0$ and $0 \leq f \leq h$. Then we have the inequality*

$$u_{f,\lambda_1}(x) \geq u_{f,\lambda_2}(x) + c_0(2n + 2C\ diam\ \Omega)^{-1}d^2(x, \partial\Omega), \tag{11}$$

where $d(x, \partial\Omega) =$ dist $(x, \partial\Omega)$ denotes the distance of x to the boundary $\partial\Omega$ and $c_0 = c_0(h, \lambda_1, \lambda_2) = \frac{(\lambda_1 - \lambda_2)}{(h+\lambda_1)^2}$ and $C = C(n, \Omega, \lambda_0, h, |\varphi|_{2,\alpha}, |f|_{0,\alpha})$.

Proof. Put $a_{ij}(p) = \delta_{ij} - \frac{p_i p_j}{1+|p|^2}$, $p \in \mathbb{R}^{\ltimes}$. Then (7) may be rewritten into $a_{ij}(Du)D_{ij}u = (f + \lambda)^{-1}$. Therefore $w := u_{f,\lambda_1} - u_{f,\lambda_2}$ satisfies

$$a_{ij}(Du_{f,\lambda_1})D_{ij}w + a_{ij}(Du_{f,\lambda_1})D_{ij}u_{f,\lambda_2} - a_{ij}(Du_{f,\lambda_2})D_{ij}u_{f,\lambda_2} = (f + \lambda_1)^{-1} - (f + \lambda_2)^{-1}$$

whence

$$A_{i_j}(x)D_{ij}w + B_i(x)D_iw = \frac{\lambda_2 - \lambda_1}{(f + \lambda_1)(f + \lambda_2)} < 0,$$

where

$$A_{i_j}(x) := a_{ij}\left(Du_{f,\lambda_1}(x)\right) \qquad \text{and}$$

$$B_i(x) := D_{kj}u_{f,\lambda_2}(x) \int_0^1 a_{kj,p_i}\left(tDu_{f,\lambda_1} + (1-t)Du_{f,\lambda_2}\right) dt.$$

(Note that by the Hopf-maximum principle for linear equations this already implies that $u_{f,\lambda_1} \geq u_{f,\lambda_2}$). Let L denote the linear operator

$$L := A_{ij}(x)D_{ij} + B_i(x)D_i$$

and take some comparison function

$$\varphi(x) := \frac{c_0(h,\lambda_1,\lambda_2)}{2n + 2C\ \Omega} \qquad [\,|x-x_0|^2 - R^2],$$

where $R = d(x_0, \partial\Omega)$, $x_0 \in \Omega$ and $|B|_{0,\Omega} \leq C = C(n, \Omega, \lambda_0, h, |\varphi|_{2,\alpha,\Omega}, |f|_{0,\alpha,\Omega})$ denotes a constant depending only on the quantities indicated (cp. (10)). Then we compute

$$\begin{aligned} L\varphi &\leq \frac{c_0}{2n + 2C \text{ diam } \Omega} \quad [2 \text{ trace } (A_{ij}) + 2|B|_{0,\Omega}\ |x - x_0|] \\ &\leq \frac{c_0}{2n + 2C \text{ diam } \Omega} \quad [2n + 2C \text{ diam } \Omega] \leq c_0. \end{aligned}$$

Concluding we get

$$L(u_{f,\lambda_1} - u_{f,\lambda_2} + \varphi) \leq \frac{\lambda_2 - \lambda_1}{(f + \lambda_1)(f + \lambda_2)} + \frac{\lambda_1 - \lambda_2}{(h + \lambda_1)^2} \leq 0$$

and $u_{f,\lambda_1} - u_{f,\lambda_2} + \varphi \geq 0$ on the boundary of Ω. Therefore

$$u_{f,\lambda_1}(x) \geq u_{f,\lambda_2}(x) - \varphi(x) = u_{f,\lambda_2}(x) + \frac{c_0}{2n + 2C \text{ diam } \Omega}[R^2 - |x - x_0|^2]$$

and, on putting $x = x_0$ we obtain

$$u_{f,\lambda_1}(x_0) \geq u_{f,\lambda_2}(x_0) + \frac{c_0 R^2}{2n + 2C \text{ diam } \Omega}. \qquad \square$$

We now show that the area of solutions $u_{f,\lambda}$ decreases as λ increases; in fact we have the following:

Lemma 3. *Let $\lambda_1 \geq \lambda_2 \geq \lambda_0$ and $0 \leq f \leq h$ be given and denote by $u_{f,\lambda_1}, u_{f,\lambda_2}$ two solutions of (7) with r.h.s. $(f+\lambda_1)^{-1}$ and $(f+\lambda_2)^{-1}$ respectively. Then there is some constant $C > 0$ depending only on $n, \Omega, \lambda_0, h, |\varphi|_{2,\alpha,\Omega}$ and $|f|_{0,\alpha,\Omega}$ such that the following estimate holds*

$$\int_\Omega \sqrt{1+|Du_{f,\lambda_1}|^2}dx + \frac{(\lambda_1-\lambda_2)}{(h+\lambda_1)^3}C \leq \int_\Omega \sqrt{1+|Du_{f,\lambda_2}|^2}dx. \tag{12}$$

Proof. Put $u_1 := u_{f,\lambda_1}$ and $u_2 := u_{f,\lambda_2}$. ¿From (7) we obtain for all $\varphi \in C_0^1(\Omega)$

$$-\int_\Omega \frac{Du_1 D\varphi}{\sqrt{1+|Du_1|^2}}dx = \int_\Omega \frac{\varphi}{(f+\lambda_1)\sqrt{1+|Du_1|^2}}dx. \tag{13}$$

We test (13) with $\varphi := u_2 - u_1 \in C_0^1(\Omega)$ and obtain

$$-\int_\Omega \frac{Du_1 D(u_2-u_1)}{\sqrt{1+|Du_1|^2}}dx = \int_\Omega \frac{u_2-u_1}{(f+\lambda_1)\sqrt{1+|Du_1|^2}}dx \quad \text{whence}$$

$$\int_\Omega \frac{|Du_1|^2}{\sqrt{1+|Du_1|^2}}dx = \int_\Omega \frac{Du_1 Du_2}{\sqrt{1+|Du_1|^2}} + \int_\Omega \frac{u_2-u_1}{(f+\lambda_1)\sqrt{1+|Du_1|^2}}dx.$$

We apply Schwarz' s inequality and Lemma 2 to abtain

$$\begin{aligned}\int_\Omega \sqrt{1+|Du_1|^2}dx \leq & \int_\Omega \sqrt{1+|Du_2|^2}dx \\ & + \frac{(\lambda_2-\lambda_1)(2n+2C \text{ diam } \Omega)^{-1}}{(h+\lambda_1)^3\sqrt{1+|Du_1|^2_{0,\Omega}}} \int_\Omega d^2(x,\partial\Omega)dx\end{aligned}.$$

Concluding we have

$$\int_\Omega \sqrt{1+|Du_1|^2}dx \leq \int_\Omega \sqrt{1+|Du_2|^2}dx + C\frac{(\lambda_2-\lambda_1)}{(h+\lambda_1)^3}, \quad \text{with}$$

some constant $C = C(n, \Omega, \lambda_0, h, |\varphi|_{2,\alpha}, |f|_{0,\alpha})$. $\square$

It is now desirable to have an explicit bound for the increment of area of the graphs of $u_0 := u_{f,\infty}$ and $u_{f,\lambda}$ respectively. Note that this estimate does not immediately follow from (12) b letting λ_1 tend to infinity.

Lemma 4. *Let $\lambda \geq \lambda_0, \quad h \geq f \geq 0$ be given and denote by $u_{f,\lambda}$ and u_0 the unique solution of the Dirichlet problem (7) and the minimal surface spanned by φ respectively: Then there exists some constant C depending only on $n, \Omega, \lambda_0, h, |\varphi|_{2,\alpha}$ and $|f|_{0,\alpha}$ such that*

$$\int_\Omega \sqrt{1+|Du_0|^2}dx + \frac{C}{(h+\lambda)^2} \leq \int_\Omega \sqrt{1+|Du_{f,\lambda}|^2}dx$$

holds true for all $\lambda \geq \lambda_0$,

Proof. Let $a(\lambda) := \int_\Omega \sqrt{1+|Du_{f,\lambda}|^2}dx$ denote the area of the graph of $u_{f,\lambda}$. Then (12) implies the inequality

$$\frac{a(\lambda_1)-a(\lambda_2)}{\lambda_1-\lambda_2} \leq \frac{-C}{(h+\lambda_1)^3} \quad \text{for } \lambda_1 \geq \lambda_2 \tag{13}$$

and some $C > 0$ independent of λ_1, λ_2. Also, $a(\lambda)$ is monotone decreasing, so $a'(\lambda)$ exists almost everywhere, $a'(\lambda) \leq \frac{-C}{(h+\lambda)^3}$ by (13) and

$$a(\infty) \leq a(\lambda) + \int_\lambda^\infty a'(\xi)d\xi. \tag{14}$$

(Note that from Schauder theory for linear equations we could even infer that $u_{f,\lambda}, Du_{f,\lambda}$ depend Lipschitz-continuously on λ, i.e. (14) is in fact an identity). From (14) we infer

$$\int_\Omega \sqrt{1+|Du_0|^2}dx + \frac{C}{(h+\lambda)^2} \leq \int_\Omega \sqrt{1+|Du_{f,\lambda}|^2}dx$$

with $C > 0$. □

Proof of the Theorem. We define the set $\mathfrak{M}$ by

$$\mathfrak{M} := \{\ f \in C^{1,\alpha}(\bar{\Omega}) : 0 \leq f \leq h,\ \sup_\Omega |Df| \leq M\ \}$$

By virtue of our C^1-estimates we may choose $M = M(n, \Omega, h, |\varphi|_{2,\Omega})$ large, so that $u_{f,\lambda} \in \mathfrak{M}$ for all $f \in \mathfrak{M}, \lambda \geq \lambda_0$. If f is restricted to $\mathfrak{M}$ then the constant C appearing in Lemma 4 only depends on $n, \Omega, h, |\varphi|_{2,\alpha}$ and $M = M(n, \Omega, h, |\varphi|_{2,\Omega})$.

We put $A_1 := A_0 + \frac{C}{(h+\lambda_0)^2}$ where $C = C(n, \Omega, h, |\varphi|_{2,\alpha}, M)$ denotes the constant in Lemma 4, and we also assume for a moment that

$$k_0 := \inf_{\partial\Omega} \geq (1 + \sqrt{2^{n+1}})\lambda_0.$$

Now fix a value $A \in (A_0, A_1]$. It follows from $a(\lambda) \to A_0$ as $\lambda \to \infty$ and from the monotonicity of $a(\lambda)$ that for $f \in \mathfrak{M}$ given there is precisely one $\lambda = \lambda(A) \geq \lambda_0$ and some unique solution $u_{f,\lambda} \in C^{2,\alpha}(\bar{\Omega})$ of (7) with prescribed area A, i.e. $A(u_{f,\lambda}) = A$.

Consider the operator T_A

$$T_A : \mathfrak{M} \to \mathfrak{M}$$

$$f \longrightarrow u_{f,\lambda(A)}.$$

It follows from the $C^{2,\alpha}$ estimate (10) and Arzela-Ascoli that T_A is compact. Furthermore T_A is continuous. In fact let f_m converge to f in $C^{1,\alpha}(\bar{\Omega})$. Then $\{u_{m,\lambda_m} = T_A f_m\}$ is precompact in $C^2(\bar{\Omega})$ and hence any subsequence in turn has a convergent subsequence. Suppose that $u_{m_j} = u_{m_j,\lambda_{m_j}} \to u$ in $C^2(\bar{\Omega})$. Then $A(u_{m_j}) = A$ implies $A(u) = A$ and

$$\sqrt{1 + |Du_{m_j}|^2} \;\operatorname{div} \frac{Du_{m_j}}{\sqrt{1 + |Du_{m_j}|^2}} = (f_{m_j} + \lambda_{m_j})^{-1}$$

implies

$$\sqrt{1 + |Du|^2} \;\operatorname{div} \frac{Du}{\sqrt{1 + |Du|^2}} = (f + \Lambda)^{-1}$$

for some $\Lambda \in \mathbb{R}$. But from $A(u) = A$ and the uniqueness of λ it follows that $\Lambda = \lambda(f)$ and $u = T_A f$. Hence $T_A f_m$ converges to u and we can apply Schauder's fixed point theorem to obtain the existence of a regular $u \in C^{2,\alpha}(\bar{\Omega})$ solving (5) and (6). Now we have to get rid of the additional assumption $k_0 \geq (1 + \sqrt{2^{n+1}})\lambda_0$. To this end we choose some number $\gamma \in \mathbb{R}$ large, so that $\varphi_\gamma := \varphi + \gamma$ satisfies $\inf_{\partial\Omega} \varphi_\gamma \geq (1 + \sqrt{2^{n+1}})\lambda_0$. Then there is some further number $\lambda \in \mathbb{R}$ and a solution $u = u_\gamma \in C^{2,\alpha}(\bar{\Omega})$ satisfying (5) and (6) and $u_\gamma = \varphi_\gamma$ on the boundary of Ω. Therefore the function $u := u_\gamma - \gamma$ has boundary values φ and fulfills the equation

$$\sqrt{1+|Du|^2}\ \mathrm{div}\ \frac{Du}{\sqrt{1+|Du|^2}} = \frac{1}{u+(\gamma+\lambda)} \qquad \text{in } \Omega, \qquad \text{and}$$
$$A(u) = A.$$

This proves the Theorem.

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