On the case of Kovalevskaya and new examples of integrable conservative systems on S^2

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Abstract. There is a well-known example of an integrable conservative system on S^2, the case of Kovalevskaya in the dynamics of a rigid body, possessing an integral of fourth degree in momenta. The aim of this paper is to construct new families of examples of conservative systems on S^2 possessing an integral of fourth degree in momenta.

1 Introduction

Let M be an n-dimensional Riemannian manifold, and $U : M \to \mathbf{R}$ be a smooth function on M. For the Lagrangian $L : TM \to \mathbf{R}$ we choose

$$L(\eta) = \frac{|\eta|^2}{2} + (U \circ \pi)(\eta)$$

where $\pi : TM \to M$ is the canonical projection. In local coordinates $q_1, \ \dots \ , q_n$, $\dot{q}_1, \ \dots \ , \dot{q}_n$ on TM we have

$$L(\eta) = \frac{1}{2} \sum g_{ij} \dot{q}^i \dot{q}^j + U(q).$$

Identifying TM and T^*M by means of the Riemannian metric, we get a Hamiltonian system with the Hamiltonian $H : T^*M \to \mathbf{R}$ which in local coordinates $q_1, \ \dots \ , q_n$, $p_1, \ \dots \ , p_n$ on T^*M has the form

$$H = \frac{1}{2} \sum g^{ij} p_i p_j + U(q) = K + U.$$

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We will call these Hamiltonian systems *conservative systems* on M.

A smooth function $F : T^*M \to \mathbf{R}$ which is an integral of the Hamiltonian system with the Hamiltonian H and which is independent of H we will call an *integral* of this system of *degree* m in momenta if in local coordinates F has the form

$$F = \sum_{k_1+ \ \dots \ +k_n \le m} a_{k_1 \ \dots \ k_n}(q) p_1^{k_1} \ \dots \ p_n^{k_n}.$$

We will say that two Hamiltonians $H_1 = K_1 + U_1$ and $H_2 = K_2 + U_2$ are *equivalent* if there exists a diffeomorphism ϕ of M and a diffeomorphism Φ of T^*M such that the diagram

$$\begin{array}{ccc} \Phi : T^*M & \to & T^*M \\ \pi' \downarrow & & \downarrow \pi' \\ \phi : M & \to & M \end{array}$$

is commutative, where Φ is linear for $p \in M$ fixed, and if there are some nonzero constants κ, $\tilde{\kappa}$ such that $\Phi^*(K_1) = \kappa K_2$, $\phi^*(U_1) = \tilde{\kappa} U_2$.

Clearly, if the Hamiltonians H_1, H_2 are equivalent and one of the corresponding systems possesses an integral of degree m in momenta, then the other system has the same property.

There is a well-known example of an integrable conservative system on S^2, the case of Kovalevskaya [4] in the dynamics of a rigid body, possessing an integral of fourth degree in momenta. The total energy (the Hamiltonian) of this system has the following form

$$H = \frac{du_1^2 + du_2^2 + 2du_3^2}{2u_1^2 + 2u_2^2 + u_3^2} - u_1, \tag{1}$$

where S^2 is given by $u_1^2 + u_2^2 + u_3^2 = 1$, (see [1]).

Goryachev proposed in [2] a family of examples of conservative systems on S^2 possessing an integral of fourth degree in momenta. The Hamiltonians of these systems have the following form

$$H = \frac{du_1^2 + du_2^2 + 2du_3^2}{2u_1^2 + 2u_2^2 + u_3^2} - 2B_1 u_1 u_2 - B_2(u_1^2 - u_2^2) - u_1 \tag{2}$$

where S^2 is given by $u_1^2 + u_2^2 + u_3^2 = 1$, B_1, B_2 are arbitrary constants. Clearly, this family reduces to the case of Kovalevskaya when $B_1 = B_2 = 0$.

The aim of this paper is to construct *new families of examples* of conservative systems on S^2 possessing an integral F of *fourth degree* in momenta, i.e. examples which are not equivalent to the cases of Kovalevskaya or Goryachev.

We will show that these families include the integrable cases from [5] and the case of Kovalevskaya. In particular, we will obtain an explicit expression for the case of Kovalevskaya from a quite different point of view.

2 New examples

In [5] the following local criterion for integrability of the geodesic flow of a Riemannian metric ds^2 by a polynomial of fourth degree in momenta has been obtained.

Theorem 2.1 *Let $ds^2 = \lambda(z,\bar{z})dzd\bar{z}$ be a metric such that there exists a function $f : \mathbf{R}^2 \mapsto \mathbf{R}$, satisfying the following conditions*

$$\lambda(z,\bar{z}) = \frac{\partial^2 f}{\partial z \partial \bar{z}}, \quad Im\left(\frac{\partial^4 f}{\partial z^4}\frac{\partial^2 f}{\partial z \partial \bar{z}} + 3\frac{\partial^3 f}{\partial z^3}\frac{\partial^3 f}{\partial z^2 \partial \bar{z}} + 2\frac{\partial^2 f}{\partial z^2}\frac{\partial^4 f}{\partial z^3 \partial \bar{z}}\right) = 0. \qquad (3)$$

Then the geodesic flow of ds^2 possesses an integral of fourth degree in momenta.

If the geodesic flow of a metric ds^2 possesses an integral which is a polynomial of fourth degree in momenta and it does not depend on the Hamiltonian and an integral of smaller degree then there exist conformal coordinates x, y and a function $f : \mathbf{R}^2 \mapsto \mathbf{R}$ such that $ds^2 = \lambda(z,\bar{z})dzd\bar{z}$ where $z = \varphi + iy$ and (3) holds.

Equation (3) in some other form has been obtained also in [3].

As in [5] we consider now the solutions of (3) of the following form:

$$f(\varphi, y) = u(y)\cos\varphi + \xi(y) + d(\varphi^2 - y^2)$$

where $u(y)$, $\xi(y)$ are some smooth functions and d is a constant. In [5] it has been shown that the geodesic flows of the metrics $ds^2 = \lambda(d\varphi^2 + dy^2)$ where

$$\lambda = \frac{1}{4}\left((u''(y) - u(y))\cos\varphi + \xi''(y)\right),$$

$$\xi'' = \frac{d_1 u(y) + c}{(u'(y))^2},\ d_1, c - const, \quad \text{for} \quad d = 0,$$

or

$$\xi'' = 2d\frac{u'^2(y) - u^2(y) + d_1(2d)^{-1}u(y) + p}{u'(y))^2},\ d_1, p - const, \quad \text{for} \quad d \neq 0,$$

and $u(y)$ satisfies

$$2u''^2 - 3u^2 + u'u''' = \frac{a}{2},\ a - const, \qquad (4)$$

possess an integral of fourth degree in momenta.

Using the well-known Maupertuis's principle and taking into account that c and d are arbitrary constants we find that the Hamiltonian systems with the Hamiltonians

$$H = \frac{d\varphi^2 + dy^2}{u'^2(y)} - (u''(y) - u(y))u'^2(y)\cos\varphi \qquad (5)$$

and

$$H_p = \frac{u'^2(y) - u^2(y) + p}{u'^2(y)}\left(d\varphi^2 + dy^2\right) - (u''(y) - u(y))\frac{u'^2(y)}{u'^2(y) - u^2(y) + p}\cos\varphi \qquad (6)$$

where u is a solution of (4), possess also an integral of fourth degree in momenta.

Now we exploit the properties of the differential equation (4).

Proposition 2.2 *The differential equation (4) is equivalent to the two-parameter family of first-order differential equations*

$$u'^4 = b + b_1 u + a u^2 + u^4 \tag{7}$$

where b, b_1 are arbitrary constants.

Proof. Multiply equation (4) by u'

$$u''' u'^2 + 2u'' u' u'' - 3u^2 u' = \frac{a}{2} u'$$

and integrate

$$u'' u'^2 - u^3 = \frac{a}{2} u + \frac{b_1}{4}$$

with some constant b_1. Multiply again with u'

$$u'' u'^3 - u^3 u' = \frac{a}{2} u u' + \frac{b_1}{4} u'$$

and integrate

$$\frac{1}{4} u'^4 - \frac{1}{4} u^4 = \frac{a}{4} u^2 + \frac{b_1}{4} u + \frac{b}{4}$$

with some constant b. This is (7).

□

Thus the family (5) can be parametrized by a, b, b_1, $u(0)$ and (6) by a, b, b_1, $u(0)$, p.

We will consider the case $b_1 = 0$ in (7). Denote $A(u, a, b) = b + a u^2 + u^4$ where a and b are constants.

Thus, the construction of our examples is based on the properties of the following differential equation

$$u'^4 = b + a u^2 + u^4. \tag{8}$$

Due to Proposition 2.2 we may obtain explicit expressions for the Hamiltonians (5) and (6) in the coordinates φ and u:

$$H = A^{-\frac{1}{2}}(u, a, b) \left(d\varphi^2 + A^{-\frac{1}{2}} du^2 \right) - \frac{u}{2} ((a + 2u^2) - 2A^{\frac{1}{2}}) \cos\varphi \tag{9}$$

and

$$H_p = \frac{A^{\frac{1}{2}} - u^2 + p}{A^{\frac{1}{2}}} \left(d\varphi^2 + A^{-\frac{1}{2}} du^2 \right) - \frac{u((a + 2u^2) - 2A^{\frac{1}{2}})}{2(A^{\frac{1}{2}} - u^2 + p)} \cos\varphi \tag{10}$$

Thus, in (u, φ) coordinates, H depends only the parameters a, b, and H_p on a, b, p.

By a substitution $u \to \alpha_1 u$, $\alpha_1 - const$ in (8) we obtain

$$u'^4 = \frac{b}{\alpha_1^4} + \frac{a}{\alpha_1^2} u^2 + u^4.$$

So we can normalize b to 1, 0 or -1 and keep a.

In this section we will consider the case $b = 1$. In the next section we will consider the case $b = 0$, $a = 1$ and show that it is in fact the case of Kovalevskaya.

The case $b = 1$, $a = 0$ has been considered in [5]. It has been shown that in this case the systems given by (5), (6) (and therefore (9), (10)) define smooth conservative systems on S^2 possessing an integral of fourth degree in momenta.

We will use the following proposition.

Proposition 2.3 *If $b = 1$ in (8) and $a > -2$, then there is a solution $u_a(y) : \mathbf{R} \to \mathbf{R}$ of (8) such that the following holds*

$$u_a'(y) = (\exp y)\nu_a(\exp(-2y)) = \exp(-y)\nu_a(\exp(2y)),$$

$$u_a'^2(y)(u_a''(y) - u_a(y)) = \exp(-y)\mu_a(\exp(-2y)) = -(\exp y)\mu_a(\exp(2y))$$

where the functions μ_a, ν_a are of class C^∞ and $\nu_a > 0$ everywhere.

Proof. By a simple computation we get that if $-2 < a < 0$, then $A(x, a, 1) \geq 1 - \frac{a^2}{4} > 0$ for all $x \in \mathbf{R}$ and if $a \geq 0$, then $A(x, a, 1) \geq 1$ for all $x \in \mathbf{R}$.

Therefore, if $b = 1$ and $a > -2$, then all solutions u of (8) exist globally. All increasing (descreasing) solutions are translates of each other. Increasing solutions run from $-\infty$ to $+\infty$, similarly for descreasing solutions.

W.l.o.g. we may only consider increasing solutions of

$$u' = (1 + au^2 + u^4)^{\frac{1}{4}}, \quad u' > 0. \tag{11}$$

Then we have for $u \geq 0$

$$u' \leq u + c_0$$

with a constant c_0 such that $c_0 \geq 1$ and $6c_0^2 \geq a$ and therefore

$$u(y) \leq (u(0) + c_0)(\exp y) - c_0$$

and

$$\frac{u(y)}{\exp y} \leq u(0) + c_0. \tag{12}$$

Now consider any such solution. Put $s = \exp(-2y)$ and

$$g(s) = \sqrt{s}u\left(-\frac{1}{2}\log s\right).$$

Then $u(y) = (\exp y)g(\exp(-2y))$. The function g is of class C^∞ on $(0, \infty)$. We normalize $u(0) = g(1) = 0$.

Equation (11) can be rewritten as a differential equation for $g(s)$, $s = \exp(-2y)$:

$$g' = -\frac{s + ag^2}{2((s^2 + ag^2 + g^4)^{\frac{1}{4}} + g)((s^2 + ag^2 + g^4)^{\frac{1}{2}} + g^2)} = \Theta_a(s, g).$$

Thus, g is decreasing, $g(0)$ is finite in view of (12), and $g(0) > g(1) = 0$.

Therefore, there is a solution u_a of (8) which can be given as

$$u_a(y) = (\exp y) g_a(\exp(-2y)) = -(\exp(-y)) g_a(\exp 2y)$$

where g_a is of class C^∞ on $[0, +\infty)$ if $a \neq 0$.

The case $a = 0$ has been considered in [5] but we can give here another proof. We can consider the function $\beta(s^2) = g_0(s)$ and prove, with the same arguments as above, that β is smooth in zero, and therefore, g_0 is smooth in zero, too.

Now the corresponding expressions for the functions ν_a and μ_a can be obtained in terms of g_a.

□

Further we will use some properties of the geodesic flows of metrics

$$ds^2 = \lambda(r^2)(r^2 d\varphi^2 + dr^2) \tag{13}$$

on S^2 which have been proved in [5]. These properties follow also from the results of Kolokol'tsov, published in his Ph.D. Dissertation, (Moscow State University, 1984).

Proposition 2.4 *The geodesic flow of a Riemannian metric (13) on S^2 does not possess a nontrivial integral quadratic in momenta (which does not depend on H and linear integrals).*

It is known that a metric of constant positive curvature has the following form in polar coordinates

$$ds^2 = \frac{C_1}{(1 + Dr^2)^2} \left(r^2 d\varphi^2 + dr^2 \right), \tag{14}$$

where $C_1, D - const.$

Proposition 2.5 *The geodesic flow of a metric (13) on S^2 possesses two independent linear integrals if and only if it is has the form (14), i.e. if it is a metric of constant positive curvature.*

Corollary 2.6 *Liouville coordinates $\varphi, y = \log r$, related to polar coordinates φ, r of a metric (13) on S^2 are unique up to shifts and the transform $y \to -y$.*

Now we will prove the main theorems.

Theorem 2.7 *Assume $b = 1$. Then for any $a > -2$ the Hamiltonian (9) where $\varphi \in [0, 2\pi)$ and $u \in (-\infty, +\infty)$ defines a conservative system on S^2 possessing an integral of fourth degree in momenta.*

If $a \neq 2$, then this integral is nontrivial, i.e. there is no quadratic or linear integral. If $a = 2$ it is a Hamiltonian of a metric of constant positive curvature.

The Hamiltonians (9) for different values of the parameter a are not equivalent.

Proof. The integrability of these systems follows immediately from Theorem 2.1, see above.

So, we have to prove only that the Hamiltonian (9), for $b = 1$ and $a > -2$, is a sum of a smooth Riemannian metric on S^2 and a smooth function on S^2.

Using Proposition 2.3 we may rewrite the corresponding Hamiltonian in polar coordinates $\varphi, r = \exp y$. By computation we obtain

$$H = \frac{1}{\nu_a^2(r^2)}(r^2 d\varphi^2 + dr^2) + \mu_a(r^2) r \cos\varphi$$

$$= \frac{1}{\nu_a^2(\tilde{r}^2)}(\tilde{r}^2 d\tilde{\varphi}^2 + d\tilde{r}^2) - \mu_a(\tilde{r}^2)\tilde{r}\cos\tilde{\varphi}$$

where $\tilde{r} = \frac{1}{r}$, $\tilde{\varphi} = -\varphi$. Since ν_a, μ_a are of class C^∞ and $\nu_a \neq 0$, see Proposition 2.3, this system is a conservative system on S^2.

Now we prove that the systems with Hamiltonians (9) where $b = 1$ and $a > -2$ do not have linear or nontrivial quadratic integrals.

Write $H = \hat{H} + V(\varphi, y)$ where $\hat{H}$ is the Hamiltonian of the geodesic flow of the metric

$$ds_1^2 = \frac{d\varphi^2 + dy^2}{u'^2(y)}$$

and

$$V(\varphi, y) = -u'^2(y)(u''(y) - u(y))\cos\varphi \tag{15}$$

where $u(y)$ satisfies (8) with the parameters $b = 1$, $a > -2$. Note that ds_1^2 has the form (13) in $\varphi, r = \exp y$ but it has the form (14) if and only if $a = 2$ (if $b = 1$). So, we can apply Propositions 2.4 and 2.5. We conclude that if $a \neq 2$, then an integral quadratic in momenta of the geodesic flow of ds_1^2 depends on the linear integral p_φ and the Hamiltonian $\hat{H}$.

Let us assume that a system from our theorem has an integral which is independent of the energy H and which is quadratic in momenta (clearly, this assumption includes the case of linear integrals).

So, there is an integral $\tilde{F}$ of (9) which is quadratic in momenta. Thus, $\tilde{F} = D(p_\varphi, p_y, \varphi, y) + B(\varphi, y)$ where $D(p_\varphi, p_y, \varphi, y)$ is a polynomial of second degree in momenta p_φ, p_y.

We write $\{\tilde{F}, H\} = \{D(p_\varphi, p_y, \varphi, y) + B(\varphi, y), \hat{H} + V\} \equiv 0$ and, therefore, $\{D(p_\varphi, p_y, \varphi, y), \hat{H}\} \equiv 0$.

Thus, the geodesic flow of ds_1^2 has an integral quadratic in momenta. As mentioned above, this integral depends on p_φ and $\hat{H}$. Since $\tilde{F}$ does not depend on H, we may put w.l.o.g. that

$$D(p_\varphi, p_y, \varphi, y) = p_\varphi^2.$$

We write now

$$\{D, V\} + \{B, \hat{H}\} = \{p_\varphi^2, V\} + \{B, \hat{H}\} \equiv 0.$$

By computation we obtain

$$\frac{\partial B}{\partial y} \equiv 0,$$

and

$$u'^2(y)\frac{\partial B}{\partial \varphi} = \frac{\partial V}{\partial \varphi}.$$

So, we get

$$V = (B(\varphi) + \alpha(y))u'^2(y)$$

for a smooth function $\alpha(y)$. Comparing now this expression with (15), we get $u''(y) - u(y) \equiv const$, that is not true if $a \neq 2$. So, there is no nontrivial quadratic integral of the system given by (9) where $b = 1$ and $-2 < a < 2$ or $a > 2$.

In order to prove that the Hamiltonians (9) for different values of the parameter a are not equivalent we will apply Corollary 2.6.

Suppose that the Hamiltonians (9) for $a = a_1$ and $a = a_2$, $(a_1 \neq a_2)$ are equivalent. Therefore, from Corollary 2.6 the following holds identically

$$\frac{du}{d\tilde{u}} = \pm \left(\frac{1 + a_1u^2 + u^4}{1 + a_2\tilde{u}^2 + \tilde{u}^4}\right)^{\frac{1}{4}}$$

and

$$\frac{1}{\sqrt{1 + a_1u^2 + u^4}} = \kappa \frac{1}{\sqrt{1 + a_2\tilde{u}^2 + \tilde{u}^4}}$$

for some constant κ. Thus, we obtain $u = \tilde{u}$ and, therefore, $a_1 = a_2$.

With the same arguments as in [5], we may prove that no Hamiltonian from our theorem is equivalent to the Hamiltonian of the cases of Kovalevskaya or Goryachev.

From Corollary 2.6 it follows immediately that in the family (2) of Goryachev we need to consider only the case $B_1 = B_2 = 0$ which is in fact the case of Kovalevskaya, see the Introduction.

Let us write the Hamiltonian (1) of the case of Kovalevskaya in polar coordinates. We obtain

$$H = \gamma_1(r^2)(r^2d\varphi^2 + dr^2) - \gamma_2(r^2)\cos\varphi$$

where $\gamma_2(r^2) \neq 0$ for all $0 < r < +\infty$. Comparing this with (9) where $u \in (-\infty, +\infty)$, we see that no Hamiltonian from our theorem is equivalent to the Hamiltonian of the case of Kovalevskaya.

□

Theorem 2.8 *Assume $b = 1$. Then for any $a > 2$, $p \in (-\infty, -\frac{a}{2}) \cup (-1, +\infty)$ and $-2 < a < 2$, $p \in (-\infty, -1) \cup (-\frac{a}{2}, +\infty)$ the Hamiltonian H_p (or $-H_p$) where H_p has the form (10) and $\varphi \in [0, 2\pi)$, $u \in (-\infty, +\infty)$ defines a conservative system on S^2 possessing an integral of fourth degree in momenta. This integral is nontrivial, i.e. there is no quadratic or linear integral.*

These Hamiltonians for different values of parameters a and p are not equivalent. No Hamiltonian from this family is equivalent to the Hamiltonians of the cases of Kovalevskaya or Goryachev.

Proof. The integrability follows immediately from Theorem 2.1.

We will use Proposition 2.3 to rewrite the corresponding Hamiltonians in polar coordinates $\varphi, r = \exp y$:

$$H_p = \frac{\xi_a(r^2) + p + 1}{\nu_a^2(r^2)}(r^2 d\varphi^2 + dr^2) + \frac{\mu_a(r^2)}{\xi_a(r^2) + p + 1} r\cos\varphi$$

$$= \frac{\xi_a(\tilde{r}^2) + p + 1}{\nu_a^2(\tilde{r}^2)}(\tilde{r}^2 d\tilde{\varphi}^2 + d\tilde{r}^2) - \frac{\mu_a(\tilde{r}^2)}{\xi_a(\tilde{r}^2) + p + 1}\tilde{r}\cos\tilde{\varphi}$$

where $\tilde{r} = \frac{1}{r}$, $\tilde{\varphi} = -\varphi$ and

$$\xi_a(t) = \int_t^1 \mu_a(s)\nu_a^{-1}(s)ds.$$

Now we must find the admissible values of the parameter p. We have

$$p > \max_{z\geq 0} f(z) \quad \text{or} \quad p < \min_{z\geq 0} f(z)$$

where $f(z) = z - \sqrt{1 + az + z^2}$. By computation we obtain $\min f(x) = f(0) = -1$, $\max f(x) = f(\infty) = -\frac{a}{2}$ if $-2 < a < 2$ and $\min f(x) = f(\infty) = -\frac{a}{2}$, $\max f(x) = f(0) = -1$ if $a > 2$.

In order to prove all other statements of the theorem one must only repeat the arguments from the proof of Theorem 2.7.

□

3 The case of Kovalevskaya

Theorem 3.1 *If $b = 0$, $a = 1$, $p = 0$, then the Hamiltonian (10) where $\varphi \in [0, 2\pi)$ and $u \in [0, +\infty)$ defines a conservative system on S^2, corresponding to the case of Kovalevskaya.*

Proof. In this case the Hamiltonian (10) has the form

$$H = \frac{1}{\sqrt{1+u^2}(\sqrt{1+u^2}+u)}\left(d\varphi^2 + \frac{du^2}{u\sqrt{1+u^2}}\right) + \frac{1}{2(\sqrt{1+u^2}+u)}\cos\varphi \qquad (16)$$

where $u \in [0, +\infty)$, $\varphi \in [0, 2\pi)$.

Let us introduce new variables

$$x = \Psi(u)\cos\varphi,\ y = \Psi(u)\sin\varphi,\ z = \pm\sqrt{1 - \Psi(u)^2}$$

where

$$\Psi(u) = \frac{1}{\sqrt{1+u^2}+u} = \sqrt{1+u^2} - u,\ 0 < \Psi(u) < 1.$$

We get

$$\sqrt{1+u^2} = \frac{1}{2}\left(\Psi(u) + \frac{1}{\Psi(u)}\right),\ u = \frac{1}{2}\left(-\Psi(u) + \frac{1}{\Psi(u)}\right).$$

Let us compute

$$dx^2 + dy^2 + 2dz^2 = \Psi'^2 du^2 + \Psi^2 d\varphi^2 + 2\frac{\Psi^2\Psi'^2}{1-\Psi^2}du^2 = \Psi^2\left(d\varphi^2 + \frac{du^2}{u\sqrt{1+u^2}}\right).$$

Then (16) can be rewritten in the form

$$H = 2\frac{\Psi^2}{\Psi^2+1}\left(d\varphi^2 + \frac{du^2}{u\sqrt{1+u^2}}\right) + \frac{1}{2}\Psi\cos\varphi = 2\frac{dx^2+dy^2+2dz^2}{2x^2+2y^2+z^2} + \frac{1}{2}x$$

where

$$x^2 + y^2 + z^2 = 1.$$

□

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