Existence of Stationary Vacuum Solutions of Einstein's Equations in an Exterior Domain*

Jürgen Klenk†

Mathematisches Institut
Universität Tübingen
72076 Tübingen, Germany
Juergen.Klenk@uni-tuebingen.de

December 28, 1994

Abstract

A proof is given for the existence and uniqueness of a stationary vacuum solution $(\mathcal{M}, g, \xi)$ of the boundary value problem consisting of Einstein's equations in an exterior domain $\mathcal{M}$ diffeomorphic to $\mathbb{R} \times \Sigma$ (where $\Sigma = \mathbb{R}^3 \backslash B(0, R)$) and boundary data depending on the Killing field ξ on $\partial\Sigma$. The boundary data must be sufficiently close to that of a stationary, spatially conformally flat vacuum solution $(\mathcal{M}, \overset{o}{g}, \overset{o}{\xi})$.

1 Introduction

It is believed that in general relativity rotating stars can be described as rigidly rotating bodies of perfect fluid in hydro– and thermodynamical equilibrium. This model has been extensively discussed in many papers, for an overview see e.g. Lindblom [1]. However, to verify the validity of this model, it is necessary to

*Supported in part by a "DAAD–Doktorandenstipendium aus den Mitteln des zweiten Hochschulsonderprogrammes"

†Current address: Centre for Mathematics and its Applications, Australian National University, Canberra ACT 0200, Australia

establish an existence proof for stationary, asymptotically flat global solutions of Einstein's equations $G = 8\pi T$, where T is the perfect fluid energy–momentum tensor.

This was partially achieved by Heilig [2] in 1994. He constructed an operator depending on three variables: the angular velocity ω of the fluid, the parameter $\nu = \frac{1}{c^2}$ of Ehler's framework theory [3], and the metric g of the whole spacetime. This operator vanishes for $\omega = 0$, $\nu = 0$, and g the well–known global solution of the corresponding static Newtonian problem. Using an implicit function theorem argument, he could then prove the existence of global solutions in the neighbourhood of the static Newtonian solution for small ω and ν. These solutions continuously depend on the two parameters ω and ν and are thus only suitable to describe "very slowly" rotating stars.

A different approach, outlined e.g. by Pfister [4] is to decompose the the global problem by taking the surface of the star as a natural boundary and solving the interior matter problem and the exterior vacuum problem seperately. The interior and exterior solutions are then fitted so that the global solution is continuously differentiable across the boundary. But until now all attempts to create an explicit global solution by extending known interior matter solutions with specific equations of state to the exterior region or by extending the Kerr solution to the interior region have failed. Therefore, to successfully carry out this program, it seems necessary to establish mathematical existence results for both the interior and exterior boundary value problem.

In this paper we establish an existence result for the stationary vacuum boundary value problem in an exterior domain, an extension of a result obtained by Reula [5] in 1989. The solutions we seek will be called 4–solutions and are defined in the following way:

Definition 1.1 *A stationary vacuum solution of Einstein's equations in an exterior domain in the 4–dimensional formulation (4–solution) is given by a triple* $(\mathcal{M}, g, \xi)$, *where*

- $\mathcal{M}$ *is a 4–dimensional manifold,*
- g *is the metric on* $\mathcal{M}$ *satisfying* $\mathrm{Ric} = 0$,
- ξ *is a smooth timelike Killing vector field on* $(\mathcal{M}, g)$ *with orbits diffeomorphic to* $\mathbb{R}$,
- *there exists a spacelike hypersurface* $\mathcal{S}$ *of* $(\mathcal{M}, g)$ *and a diffeomorphism* $\chi : \Sigma := \mathbb{R}^3 \backslash B(0, R) \to \mathcal{S}$ *such that the map*

$$\Psi : \mathbb{R} \times \Sigma \to \mathcal{M}\ ,\ (t, p) \mapsto \Psi(t, p) := \phi(t, \chi(p))$$

is a diffeomorphism, where $\phi : \mathbb{R} \times \mathcal{M} \to \mathcal{M}$ *is the flow of the Killing vector field* ξ,

- *the metric* g *lies in a suitable weighted Sobolev space*[1] *that guarantees its asymptotic flatness.*

With this definition, $\mathcal{M}$ is diffeomorphic to $\mathbb{R} \times \Sigma$, where $\Sigma = \mathbb{R}^3 \backslash B(0, R)$ for a ball of fixed radius R. Thus we will eventually have to solve a fixed boundary value problem rather than a more difficult free boundary value problem, which would have been the case had we used the exterior region of the star as a domain.

In section 2 we will prove that for any 4–solution there exists a corresponding 3+1–solution and vice versa. The 3+1–solutions are defined in the following way:

Definition 1.2 *A stationary vacuum solution of Einstein's equations in an exterior domain in the 3+1–decomposed formulation (3+1–solution) is given by a triple* $(\mathcal{S}, \gamma, \tau)$, *where*

- $\mathcal{S}$ *is a 3–dimensional manifold,*
- γ *is the metric and* τ *a complex scalar field on* $\mathcal{S}$,
- *there exists a diffeomorphism* $\chi : \Sigma := \mathbb{R}^3 \backslash B(0, R) \to \mathcal{S}$,
- γ *and* τ *satisfy the following equations:*

$$\Delta\tau - 2(\tau + \bar{\tau})^{-1}\gamma(\nabla\tau, \nabla\tau) = 0, \tag{1}$$

$$\mathcal{R}ic - (\tau + \bar{\tau})^{-2}(d\tau \otimes d\bar{\tau} + d\bar{\tau} \otimes d\tau) = 0. \tag{2}$$

Introducing harmonic coordinates we will show in section 3 that in a neighbourhood of the Minkowski background metric, equations (1) and (2) together with suitable boundary conditions can be reduced to a quasilinear elliptic boundary value problem, for which we will establish an existence and uniqueness result. Finally, in section 4 we will extend this result to the case of an arbitrary stationary, spatially conformally flat vacuum solution as background metric.

The main theorem can then be expressed in the following way[2]:

Theorem 1.3 *Let* $(\mathcal{M}, \overset{o}{g}, \overset{o}{\xi})$ *be a spatially conformally flat 4–solution and let* $\overset{o}{\tau} = \overset{o}{\lambda} + i\overset{o}{\omega}$ *be the complex scalar field consisting of the norm* $\overset{o}{\lambda}$ *and the scalar potential* $\overset{o}{\omega}$ *of the twist of the Killing field* $\overset{o}{\xi}$. *Let* $\hat{\tau}$ *be a complex scalar field on* $\partial\Sigma$ *sufficiently close to* $\overset{o}{\tau}|_{\partial\Sigma}$. *Then there exists a unique (up to diffeomorphism) 4–solution* $(\mathcal{M}, g, \xi)$ *such that* $\tau|_{\partial\Sigma} = \hat{\tau}$, *where* τ *is the complex scalar field defined by the Killing field* ξ.

[1] For a discussion of these function spaces see e.g. Bartnik [6] and Adams [7].

[2] See Definition 2.6 for the precise definition of the expressions used in this theorem.

For an application of this theorem, we note that the Schwarzschild metric on $\mathcal{M}$ is spatially conformally flat and its Killing field has vanishing twist, i.e. it describes the exterior region of a static (nonrotating) star. According to Theorem 1.3 we then choose boundary data with non–vanishing twist from a neighbourhood of the boundary data of the Schwarzschild metric. Thus the solution corresponding to this prescribed boundary data has a Killing field with small but non–vanishing twist and, therefore, is suitable to describe the exterior region of a "slowly" rotating star.

Finally, because rotating stars are axisymmetric, we note that for axisymmetrically prescribed boundary data, the solution possesses the same symmetry property. This follows from the uniqueness of the solution and the invariance of Einstein's equations under rotations.

2 The 3+1–decomposition

2.1 Preliminaries

The original motivation to derive a 3+1–decomposition for Einstein's equations was to develop an initial value formulation for general relativity so that certain existence and uniqueness results could be established. For an overview see the paper of Choquet–Bruhat and York on the Cauchy problem [8]. By assuming global hyperbolicity, they arrive at a natural foliation for the spacetime with Cauchy surfaces as cross–sections. These Cauchy surfaces are 3–dimensional spacelike submanifolds of the spacetime. The unit normal fields on them form a timelike, hypersurface–orthogonal vector field on the whole spacetime with orbits diffeomorphic to $\mathbb{R}$.

Contrary to this approach, we use the Killing vector field ξ rather than a unit normal field to create a foliation of $\mathcal{M}$. The advantage of this construction is that if $(\mathcal{M}, g, \xi)$ is a 4–solution, then all significant physical quantities are "conserved" along the orbits of ξ. Thus the cross–sections contain all physical information of the 4–dimensional system, and the problem to solve Einstein's equations can be reduced to a 3–dimensional one. However, in the stationary (non–static) case the Killing field is not hypersurface–orthogonal to some spacelike submanifold $\mathcal{S}$ of $\mathcal{M}$. Therefore, to be able to carry out the projection formalism, we must regard $\mathcal{S}$ as a quotient space of $\mathcal{M}$.

First we establish some general properties of quotient manifolds.

Definition 2.1 *An equivalence relation $\sim$ on a finite–dimensional manifold $\mathcal{M}$ is called regular if the quotient space $\mathcal{S} := \mathcal{M}/\sim$ carries a manifold structure* *such that the canonical projection $\pi : \mathcal{M} \to \mathcal{S}$ is a submersion, i.e. such that for all $x \in \mathcal{M}$, the tangent map $\pi_*|_x : T_x\mathcal{M} \to T_{\pi(x)}\mathcal{S}$ is surjective.*

Definition 2.2 *Let $\mathcal{M}$ be an n–dimensional manifold and $\Phi = \{\phi_\alpha\}_{\alpha\in I}$ a partition of $\mathcal{M}$ into disjoint connected sets. The partition Φ is called a p–dimensional foliation if each point of $\mathcal{M}$ has a chart $\varphi : U \to U' \times V' \subset \mathrm{I\!R}^p \oplus \mathrm{I\!R}^{n-p}$ such that for each ϕ_α, the connected components $(U \cap \phi_\alpha)^\beta$ of $U \cap \phi_\alpha$ are given by $\varphi((U \cap \phi_\alpha)^\beta) = U' \times \{c_\alpha^\beta\}$, where $c_\alpha^\beta \in V'$.*

Each foliation $\Phi = \{\phi_\alpha\}_{\alpha\in I}$ determines an equivalence relation $\sim$ on $\mathcal{M}$ by $x \sim y$ if x and y belong to the same leaf ϕ_α. The following proposition is a useful criterion to determine whether this equivalence relation is regular, i.e. whether the quotient space $\mathcal{S}$ is a manifold.

Proposition 2.3 *Let Φ be a foliation of a finite–dimensional manifold $\mathcal{M}$ and let $\sim$ be the equivalence relation determined by Φ. Then $\sim$ is regular iff for every point $x \in \mathcal{M}$ there exists a local submanifold Σ_x of $\mathcal{M}$ containing x such that Σ_x intersects every leaf in at most one point, and $T_x\Sigma_x \oplus T_x\phi_x = T_x\mathcal{M}$, where ϕ_x is the leaf containing the point x.*

Proof: This is Proposition 4.4.9 in [9]. $\square$

The local submanifolds Σ_x are called the *slices* or *local cross–sections* for the foliation.

Proposition 2.4 *Let $\mathcal{M}$ be an n–dimensional manifold and ξ a smooth vector field on $\mathcal{M}$ with orbits diffeomorphic to $\mathrm{I\!R}$. Assume there exists a hypersurface $\mathcal{S} \subset \mathcal{M}$ and a diffeomorphism $\chi : U \subset \mathrm{I\!R}^{n-1} \to \mathcal{S}$ such that the map*

$$\Psi : \mathrm{I\!R} \times U \to \mathcal{M}\ ,\ \ (t,p) \mapsto \Psi(t,p) := \phi(t, \chi(p))$$

is a diffeomorphism, where $\phi : \mathrm{I\!R} \times \mathcal{M} \to \mathcal{M}$ is the flow of the vector field ξ. Let Φ be the set of all orbits ϕ_x. Then Φ is a 1–dimensional foliation of $\mathcal{M}$ and its equivalence relation $\sim$ is regular, i.e. the induced quotient space $\mathcal{S}$ is a 3–dimensional manifold.

Proof: We construct local cross–sections for the foliation Φ. For $x = \Psi(t,p) \in \mathcal{M}$ and $V \subset U$ a neighbourhood of p let $\Sigma_x := \Psi(t, V)$. For sufficiently small V the criterion of Proposition 2.3 is satisfied and thus $\sim$ is regular. $\square$

From now on we assume that $\mathcal{M}$ and $\mathcal{S}$ are given as in Proposition 2.4. Note that for the following results it is not necessary to equip $\mathcal{M}$ with a metric. We establish some properties of the push–forward and pull–back associated with the canonical projection $\pi : \mathcal{M} \to \mathcal{S}$.

1. The push–forward of a vector $v \in T_x\mathcal{M}$ is given by its image under the tangent map, i.e. $\pi_*|_x(v)$.

2. The pull–back of a covector $\omega \in T_y^*\mathcal{S}$ is given by $\pi^*|_y(\omega) = \hat{\omega} \in T_x^*\mathcal{M}$, where $\hat{\omega}(v) := \omega(\pi_*|_x(v))$ for any $v \in T_x\mathcal{M}$ and $x \in \pi^{-1}(y)$.

The push–forward and pull–back can be extended to vector fields on $\mathcal{M}$ and covector fields on $\mathcal{S}$, respectively. This is achieved by applying the pointwise definition above. In addition, they can be extended to tensor fields on $\mathcal{M}$ of rank $(0, r)$ and tensor fields on $\mathcal{S}$ of rank $(s, 0)$, respectively, by the properties of the tensor product. Finally, scalar fields on $\mathcal{M}$ can be mapped onto scalar fields on $\mathcal{S}$ and vice versa by composition of the scalar field with π or π^{-1}. It is important to keep in mind that none of these definitions requires any metric on $\mathcal{M}$ or $\mathcal{S}$. However, the following lemma is an important restriction for the definition of the push–forward.

Lemma 2.5 *The push–forward of a tensor field T on $\mathcal{M}$ of rank $(0, r)$ $(r \geq 0)$ is well–defined if and only if $\mathcal{L}_\xi T = 0$.*

Proof: We only give the proof for vector fields. The push–forward of a vector field $V \in T_x\mathcal{M}$ can be well–defined iff V is constant along the orbits of the vector field ξ, i.e. iff $V = \phi_t^* V$ for all $t \in \mathbb{R}$ (where ϕ_t^* is the pull–back of the diffeomorphism ϕ_t), or in other words, ϕ_t is a symmetry transformation[3] for V. The dynamical characterisation of the Lie derivative, $(\mathcal{L}_\xi V)(x) = \lim_{t\to 0} \frac{1}{t}((\phi_t^* V)(x) - V(x))$ yields the desired result. □

In view of this Lemma, we adopt the following notation:

$$\overset{o}{\mathcal{T}}{}^r_s\mathcal{M} := \{T \in \mathcal{T}^r_s\mathcal{M} : \mathcal{L}_\xi T = 0\}$$

the set of all tensor fields on $\mathcal{M}$ of rank (s, r) which are constant along the orbits of the vector field ξ.

We now want to extend the push–forward to tensor fields on $\mathcal{M}$ of arbitrary rank. Only from now on do we assume that $\mathcal{M}$ is equipped with a metric g and that the smooth vector field ξ on $\mathcal{M}$ is a Killing field with respect to g.

Definition 2.6 *The norm $\hat{\lambda}$, and the twist $\hat{\Omega}$, of the Killing field ξ are defined by $\hat{\lambda} := g(\xi, \xi)$, and $\hat{\Omega} := (*(\xi^\flat \wedge d\xi^\flat))^\sharp$, respectively.*

[3] This is the precise definition of the expression "V is constant along the orbits of the vector field ξ".

In the following we will also adopt the well–known identification of the tangent space $T_y\mathcal{S}$ at a point y of the abstract quotient manifold $\mathcal{S}$ with the subspace $\xi(x)^\perp := \{v \in T_x\mathcal{M} : g|_x(v, \xi(x)) = 0\} \subset T_x\mathcal{M}$ at a point $x \in \pi^{-1}(y)$ of the manifold $\mathcal{M}$. The push–forward can then be extended to tensor fields T on $\mathcal{M}$ of arbitrary rank which satisfy $\mathcal{L}_\xi T = 0$ by making use of the metric g and the following pointwise definition for vectors:

$$\pi_*|_x : T_x\mathcal{M} \to T_{\pi(x)}\mathcal{S}\ ,\ v \mapsto v - \hat{\lambda}^{-1} g|_x(v, \xi(x))\, \xi(x).$$

It is easy to show that $\mathcal{L}_\xi\hat{\lambda} = 0$ and $\mathcal{L}_\xi\hat{\Omega} = 0$. Thus the projections of these fields onto $\mathcal{S}$ are defined. We denote them by $\lambda := \pi_*\hat{\lambda}$, and $\Omega := \pi_*\hat{\Omega}$, respectively. Because $g(\hat{\Omega}, \xi) = 0$ we obtain with the above mentioned identification the simple relations $\hat{\lambda} = \lambda$ and $\hat{\Omega} = \Omega$.

The following Proposition is fundamental for the equivalence of 4–solutions and 3+1–solutions.

Proposition 2.7

1. *We denote by* $\underset{o}{\tilde{\gamma}} := \pi_* g = \underset{o}{g} - \hat{\lambda}^{-1}\xi^\flat \otimes \xi^\flat$ *the projection of the metric g onto $\mathcal{S}$. Let* $\hat{V} \in \mathcal{T}_0^1\mathcal{M}$, $\hat{T} \in \mathcal{T}_s^r\mathcal{M}$, *and D be the Levi–Civita connection on* $(\mathcal{M}, g)$. *Let* $V = \pi_*(\hat{V})$, $T = \pi_*(\hat{T})$. *Then the Levi–Civita connection* $\tilde{\nabla}$ *on* $(\mathcal{S}, \tilde{\gamma})$ *is given by*
$$\tilde{\nabla}_V T = \pi_*(D_{\hat{V}}\hat{T}). \tag{3}$$
Differential operators associated with the Levi–Civita connection $\tilde{\nabla}$ *are:*
   (a) *the covariant differential* $\tilde{\nabla}$,
   (b) *the gradient* $\tilde{\nabla}$,
   (c) *the divergence* $\widetilde{\mathrm{div}}$,
   (d) *the Laplacian* $\tilde{\Delta}$.

   In addition, we have the exterior derivative $\tilde{d} : \bigwedge^k \mathcal{S} \to \bigwedge^{k+1}\mathcal{S}$, *which is independent of* $\tilde{\nabla}$.

2. *For the norm and the twist of the Killing field, we have the following equations*

$$\begin{aligned}
\tilde{d}\Omega^\flat &= \pi_*(*(\xi^\flat \wedge \mathrm{Ric}(\xi, .))) && (4)\\
\widetilde{\mathrm{div}}\,\Omega &= \frac{3}{2}\lambda^{-1}\tilde{\gamma}(\Omega, \tilde{\nabla}\lambda) && (5)\\
\tilde{\Delta}\lambda &= \frac{1}{2}\lambda^{-1}\tilde{\gamma}(\tilde{\nabla}\lambda, \tilde{\nabla}\lambda) - \lambda^{-1}\tilde{\gamma}(\Omega, \Omega) - 2\pi_*(\mathrm{Ric}(\xi, \xi)). && (6)
\end{aligned}$$

3. *Let* $\mathrm{II} : \mathcal{T}_0^1\mathcal{M} \times \mathcal{T}_0^1\mathcal{M} \to \mathcal{T}_0^0\mathcal{M}$ *be defined by* $\mathrm{II}(U,V) := (-\hat{\lambda})^{-\frac{1}{2}}\, g(D_U V, \xi)$. *We note that* II *can be regarded as the second fundamental form on the spacelike hypersurface* $\mathcal{S}$. *Let* Riem *be the Riemannian curvature tensor on* $(\mathcal{M}, g)$, $\hat{W}, \hat{X}, \hat{Y}, \hat{Z} \in \overset{o}{\mathcal{T}}{}_0^1\mathcal{M}$, *and* $W, X, Y, Z \in \mathcal{T}_0^1\mathcal{S}$ *their respective projections onto* $\mathcal{S}$. *Then the Riemannian curvature tensor* $\widetilde{\mathcal{R}iem}$ *on* $(\mathcal{S}, \tilde{\gamma})$ *is given by*

$$\begin{aligned}\tilde{\gamma}\Big(\widetilde{\mathcal{R}iem}(W,X)Y,Z\Big) = \pi_*\Big(&g(\mathrm{Riem}(\hat{W},\hat{X})\hat{Y},\hat{Z}) \\ &+ \mathrm{II}(\hat{X},\hat{Y})\mathrm{II}(\hat{W},\hat{Z}) - \mathrm{II}(\hat{W},\hat{Y})\mathrm{II}(\hat{X},\hat{Z})\Big).\end{aligned} \tag{7}$$

4. *Let* Ric *be the Ricci tensor on* $(\mathcal{M}, g)$. *Then the Ricci tensor* $\widetilde{\mathcal{R}ic}$ *on* $(\mathcal{S}, \tilde{\gamma})$ *is given by*

$$\begin{aligned}\widetilde{\mathcal{R}ic} \;=\; &\frac{1}{2}\lambda^{-2}\Omega^\flat \otimes \Omega^\flat - \frac{1}{2}\lambda^{-2}\tilde{\gamma}(\Omega,\Omega)\tilde{\gamma} \\ &+\frac{1}{2}\lambda^{-1}\tilde{\nabla}(\tilde{\nabla}\lambda) - \frac{1}{4}\tilde{d}\lambda \otimes \tilde{d}\lambda + \pi_*\mathrm{Ric}.\end{aligned} \tag{8}$$

5. *Let* G *be the Einstein tensor on* $(\mathcal{M}, g)$. *Then the Einstein tensor* $\tilde{\mathcal{G}}$ *on* $(\mathcal{S}, \tilde{\gamma})$ *is given by*

$$\begin{aligned}\tilde{\mathcal{G}} \;=\; &\frac{1}{2}\lambda^{-2}\Omega^\flat \otimes \Omega^\flat - \frac{1}{4}\lambda^{-2}\tilde{\gamma}(\Omega,\Omega)\tilde{\gamma} + \frac{1}{2}\lambda^{-1}\tilde{\nabla}(\tilde{\nabla}\lambda) - \frac{1}{4}\lambda^{-2}\tilde{d}\lambda \otimes \tilde{d}\lambda \\ &-\frac{1}{2}\lambda^{-1}(\tilde{\Delta}\lambda)\tilde{\gamma} + \frac{1}{4}\lambda^{-2}\tilde{\gamma}(\tilde{\nabla}\lambda,\tilde{\nabla}\lambda)\tilde{\gamma} + \pi_*\mathrm{G}.\end{aligned} \tag{9}$$

Proof:

1. The operator $\tilde{\nabla}$ defined in equation (3) is the induced connection on $(\mathcal{S}, \tilde{\gamma})$, thus the five Levi–Civita properties hold.

2. To prove equations (4) – (6) we note that for any Killing field ξ,

$$\mathrm{Riem}(\xi, U)V = D_U D_V \xi - D_{D_U V}\xi.$$

   For further details see e.g. Lindblom [10].

3. This is Gauß' equation.

4. Equation (8) can be derived by using the definition $\widetilde{\mathcal{R}ic} = C_3^1\widetilde{\mathcal{R}iem}$ (where $C_3^1\widetilde{\mathcal{R}iem}$ is the contraction of $\widetilde{\mathcal{R}iem}$ over 1 and 3), equation (7), and the definition of the norm and the twist of the Killing field.

5. From equations (6) and (8) we calculate the scalar curvature $\widetilde{\mathcal{R}}$ on $(\mathcal{S}, \tilde{\gamma})$ (where R is the scalar curvature on $(\mathcal{M}, g)$):

$$\widetilde{\mathcal{R}} = -\frac{1}{2}\lambda^{-2}\tilde{\gamma}(\Omega,\Omega) - \frac{1}{2}\lambda^{-2}\tilde{\gamma}(\tilde{\nabla}\lambda,\tilde{\nabla}\lambda) + \lambda^{-1}\tilde{\Delta}\lambda + \pi_*\mathrm{R}. \tag{10}$$

   Then the Einstein tensor can be computed from its definition. □

2.2 The decomposition

Theorem 2.8 *There is a unique (up to diffeomorphism) correspondence between 4–solutions and 3+1–solutions.*

1. *Let $(\mathcal{M}, g, \xi)$ be a 4–solution. The corresponding 3+1–solution $(\mathcal{S}, \gamma, \tau)$ can be obtained as follows:*

   Let $\mathcal{S} := \mathcal{M}/\sim$, where $\sim$ is the equivalence relation determined by the foliation $\Phi = \{\phi_x\}$ of the orbits of the Killing field ξ. Let λ and Ω be the projections of the norm and the twist of the Killing field ξ onto $\mathcal{S}$, and $\tilde{\gamma} := \pi_* g$ the projection of the metric onto $\mathcal{S}$. Define the scalar field ω by $\tilde{\nabla}\omega := \Omega$.

   Then the metric γ and the complex scalar field τ on $\mathcal{S}$ are given by

$$
\begin{aligned}
\gamma &:= \lambda\tilde{\gamma} \\
\tau &:= \lambda + i\omega.
\end{aligned}
$$

2. *Let $(\mathcal{S}, \gamma, \tau)$ be a 3+1–solution. The corresponding 4–solution $(\mathcal{M}, g, \xi)$ can be obtained as follows:*

   Let $\mathcal{M}$ be a 4–dimensional manifold diffeomorphic to $\mathrm{I\!R} \times \Sigma$ and let ξ be a vector field on $\mathcal{M}$ with orbits diffeomorphic to $\mathrm{I\!R}$ such that $\mathcal{S} \cong \mathcal{M}/\sim$, where $\sim$ is the equivalence relation determined by the foliation $\Phi = \{\phi_x\}$ of the orbits of the vector field ξ. On $\mathcal{M}$ choose a basis field $\{\mathrm{e}_0, \mathrm{e}_1, \mathrm{e}_2, \mathrm{e}_3\}$ such that $\mathrm{e}_0 = \xi$. For any point $y \in \mathcal{S}$ identify the tangent space $T_y\mathcal{S}$ with the subspace $\mathrm{span}\{\mathrm{e}_1(\mathrm{x}), \mathrm{e}_2(\mathrm{x}), \mathrm{e}_3(\mathrm{x})\}$ *of $T_x\mathcal{M}$, where $x \in \pi^{-1}(y)$. Define*

$$
\begin{aligned}
\lambda &:= \mathrm{Re}\,\tau \\
\omega &:= \mathrm{Im}\,\tau \\
\tilde{\gamma} &:= \lambda^{-1}\gamma \\
\Omega &:= \tilde{\nabla}\omega \\
\tilde{d}A^\flat &:= (-\lambda)^{-\frac{3}{2}} *(\Omega^\flat).
\end{aligned}
$$

   Let $\hat{\lambda} := \pi^*\lambda$ and $\hat{\gamma} := \pi^*\tilde{\gamma}$. Furthermore, for any $x \in \mathcal{M}$ define the covector $\xi^\flat(x)$ in the following way:

$$
\begin{aligned}
\xi^\flat(x)\ (\xi(x)) &= \lambda(\pi(x)) \\
\xi^\flat(x)\ (e_i(x)) &= \Big(\pi^*(\lambda A^\flat)\Big)(x)\ (e_i(x)) \quad \textit{for } i = 1,2,3.
\end{aligned}
$$

   Then the metric g on $\mathcal{M}$ is given by

$$
g = \hat{\gamma} + \hat{\lambda}^{-1}\,\xi^\flat \otimes \xi^\flat.
$$

Proof:

1. Let $(\mathcal{M}, g, \xi)$ be a 4–solution. Since $\mathrm{Ric} = 0$, equation (4) simply becomes $\tilde{d}\Omega^\flat = 0$, i.e. $\Omega^\flat$ is a closed 1–form. Applying Poincaré's Lemma on the contractible manifold $\mathcal{S}$, the 1–form $\Omega^\flat$ is exact, i.e. $\Omega^\flat = \tilde{d}\omega$ or $\Omega = \tilde{\nabla}\omega$ for a function ω on $\mathcal{S}$. Equations (5), (6), and (8) then become

$$\begin{aligned}
\tilde{\Delta}\omega &= \frac{3}{2}\lambda^{-1}\tilde{\gamma}(\tilde{\nabla}\omega, \tilde{\nabla}\lambda) && (11)\\
\tilde{\Delta}\lambda &= \frac{1}{2}\lambda^{-1}\tilde{\gamma}(\tilde{\nabla}\lambda, \tilde{\nabla}\lambda) - \lambda^{-1}\tilde{\gamma}(\tilde{\nabla}\omega, \tilde{\nabla}\omega) && (12)\\
\widetilde{\mathcal{R}ic} &= \frac{1}{2}\lambda^{-2}\tilde{d}\omega \otimes \tilde{d}\omega - \frac{1}{2}\lambda^{-2}\tilde{\gamma}(\tilde{\nabla}\omega, \tilde{\nabla}\omega)\tilde{\gamma}\\
&\quad + \frac{1}{2}\lambda^{-1}\tilde{\nabla}(\tilde{\nabla}\lambda) - \frac{1}{4}\tilde{d}\lambda \otimes \tilde{d}\lambda. && (13)
\end{aligned}$$

   Applying both the conformal transformation $\gamma := \lambda\tilde{\gamma}$ and the substitution $\tau := \lambda + i\omega$ we find that equations (1) and (2) are satisfied. Thus $(\mathcal{S}, \gamma, \tau)$ is a 3+1–solution.

2. Let $(\mathcal{S}, \gamma, \tau)$ be a 3+1–solution. Equations (11) – (13) follow directly from equations (1) and (2) when applying the conformal transformation $\tilde{\gamma} = \lambda^{-1}\gamma$ and the substitution $\lambda + i\omega := \tau$. With $\Omega := \tilde{\nabla}\omega$, equation (11) can be rewritten as $\tilde{d}(*(\Omega^\flat)) = \frac{3}{2}\lambda^{-1}\tilde{d}\lambda(\Omega) *1 = \frac{3}{2}\lambda^{-1}\tilde{d}\lambda\wedge(*(\Omega^\flat))$, where the second equation holds due to the properties of the Hodge–star operator. We will use this equation to prove that $(-\lambda)^{-\frac{3}{2}}(*(\Omega^\flat))$ is closed, hence exaxt with Poincaré's Lemma applied on the contractible manifold $\mathcal{S}$. This proves the existence of the 1–form $A^\flat$. We have

$$\begin{aligned}
\tilde{d}((-\lambda)^{-\frac{3}{2}}(*(\Omega^\flat))) &= \tilde{d}((-\lambda)^{-\frac{3}{2}}) \wedge (*(\Omega^\flat)) + (-\lambda)^{-\frac{3}{2}}\tilde{d}(*(\Omega^\flat))\\
&= \frac{3}{2}(-\lambda)^{-\frac{5}{2}}\tilde{d}\lambda \wedge (*(\Omega^\flat)) + (-\lambda)^{-\frac{3}{2}}\frac{3}{2}\lambda^{-1}\tilde{d}\lambda \wedge (*(\Omega^\flat))\\
&= 0.
\end{aligned}$$

   With the given definition of the metric g, $\xi^\flat$ is the covector field associated with the vector field ξ. It is a straightforward computation to show that ξ is a timelike Killing field, and that λ and Ω are the projections of the norm and the twist of ξ, respectively. Thus we can apply Proposition 2.7 to prove that $\mathrm{Ric} = 0$.

   Indeed by inserting equations (12) in (6) and (13) in (8) and using equation (4) together with $\tilde{d}\Omega^\flat = \tilde{d}^2\omega = 0$, ξ, $\xi^\flat \neq 0$ everywhere, and $\xi^\flat(e_i) \neq 0$ for $i = 1, 2, 3$, we find that $\mathrm{Ric} = 0$. Thus $(\mathcal{M}, g, \xi)$ is a 4–solution. □

3 Reduction, existence, and uniqueness

In this section we reduce equations (1) and (2) along with suitable boundary conditions on $\partial\mathcal{S}$ to formulate a quasilinear elliptic boundary value problem. For this problem we establish an existence and uniqueness result provided the boundary data is sufficiently close to that of the Minkowski metric (serving as "background metric"). This is achieved by reproving a result of Reula [5], making use of a different theorem for elliptic boundary value problems. In the next section we will extend this result to a larger class of background solutions.

Using equations (8), (9), and (10), we replace (2) by the equivalent expression for the Einstein tensor $\mathcal{G}$ on $\mathcal{S}$. From now on we also use the index notation because all quantities will be expressed in the global chart $\mathcal{S} \cong \Sigma$.

3.1 Preliminaries

We state an implicit function theorem and an existence and uniqueness theorem for elliptic boundary value problems, which will be used several times in the proofs of the reduction, existence, and uniqueness results of this section.

Theorem 3.1 *Let X, Y, Z be Banach spaces, let $U \subset X$ and $V \subset Y$ be open subsets, and let $f : U \times V \to Z$. Let $(x_0, y_0) \in U \times V$ be such that $f(x_0, y_0) = 0$. Let f be Fréchet differentiable at (x_0, y_0) with respect to the first component and $D_1 f|_{(x_0,y_0)} : X \to Z$ be either surjective with complemented kernel[4] or bijective.*

Then there exist neighbourhoods $V_0 \subset V$ of y_0 and $U_0 \subset U$ of x_0 and a map $g : V_0 \to U_0$ (which is unique if $D_1 f|_{(x_0,y_0)}$ is bijective) such that for all $y \in V_0$ we have $f(g(y), y) = 0$.

Proof: The case of the bijective Fréchet differential is Theorem 2.5.7 in [9]. The extension to the surjective case is a straightforward calculation. □

Definition 3.2 *Let D be a differential operator on a domain $\Sigma \subset \mathbb{R}^n$ with the flat Laplacian ∂^2 as principal part[5] and let B be a boundary operator on $\partial\Sigma$ with principal part B^p. Let $t^a \neq 0$ be a tangent vector field on $\partial\Sigma$, and n^a the normal vector field on $\partial\Sigma$ with respect to the flat metric. Let $\tau^+(t^a(x))$ be the root with positive imaginary part of the polynomial $\sum_{a=1}^{n}(t^a(x) + \tau n^a(x))^2$ for all $x \in \partial\Sigma$, and define the polynomials $p_{t,x}(\tau) := \tau - \tau^+(t^a(x))$ and $q_{t,x}(\tau) := B^p(x, t^a(x) + \tau n^a(x))$.*

[4] This means that the Banach space X can be written as the direct sum of the kernel of $D_1 f|_{(x_0,y_0)}$ and some closed subspace of X.

[5] See Definition 2.1 in [11].

Then the boundary value problem (D,B) in Σ is called elliptic if for all $x \in \partial\Sigma$ and $t^a(x) \neq 0$ the set $\{p_{t,x}(\tau), q_{t,x}(\tau)\}$ is linearly independent.

Theorem 3.3 *Let $n \geq 2, s \geq 2, \delta \in \mathbb{R}, \delta \neq -2+\frac{n}{2}+m$, $\delta \neq -\frac{n}{2}-m$ for $m \in \mathbb{N}$. Let (D,B) be an elliptic boundary value problem in Σ. Then the map*

$$(D,B) : H_{s,\delta}(\Sigma) \to H_{s-2,\delta+2}(\Sigma) \times H_{s-k-\frac{1}{2}}(\partial\Sigma)$$

is Fredholm. Moreover, for $n \geq 3$ and

1. *for $-\frac{n}{2} < \delta < -2 + \frac{n}{2}$ it is bijective,*
2. *for $-2 + \frac{n}{2} + m < \delta < -2 + \frac{n}{2} + m + 1$ $(m \in \mathbb{N})$ it is surjective,*
3. *for $-\frac{n}{2} - m - 1 < \delta < -\frac{n}{2} - m$ $(m \in \mathbb{N})$ it is injective.*

Proof: The proof can be found in McOwen [12]. □

The Sobolev spaces used in this theorem are weighted and of fractional differentiability. An introduction to the former can be found in Bartnik [6] and of the latter in Adams [7].

3.2 The reduction

We introduce a background metric and a vector field $(\overset{o}{g}_{ab}, \overset{o}{\xi}{}^a)$ on $\mathcal{M}$ or, equivalently, a background metric and a complex scalar field $(\overset{o}{\gamma}_{ab}, \overset{o}{\tau})$ on $\mathcal{S}$, which are 4–solutions or 3+1–solutions, respectively. In this section we choose $(\overset{o}{g}_{ab}, \overset{o}{\xi}{}^a)$ to be the Minkowski–solution $(\eta_{ab}, (\partial_t)^a)$, and thus $(\overset{o}{\gamma}_{ab}, \overset{o}{\tau}) = (\delta_{ab}, 1)$.

Then we replace (γ_{ab}, τ) by (ϕ_{ab}, u), the distance to the background metric:

$$\begin{aligned} \phi_{ab} &:= \sqrt{\gamma}\,\gamma_{ab} - \sqrt{\overset{o}{\gamma}}\,\overset{o}{\gamma}_{ab} = \sqrt{\gamma}\,\gamma_{ab} - \delta_{ab}, \\ u &:= \tau - \overset{o}{\tau} = \tau - 1, \end{aligned}$$

where $\gamma = \det(\gamma_{ab})$. Furthermore we define

∂	the Levi–Civita connection on $(\mathcal{S}, \delta_{ab})$,
$\psi^b := \partial_a \phi^{ab}$	the divergence of ϕ^{ab} which vanishes only in harmonic coordinates,
$\varphi := \delta_{ab}\phi^{ab}$	the trace of ϕ^{ab}.

Finally we introduce a vector field n^a on $\mathcal{S}$ by writing (in the global chart) $\Sigma = \bigcup_{r \geq R} \partial B(0,r)$ and defining $n^a(x)$ as the outward[6] unit normal vector at $x \in \partial B(0,r)$ with respect to δ_{ab}. With this unit normal vector field, we define

[6] The outward region of $\partial B(0,r)$ is $\mathbb{R}^3 \backslash \overline{B(0,r)}$.

$\hat{\delta}_{ab} := \delta_{ab} - n_a n_b$	the tangential component of δ_{ab},
$\hat{\partial}$	the Levi–Civita connection on $(\partial\mathcal{S}, \hat{\delta}_{ab}\vert_{\partial\Sigma})$,
$\sigma_{ab} := \hat{\delta}_{ac}\hat{\delta}_{bd}\phi^{cd} - \frac{1}{2}\hat{\delta}_{ab}\hat{\delta}_{cd}\phi^{cd}$	the traceless tangential component of ϕ_{ab},
${}_n\dot{\psi} := n^a\partial_a(n_b\psi^b)$	the normal deriv. of the normal component of ψ^b,
${}_t\psi^b := \hat{\delta}^b_a\psi^a$	the tangential component of ψ^b.

Theorem 3.4 *Let $\hat{u} \in H_2(\partial\Sigma)$. For $(u, \phi^{ab}) \in H_{\frac{5}{2},-\frac{3}{4}}(\Sigma) \times H_{\frac{5}{2},-\frac{3}{4}}(\Sigma)$ denote by*

- *system 1:*

$$\begin{aligned}
&\Delta u - 2(u + \bar{u} + 2)^{-1}\gamma^{cd}\,\partial_c u\,\partial_d u = 0\\
&\mathcal{G}^{ab} - (u + \bar{u} + 2)^{-2}(\partial^a u\,\partial^b \bar{u} + \partial^a \bar{u}\,\partial^b u - \gamma^{cd}\,\partial_c u\,\partial_d \bar{u}\gamma^{ab}) = 0\\
&u|_{\partial\Sigma} - \hat{u} = 0\\
&\psi^b = 0
\end{aligned}$$

- *system 2:*

$$\begin{aligned}
&E(u, \phi^{ab}) = \gamma^{cd}\partial_c\partial_d u + (\textit{terms of lesser order}) = 0\\
&E^{ab}(u, \phi^{ab}) = \gamma^{cd}\partial_c\partial_d\phi^{ab} + (\textit{terms of lesser order})^{ab} = 0\\
&u|_{\partial\Sigma} - \hat{u} = 0\\
&\varphi|_{\partial\Sigma} = 0\\
&\sigma_{ab}|_{\partial\Sigma} = 0\\
&{}_n\dot{\psi}|_{\partial\Sigma} = 0\\
&{}_t\psi^b|_{\partial\Sigma} = 0
\end{aligned}$$

where the precise definitions of E and E^{ab} will be given in the first part of the proof.

Then there exists $\varepsilon(\hat{u}) > 0$ such that for all (u, ϕ^{ab}) with $\|(u, \phi^{ab})\| < \varepsilon(\hat{u})$ we have: (u, ϕ^{ab}) is a solution of system 1 if and only if it is a solution of system 2 (up to diffeomorphism).

Proof: "if": Let (u, ϕ^{ab}) be a solution of system 1. The Laplacian on $(\mathcal{S}, \gamma_{ab})$ can be written as $\Delta u = \gamma^{cd}\partial_c\partial_d u + (\textit{terms of lesser order})$, so $E(u, \phi^{ab})$ is easily defined to match the first equation of system 1. Since $\psi^b = 0$, the solution is given in harmonic coordinates, and thus the Einstein tensor takes the quasilinear elliptic form $\mathcal{G}^{ab} = \gamma^{cd}\partial_c\partial_d\phi^{ab} + (\textit{terms of lesser order})^{ab}$. Therefore, $E^{ab}(u, \phi^{ab})$ can be defined to match the second equation of system 1. All terms of lesser order depend continuously on (u, ϕ^{ab}) and vanish for $(u, \phi^{ab}) = (0, 0)$.

To obtain the boundary values of system 2, we consider the map

$$G : U \times V \to Z \ , \ (d, \phi^{ab}) \mapsto \left(\partial_a d(\phi^{ab}), d(\varphi)|_{\partial\Sigma}, \hat{\partial}_a d(\sigma^{ab})|_{\partial\Sigma}\right),$$

where $X := H_{\frac{7}{2},-\frac{7}{4}}(\Sigma), Y := H_{\frac{5}{2},-\frac{3}{4}}(\Sigma), Z := H_{\frac{3}{2},\frac{1}{4}}(\Sigma) \times H_2(\partial\Sigma) \times (H_1(\partial\Sigma)/\{$infinitesimal conformal isometries of $\hat{\delta}_{ab}\}), U := \{d \in X : \|d - id\| < \varepsilon\}, V := Y$ and ε is such that all $d \in U$ are diffeomorphisms. We identify each diffeomorphism $d : \Sigma \to \Sigma, x^b \mapsto d^a(x^b) = x^a + \zeta^a(x^b)$ with the vector field $\zeta^a \in U$. Since d is a diffeomorphism, $(d(u), d(\phi^{ab}))$ satisfies the (transformed) differential equations of system 2. We also have $G(id, 0) = (0, 0, 0)$. The definition of the Fréchet differential yields

$$D_1G|_{(id,0)} : X \to Z \ , \ \zeta^a \mapsto D_1G|_{(id,0)}(\zeta^a) \,,$$

where

$$\begin{aligned}
D_1G|_{(id,0)}(\zeta^a) &= \begin{pmatrix} \partial_a\partial^a\zeta^b \\ (\delta_{ab}\partial^a\zeta^b)|_{\partial\Sigma} \\ \hat{\partial}_a(\hat{\delta}^a_c\hat{\delta}^b_d\partial^{(c}\zeta^{d)} - \hat{\delta}^{ab}\hat{\delta}_{cd}\partial^c\zeta^d)|_{\partial\Sigma} \end{pmatrix} \\
&= \begin{pmatrix} \partial^2\zeta^b \\ \partial^a\left((\hat{\delta}_{ab} + n_a n_b)\zeta^b\right)|_{\partial\Sigma} \\ \hat{\partial}_a(\hat{\delta}^{(a}_c\hat{\delta}^{b)}_d\hat{\partial}^c\zeta^d|_{\partial\Sigma} - \hat{\delta}^{ab}\hat{\delta}_{cd}\hat{\partial}^c\zeta^d|_{\partial\Sigma}) \end{pmatrix} \\
&= \begin{pmatrix} \partial^2\zeta^b \\ (\partial^a n_a)|_{\partial\Sigma}\, {}_n\zeta|_{\partial\Sigma} + (n_a\partial^a{}_n\zeta|_{\partial\Sigma})|_{\partial\Sigma} + \hat{\partial}^a{}_t\zeta_a|_{\partial\Sigma} \\ \hat{\partial}_a(\hat{\partial}^{(a}{}_t\zeta^{b)}|_{\partial\Sigma} - \hat{\delta}^{ab}\hat{\partial}^c{}_t\zeta_c|_{\partial\Sigma}) \end{pmatrix},
\end{aligned}$$

so that $D_1G|_{(id,0)}$ defines an elliptic boundary value problem in Σ. According to Theorem 3.3, $D_1G|_{(id,0)} : X \to Z$ is a Fredholm operator. Thus it has finite–dimensional (complemented) kernel. It is also surjective since $\delta < -\frac{3}{2}$.

Applying Theorem 3.1, we find that for any solution (u, ϕ^{ab}) of system 1 with $\|(u, \phi^{ab})\| < \varepsilon$ there exists a diffeomorphism d such that the transformed solution $(d(u), d(\phi^{ab}))$ satisfies $d(\psi^b) = 0$ in Σ, $d(\varphi) = 0$ on $\partial\Sigma$, and $\hat{\partial}_a d(\sigma^{ab}) = 0$ on $\partial\Sigma$. Thus we also have $d(\sigma^{ab}) = 0$ on $\partial\Sigma$ since any traceless, symmetric, divergence free tensor field of a 2–dimensional manifold vanishes. Therefore, the boundary conditions of system 2 hold and $(d(u), d(\phi^{ab}))$ is a solution of system 2.

“only if”: Let (u, ϕ^{ab}) be a solution of system 2. With the given definition of E and E^{ab} it remains to be shown that $\psi^b = 0$ if $\|(u, \phi^{ab})\| < \varepsilon$, i.e. the solution is given in harmonic coordinates.

We multiply the divergence of E^{ab} with $\sqrt{\gamma}$ and rewrite the result as a differential equation for $\psi^b = \partial_a\phi^{ab}$:

$$L(u, \phi^{ab})\psi^b = \partial^2\psi^b + R(u, \phi^{ab})\psi^b = 0, \text{ where } R(0, 0) = 0.$$

Since $L(0,0) = \partial^2$ (the flat Laplacian), it is injective. $R(u,\phi^{ab})$ depends continuously on (u,ϕ^{ab}) and can be estimated such that $L(u,\phi^{ab})$ remains injective in a neighbourhood of $(0,0)$. Therefore, for $\|(u,\phi^{ab})\| < \varepsilon$, the differential equation $L(u,\phi^{ab})\psi^b = 0$, together with the boundary conditions ${}_n\dot{\psi}|_{\partial\Sigma} = 0$, ${}_t\psi^b|_{\partial\Sigma} = 0$, yield $\psi^b = 0$ in Σ. □

3.3 The existence and uniqueness theorem

Theorem 3.5 *There exists $\varepsilon > 0$ such that for all $\hat{u} \in H_2(\partial\Sigma)$ with $\|\hat{u}\| < \varepsilon$ there is exactly one solution $(u,\phi^{ab}) \in H_{\frac{5}{2},-\frac{3}{4}}(\Sigma) \times H_{\frac{5}{2},-\frac{3}{4}}(\Sigma)$ of system 2.*

Proof: We rewrite system 2 implicitly as the trivial solution of the map

$$\begin{aligned} &F : X \times Y \to Z \\ &\big((u,\phi^{ab}),\hat{u}\big) \mapsto \big(E(u,\phi^{ab}), E^{ab}(u,\phi^{ab}), u|_{\partial\Sigma} - \hat{u}, \varphi|_{\partial\Sigma}, \sigma_{ab}|_{\partial\Sigma}, {}_n\dot{\psi}|_{\partial\Sigma}, {}_t\psi^b|_{\partial\Sigma}\big)\,, \end{aligned}$$

where $X = H_{\frac{5}{2},-\frac{3}{4}}(\Sigma) \times H_{\frac{5}{2},-\frac{3}{4}}(\Sigma)$, $Y = H_2(\partial\Sigma)$, and $Z = H_{\frac{1}{2},\frac{5}{4}}(\Sigma) \times H_{\frac{1}{2},\frac{5}{4}}(\Sigma) \times H_2(\partial\Sigma) \times H_2(\partial\Sigma) \times H_2(\partial\Sigma) \times H_0(\partial\Sigma) \times H_1(\partial\Sigma)$.

Obviously $F((0,0),0) = (0,0,0,0,0,0,0)$ (this is the background solution), and for the Fréchet differential $D_1F|_{((0,0),0)}$ we calculate

$$D_1F|_{((0,0),0)} : X \to Z, (u,\phi^{ab}) \mapsto (\partial^2 u, \partial^2\phi^{ab}, u|_{\partial\Sigma}, \varphi|_{\partial\Sigma}, \sigma_{ab}|_{\partial\Sigma}, {}_n\dot{\psi}|_{\partial\Sigma}, {}_t\psi^b|_{\partial\Sigma}),$$

so that $D_1F|_{((0,0),0)}$ decouples into two elliptic boundary value problems for u and ϕ^{ab}. The first is simply a Dirichlet problem ($u \mapsto (\partial^2 u, u|_{\partial\Sigma})$), and Theorem 3.3 shows that this map is bijective since $\delta = -\frac{3}{4}$.

Accordingly, we can formulate an elliptic boundary value problem for ϕ^{ab}. This is achieved by decomposing ϕ^{ab} as $\phi^{ab} = \nu\, n^a n^b + n^a\chi^b + n^b\chi^a + \frac{\sigma}{2}\hat{\delta}^{ab} + \sigma^{ab}$, where $\sigma = \hat{\delta}_{cd}\phi^{cd}$ and χ^a is a vector field orthogonal to n^a with respect to δ_{ab}. Rewriting the boundary value problem in terms of these quantities, the boundary conditions become

$$\begin{aligned} \varphi|_{\partial\Sigma} &= (\nu+\sigma)|_{\partial\Sigma} \\ {}_n\dot{\psi}|_{\partial\Sigma} &= \left(n^a n^c \partial_a\partial_c\left(\nu + \frac{\sigma}{2}\right) + \frac{1}{2}(\partial_a n^a)(n^c\partial_c\sigma) - \frac{1}{2}(\partial_a n^a)(\partial_c n^c)\left(\nu - \frac{\sigma}{2}\right)\right)\Big|_{\partial\Sigma} \\ {}_t\psi^b|_{\partial\Sigma} &= \left(\frac{1}{2}(\hat{\partial}^b\sigma) + (n^c\partial_c\chi^b) + \frac{3}{2}(\partial_c n^c)\chi^b\right)\Big|_{\partial\Sigma} \end{aligned}$$

and $\sigma^{ab}|_{\partial\Sigma}$ remains unchanged.

Again applying Theorem 3.3 we see that the elliptic boundary value problem for ϕ^{ab} is a bijective map since $\delta = -\frac{3}{4}$.

Thus $D_1F|_{((0,0),0)}$ is bijective and the implicit function theorem 3.1 yields the desired result. □

4 Spatially conformally flat metrics

We now extend the results of the previous section to the case of arbitrary spatially conformally flat background metrics.

Definition 4.1 *A stationary metric (g_{ab}, ξ^a) on a spacetime $\mathcal{M}$ is called spatially conformally flat if the associated metric (γ_{ab}, τ) on $\mathcal{S}$ is conformally flat, i.e. if $\gamma_{ab} = \mu\delta_{ab}$ for a positive function $\mu \in \mathcal{T}_0^0\mathcal{S}$.*

Theorem 4.2 *Let $(\mathcal{M}, \overset{o}{g}_{ab}, \overset{o}{\xi}{}^a)$ be a stationary, spatially conformally flat vacuum solution of Einstein's equations. Then Theorems 3.4 and 3.5 hold for $(\overset{o}{g}_{ab}, \overset{o}{\xi}{}^a)$ instead of $(\eta_{ab}, (\partial_t)^a)$ as background metric and vector field.*

Proof: We have $\overset{o}{\gamma}_{ab} = \mu\delta_{ab}$ and $\bar{\gamma}_{ab} = \mu\gamma_{ab}$, thus $\bar{\phi}^{ab} = \mu^{-1}\phi^{ab}$ and $\bar{u} = \mu^{-1}u$. Denote by $\overset{o}{\partial}$ the Levi–Civita connection on $(\mathcal{S}, \overset{o}{\gamma}_{ab})$. Then we have the following transformation relations:

$$\begin{aligned} \gamma^{cd}\,\partial_c\partial_d u &= \bar{\gamma}^{cd}\,\overset{o}{\partial}_c\overset{o}{\partial}_d\bar{u} + (\textit{terms of lesser order}) \\ \gamma^{cd}\,\partial_c\partial_d\phi^{ab} &= \bar{\gamma}^{cd}\,\overset{o}{\partial}_c\overset{o}{\partial}_d\bar{\phi}^{ab} + (\textit{terms of lesser order})^{ab} \end{aligned}$$

Thus the principal parts of the differential equations and the boundary conditions of both systems 1 and 2 remain unchanged.

The two crucial parts in the proof of the reduction theorem 3.4 are

1. the Fréchet differential $D_1\bar{G}|_{(id,0)}$. With the above transformation relations it still defines an elliptic boundary value problem with the flat Laplacian as differential operator. Thus the proof of the "if"–case remains unchanged.

2. the remainder $\bar{R}(\bar{u}, \bar{\phi}^{ab})$. We still have the continuous dependence of $\bar{R}$ on $(\bar{u}, \bar{\phi}^{ab})$, and $\bar{R}(0,0) = 0$. $\bar{L}(0,0)$ also remains the flat Laplacian and so the injectivity argument remains valid.

The crucial part in Theorem 3.5 is the Fréchet differential $D_1\bar{F}|_{((0,0),0)}$. Just as in the case of the Fréchet differential $D_1\bar{G}|_{id,0}$, it follows from the above transformation relations that we again have two elliptic boundary value problems for $\bar{u}$ and $\bar{\phi}^{ab}$, and thus the proof remains unchanged. □

Proposition 4.3 *Define the following tensor field on $(\mathcal{S}, \gamma_{ab})$:*

$$\mathcal{C}_{ijk} := \nabla_k\,\mathcal{R}ic_{ij} - \nabla_j\,\mathcal{R}ic_{ik} + \frac{1}{4}\left(\gamma_{ik}\nabla_j\,\mathcal{R} - \gamma_{ij}\nabla_k\,\mathcal{R}\right).$$

Then γ_{ab} is conformally flat if and only if $\mathcal{C}_{ijk} = 0$.

Proof: See e.g. [13] pp. 91–92. □

Proposition 4.4 *Both the Schwarzschild and the Reissner–Nordstrøm metric are spatially conformally flat. The Kerr–Newman metric is not spatially conformally flat.*

Proof: The tensor field $\mathcal{C}_{ijk}$ vanishes for the spatial parts of both the Schwarzschild and the Reissner–Nordstrøm metric. For the Kerr–Newman metric we find that $\mathcal{C}_{112} \neq 0$. The calculations were done with *Mathematica*™. □

5 Summary

To prove Theorem 1.3, we first choose a stationary, spatially conformally flat vacuum solution $(\mathcal{M}, \overset{o}{g}_{ab}, \overset{o}{\xi}{}^a)$ of Einstein's equations, and use the 3+1–decomposition to rewrite it in the equivalent form $(\mathcal{S}, \overset{o}{\gamma}_{ab}, \overset{o}{\tau})$. Then by choosing sufficiently small data on the boundary $\hat{u} = (\tau - \overset{o}{\tau})|_{\partial\Sigma}$, Theorems 3.4 and 3.5 show that a solution $(\mathcal{S}, \phi^{ab}, u)$ of system 2 and thus a corresponding solution of system 1 exists. With the 3+1–decomposition we finally obtain the desired stationary vacuum solution $(\mathcal{M}, g_{ab}, \xi^a)$ of Einstein's equations, which lies in the neighbourhood of the background metric.

Using a more general theorem for elliptic boundary value problems, it should be possible to extend the result to non–spatially conformally flat, stationary vacuum solutions as background metrics.

Acknowledgements

I would like to thank Robert Bartnik for pointing out the above mentioned extension. I also thank Gerhard Huisken, Helmut Kaul, and Ben Evans for reading the manuscript, and the Centre for Mathematics and its Applications in Canberra for its hospitality.

References

[1] Lindblom, L.: *Stationary stars are axisymmetric.* Astrophys. J. **208** (1976), 873–880.

[2] Heilig, U.: *On the existence of rotating stars in general relativity.* To appear in Comm. Math. Phys..

[3] Ehlers, J.: *On limit relations between, and approximate explanations of, physical theories*. In *Logic methodology and philosophy of science VII*, eds. R. Barcan Marcus, J.G.W. Dorn, P. Weingartner, Amsterdam, North Holland, 1986, 387–403.

[4] Pfister, H.: *Zur Frage nach globalen Lösungen der Einsteinschen Feldgleichungen für rotierende Sterne*. [The quest for global solutions of Einstein's field equations for rotating stars.] Wiss. Zs. Friedrich Schiller Univ. Jena Natur. Reihe **39** (1990), 152–160.

[5] Reula, O.: *On existence and behaviour of asymptotically flat solutions to the stationary Einstein equations*. Comm. Math. Phys. **122** (1989), 615–624.

[6] Bartnik, R.: *The mass of an asymptotically flat manifold*. Comm. Pure Appl. Math. **34** (1986), 661–693.

[7] Adams, R.A.: *Sobolev spaces*. Academic Press, 1975.

[8] Choquet–Bruhat, Y. & York, J.W.: *The Cauchy problem*. In *General relativity and gravitation 1*, ed. A. Held, Plenum Press, 1979, 99–172.

[9] Abraham, R. & Marsden, J. & Ratiu, T.: *Manifolds, tensor analysis, and applications*. Second edition. Applied mathematical sciences vol. 75, Springer Verlag, 1988.

[10] Lindblom, L.: *Fundamental properties of equilibrium stellar models*. Dissertation, University of Maryland, 1978.

[11] Renardy, M. & Rogers, R.C.: *An introduction to partial differential equations*. Springer Verlag, 1993.

[12] McOwen, R.C.: *Boundary value problems for the Laplacian in an exterior domain*. Comm. Part. Diff. Eqns. **6** (1981), 783–798.

[13] Eisenhart, L.P.: *Riemannian geometry*. Second printing. Princeton University Press, 1949.