Geometric concepts for the mass in General Relativity

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Abstract. In General Relativity the total energy of an isolated gravitating system is described by a geometric invariant of asymptotically flat Riemannian 3–manifolds. One–parameter families of hypersurfaces foliating such a manifold can be used to encode and study global geometrical and physical properties such as the center of mass and energy inequalities. The article summarizes three lectures given at the conference on "Trends in Mathematical Physics" in Knoxville 1998.

1 Asymptotically flat 3–manifolds

The space–time of an isolated gravitating system such as a star, black hole or galaxy is modelled by a Lorentzian 4–manifold $(L^4, \bar{g})$ of signature $(- + ++)$ satisfying Einsteins equations

$$R^4_{\alpha\beta} - \frac{1}{2} R^4 \bar{g}_{\alpha\beta} = 8\pi T_{\alpha\beta} \tag{1.1}$$

and approaching the geometry fo Minkowski space near infinity in an appropriate sense. Here $\{R^4_{\alpha\beta}\}$, $0 \leq \alpha, \beta \leq 3$, and R^4 denote the Ricci tensor and the scalar curvature of the Lorentzian metric $\bar{g}$ and the energy momentum Tensor $\{T_{\alpha\beta}\}$ is often assumed to satisfy a physically motivated positivity condition such as the weak energy condition

$$T_{\alpha\beta} X^\alpha X^\beta \geq 0 \qquad \text{for } X \in TL \text{ timelike.} \tag{1.2}$$

When decomposing the space–time $(L^4, \bar{g})$ along some natural time function into a family of spacelike hypersurfaces one is naturally led to the study of asymptotically flat complete Riemannian 3–manifolds (M^3, g, K) carrying an induced metric $g = \{g_{AB}\}$, $1 \le A, B \le 3$, and a second fundamental form $K = \{K_{AB}\}$. The Einstein equations can then be rewritten as a Hamiltonian system for the tensors g and K, compare [?], [?]. Moreover, under very general assumptions the hypersurfaces (M^3, g, K) can be chosen to be in "maximal gauge", ie with vanishing trace of the second fundamental form, as was shown by Bartnik in [?]. In this case the energy condition (??) implies that the scalar curvature R^3 of the induced metric g is nonnegative. In the more special case of a "timesymmetric" hypersurface (M^3, g, K) the second fundamental K vanishes completely, the hypersurface is totally geodesic and all geometric information is encoded in the metric g. The horizon of a black hole is then modelled by a two dimensional minimal surface in (M^3, g).
Having this motivation in mind, it is important to understand how much geometric information is already encoded in complete riemannian 3–manifolds (M^3, g) of nonnegative scalar curvature, which approach the geometry of Euclidean $\mathbb{R}^3$ near infinity. We use the following notion of asymptotically flat.

Definition 1.1 *i) A 3–dimensional asymptotically flat end is a 3–manifold (M^3, g) with Riemannian metric g, which is diffeomorphic to $\mathbb{R}^3 \setminus K$ for some compact set K, such that in the Euclidean coordinates $\{x^A\}$ the metric approaches the flat metric $\delta = \{\delta_{AB}\}$ for $r = |x| \to \infty$:*

$$|g_{AB} - \delta_{AB}| \le \frac{C_0}{r}, \quad |\partial_A g_{BC}| \le \frac{C_1}{r^2}, \quad Ric \ge -\frac{C_2}{r^2}. \tag{1.3}$$

ii) A complete 3–manifold (M^3, g) without boundary is called asymptotically flat if it can be decomposed as a disjoint union of a compact set C and finately many asymptotically flat ends (M_i^3, g), $1 \le i \le q$:

$$M^3 = C \cup M_1^3 \cup \cdots \cup M_q^3. \tag{1.4}$$

Apart from empty space $\mathbb{R}^3$ the most important example of an asymptotically flat 3–manifold is the spatial Schwarzschild manifold given by $({}^S M^3, g^S) = (\mathbb{R}^3 - \{0\}, g^S)$ with

$$g^S_{AB} = \delta_{AB}(1 + \frac{m}{2r})^4, \quad 0 < m = \text{const.} \tag{1.5}$$

It arises as the $\{t = 0\}$–slice of the Schwarzschild space–time modelling a spherically symmetric, static vacuum black hole. It has two asymptotically flat ends for $r \to \infty$ and $r \to 0$, which are isometric under reflection in the totally geodesic 2–sphere $\{r = m/2\}$. The positive constant m represents the total mass of the black hole.
For a general asymptotically flat end (M^3, g) as in Definition 1.1 the total energy or ADM–mass is defined according to Arnowitt, Deser and Misner [?] by a flux integral at infinity,

$$m_{ADM}(M^3, g) := \lim_{R \to \infty} \frac{1}{16\pi} \int_{\partial B_R(0)} (g_{ii,j} - g_{ij,i}) n^j \, d\mu,$$

in agreement with Schwarzschild manifold above, $m_{ADM}({}^{S}M^3, g^S) = m$. Although the definition is given with respect to specific Euclidean coordinates, it has been shown by Bartnik [?] and Chrusciel [?] that m_{ADM} is a geometric invariant of the manifold (M^3, g). From a physical point of view the mass defined here is supposed to play the role of a total energy and measure both the matter content of the system and the gravitational energy. The "Positive Mass Theorem" by Schoen and Yau [?] shows that the local energy condition (??) is consistent with this concept of global energy, a basic version states:

Theorem 1.2 (Schoen-Yau 1979) *If (M^3, g) is an asymptotically flat 3–manifold with nonnegative scalar curvature, then the mass of each end is nonnegative. If the mass is zero in one end, then (M^3, g) is isometric to flat space $(\mathbb{R}^3, \delta)$.*

Asymptotically flat 3–manifolds and the question of positivity for the mass were also studied by many other authors including Geroch, Jang, Kijowski, Jezierski, Chrusciel, Witten, Choquet–Bruhat, Parker–Taubes, Reula, Tod and Lohkamp. See [?] and the survey article of Lee–Parker [?] for additional references to asymptotically flat 3–manifolds and the concept of total energy.

2 Foliations of constant mean curvature and the center of mass

In addition to a consistent concept of total energy provided above it is desirable to extend the notion of "center of mass" from Newtonian mechanics to the relativistic setting. Since no distinguished Euclidean structure is available in a natural way and since alraedy the spatial Schwarzschild manifold illustrates that a "center" cannot be provided by a point of the manifold, a different, geometrically motivated concept is necessary.
Using the spatial Schwarzschild manifold $({}^{S}M^3, g^S)$ as a guide, we observe that in that case every concentric coordinate sphere $\{r = \text{const}\}$ is a solution to the isoperimetric problem: It is a constant mean curvature surface and thus a critical point of the area functional with respect to volume preserving variations of the surface (enclosed volume measured eg with respect to some reference surface), and it is strictly stable in the sense that the second variation of area with respect to any such volume preserving variation is positive. This is in marked contrast to Euclidean space where the translations of $\mathbb{R}^3$ provide Jacobi fields for this variational problem. The idea is then to show that each asymptotically flat end of a 3–manifold sufficiently close to the Schwarzschild example admits a unique foliation by strictly stable 2–spheres of constant mean curvature near infinity which breaks the translation invariance of the system and provides a canonical concept for the "center of mass".
To make this precise we say that an asymptotically flat end is *strongly asymptotically flat* if in a Euclidean coordinate system the metric satisfies

$$g_{AB} = \delta_{AB}(1 + \frac{m}{2r})^4 + P_{AB},$$

$$|P_{AB}| \leq \frac{C_0}{r^2},\ |\partial^j P_{AB}| \leq \frac{C_j}{r^{2+j}},\ j = 1, \cdots, 4. \tag{2.1}$$

Then we have the following existence result [?]:

Theorem 2.1 (Huisken-Yau 1996) *Let (M^3, g) be a strongly asymptotically flat end of a Riemannian 3–manifold with strictly positive mass $m > 0$. Then there are constants $R_0 > 0$, $\tau_0 > 0$ depending only on the mass m and the constants in (??) together with a family of 2–spheres $\Sigma^2_\tau, 0 < \tau < \tau_0$, of constant mean curvature τ providing a regular foliation of the exterior regions $M^3 \setminus B_{R_0}(0)$.*

The construction of the constant mean curvature surfaces above in [?] starts out with a large coordinate sphere $F_0 : S^2 \mapsto (M^3, g)$ and then solves a parabolic evolution equation for a deformation $F : S^2 \times [0, \infty) \mapsto (M^3, g)$ of this initial sphere:

$$\frac{d}{dt} F(p,t) = -(H - h)\nu(p,t),\ p \in S^2, t \in [0, \infty). \tag{2.2}$$

In this equation $H(p,t)$ and $\nu(p,t)$ denote the mean curvature and the unit normal of the surface at $F(p,t)$ respectively such that $-H \cdot \nu = \Delta_t F$ is the mean curvature vector and $h = \int H d\mu / \int d\mu$ denotes the average of the mean curvature. The flow (??) decreases the area of the evolving surfaces and fixes the enclosed volume (with respect to some reference surface). using the strict positivity of the mass m it can be shown that the solution of (??) remains smooth and 'centered' and converges exponentially fast to a constant mean curvature sphere in (M^3, g). The exponential convergence and the properties of a foliation for the resulting family of constant mean curvature spheres follow from the stability already mentioned above: It can be seen that a constant mean curvature surface is stable for the area functional with respect to volume preserving variations exactly when the inequality

$$\int_{\Sigma^2} (|A|^2 + Ric(\nu, \nu)) f^2\, d\mu \leq \int_{\Sigma^2} |\nabla f|^2\, d\mu \tag{2.3}$$

holds for all functions f satisfying $\int f\, d\mu = 0$. Strict stability then means that the first eigenvalue of the Jacobi operator

$$Lu = -\Delta u - u(|A|^2 + Ric(\nu, \nu)) \tag{2.4}$$

when restricted to functions f with $\int f\, d\mu = 0$ is strictly positive. This is sufficient to derive the necessary a priori estimates, compare [?] for the details. Note also that [?] presents a different approach to the existence result usung the implicit function theorem.

In order to have a physically meaningful "center of mass", it is necessary to prove uniqueness of the foliation constructed above. Since in general there may be many closed constant mean curvature surfaces in 3–space, an extra condition is required to enforce uniqueness. It turns out that strict stability is just the right criterion; in fact it is shown that each leaf of the foliation Σ_τ constructed above is the unique stable constant mean curvature surface in a properly defined exterior region of (M^3, g):

Theorem 2.2 (Huisken-Yau 1996) *Let* (M^3, g) *be a strongly asymptotically flat end with positive mass* $m > 0$*. Then for each* $1/2 < q \leq 1$ *there is a constant* $\tau_1 > 0$ *depending only on* q, m *and the constants in (??) such that for each* $0 < \tau < \tau_1$ *the surface* Σ^2_τ *is the only stable constant mean curvature surface contained in* $M^3 \setminus B_{\tau^{-q}}(0)$.

Notice that τ^{-q} for q close to $1/2$ is a much smaller radius than expected from the scaling of the mean curvature. The strategy of proof aims first for some initial integral curvature estimates on a candidate surface based on the stability inequality. Using then techniques from minimal surface theory [?] this is extended to stronger estimates showing that the surface has to be approximately round and close to some coordinate sphere. The positivity of the mass can then be used to control the position of such an approximate sphere such that ultimately the implicit function theorem applies in a neighbourhood of the original leaf of the foliation.

Based on this uniqueness result it is also possible to construct a canonical coordinate system for the exterior region representing the center of mass in each strongly asymptotically flat 3–manifold. Using the inverse of the stability operator and Einsteins equations it is then possible to track the constant mean curvature foliation through a family of 3–dimensional timeslices (M^3_t, g_t) in space–time. Notice also that in recent work Bray [?] used variational techniques to extend the family of constant mean curvature surfaces above to the interior of the manifold (M^3, g), providing another approach to the Penrose inequality which is discussed in the next section.

3 Inverse mean curvature flow and a lower bound for the mass

While Theorem 1.2 by Schoen and Yau shows that the total energy of an isolated system is nonnegative and that in some sense Minkowski space is the absolute ground state of minimum energy, it was already conjectured earlier by Penrose [?] in 1973 that there should be a positive lower bound for the total energy in terms of the size of the outermost black hole present in the system, with the Schwarzschild metric as the sharp limiting ground state for such an inequality.
In recent work by Tom Ilmanen and the author [?], [?] such a lower bound for the total energy is proved if the outermost horizon of a black hole is represented by a minimal surface in an asymptotically flat 3–manfifold (M^3, g) of nonnegative scalar curvature. Recall that the scalar curvature will be nonnegative on any maximal slice of a Lorentzian manifold satisfying the weak energy condition (??), and that a minimal surface represents a horizon if the second fundamental form of a 3–manifold given as a spacelike hypersurface in space–time vanishes on the surface.
To give a precise formulation of the Theorem, we call an asymptotically flat end (M^3, g) an *exterior region* if the boundary of (M^3, g) is compact and consists of minmal surfaces, and (M^3, g) contains no other minmal surfaces.

Theorem 3.1 (Huisken-Ilmanen 1997) *Let (M^3, g) be an asymptotically flat exterior region with nonnegative scalar curvature. Then the total mass m of (M^3, g) is nonnegative and*

$$16\pi m^2 \geq |\Sigma^2|,$$

where $|\Sigma^2|$ is the area of any connected component of ∂N^3. Equality holds if and only if N^3 is one-half of the spatial Schwarzschild manifold.

It is conjectured that the supremum of all areas of connected components can be replaced by the sum of the areas of the boundary components. Notice that the exclusion of other compact minimal surfaces in the definition of an exterior region is necessary to account for the possibility of large black holes being hidden behind small ones.

An approach to this result based on 2–dimensional surfaces flowing along the inverse of the mean curvature was originally suggested by Geroch [?], see also [?], [?]. Other approaches are due to Gibbons [?], [?], Herzlich [?], Bartnik [?], Jezierski [?], [?] and Bray [?]. See [?], [?] for additional references and a more detailed account.
The approach suggested by Geroch is based on the fact that a smooth solution $F : S^2 \times [0, T) \mapsto (M^3, g)$ of the inverse mean curvature flow

$$\frac{d}{dt}F(p,t) = \frac{1}{H}\nu(p,t), \quad p \in S^2,\, t \in [0,T) \tag{3.1}$$

acts monotone on the Hawking–Mass

$$m_H(\Sigma^2) := \frac{|\Sigma^2|^{1/2}}{(4\pi)^{3/2}}\left(16\pi - \int_{\Sigma^2} H^2\, d\mu\right) \tag{3.2}$$

of the flowing surfaces: If the scalar curvature of the ambient manifold (M^3, g) is nonnegative, it turns out that on connected surfaces Σ_t^2 solving (**??**) the Hawking–Mass is monotone:

$$\frac{d}{dt}m_H(\Sigma_t^2) \geq 0. \tag{3.3}$$

Since near an initial minimal surface the Hawking–Mass yields the necessary area term and for large coordinate spheres approaches the total energy, the desired inequality is proved provided the existence of a solution to (**??**) is established. Without extra assumptions on the ambient manifold and the initial data it is quite clear that singularities have to occur in certain situations. For example, the solution evolving from a thin symmetric torus can not exist forever; due to un upper bound that can be established on the mean curvature H, some blowup in the speed H^{-1} must occur for such initial data. Similar examples can be constructed in the class of 2–spheres making it clear that there cannot be a smooth solution for the flow in the general situations that are of natural interest in physics.

In [?] the problem of singularities for solutions of this flow is addressed by using a level–set formulation where the surfaces Σ_t^2 solving (**??**) are replaced by level–sets

$$\Sigma_t^2 = \partial\{x \in M^3 | u(x) < t\},$$

of a scalar function $u : M^3 \mapsto \mathbb{R}$, which satisfies a degenerate elliptic boundary value problem

$$\operatorname{div}\Big(\frac{Du}{|Du|}\Big) = |Du|, \quad u\Big|_{\Sigma_0^2} = 0,$$

replacing the parabolic equation (**??**). In combination with elliptic regularisation and a minmization principle it is possible to construct a particular unique weak solution of this degenerate equation which allows for jumps of the surfaces in time while still retaining continuity of the area of the surfaces and the crucial monotonicity of the Hawking–Mass in (**??**). This monotonicity is one of the main analytic difficulties in this approach, see [**?**] for the technical details.

References

[1] R. Arnowitt, S. Deser and C. Misner, *Coordinate Invariance and Energy Expressions in General Relativity*, Phys. Rev. **122** (1961) 997–1006.

[2] R. Bartnik, *Existence of Maximal Surfaces in Asymptotically Flat Spacetimes*, Comm. Math. Phys. **94** (1984) 155–175.

[3] R. Bartnik, *The Mass of an Asymptotically Flat Manifold*, Comm. Pure Appl. Math. **39** (1986) 661–693.

[4] R. Bartnik, *Quasi-Spherical Metrics and Prescribed Scalar Curvature*, J. Diff. Geom. **37** (1995) 31–71.

[5] H. Bray, *The Penrose Inequality in General Relativity and Volume Comparison Theorems Involving Scalar Curvature*, thesis, Stanford University, 1997.

[6] D. Christodoulou and S. Klainerman, *The Global Nonlinear Stability of Minkowski Space*, Princeton Math. Series #41, Princ. Univ. Press, Princeton, 1993.

[7] P. T. Chruściel, *Boundary Conditions at Spatial Infinity from a Hamiltonian Point of View*, Topological Properties and Global Structure of Space-Time (Erice, 1985), NATO Adv. Sci. Inst. Ser. B: Phys. #138, Plenum, New York, 1986, 49–59.

[8] R. Geroch, *Energy Extraction*, Ann. New York Acad. Sci. **224** (1973) 108–17.

[9] G. W. Gibbons, thesis, Cambridge University, 1973.

[10] G. W. Gibbons, *Collapsing Shells and the Isoperimetric Inequality for Black Holes*, Class. Quant. Grav. **14** (1997) 2905-2915, also hep-th/9701049.

[11] S. W. Hawking, *Gravitational Radiation in an Expanding Universe*, J. Math. Phys. **9** (1968) 598–604.

[12] M. Herzlich, *A Penrose-like Inequality for the Mass of Riemannian Asymptotically Flat Manifolds*, Comm. Math. Phys. **188** (1997) 121-133.

[13] G. Huisken and T. Ilmanen, *The Riemannian Penrose Inequality*, Int. Math. Res. Not. **20** (1997) 1045–1058, also http://www.math.nwu.edu/∼ilmanen.

[14] G. Huisken and T. Ilmanen, *The inverse mean curvature flow and the Riemannian Penrose inequality*, to appear J.Diff.Geom.

[15] G. Huisken and S.-T. Yau, *Definition of Center of Mass for Isolated Physical Systems and Unique Foliations by Stable Spheres with Constant Mean Curvature*, Invent. Math. **124** (1996) 281–311.

[16] P. S. Jang, *On the Positive Energy Conjecture*, J. Math. Phys. **17** (1976) 141–145.

[17] P. S. Jang, *On the Positivity of Energy in General Relativity*, J. Math. Phys. **19** (1978) 1152–1155.

[18] J. Jezierski, *Positivity of Mass for Certain Spacetimes with Horizons*, Class. Quant. Grav. **6** (1989) 1535–1539.

[19] J. Jezierski, *Perturbation of Initial Data for Spherically Symmetric Charged Black Hole and Penrose Conjecture*, Acta Phys. Pol. B **25** (1994) 1413–17.

[20] J. M. Lee and T. H. Parker, *The Yamabe Problem*, Bull. Amer. Math. Soc. **17** (1987) 37–91.

[21] R. Penrose, *Naked singularities*, Ann.New York Acad.Sci. **224** (1973) 125–134.

[22] R. Schoen, L. M. Simon, and S.-T. Yau, *Curvature Estimates for Minimal Hypersurfaces*, Acta Math. **134** (1975) 275–288.

[23] R. Schoen and S.-T. Yau, *On the Proof of the Positive Mass Conjecture in General Relativity*, Comm. Math. Phys. **65** (1979) 45–76.

[24] R. Ye, *Foliation by constant mean curvature spheres on asymptotically flat manifolds*, in: Geometric Analysis and the Calculus of Variations, Ed.J.Jost, International Press, 1996.

[25] J. W. York Jr., *Kinematics and Dynamics of General Relativity*, Sources of Gravitational Radiation, ed. L. Smarr, Camb. Univ. Press, 1979, p.83.