A Distance Comparison Principle for Evolving Curves

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Abstract. A lower bound for the ratio of extrinsic and intrinsic distance is proved for embedded curves evolving by curve shortening flow in the plane. The estimates yield a new approach to longtime existence results and can be applied to related evolution equations.

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1. Introduction

Let $F : \Gamma \times [0,T] \to \mathbf{R}^2$ be a smooth family of embedded curves, where Γ is either S^1 or an interval. We say that Γ moves by the curve shortening flow (or mean curvature flow) if

$$\frac{d}{dt}F(p,t) = -\kappa\ \nu(p,t), \quad p \in \Gamma, \quad t \in [0,T], \tag{1.1}$$

where ν and κ are a choice of unit normal and corresponding geodesic curvature respectively, such that $-\kappa\nu = \overrightarrow{\kappa}$ is the curvature vector. If we let $s = s_t$ be the arclength parameter on $\Gamma_t = F(\cdot,t)(\Gamma)$, then $\overrightarrow{\kappa} = (d^2/ds^2)F$ and equation (1.1) can be written in the form

$$(\frac{d}{dt} - \frac{d^2}{ds_t^2})F = 0, \tag{1.2}$$

making the quasilinear parabolic nature of the equation apparent. The existence, regularity and long-term behaviour of solutions to this system have been studied extensively by Mullins [Mu], Gage [Ga], Gage-Hamilton [GH], Grayson [Gr1,Gr2], Abresch-Langer [AL], Angenent [An1] and Hamilton [Ha].

In this article we describe a new approach to the curve shortening flow and related geometric evolution equations based on comparison principles for intrinsic and extrinsic distance functions. Define

$$d, l\ :\ \Gamma \times \Gamma \times [0,T] \to \mathbf{R} \tag{1.3}$$

by setting

$$\begin{aligned} d(p,q,t) &:= \big|F(p,t) - F(q,t)\big|_{R^2}, \\ l(p,q,t) &:= \int_p^q ds_t. \end{aligned} \tag{1.4}$$

The ratio d/l can be considered as a measure for the straightness of an embedded curve and it is demonstrated in theorem 2.1 that under the curve shortening flow this ratio improves at a local minimum. We therefore obtain a quantitative version of the wellknown fact that embedded curves stay embedded.

On closed curves we use $\psi(l) = \frac{l}{\pi}\sin(\frac{l\pi}{L})$ instead of l, where L is the total length of the curve. We show that $\min\left(\frac{d}{\psi}\right)$ is nondecreasing under the curve shortening flow, theorem 2.2, and therefore plays the role of an improving isoperimetric ratio which measures the deviation of the evolving curve from a round circle.

The distance comparison principles thus established immediately rule out slowly forming (type II) singularities for the flow, where the ratios estimated above are known to tend to zero. Thus theorem 2.2 yields a new proof of Grayson's longtime existence and convergence result for embedded curves in the plane [Gr1], based only on the known classification of possible singularities. Note that a similar strategy was employed by Hamilton [Ha], where an isoperimetric ratio related to area content was estimated. Theorem 2.1 can also be used to prove longtime existence results for noncompact evolving curves and solutions to boundary value problems, see section 2.

In a forthcoming paper we show that these techniques can be extended to curveshortening flow on surfaces and related more general evolution equations for curves.

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2. Curves evolving in the plane

Let $F : \Gamma \times [0,T] \to \mathbf{R}^2$ be a smooth embedded solution of the curve shortening flow (1.1), and let d and l be the extrinsic and intrinsic distances as defined in section 1. Notice that always $d/l \leq 1$ with equality on the diagonal of $\Gamma \times \Gamma$ and $d/l \equiv 1$ if and only if Γ is a straight line. We begin with a distance comparison principle for non-closed curves, where l is globally defined as a smooth function.

2.1 Theorem. *Let $F : \Gamma \times [0,T] \to \mathbf{R}^2$ be a smooth embedded solution of the curve shortening flow* (1.1)*. Let $\Gamma \neq S^1$, such that l is smoothly defined on $\Gamma \times \Gamma$. Suppose d/l attains a local minimum at (p,q) in the interior of $\Gamma \times \Gamma$ at time $t_0 \in [0,T]$. Then*

$$\frac{d}{dt}(d/l)(p,q,t_0) \geq 0$$

with equality if and only if Γ is a straight line.

Proof. Since d/l has its global maximum on the diagonal of $\Gamma \times \Gamma$, we may assume w.l.o.g. that $p \neq q$ and $s(p) > s(q)$ at time t_0. By assumption we have

$$\delta(\xi)(d/l)(p,q,t_0) = 0, \quad \delta^2(\xi)(d/l)(p,q,t_0) \geq 0, \tag{2.1}$$

where $\delta(\xi)$ and $\delta^2(\xi)$ denote the first and second variation with regard to a variation vector $\xi \in T_p\Gamma_{t_0} \bigoplus T_q\Gamma_{t_0}$. To exploit this information, define

$$e_1 := \frac{d}{ds}F(p,t_0), \quad e_2 := \frac{d}{ds}F(q,t_0) \quad \textit{and} \quad \omega := d^{-1}(p,q,t_0)(F(q,t_0) - F(p,t_0)).$$

Then the vanishing of the first variation for $\xi = e_1 \oplus 0$ yields

$$0 = \delta(e_1 \oplus 0)(d/l) = \frac{d}{l^2} - \frac{1}{l}\langle \omega, e_1 \rangle,$$

such that at p

$$\langle \omega, e_1 \rangle = d/l. \tag{2.2}$$

Similarly we obtain from $0 = \delta(0 \oplus e_2)$ that at q

$$\langle \omega, e_2 \rangle = d/l. \tag{2.3}$$

In view of relations (2.2) and (2.3) we now have two possibilities:

<u>Case1</u> : $e_1 = e_2$. Then we choose $\xi = e_1 \oplus e_2$ in the second variation inequality and obtain from $\delta(\xi)(l) = 0$ that

$$0 \leq \delta^2(e_1 \oplus e_2)(d/l) = \frac{1}{l}\langle \omega, \overrightarrow{\kappa}(q, t_0) - \overrightarrow{\kappa}(p, t_0) \rangle. \tag{2.4}$$

<u>Case2</u> : $e_1 \neq e_2$. In this case we use $\xi = e_1 \ominus e_2$ in the second variation inequality and note that now $(e_1 + e_2)$ is parallel to ω. We compute with $\delta(e_1 \ominus e_2)(l) = -2$

$$0 = \delta(e_1 \ominus e_2)(d/l) = \frac{2d}{l^2} - \frac{1}{l}\langle \omega, e_1 + e_2 \rangle,$$

and consequently from (2.2) and (2.3)

$$\begin{aligned} 0 \leq \delta^2(e_1 \ominus e_2)(d/l) = & \frac{1}{l}\langle \omega, \overrightarrow{\kappa}(q, t_0) - \overrightarrow{\kappa}(p, t_0) \rangle \\ & + \frac{1}{dl}|e_1 + e_2|^2 - \frac{1}{dl}\langle \omega, e_1 + e_2 \rangle^2. \end{aligned}$$

Since now ω is parallel to (e_1+e_2), the last terms cancel, and we obtain the same conclusion as in the first case

$$0 \leq \delta^2(e_1 \ominus e_2)(d/l) = \frac{1}{l}\langle \omega, \overrightarrow{\kappa}(q, t_0) - \overrightarrow{\kappa}(p, t_0) \rangle. \tag{2.5}$$

We can now use (2.4), (2.5) and the original evolution equation (1.1) to compute

$$\frac{d}{dt}(d/l)(p, q, t_0) = \frac{1}{l}\langle \omega, \overrightarrow{\kappa}(q, t_0) - \overrightarrow{\kappa}(p, t_0) \rangle - \frac{d}{l^2}\frac{d}{dt}(l) \geq -\frac{d}{l^2}\frac{d}{dt}(l).$$

Now notice that $(d/dt)(ds) = -\kappa^2(ds)$ in the curve shortening flow, such that

$$\frac{d}{dt}(d/l)(p, q, t_0) \geq \frac{d}{l^2}\int_p^q \kappa^2 ds_{t_0},$$

completing the proof of theorem 2.1.

2.2 Remark. If $-\pi/2 \le \alpha < \pi/2$ is such that $\langle \omega, e_1 \rangle = \cos\alpha$, then the Hölder-inequality

$$\int_p^q \kappa^2 ds \ge \frac{1}{l}\Big(\int_p^q |\kappa| ds\Big)^2 \tag{2.6}$$

implies a lower bound on the rate of improvement of d/l, since $\int |\kappa| ds \ge 2|\alpha|$:

$$\frac{d}{dt}(d/l)(p,q,t_0) \ge \frac{d}{l^2}\int_p^q \kappa^2 ds \ge \frac{4d}{l^3}\alpha^2 \ge \frac{4d}{l^2}\sin^2\alpha = \frac{4d}{l^2}(1-\cos^2\alpha) = \frac{4d}{l^2}(1-\frac{d^2}{l^2}).$$

Now let $F : S^1 \times [0,T] \to \mathbf{R}^2$ be a closed embedded curve moving by the curve shortening flow. If $L = L(t)$ is the total length of the curve, the intrinsic distance function l is now only smoothly defined for $0 \le l < L/2$, with conjugate points where $l = L/2$. We therefore define a smooth function $\psi : S^1 \times S^1 \times [0,T] \to \mathbf{R}$ by setting

$$\psi := \frac{L}{\pi}\sin(\frac{l\pi}{L}). \tag{2.7}$$

With this choice of ψ the isoperimetric ratio d/ψ approaches 1 on the diagonal of $S^1 \times S^1$ for any smooth imbedding of S^1 in $\mathbf{R}^2$ and the ratio d/ψ is identically one on any round circle.

2.3 Theorem. *Let $F : S^1 \times [0,T] \to \mathbf{R}^2$ be a smooth embedded solution of the curve shortening flow (1.1). Then the minimum of d/ψ on S^1 is nondecreasing; it is strictly increasing unless $d/\psi \equiv 1$ and $F(S^1)$ is a round circle.*

Proof. Since d and ψ are smooth functions outside the diagonal in $S^1 \times S^1$, it is sufficient to show that whenever d/ψ attains a (spatial) minimum $(d/\psi)(p,q,t_0) < 1$ at some pair of points $(p,q) \in S^1 \times S^1$ and some time $t_0 \in [0,T]$, then

$$\frac{d}{dt}(d/\psi)(p,q,t_0) \ge 0. \tag{2.8}$$

Let again s be the arclength parameter at time t_0 and without loss of generality $0 \le s(p) < s(q) \le L(t_0)/2$, such that $l(p,q,t_0) = s(q) - s(p)$.

We use again the notation from the proof of theorem 2.1 and note that the first and second variation of d/ψ satisfy

$$\delta(\xi)(d/\psi)(p,q,t_0)=0, \quad \delta^2(\xi)(d/l)(p,q,t_0)\geq 0 \tag{2.9}$$

for variations $\xi\in T_pS^1_{t_0}\oplus T_qS^1_{t_0}$. From the vanishing of the first variation for $\xi=e_1\oplus 0$ as well as $\xi=0\oplus e_2$ we then deduce

$$\langle\omega,e_1\rangle=\langle\omega,e_2\rangle=\frac{d}{\psi}\cos(\frac{l\pi}{L}). \tag{2.10}$$

As before we consider two possibilities.

<u>Case1</u> : $e_1=e_2$. We choose $\xi=e_1\oplus e_2$ in the second variation inequaility and compute from (2.10)

$$0\leq\delta^2(e_1\oplus e_2)(d/\psi)=\frac{1}{\psi}\langle\omega,\overrightarrow{\kappa}(q,t_0)-\overrightarrow{\kappa}(p,t_0)\rangle. \tag{2.11}$$

<u>Case2</u> : $e_1\neq e_2$. Using again that now (e_1+e_2) is parallel to ω and employing the second variation inequality with $\xi=e_1\ominus e_2$ we deduce similar to case 2 in theorem 2.1

$$\delta(e_1\ominus e_2)(d/\psi)=\frac{2d}{\psi^2}-\frac{1}{\psi}\langle\omega,e_1+e_2\rangle$$

and then at (p,q,t_0)

$$0\leq\delta^2(e_1\ominus e_2)(d/\psi)=\frac{1}{\psi}\langle\omega,\overrightarrow{\kappa}(q,t_0)-\overrightarrow{\kappa}(p,t_0)\rangle+\frac{4\pi^2}{L^2}\frac{d}{\psi}.$$

Here we used that $|e_1+e_2|^2=4(d^2/\psi^2)\cos^2(l\pi/L)$ in view of (2.10).

We are now ready to estimate the time derivative of (d/ψ). Using the original evolution equation and the time derivative of the arclength $(d/dt)(ds)=-\kappa^2(ds)$ we compute

$$\begin{aligned}\frac{d}{dt}(\frac{d}{\psi})=&\frac{1}{d\psi}\langle F(q,t_0)-F(p,t_0),\overrightarrow{\kappa}(q,t_0)-\overrightarrow{\kappa}(p,t_0)\rangle-\frac{d}{\psi^2}\frac{d}{dt}(\frac{L}{\pi}sin(\frac{l\pi}{L}))\\ =&\frac{1}{\psi}\langle\omega,\overrightarrow{\kappa}(q,t_0)-\overrightarrow{\kappa}(p,t_0)\rangle+\frac{d}{\psi^2\pi}sin\alpha\int_{S^1}\kappa^2ds\\ &+\frac{d}{\psi^2}\cos\alpha\int_p^q\kappa^2ds-\frac{dl}{\psi^2L}\cos\alpha\int_{S^1}\kappa^2ds,\end{aligned}$$

where we introduced the notation $\alpha = (l\pi/L)$, $0 < \alpha \leq \pi/2$. To proceed we distinguish the two cases above:

<u>Case1</u> : $e_1 = e_2$. From (2.12) we deduce

$$\begin{aligned}\frac{d}{dt}(d/\psi) &\geq \frac{d}{\psi L}\int_{S^1}\kappa^2 ds + \frac{d}{\psi^2}\cos\alpha\int_p^q \kappa^2 ds - \frac{dl}{\psi^2 L}\cos\alpha\int_{S^1}\kappa^2 ds \\ &= \frac{d}{\psi L}(1 - \frac{l}{\psi}\cos\alpha)\int_{S^1}\kappa^2 ds + \frac{d}{\psi^2}\cos\alpha\int_p^q \kappa^2 ds.\end{aligned}$$

Now notice that $\frac{l}{\psi}\cos\alpha = \alpha\tan^{-1}\alpha < 1$ since $0 < \alpha \leq \pi/2$ by assumption. This proves the required strict positivity of the time derivative.

<u>Case2</u> : $e_1 \neq e_2$. In this case we deduce as in the argument for case 1 from (2.13), the Hölder-inequality and $\int_{S^1}\kappa\, ds = 2\pi$ that

$$\begin{aligned}\frac{d}{dt}(d/\psi) \geq &\frac{d}{\psi L}(1 - \frac{l}{\psi}\cos\alpha)\int_{S^1}\kappa\, ds - \frac{4\pi^2}{L^2}\frac{d}{\psi} \\ &+ \frac{d}{\psi^2}\cos\alpha\int_p^q \kappa^2 ds \geq \frac{d}{\psi^2 l}\cos\alpha\big(l\int_p^q \kappa^2 ds - \frac{4\pi^2 l^2}{L^2}\big).\end{aligned}$$

But now notice that

$$l\int_p^q \kappa^2 ds \geq \big(\int_p^q \kappa\, ds\big)^2 = \beta^2,$$

where $0 < \beta \leq \frac{\pi}{2}$ is the angle between e_1 and e_2; since $(e_1 + e_2)$ is parallel to ω and $\langle\omega, e_1\rangle = \langle\omega, e_2\rangle$ we conclude that

$$\cos(\frac{\beta}{2}) = \langle\omega, e_1\rangle = \langle\omega, e_2\rangle = \frac{d}{\psi}\cos\alpha$$

by equation (2.10). Since by assumption at (p, q, t_0) we have $d/\psi < 1$ we obtain from $\cos(\beta/2) < \alpha$ that $\alpha < \beta/2$ and hence

$$l\int_p^q \kappa^2 ds \geq \beta^2 > 4\alpha^2 = \frac{4\pi^2 l^2}{L^2}.$$

This completes the proof of theorem 2.2.

In view of the known structure of possible singularities of the curve shortening flow theorems 2.1 and 2.2 are the only global estimates needed for longtime existence results. As a first application we give a short proof of Grayson's convergence result for embedded curves.

Theorem 2.4. *(Grayson [Gr1]) If the initial curve $F_0 : S^1 \to \mathbf{R}^2$ is embedded, then the curve shortening flow* (1.1) *has a smooth solution on a finite time interval* $[0,T)$ *and contracts smoothly to a "round" point as $t \to T$.*

Proof. Consider the first time $T > 0$ where the curvature on S^1_t becomes unbounded for $t \to T$. If the singularity is a type I - singularity where $\sup_t |\kappa|^2(T-t)$ remains bounded, then the curve approaches a selfsimilar homothetically shrinking solution for $t \to T$ in view of the general monotonicity formula for the mean curvature flow [Hu]. This yields the desired conclusion since the only embedded homothetically shrinking solutions are circles, see Mullins [Mu] and Abresch-Langer [AL]. If $\sup_t |\kappa|^2(T-t)$ is unbounded, it is known that the curve can be rescaled to a selftranslating "grim reaper curve" given by $y = t - \log\cos x$, compare Altschuler [Al]. But on such a curve the ratio between extrinsic and intrinsic distance becomes arbitrarily small, contradicting theorem 2.2 and completing the argument.

Just as theorem 2.2 simplifies the regularity result for closed embedded curves, theorem 2.1 can be used to give brief arguments for the longtime existence and convergence of certain complete curves and curves with boundary. The following result was originally obtained by Polden [Po] from an extension of Grayson's methods.

2.5 Theorem. *(Polden [Po]) Suppose $F_0 : \mathbf{R} \to \mathbf{R}^2$ is a smooth embedded curve which is for $s \to \infty$ and for $s \to -\infty$ asymptotic (say in C^1) to two distinct halflines L_1 and L_2. Then the curve shortening flow* (1.1) *has a smooth solution for all $t > 0$ and the solution asymptotically approaches the selfsimilar solution associated with the two halflines. In particular, a) if L_1 and L_2 form an angle different from 0 and π, then $F(\cdot,t)$ approaches a selfsimilar homothetically expanding solution of the flow. b) if L_1 and L_2 form an angle π, then $F(\cdot,t)$ becomes a bounded graph in finite time and subsequences converge to a straight line parallel to L_1 and L_2. c) if L_1 and L_2 are parallel and point in the same direction, then the solution $F(\cdot,t)$ approaches the translating solution associated with L_1*

and L_2.

Proof. In cases a) and b) it is clear that near infinity d/l is bounded below. Therefore theorem 2.1 yields a global uniform lower bound for d/l and therefore longtime regularity as in the proof for theorem 2.3. Note that near infinity the curve remains regular e.g. due to the interior regularity estimates of Ecker-Huisken [EH1]. In case c) the assumptions are such that one can put a translating solution of slightly smaller width then the distance from L_1 to L_2 in between the two halflines, disjointly from the initial curve. This translating solution then acts as a separating barrier curve between the two ends of the evolving curve asymptotic to L_1 and L_2. Together with theorem 2.1 this yields for each given $T > 0$ a unifom lower bound for d/l on any bounded section of the curve, again implying longtime existence. Once longtime existence is established, a theorem of Angenent [An1] concerning the decrease in the number of intersections of different solutions to the curve shortening flow can be applied to suitable comparison curves to establish that all solutions above eventually become graphs, see [Po]. Asymptotic convergence then follows from results of Ecker-Huisken [EH2] on the evolution of graphs.

We also give one application of theorem 2.1 to curves with boundary.

2.6 Theorem. *Suppose* $\Omega \subset \mathbf{R}^2$ *is an open strip, that is* $\Omega = \{(x,y) \in \mathbf{R}^2 | 0 < x < M\}$ *with boundaries* $\Gamma_1 = \{(x,y) \in \mathbf{R}^2 | x = 0\}$ *and* $\Gamma_2 = \{(x,y) \in \mathbf{R}^2 | x = M\}$. *Let* $F_0 : [a,b] \subset \mathbf{R} \to \mathbf{R}^2$ *be a smooth embedded curve with* $F_0(a) \in \Gamma_1, F_0(b) \in \Gamma_2$ *and* $F_0((a,b)) \subset \Omega$. *Then the curve shortening flow for the initial curve* F_0 *with boundary conditions* $F_t(a) = F_0(a)$, $F_t(b) = F_0(b)$ *has a smooth solution for all times* $t > 0$, *which for* $t \to \infty$ *converges to the straight line connecting* $F_0(a)$ *and* $F_0(b)$.

Proof. In view of the strong maximum principle F_t will be transverse to Γ_1 and Γ_2 at $F_0(a)$ and $F_0(b)$ for all $t > 0$. So there exists $t_0 > 0$ and $\epsilon > 0$ such that $F_{t_0}([a,b]) \cap \{(x,y) \in \mathbf{R}^2 | 0 \le x \le \epsilon\}$ and $F_{t_0}([a,b]) \cap \{(x,y) \in \mathbf{R}^2 | M - \epsilon \le x \le M\}$ can be written as graphs over the x–axis. In view of the maximum principle due to Angenent [An1] this remains so with the same ϵ for all $t \geq t_0$. The regularity results in [EH1] then imply that the curve remains completely regular in the boundary strips of size $\epsilon/2$, in particular with a uniform gradient bound. Therefore d/l has a uniform lower bound near the boundary of $[a,b] \times [a,b]$. Since by theorem 2.1 the ratio d/l cannot attain a new interior minimum, this implies that d/l is uniformly bounded below for all times. From this we may conclude as in theorem 2.3 that no singularity occurs in finite time, thus proving longtime regularity. Convergence to a straight line can then be deduced from

$$\frac{d}{dt}L(t) = -\int_a^b \kappa^2 ds$$

by standard arguments; in fact it is easy to see that there exists $t_1 > 0$ such that $F_t([a,b])$ is a graph over the x-axis for all $t \geq t_1$.

2.7 Remark a) Similar arguments can be applied to curves contained in other convex subsets of the plane or to curves modelled on $\mathbf{R}_+$ having one fixed endpoint on the boundary of a convex set and being asymptotic to a halfline at infinity.
b) Neumann boundary conditions on Γ_1 or Γ_2 can be treated with the same method. In particular, the convergence result by Stahl [St] for convex curves having orthonormal boundary contact with some other fixed convex curve can in this way be extended to embedded initial curves.

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