Mean curvature flow singularities for mean convex surfaces

Gerhard Huisken
Mathematisches Institut, Universität Tübingen
Auf der Morgenstelle 10, 72076 Tübingen, Germany
e-mail: gerhard.huisken@uni-tuebingen.de

Carlo Sinestrari
Dipartimento di Matematica, Università di Roma "Tor Vergata"
Via della Ricerca Scientifica, 00133 Roma, Italy
e-mail: sinestra@mat.utovrm.it

Abstract. We study the evolution by mean curvature of a smooth n–dimensional surface $\mathcal{M} \subset \mathbb{R}^{n+1}$, compact and with positive mean curvature. We first prove an estimate on the negative part of the scalar curvature of the surface. Then we apply this result to study the formation of singularities by rescaling techniques, showing that there exists a sequence of rescaled flows converging to a smooth limit flow of surfaces with nonnegative scalar curvature. This gives a classification of the possible singular behaviour for mean convex surfaces in the case $n = 2$.

1 Introduction

Let $\mathcal{M}$ be a compact n–dimensional manifold without boundary and let $F_0 : \mathcal{M} \to \mathbb{R}^{n+1}$ be a smooth immersion of $\mathcal{M}$ as a hypersurface. We want to study the evolution of $\mathcal{M}_0 = F_0(\mathcal{M})$ by mean curvature flow; that is, the family of immersions $F(\cdot, t)$ satisfying

$$\frac{\partial F}{\partial t}(p,t) = -H(p,t)\nu(p,t), \qquad p \in \mathcal{M}, t \geq 0, \tag{1.1}$$

$$F(\cdot, 0) = F_0, \tag{1.2}$$

where $H(p,t)$ and $\nu(p,t)$ are the mean curvature and the outer normal respectively at the point $F(p,t)$ of the surface $\mathcal{M}_t = F(\cdot,t)(\mathcal{M})$. The signs are chosen such that $-H\nu = \vec{H}$ is the mean curvature vector and the mean curvature of a convex surface is positive.

It is well known that problem (1.1)–(1.2) is a quasilinear parabolic system and that the mean curvature flow is well defined up to a finite critical time T at which the curvature of the surface becomes unbounded. Moreover, various geometric properties (for instance convexity or positive mean curvature) are invariant under the flow.

The structure of the singularity of $\mathcal{M}_t$ when t approaches the critical time has been of great interest. For instance, when the initial surface $\mathcal{M}_0$ is convex, then as $t \to T$ the surfaces $\mathcal{M}_t$ become spherical and contract to a point. This result was obtained by Gage and Hamilton [5] in the case $n = 1$ and by Huisken [10] when $n \geq 2$; it was then extended by Grayson [6] to arbitrary embedded curves. On the other hand, for $n \geq 2$ nonconvex surfaces may become singular without shrinking to a point also in the embedded case. Then one can analyse the behaviour near a singularity with rescaling techniques. In [11] Huisken showed that singularities having a certain maximal blow–up rate (type I singularities) are asymptotically self–similar. For surfaces of positive mean curvature the possible limiting profiles were then classified in [12]. The singularities of rotationally symmetric surfaces were described by Altschuler, Angenent and Giga [2] and by Angenent and Velazquez [3], while other rescalings of mean curvature flow in the case $n = 2$ were recently studied by Ilmanen [14] and White [17].

In this paper we consider a general surface with positive mean curvature and with $n \geq 2$. Without any assumption on the blow–up rate, we prove a new a priori estimate, Theorem 3.1, showing that the scalar curvature of the surface is asymptotically nonnegative near a singularity. This implies that a sequence of rescaled flows tends to a limit flow of surfaces with nonnegative scalar curvature (Theorem 4.5). Such a property is particularly interesting for $n = 2$ because then positive scalar curvature is equivalent to convexity. This leads to a classification of the possible limiting shapes both for type I and type II singularities (Corollary 4.7).

2 Preliminaries

In the next sections we study the mean curvature flow (1.1)–(1.2) assuming that the initial hypersurface $\mathcal{M}_0 = F_0(\mathcal{M}) \subset \mathbb{R}^{n+1}$ $(n \geq 2)$ is smooth, compact without boundary and has positive mean curvature. It is well known (see [10, Theorem 8.1]) that there exists a smooth solution to problem (1.1)–(1.2) up to a critical time T at which the curvature of the evolving surface becomes unbounded. This critical time is finite for any compact $\mathcal{M}_0$, and satisfies

$$T \leq \frac{(\mathrm{diam}\ \mathcal{M}_0)^2}{8n}. \tag{2.1}$$

We recall now the equations for some geometric quantities associated with the evolving surface and other identities which we shall need in the sequel. We shall follow the notations of [10]; in particular $g = (g_{ij})$ and $A = (h_{ij})$ $(i, j = 1, \ldots, n)$ will denote the metric tensor and the second fundamental form on $\mathcal{M}$ induced by the immersion, while $H = \mathrm{tr}(h_{ij})$ and $R = H^2 - |A|^2$ are the mean curvature and the scalar curvature respectively. We also denote by $d\mu$ the volume element on $\mathcal{M}$. All these quantities depend on x, t (where x is a local coordinate on $\mathcal{M}$), but this dependence will not be written explicitly unless necessary.

Proposition 2.1 *We have the evolution equations*

(i) $\dfrac{\partial H}{\partial t} = \Delta H + |A|^2 H,$

(ii) $\dfrac{\partial |A|^2}{\partial t} = \Delta |A|^2 - 2|\nabla A|^2 + 2|A|^4,$

(iii) $\dfrac{\partial}{\partial t}\, d\mu = -H^2 d\mu.$

Proof. See [10]. ∎

¿From (i) and the maximum principle we obtain that the mean curvature of our surface remains positive during the evolution. Let us now introduce, for $\eta \in \mathbb{R}$ and $\sigma \in [0,2]$, the function

$$g_{\sigma,\eta} = \frac{|A|^2 - (1+\eta)H^2}{H^{2-\sigma}}. \tag{2.2}$$

Then we compute, using (i) and (ii) from the previous proposition,

$$\begin{aligned} \frac{\partial g_{\sigma,\eta}}{\partial t} &= \Delta g_{\sigma,\eta} + \frac{2(1-\sigma)}{H}\langle \nabla H, \nabla g_{\sigma,\eta}\rangle - \frac{\sigma(1-\sigma)}{H^2} g_{\sigma,\eta}|\nabla H|^2 \\ &\quad - \frac{2}{H^{4-\sigma}}|H\nabla_i h_{kl} - \nabla_i H\, h_{kl}|^2 + \sigma |A|^2 g_{\sigma,\eta}. \end{aligned} \tag{2.3}$$

In particular, for $\sigma = 0$, we find

$$\frac{\partial}{\partial t}\frac{|A|^2}{H^2} = \Delta \frac{|A|^2}{H^2} + \frac{2}{H}\langle \nabla H, \nabla \frac{|A|^2}{H^2}\rangle - \frac{2}{H^4}|H\nabla_i h_{kl} - \nabla_i H\, h_{kl}|^2. \tag{2.4}$$

By the maximum principle and by the compactness of $\mathcal{M}$ we obtain that $|A|^2/H^2$ is uniformly bounded from above by its initial data. Thus, if we call c_0 the maximum of $|A|^2/H^2$ on $\mathcal{M}_0$, we have

$$|A|^2 \leq c_0 H^2 \text{ on } \mathcal{M}_t, \forall\, t \in [0, T[\,. \tag{2.5}$$

Let us also recall the identity (see [10])

$$\frac{1}{2}\Delta |A|^2 = \langle h_{ij}, \nabla_i \nabla_j H\rangle + |\nabla A|^2 + Z, \tag{2.6}$$

where $Z = H\,\mathrm{tr}(A^3) - |A|^4$. The following elementary inequality will be useful to derive estimates on Z.

Lemma 2.2 *We have* $|tr(A^3)| \leq |A|^3$.

Proof. If we denote by $\lambda_1, \ldots, \lambda_n$ the eigenvalues of the second fundamental form, the above inequality can be written as

$$\left|\sum_{i=1}^n \lambda_i^3\right| \leq \left(\sum_{i=1}^n \lambda_i^2\right)^{3/2}.$$

It is not restrictive to assume $|A| > 0$. If we set

$$\mu_i = \frac{|\lambda_i|}{\left(\sum_{j=1}^n \lambda_j^2\right)^{1/2}}, \qquad i = 1, \ldots, n,$$

then $|\mu_i| \leq 1$, and therefore $\mu_i^3 \leq \mu_i^2$. It follows

$$\left|\sum_{i=1}^n \lambda_i^3\right| \leq \left(\sum_{i=1}^n \mu_i^3\right)\left(\sum_{j=1}^n \lambda_j^2\right)^{3/2} \leq \left(\sum_{i=1}^n \mu_i^2\right)\left(\sum_{j=1}^n \lambda_j^2\right)^{3/2} = \left(\sum_{j=1}^n \lambda_j^2\right)^{3/2}. \blacksquare$$

3 An estimate on the scalar curvature

The goal of this section is to prove the following result.

Theorem 3.1 *Let $\mathcal{M}_t$, $t \in [0, T[$ be a smooth solution of the mean curvature flow (1.1)–(1.2), with $n \geq 2$ and the initial manifold $\mathcal{M}_0$ compact and of positive mean curvature. Then, for any $\eta > 0$ there exists a constant $C_\eta > 0$, depending only on n, η and $\mathcal{M}_0$ such that*

$$R \geq -\eta H^2 - C_\eta$$

on $\mathcal{M}_t$ for any $t \in [0, T[$.

Roughly speaking, this result shows that the negative part of the scalar curvature cannot grow as fast as H^2 when the latter tends to infinity. Following the ideas of Hamilton [7] and Huisken [10] one is led to look for an upper bound for the function $(|A|^2 - H^2)/H^{2-\sigma}$ for some small positive σ. However, the method of [10] does not carry on to the present context, because it relies on some estimates which hold only for convex surfaces. Therefore we have to introduce an additional parameter η and to study the function $g_{\sigma,\eta}$ defined in (2.2). Exploiting the presence of this parameter we obtain some inequalities (see Lemma 3.2) which allows us to bound $g_{\sigma,\eta}$ under the only hypothesis of positive mean curvature. This bound will yield Theorem 3.1.

The following estimate will be used in the proof of our theorem; we remark that it does not depend on our assumptions on $\mathcal{M}_t$ and is valid for any smooth hypersurface.

Lemma 3.2 *Suppose $(1+\eta)H^2 \leq |A|^2 \leq c_0 H^2$ for some $\eta, c_0 > 0$ at some point of $\mathcal{M}_t$. Then we also have*

(i) $-2Z \geq \eta H^2 |A|^2$;

(ii) $|H\nabla_i h_{kl} - \nabla_i H\, h_{kl}|^2 \geq \dfrac{\eta^2}{4n(n-1)^2 c_0} H^2 |\nabla H|^2$.

Proof. **(i)** Taking into account Lemma 2.2 and the assumption $|A| \geq \sqrt{1+\eta}\,|H|$ we obtain

$$\begin{aligned} -2Z &= 2(|A|^4 - \mathrm{tr}(A^3)H) \geq 2|A|^3(|A| - |H|) \\ &\geq 2(1+\eta - \sqrt{1+\eta})|A|^2 H^2 \\ &\geq \eta |A|^2 H^2. \end{aligned}$$

(ii) We have (see [10, Lemma 2.3(ii)])

$$\begin{aligned} |H\nabla_i h_{kl} - \nabla_i H\, h_{kl}|^2 &\geq \frac{1}{4}|\nabla_i H\, h_{kl} - \nabla_k H\, h_{il}|^2 \\ &= \frac{1}{2}(|A|^2|\nabla H|^2 - |\nabla^i H h_{il}|^2). \end{aligned}$$

Let us denote with $\lambda_1, \ldots, \lambda_n$ the eigenvalues of A in such a way that λ_n is an eigenvalue with the largest modulus. Then we have $|\nabla^i H h_{il}|^2 \leq \lambda_n^2 |\nabla H|^2$ and

$$\begin{aligned} |H\nabla_i h_{kl} - \nabla_i H\, h_{kl}|^2 &\geq \frac{1}{2}\sum_{i=1}^{n-1} \lambda_i^2 |\nabla H|^2 = \sum_{i=1}^{n-1} \lambda_i^2 \lambda_n^2 \frac{|\nabla H|^2}{2\lambda_n^2} \\ &\geq \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \lambda_i^2 \lambda_j^2 \frac{|\nabla H|^2}{2(n-1)|A|^2} \\ &\geq \left(\sum_{\substack{i,j=1 \\ i<j}}^{n} \lambda_i \lambda_j \right)^2 \frac{|\nabla H|^2}{n(n-1)^2|A|^2} \\ &= \frac{(|A|^2 - H^2)^2}{4n(n-1)^2|A|^2}|\nabla H|^2 \geq \frac{\eta^2 H^2}{4n(n-1)^2 c_0}|\nabla H|^2. \ \blacksquare \end{aligned}$$

Let us now consider the function $g_{\sigma,\eta}$ defined in (2.2). We assume that $\sigma, \eta \in]0,1[$. In the following we denote by c_i any positive constant depending only on n, η and $\mathcal{M}_0$, and we assume that these constants are chosen larger than 1. We shall write for simplicity $g_{\sigma,\eta} = g$ as long as σ, η are kept fixed. We observe that (2.5) implies

$$g_{\sigma,\eta} \leq c_0 H^\sigma. \tag{3.1}$$

Remark 3.3 Let g_+ denote the positive part of g, i.e. $g_+(x,t) = \max\{g(x,t), 0\}$. Then it is easily seen that $g_+^p \in C^1(\mathcal{M} \times [0,T[\,)$ for any $p > 1$ and

$$\frac{\partial g_+^p}{\partial t} = p\, g_+^{p-1} \frac{\partial g}{\partial t}, \qquad \nabla(g_+^p) = p\, g_+^{p-1} \nabla g.$$

Lemma 3.4 *There exist constants c_2, c_3 such that*

$$\begin{aligned} \frac{d}{dt}\int_{\mathcal{M}_t} g_+^p d\mu &\leq -\frac{p(p-1)}{2}\int_{\mathcal{M}_t} g_+^{p-2}|\nabla g|^2 d\mu - \frac{p}{c_3}\int_{\mathcal{M}_t} \frac{g_+^{p-1}}{H^{2-\sigma}}|\nabla H|^2 d\mu \\ &\quad -p\int_{\mathcal{M}_t} \frac{g_+^{p-1}}{H^{4-\sigma}}|H\nabla_i h_{kl} - \nabla_i H\, h_{kl}|^2 d\mu + p\sigma \int_{\mathcal{M}_t} |A|^2 g_+^p d\mu \end{aligned}$$

for any $p \geq c_2$.

Proof. By Remark 3.3, Proposition 2.1(iii) and equation (2.3) we have for $p \geq 2$

$$\begin{aligned}
\frac{d}{dt}\int g_+^p d\mu &= \int\left(\frac{\partial g_+^p}{\partial t} - H^2 g_+^p\right)d\mu \\
&= \int\left(p g_+^{p-1}\frac{\partial g}{\partial t} - H^2 g_+^p\right)d\mu \\
&\leq -p(p-1)\int g_+^{p-2}|\nabla g|^2 d\mu + 2(1-\sigma)p\int \frac{g_+^{p-1}}{H}\langle \nabla H, \nabla g\rangle d\mu \\
&\quad -2p\int \frac{g_+^{p-1}}{H^{4-\sigma}}|H\nabla_i h_{kl} - \nabla_i H\, h_{kl}|^2 d\mu + p\sigma\int |A|^2 g_+^p d\mu. \qquad (3.2)
\end{aligned}$$

¿From Lemma 3.2(ii) and inequality (3.1) we deduce, if $c_1 \geq 4n(n-1)^2 c_0 \eta^{-2}$,

$$\begin{aligned}
\frac{g_+^{p-1}}{H^{4-\sigma}}|H\nabla_i h_{kl} - \nabla_i H\, h_{kl}|^2 &\geq \frac{g_+^{p-1}}{c_1 H^{2-\sigma}}|\nabla H|^2 \\
&\geq \frac{g_+^{p-1}}{2c_1 H^{2-\sigma}}|\nabla H|^2 + \frac{1}{2c_0c_1}\frac{g_+^p}{H^2}|\nabla H|^2.
\end{aligned}$$

Therefore, if $p \geq \max\{2, 1+4c_0c_1\}$ we obtain

$$\begin{aligned}
2(1-\sigma)p\frac{g_+^{p-1}}{H}\langle \nabla H, \nabla g\rangle &\leq 2p\frac{g_+^{p-1}}{H}|\nabla H||\nabla g| \\
&\leq \frac{p}{2c_0c_1}\frac{g_+^p}{H^2}|\nabla H|^2 + 2c_0c_1 p g_+^{p-2}|\nabla g|^2 \\
&\leq p\frac{g_+^{p-1}}{H^{4-\sigma}}|H\nabla_i h_{kl} - \nabla_i H\, h_{kl}|^2 + \frac{p(p-1)}{2}g_+^{p-2}|\nabla g|^2 \\
&\quad -p\frac{g_+^{p-1}}{2c_1 H^{2-\sigma}}|\nabla H|^2.
\end{aligned}$$

Substituting in inequality (3.2) we obtain the conclusion. ∎

Lemma 3.5 *There exists a constant c_4 such that*

$$\begin{aligned}
\frac{1}{c_4}\int_{\mathcal{M}_t}|A|^2 g_+^p d\mu &\leq \left(p+\frac{p}{\beta}\right)\int_{\mathcal{M}_t} g_+^{p-2}|\nabla g|^2 d\mu + (1+\beta p)\int_{\mathcal{M}_t}\frac{g_+^{p-1}}{H^{2-\sigma}}|\nabla H|^2 d\mu \\
&\quad + \int_{\mathcal{M}_t}\frac{g_+^{p-1}}{H^{4-\sigma}}|H\nabla_i h_{kl} - \nabla_i H\, h_{kl}|^2 d\mu
\end{aligned}$$

for any $\beta > 0$, $p > 2$.

Proof. Let us set $\alpha = 2-\sigma$. We compute

$$\begin{aligned}
\Delta g &= \Delta\left(\frac{|A|^2}{H^\alpha}\right) - (1+\eta)\Delta H^{2-\alpha} \\
&= \frac{\Delta |A|^2}{H^\alpha} - \alpha\frac{|A|^2}{H^{\alpha+1}}\Delta H - \alpha(\alpha-1)\frac{|A|^2}{H^{\alpha+2}}|\nabla H|^2 \\
&\quad -\frac{2\alpha}{H}\langle \nabla H, \nabla\frac{|A|^2}{H^\alpha}\rangle + (1+\eta)(2-\alpha)\left(\frac{\alpha-1}{H^\alpha}|\nabla H|^2 - \frac{1}{H^{\alpha-1}}\Delta H\right).
\end{aligned}$$

Using formula (2.6) can rewrite the above equality as

$$\begin{aligned}
\Delta g &= \frac{2}{H^\alpha}\langle h_{ij},\nabla_i\nabla_j H\rangle + \frac{2}{H^\alpha}Z + \frac{2}{H^{\alpha+2}}|H\nabla_i h_{kl} - \nabla_i H\, h_{kl}|^2 \\
&\quad - \left(\frac{\alpha g}{H} + 2(1+\eta)H^{1-\alpha}\right)\Delta H + \frac{(2-\alpha)(\alpha-1)}{H^2}g|\nabla H|^2 \\
&\quad - \frac{2(\alpha-1)}{H}\langle\nabla H,\nabla g\rangle.
\end{aligned}$$

We multiply this equality by $g_+^p H^{\alpha-2}$ and integrate on $\mathcal{M}_t$. Taking into account Remark 3.3 and the Codazzi equation we can integrate by parts and obtain

$$\begin{aligned}
\int -\frac{2Z}{H^2}g_+^p d\mu &= p\int \frac{g_+^{p-1}}{H^{2-\alpha}}|\nabla g|^2 d\mu - 2p\int \frac{g_+^{p-1}}{H^2}\langle h_{ij},\nabla_i g\nabla_j H\rangle d\mu \\
&\quad +4\int \frac{g_+^p}{H^3}\langle h_{ij},\nabla_i H\nabla_j H\rangle d\mu + 2\int \frac{g_+^p}{H^4}|H\nabla_i h_{kl} - \nabla_i H\, h_{kl}|^2 d\mu \\
&\quad +\int\left(\alpha p\frac{g_+^p}{H^{3-\alpha}} + 2p(1+\eta)\frac{g_+^{p-1}}{H}\right)\langle\nabla g,\nabla H\rangle d\mu \\
&\quad -2\int\left(\frac{g_+^{p+1}}{H^{4-\alpha}} + (2+\eta)\frac{g_+^p}{H^2}\right)|\nabla H|^2 d\mu.
\end{aligned}$$

Using repeatedly inequality (3.1) and the assumptions $\alpha \in]1,2[\,$, $\eta\in]0,1[$ we obtain

$$\begin{aligned}
\int -\frac{2Z}{H^2}g_+^p d\mu &\leq c_0 p\int g_+^{p-2}|\nabla g|^2 d\mu \\
&\quad +4p(c_0+1)\int \frac{g_+^{p-1}}{H}|\nabla H|\,|\nabla g| d\mu \\
&\quad +4c_0^2\int \frac{g_+^{p-1}}{H^\alpha}|\nabla H|^2 d\mu + 2c_0\int \frac{g_+^{p-1}}{H^{2+\alpha}}|H\nabla_i h_{kl} - \nabla_i H\, h_{kl}|^2 d\mu.
\end{aligned}$$

In addition we have, for any $\beta > 0$

$$\begin{aligned}
2\frac{g_+^{p-1}}{H}|\nabla H|\,|\nabla g| &\leq \frac{g_+^{p-2}}{\beta}|\nabla g|^2 + \beta\frac{g_+^p}{H^2}|\nabla H|^2 \\
&\leq \frac{g_+^{p-2}}{\beta}|\nabla g|^2 + c_0\beta\frac{g_+^{p-1}}{H^\alpha}|\nabla H|^2.
\end{aligned}$$

The assertion follows from the above inequalities and from Lemma 3.2(i). ∎

Proposition 3.6 *Given $\eta \in]0,1[\,$, there exist constants c_5, c_6 such that the $L^p(\mathcal{M})$ norm of $(g_{\sigma,\eta})_+$ is a nonincreasing function of t for any p,σ such that*

$$p \geq c_5, \qquad \sigma \leq (c_6 p)^{-1/2}.$$

Proof. Let c_2, c_3, c_4 be as in Lemmas 3.4, 3.5. We recall that we assumed to choose all these constants greater than 1. Suppose that p, σ satisfy

$$p \geq \max\left\{4, c_2, \frac{c_3}{2}\right\}, \qquad \sigma \leq \frac{1}{c_4}\sqrt{\frac{1}{8c_3 p}}.$$

If we set $\beta = 4c_4\sigma$, then

$$\begin{cases} p^2\sigma c_4\left(\beta + \dfrac{1}{\beta}\right) = \dfrac{p^2}{4}(\beta^2+1) \leq \dfrac{p^2}{4} + \dfrac{p}{2} \leq \dfrac{p(p-1)}{2} \\ p\sigma c_4(1+\beta p) \leq \sqrt{\dfrac{p}{8c_3}}\left(1 + \sqrt{\dfrac{2p}{c_3}}\right) \leq 2\sqrt{\dfrac{p}{8c_3}}\sqrt{\dfrac{2p}{c_3}} = \dfrac{p}{c_3} \\ p\sigma c_4 \leq \sqrt{\dfrac{p}{8c_3}} \leq p. \end{cases}$$

Thus, by Lemma 3.5,

$$\begin{aligned} p\sigma \int |A|^2 g_+^p d\mu \;\; &\leq \;\; \frac{p(p-1)}{2}\int g_+^{p-2}|\nabla g|^2 d\mu + \frac{p}{c_3}\int \frac{g_+^{p-1}}{H^{2-\sigma}}|\nabla H|^2 d\mu \\ &\quad + p\int \frac{g_+^{p-1}}{H^{4-\sigma}}|H\nabla_i h_{kl} - \nabla_i H\, h_{kl}|^2 d\mu. \end{aligned}$$

By Lemma 3.4 we conclude

$$\frac{d}{dt}\int g_+^p d\mu \leq 0. \;\blacksquare$$

Proof of Theorem 3.1. From the L^p–estimate of the previous Proposition one can derive a uniform bound on the supremum of $g_{\sigma,\eta}$ with the procedure of [10, Theorem 5.1]. We give a sketch of this method for the reader's convenience. Given any $k \geq k_0$, where

$$k_0 := \sup_{\sigma\in[0,1]} \sup_{\mathcal{M}_0} g,$$

we set

$$v = (g_{\sigma,\eta} - k)_+^{p/2}, \qquad A(k,t) = \{x \in \mathcal{M}_t \;:\; v(x,t) > 0\}.$$

Arguing as in the proof of Lemma 3.4 we obtain, for p large enough,

$$\frac{d}{dt}\int v^2 d\mu + \int |\nabla v|^2 d\mu \leq c_0\sigma p \int_{A(k,t)} H^2 g_{\sigma,\eta}^p d\mu. \tag{3.3}$$

On the other hand we have the following Sobolev–type inequality (see [15]) valid for any Lipschitz function on $\mathcal{M}_t$:

$$\left(\int v^{2q} d\mu\right)^{1/q} \leq c_7 \int |\nabla v|^2 d\mu + c_7 \left(\int_{A(k,t)} H^n d\mu\right)^{2/n} \left(\int v^{2q} d\mu\right)^{1/q}. \tag{3.4}$$

Here $q = n/(n-2)$ if $n > 2$ and an arbitrary number greater than 1 if $n = 2$. Now we observe that, if

$$p \geq \max\{c_5, 4n^2 c_6\}, \qquad \sigma \leq (4c_6 p)^{-1/2}$$

then we have

$$\int_{\mathcal{M}_t} H^n g^p_{\sigma,\eta} d\mu = \int_{\mathcal{M}_t} g^p_{\sigma',\eta} d\mu$$

with

$$\sigma' = \sigma + \frac{n}{p} \leq \frac{1}{2\sqrt{c_6 p}} + \frac{1}{\sqrt{p}}\frac{n}{\sqrt{p}} \leq \frac{1}{\sqrt{c_6 p}}.$$

Thus, by Proposition 3.6,

$$\begin{aligned}
\left(\int_{A(k,t)} H^n d\mu\right)^{2/n} &\leq k^{-2p/n}\left(\int_{A(k,t)} H^n g^p_{\sigma,\eta} d\mu\right)^{2/n} \\
&\leq k^{-2p/n}\left(\int_{\mathcal{M}_0} g^p_{\sigma',\eta} d\mu\right)^{2/n} \qquad (3.5)\\
&\leq \left(\frac{(1+|\mathcal{M}_0|)k_0}{k}\right)^{2p/n}.
\end{aligned}$$

Thus, we can fix $k_1 > k_0$ large enough such that, for any $k \geq k_1$ we may absorb the last term in (3.4) and then exploit the $|\nabla v|$ term in (3.3) to obtain

$$\frac{d}{dt}\int v^2 d\mu + \frac{1}{c_8}\left(\int v^{2q} d\mu\right)^{1/q} \leq c_0 \sigma p \int_{A(k,t)} H^2 g^p_{\sigma,\eta} d\mu. \qquad (3.6)$$

After a time integration and a computation involving the Hölder inequality and the interpolation inequality for L^p spaces (see [10] for details) we obtain, for any $r > 1$,

$$\begin{aligned}
&\int_0^T \int_{A(k,t)} v^p d\mu\, dt \\
&\leq c_9 \sigma p \left(\int_0^T \int_{A(k,t)} d\mu\, dt\right)^{1+b-1/r} \left(\int_0^T \int_{A(k,t)} H^{2r} g^{pr}_{\sigma,\eta} d\mu\, dt\right)^{1/r},
\end{aligned}$$

where $b = (q-1)/(2q-1)$. Let us now choose r large enough such that $\gamma := 1+b-1/r > 1$. With an argument as in (3.5) we can estimate the second factor on the right hand side provided p, σ^{-1} are larger than suitable constants depending only on $n, \eta, \mathcal{M}_0$. Thus we find a constant c_{10} such that, for all $h > k \geq k_1$,

$$\begin{aligned}
|h-k|^p \int_0^T \int_{A(h,t)} d\mu\, dt &\leq \int_0^T \int_{A(k,t)} v^p d\mu\, dt \\
&\leq c_{10}^p \sigma p \left(\int_0^T \int_{A(k,t)} d\mu\, dt\right)^{\gamma}
\end{aligned}$$

for all $h > k \geq k_1$. By a well–known result (see e.g. [16, Lemma 4.1]) we conclude

$$\int_0^T \int_{A(k,t)} d\mu\, dt = 0, \quad \forall\, k > k_1 + d^{1/p},$$

where

$$d = c_{10}^p \sigma p 2^{p\gamma/(\gamma-1)} \left(\int_0^T \int_{A(k_1,t)} d\mu \, dt \right)^{\gamma-1}.$$

Here we use the properties that the critical time T is finite and that the area of $\mathcal{M}_t$ decreases with t, which follow from (2.1) and Proposition 2.1(iii) respectively. We obtain, by the definition of $A(k,t)$,

$$|A|^2 \leq (1+\eta)H^2 + (k_1 + d^{1/p})H^{2-\sigma}.$$

This implies that

$$|A|^2 \leq (1+2\eta)H^2 + K_\eta$$

for some K_η depending only on $\eta, n, \mathcal{M}_0$. Setting $C_\eta = K_{\eta/2}$ we conclude that $|A|^2 - H^2 \leq \eta H^2 + C_\eta$, which proves the theorem. ∎

4 Asymptotic behaviour

We now apply the estimate of the previous section to study the asymptotic behaviour of our family of hypersurfaces near a singularity. It is customary to divide the singularities in two classes, depending on the rate at which the curvature becomes unbounded.

Definition 4.1 *We say that our surface develops a singularity of* type I *(or a* fast singularity*) as $t \to T$ if there exists a constant $C > 0$ such that*

$$\max_{\mathcal{M}_t} |A|^2 \leq \frac{C}{T-t} \tag{4.1}$$

Otherwise we say that the singularity is of type II *(or* slow*).*

Remark 4.2 (i) By a comparison argument using the evolution equation for $|A|^2$ (see Proposition 2.1(ii)) it is easily seen that $\max_{\mathcal{M}_t} |A|^2 \geq [2(T-t)]^{-1}$.
(ii) From inequality (2.5) and from the general property $H^2 \leq n|A|^2$ we deduce that under our hypotheses H^2 and $|A|^2$ have the same blow–up rate.

To study the shape of the surface near a singularity we define now a family of rescaled flows. Since singularities of type I have already been classified in [11], [12], we consider the case of type II singularities. Following an idea of Hamilton [8], we choose a sequence $\{(x_k, t_k)\}$ as follows. For any integer $k \geq 1$ let $t_k \in [0, T - 1/k]$, $x_k \in \mathcal{M}$ be such that

$$H^2(x_k, t_k)(T - \frac{1}{k} - t_k) = \max_{\substack{t \leq T-1/k \\ x \in \mathcal{M}}} H^2(x,t)(T - \frac{1}{k} - t). \tag{4.2}$$

Furthermore we set

$$L_k = H(x_k, t_k), \quad \alpha_k = -L_k^2 t_k, \quad \omega_k = L_k^2(T - t_k - 1/k). \tag{4.3}$$

Lemma 4.3 *If the singularity is of type II then we have, as* $k \to \infty$,

$$t_k \to T, \quad L_k \to \infty, \quad \alpha_k \to -\infty, \quad \omega_k \to \infty.$$

Proof. Let $M > 0$ be given. Since we are assuming that the singularity is of type II (see also Remark 4.2(ii)) there exist $\bar{t} < T$ and $\bar{x} \in \mathcal{M}$ such that $H^2(\bar{x}, \bar{t})(T - \bar{t}) > 2M$. For k large enough we have

$$\bar{t} < T - 1/k, \quad H^2(\bar{x}, \bar{t})(T - \bar{t} - 1/k) > M.$$

It follows

$$\omega_k = H^2(x_k, t_k)(T - t_k - 1/k) \geq H^2(\bar{x}, \bar{t})(T - \bar{t} - 1/k) > M.$$

Since ω_k is increasing and $M > 0$ is arbitrary, this proves that $\omega_k \to \infty$. Furthermore, it follows from the definitions that

$$\omega_k \to \infty \quad \Rightarrow \quad L_k \to \infty \quad \Rightarrow \quad t_k \to T, \ \alpha_k \to -\infty. \ ■$$

Now consider, for any $k \geq 1$, the family of surfaces $\mathcal{M}_{k,\tau}$ defined by the immersions

$$F_k(\cdot, \tau) = L_k(F(\cdot, L_k^{-2}\tau + t_k) - F(x_k, t_k)), \qquad \tau \in [\alpha_k, \omega_k]. \tag{4.4}$$

We denote by A_k and H_k the second fundamental form and the mean curvature associated with the immersion F_k.

Lemma 4.4 *If the singularity is of type II then the rescaled immersions satisfy the following properties:*

(i) $F_k(x_k, 0) = 0$, $H_k(x_k, 0) = 1$;

(ii) *for any* $\eta > 0$ *there exists* C_η *such that* $|A_k|^2 \leq (1 + \eta)H_k^2 + C_\eta L_k^{-2}$;

(iii) *for any* $\varepsilon > 0$ *and* $\bar{\omega} > 0$ *there exists* k_0 *such that* $\max H_k(\cdot, \tau) \leq 1 + \varepsilon$ *for any* $k \geq k_0$, $\tau \in [\alpha_{k_0}, \bar{\omega}]$.

Proof. We have by the definition of F_k

$$A_k(x, \tau) = L_k^{-1} A(x, L_k^{-2}\tau + t_k).$$

¿From this we deduce (i) and (ii), by virtue of Theorem 3.1. In addition, our choice of (x_k, t_k) implies

$$H_k^2(x, \tau) \leq \frac{T - \frac{1}{k} - t_k}{T - \frac{1}{k} - t_k - L_k^2 \tau} = \frac{\omega_k}{\omega_k - \tau}$$

for any k and $\tau \in [\alpha_k, \omega_k[$. Since $\omega_k \to \infty$ we obtain (iii). ■

Theorem 4.5 *Let $\mathcal{M}_t$, $t \in [0, T[$ be a smooth solution of the mean curvature flow (1.1)–(1.2), with $n \geq 2$. Assume that the initial manifold $\mathcal{M}_0$ is compact and of positive mean curvature, and that the flow develops a singularity of type II as $t \to T$. Then a subsequence of the flows $\mathcal{M}_{k,\tau}$ converges smoothly to a mean curvature flow $\widetilde{\mathcal{M}}_\tau$, defined for $\tau \in \mathbb{R}$. The mean curvature $\widetilde{H}$ of the limit flow satisfies $0 < \widetilde{H} \leq 1$ and is equal to 1 at least at one point. Furthermore, either $\widetilde{\mathcal{M}}_\tau$ has positive scalar curvature everywhere or (up to a rigid motion) $\widetilde{\mathcal{M}}_\tau = \mathbb{R}^{n-1} \times \overline{\Gamma}_\tau$, where $\overline{\Gamma}_\tau$ is the "grim reaper" curve given by $x = -\ln \cos y + \tau$.*

Proof. The curvature bound in (iii) of the previous lemma implies analogous bounds on the second fundamental form and all its covariant derivatives. Then by standard methods, based on the Arzelà-Ascoli theorem, we can extract a subsequence of the $\mathcal{M}_{k,\tau}$ converging uniformly on compact subsets of $\mathbb{R}^{n+1} \times \mathbb{R}$ to a limiting solution $\widetilde{\mathcal{M}}_\tau$ of the mean curvature flow, whose mean curvature satisfies $\widetilde{H} \leq 1$. Since $0 \in \mathcal{M}_{k,0}$ for any k and $H_k = 1$ at that point, the same holds for the limit flow. Being the limit of surfaces with positive mean curvature, the limit flow has nonnegative mean curvature. But the maximum principle implies that $\widetilde{H}$ is actually eveywhere positive, since it satisfies the equation in Proposition 2.1(i) and is not identically zero.

Furthermore, the estimate in Lemma 4.4(ii) implies that $|\widetilde{A}|^2 \leq \widetilde{H}^2$. Again the maximum principle, together with equation (2.4), implies that either $|\widetilde{A}|^2 < \widetilde{H}^2$ everywhere (i.e. the limiting surface has positive scalar curvature), or

$$|\widetilde{A}|^2 \equiv \widetilde{H}^2, \qquad |\widetilde{H}\nabla_i \widetilde{h}_{kl} - \nabla_i \widetilde{H}\, \widetilde{h}_{kl}|^2 \equiv 0.$$

In the latter case, arguing as in [12, Theorem 5.1], we obtain that (up to rigid motions) $\widetilde{\mathcal{M}}_\tau = \mathbb{R}^{n-1} \times \Gamma_\tau$, where Γ_τ is a convex eternal solution to the mean curvature flow in the plane, whose mean curvature assumes its maximum at at least one point. By Hamilton's characterization of convex eternal solutions [9, Theorem 1.3] we deduce that Γ_τ is a translating soliton, and it is well known that the only translating soliton in the plane is the grim reaper. ∎

Remark 4.6 A similar procedure can be followed in the case of a surface developing a singularity of type I. In this case we choose a sequence $\{(x_k, t_k)\}$ such that $t_k \to T$ and $H(x_k, t_k) = \max_{\mathcal{M}} H(\cdot, t_k)$. Then we define the rescaled flows as in (4.3)–(4.4). In this case we obtain convergence to a limit flow $\widetilde{\mathcal{M}}_\tau$ defined in a maximal interval of the form $]-\infty, \Omega[$ for some $\Omega < +\infty$. The limit flow has again nonnegative scalar curvature; however for singularities of type I this property follows already from the selfsimilarity result in [11, Theorem 3.5] and the classification in [12, Theorem 5.1].

For the two–dimensional case we now obtain a detailed description of possible singular behaviour.

Corollary 4.7 *Suppose $F_t : \mathcal{M}^2 \to \mathbb{R}^3$ is a smooth solution of (1.1)–(1.2) compact and with positive mean curvature on the maximal time interval $[0, T[$.*

(i) *If the singularity for $t \to T$ is of type I, the only possible limiting flows under the rescaling procedure (4.4) are the homothetically shrinking solutions associated with $S^2, \mathbb{R} \times S^1$ and $\mathbb{R} \times \Gamma$, where Γ is one of the selfsimilar immersed curves introduced by Mullins [13] (see also Abresch–Langer [1]).*

(ii) *If the singularity is of type II, then the limiting flow is either a strictly convex translating soliton or the translating solution given by $\mathbb{R} \times \bar{\Gamma}_\tau$, where $\overline{\Gamma}_\tau$ is the "grim reaper" curve given by $x = -\ln \cos y + \tau$.*

Proof. The first part is a consequence of the results quoted in Remark 4.6. The second part follows from Theorem 4.5 and Hamilton's result [9] characterizing convex eternal solutions, since for $n = 2$ positive scalar curvature is equivalent to convexity. ∎

Acknowledgments. This research was done while the second author was visiting the University of Tübingen, whose ospitality he gratefully acknowledges.

References

[1] U. Abresch, J. Langer, *The normalized curve shortening flow and homothetic solutions*, J. Differential Geom. **23** (1986), 175–196.

[2] S. Altschuler, S.B. Angenent, Y. Giga, *Mean curvature flow through singularities for surfaces of rotation*, J. Geom. Analysis **5** (1995), 293–358.

[3] S.B. Angenent, J.J.L. Velazquez, *Degenerate neckpinches in mean curvature flow*, J. Reine Angew. Math. **482** (1997), 15–66.

[4] K.A. Brakke, *The motion of a surface by its mean curvature*, Math. Notes, Princeton University Press, Princeton, NJ, 1978.

[5] M. Gage, R.S. Hamilton, *The shrinking of convex plane curves by the heat equation*, J. Differential Geom. **23** (1986), 69–96.

[6] M.A. Grayson, *The heat equation shrinks embedded plane curves into points*, J. Differential Geom. **26** (1987), 285–314.

[7] R.S. Hamilton, *Three–manifolds with positive Ricci curvature*, J. Differential Geom. **17** (1982), 255–306.

[8] R.S. Hamilton, *The formation of singularities in the Ricci Flow*, Surveys in Differential Geometry Vol. II, International Press, Cambridge MA (1993), 7–136.

[9] R.S. Hamilton, *Harnack estimate for the mean curvature flow*, J. Differential Geom. **41** (1995), 215–226.

[10] G. Huisken, *Flow by mean curvature of convex surfaces into spheres*, J. Diff. Geometry **20** (1984), 237–266.

[11] G. Huisken, *Asymptotic behaviour for singularities of the mean curvature flow*, J. Diff. Geometry **31** (1990), 285–299.

[12] G. Huisken, *Local and global behaviour of hypersurfaces moving by mean curvature*, Proceedings of Symposia in Pure Mathematics **54** (1993), 175–191.

[13] W. Mullins, *Two dimensional motion of idealised grain boundaries*, J. Appl. Physics **27** (1956), 900-904.

[14] T. Ilmanen, *Singularities of mean curvature flow of surfaces*, preprint, Northwestern University .

[15] J. H. Michael, L. M. Simon, *Sobolev and mean value inequalities on generalized submanifolds of* $\mathbb{R}^n$, Comm. Pure Appl. Math. **26** (1973), 361–379.

[16] G. Stampacchia, *Equations elliptiques au second ordre à coéfficients discontinues*, Sém. Math. Sup. 16, Les Presses de l'Université de Montreal, Montreal, 1966.

[17] B. White, *Partial regularity of mean convex hypersurfaces flowing by mean curvature*, preprint, Stanford University.