New families of conservative systems on S^2 possessing an integral of fourth degree in momenta

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Abstract. There is a well-known example of an integrable conservative system on S^2, the case of Kovalevskaya in the dynamics of a rigid body, possessing an integral of fourth degree in momenta. In this paper we propose *new families* of examples of conservative systems on S^2 possessing an integral of *fourth degree* in momenta.

Key words: Integrable Hamiltonian system, Riemannian metric on S^2.

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1 Introduction

Let M be an n-dimensional smooth Riemannian manifold, and $U : M \to \mathbf{R}$ be a smooth function on M. For the Lagrangian $L : TM \to \mathbf{R}$ we pick

$$L(\eta) = \frac{|\eta|^2}{2} + (U \circ \pi)(\eta)$$

where $\pi : TM \to M$ is the canonical projection. In local coordinates $q_1, \ldots, q_n$, $\dot{q}_1, \ldots, \dot{q}_n$ on TM we have

$$L(\eta) = \frac{1}{2} \sum g_{ij} \dot{q}^i \dot{q}^j + U(q).$$

The function U is reffered to as a potential function of the force field $-\nabla U$ on M.

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Identifying TM and T^*M by means of the Riemannian metric, we get the Hamiltonian system with the Hamiltonian $H : T^*M \to \mathbf{R}$ which in local coordinates $q_1, \ldots, q_n, p_1, \ldots, p_n$ on T^*M has the form

$$H = \frac{1}{2}\sum g^{ij}p_ip_j + U(q) = K + U$$

We will call these Hamiltonian systems *conservative systems* on M.

A smooth function $F : T^*M \to \mathbf{R}$ which is an integral of the Hamiltonian system with the Hamiltonian H and independent of H we will call an *integral* of this system of *degree* m in momenta if in local coordinates F has the form

$$F = \sum_{k_1 + \ldots + k_n \le m} a_{k_1 \ldots k_n} p_1^{k_1} \ldots p_n^{k_n}.$$

We will say that two Hamiltonians $H_1 = K_1 + U_1$ and $H_2 = K_2 + U_2$ are *equivalent* if there exist a diffeomorphism ϕ of M and a diffeomorphism Φ of T^*M such that the diagram

$$\begin{array}{ccc} \Phi : T^*M & \to & T^*M \\ \pi' \downarrow & & \downarrow \pi' \\ \phi : M & \to & M \end{array}$$

is commutative where Φ is linear for a fixed $p \in M$ and $\Phi^*(K_1) = \kappa K_2$, $\phi^*(U_1) = \tilde{\kappa} U_2$ for some nonzero constants κ, $\tilde{\kappa}$.

Clearly, if the Hamiltonians H_1 and H_2 are equivalent and the Hamiltonian system with the Hamiltonian H_1 possesses an integral of degree m in momenta, then the Hamiltonian system with the Hamiltonian H_2 possesses an integral of degree m in momenta, too.

There is a well-known example of an integrable conservative system on S^2, the case of Kovalevskaya [10] in the dynamics of a rigid body, possessing an integral of fourth degree in momenta. The total energy (the Hamiltonian) of this system has the following form

$$H = \frac{du_1^2 + du_2^2 + 2du_3^2}{2u_1^2 + 2u_2^2 + u_3^2} - u_1, \tag{1}$$

where S^2 is given by $u_1^2 + u_2^2 + u_3^2 = 1$, (see [4]).

Goryachev proposed in [5] a family of examples of conservative systems on S^2 possessing an integral of fourth degree in momenta. The Hamiltonians of these systems have the following form

$$H = \frac{du_1^2 + du_2^2 + 2du_3^2}{2u_1^2 + 2u_2^2 + u_3^2} - 2B_1u_1u_2 - B_2(u_1^2 - u_2^2) - u_1 \tag{2}$$

where S^2 is given by $u_1^2 + u_2^2 + u_3^2 = 1$, B_1, B_2 are some constants. Clearly, this family reduces to the case of Kovalevskaya when $B_1 = B_2 = 0$.

In this paper we propose *new families of examples* of conservative systems on S^2 possessing an integral F of *fourth degree* in momenta, i.e. examples which are not equivalent to the cases of Kovalevskaya or Goryachev.

In our examples we use the solution ψ_0 of the following initial value problem

$$\psi'''\psi' + 2\psi''^2 - 3\psi^2 = 0, \quad \psi(0) = 0, \psi'(0) = 1, \psi''(0) = 0. \tag{3}$$

We set $\Psi_0(r) = \psi_0(\log r)$, $\Psi_1(r) = \psi_0'(\log r)$, $\Psi_2(r) = \psi_0''(\log r)$.

We prove the following theorems.

Theorem 1.1 *The Hamiltonian*

$$H = \frac{1}{\Psi_1^2(r)} \left(d\varphi^2 + \frac{dr^2}{r^2} \right) - \Psi_1^2(r)(\Psi_2(r) - \Psi_0(r)) \cos\varphi \tag{4}$$

defines a conservative system on S^2 in polar coordinates. This system possesses an integral of fourth degree and does not possess an integral quadratic or linear in momenta.

The Hamiltonian (4) are not equivalent to (1) or (2).

Theorem 1.2 *There are some constants $p_2 < p_1$ such that for any $p > p_1$ or $p < p_2$ the Hamiltonian H_p or $-H_p$ correspondingly, where*

$$H_p = \frac{\Psi_1^2(r) - \Psi_0^2(r) + p}{\Psi_1^2(r)} \left(d\varphi^2 + \frac{dr^2}{r^2} \right) - \frac{\Psi_1^2(r)(\Psi_2(r) - \Psi_0(r)) \cos\varphi}{\Psi_1^2(r) - \Psi_0^2(r) + p}, \tag{5}$$

defines a conservative system on S^2. Any system from these families possesses an integral of fourth degree in momenta and does not possess an integral quadratic or linear in momenta.

The Hamiltonians (5) for $p > p_1$ or $p < p_2$ are not equivalent to (1) or (2).

In [11] we proposed a new *one-parameter* family of conservative systems on S^2 possessing an integral *cubic* in momenta.

The paper is organized as follows. In the first chapter we consider the initial value problem (3), show the existence of the solution ψ_0 on $\mathbf{R}$ and investigate its asymptotic behaviour at infinity. In the second chapter we obtain a local criterion for the integrability of a geodesic flow with an integral of fourth degree and find a class of metrics with integrable geodesic flows. Finally, in the third chapter we prove Theorem 1.1 and Theorem 1.2.

2 Existence and asymptotic behaviour of the solution

In this chapter we consider the initial value problem (3), prove the existence of the solution ψ_0 on $\mathbf{R}$ and investigate its asymptotic behaviour at infinity.

Theorem 2.1 *The solution $\psi_0(y)$ of the initial value problem (3) exists for all $y \in \mathbf{R}$ and, moreover, there are C^∞ functions ν and μ such that*

$$\psi_0'(y) = (\exp y)\nu(\exp(-2y)) = \exp(-y)\nu(\exp(2y)),$$

$$\psi_0'^2(y)(\psi_0''(y) - \psi_0(y)) = \exp(-y)\mu(\exp(-2y)) = -(\exp y)\mu(\exp(2y))$$

and $\nu(t) > 0$ for any t.

Proof. The initial value problem (3) has a unique solution $\psi(y) = \psi_0(y)$ which is positive on $(0, \varepsilon)$ and negative on $(-\varepsilon, 0)$ for sufficiently small ε.

Let us consider the case $y > 0$.

Let $R(y) = \ln \psi(y)$ and $q(y) = R'(y)$. Then the differential equation from (3) is equivalent to the following system of differential equations of the first order:

$$\dot{q} = p, \quad \dot{p} = \frac{1}{q}\left(-3q^4 - 7q^2p - 2p^2 + 3\right). \tag{6}$$

The system (6) has two singular points: the nodes $p = 0, q = \pm 1$.

Since the system (6) is symmetric with respect to $q \mapsto -q$, $y \mapsto -y$, it suffices to consider the case $q > 0$.

Show that orbit Γ of (6), related to $\psi_0(y)$ for $y \geq 0$, converges to the point $q = 1, p = 0$ as $y \to +\infty$.

For $\psi_0(y)$ we have $R(y) \to -\infty$ as $y \to 0+$ and then $q(y) = R'(y) \to +\infty$ as $y \to 0+$.

Introduce the function g such that $\Gamma = \{(q,p) | p = -q^2 + qg(q^{-1})\}$ and denote $s = q^{-1}$.

So, we may rewrite (3) as the following initial value problem

$$g'(1 - gs) + 3g^2 - 3s^2 = 0, \quad g(0) = \lim_{q \to +\infty} \frac{p + q^2}{q} = \frac{\psi_0''(0)}{\psi_0'(0)} = 0 \tag{7}$$

The solution of (7) is $g(s) = s^3$. So,

$$\Gamma = \{(q,p) | p = -q^2 + \frac{1}{q^2}\} \tag{8}$$

Since $p(1) = 0$ in (8), the orbit Γ converges to the point $q = 1, p = 0$.

The Jacobian of (6) at $(q,p) = (1,0)$ has the eigenvalues -3, -4. Therefore (6) is C^∞-conjugate to their linear system in a neighborhood of $q = 1, p = 0$, see [8]. Taking into account (8), we conclude that there exists C^∞-functions ξ_1, ξ_2 such that the solution of (6), related to $\psi_0(y)$, $y > 0$, can be written as

$$q = 1 + \xi_1(\exp(-4y)),$$

$$p = \xi_2(\exp(-4y))$$

where $\xi_1(0) = \xi_2(0) = 0$.

So, we get

$$\psi_0(y) = (\exp y)\beta(\exp(-2y))$$

where

$$\beta(u) = \exp(-\frac{1}{2}\int^u \frac{\xi_1(t^2)}{t}dt).$$

Thus, β is of class C^∞.

Since

$$\psi_0'(y) = \psi_0(y)q = (\exp y)\beta(\exp(-2y))(1+\xi_1(\exp(-4y)) = (\exp y)\nu(\exp(-2y)),$$

ν is of class C^∞ and $\nu(t) > 0$ for any $t \geq 0$.

Then we compute

$$\psi_0'^2(y)(\psi_0''(y) - \psi_0(y)) = \psi_0'^2(y)\psi_0(y)(p + q^2 - 1) =$$

$$\exp(-y)\nu^2(\exp(-2y))\beta(\exp(-2y))\frac{\xi_2(\exp(-4y)) + 2\xi_1(\exp(-4y)) + \xi_1^2(\exp(-4y))}{\exp(-4y)}$$

$$= \exp(-y)\mu(\exp(-2y))$$

and, therefore, μ is of class C^∞.

Since $\psi_0(y) = -\psi_0(-y)$, we get

$$\psi_0'(y) = (\exp y)\nu(\exp(-2y)) = \exp(-y)\nu(\exp(2y))$$

and

$$\psi_0'^2(y)(\psi_0''(y) - \psi_0(y)) = \exp(-y)\mu(\exp(-2y)) = -(\exp y)\mu(\exp(2y)).$$

□

3 A criterion for the integrability

Consider a metric $ds^2 = \Theta(u,v)(du^2 + dv^2)$ in conformal coordinates u, v. It can also be written as

$$ds^2 = \theta(w,\bar{w})dwd\bar{w} \tag{9}$$

where $w = u+iv$. The geodesic flow of ds^2 is a Hamiltonian system with Hamiltonian

$$H = \frac{p_w p_{\bar{w}}}{4\theta(w,\bar{w})}. \tag{10}$$

A polynomial F in momenta p_u, p_v can be also written as

$$F = \sum_{k=0}^{n} b_k(w,\bar{w})p_w^k p_{\bar{w}}^{n-k}$$

where

$$b_k = \overline{b_{n-k}}, \quad k = 0, ..., n.$$

If the polynomial F is an additional integral of the geodesic flow with the Hamiltonian (10), then $\{F, H\} = 0$ and the following holds

$$\theta \frac{\partial b_{k-1}}{\partial w} + (n - (k-1)) b_{k-1} \frac{\partial \theta}{\partial w} + \theta \frac{\partial b_k}{\partial \bar{w}} + k b_k \frac{\partial \theta}{\partial \bar{w}} = 0, \tag{11}$$

where $k = 0, ..., n+1$ and $b_{-1} = b_{n+1} = 0$. Substituting $k = 0$ and $k = n+1$ in (11) we get immediately

$$\frac{\partial}{\partial \bar{w}} b_0 \equiv 0$$

and

$$\frac{\partial}{\partial w} b_{n+1} \equiv 0.$$

Note that in another conform coordinate system

$$\hat{b}_0(\hat{w}) = b_0(w) \left(\frac{d\hat{w}}{dw} \right)^n. \tag{12}$$

Proposition 3.1 *The geodesic flow of a Riemannian metric on* S^2

$$ds^2 = \lambda(r^2)(r^2 d\varphi^2 + dr^2) \tag{13}$$

does not possess a nontrivial integral quadratic in momenta (which does not depend on H *and linear integrals).*

Proof. Let us introduce the conform coordinate system $u = r\cos\varphi, v = r\sin\varphi$. Denote $w = u + iv$ and consider $\hat{w} = w^{-1}$ for ds^2.

Let $F = b_0(w, \bar{w}) p_w^2 + b_1(w, \bar{w}) p_w p_{\bar{w}} + b_2(w, \bar{w}) p_{\bar{w}}^2$ be an integral of the geodesic of (13). As above, $b_0 = b_0(w)$, $b_2 = b_2(\bar{w}) = \bar{b}_0(w)$ and $\mathrm{Im}\, b_1(w, \bar{w}) = 0$. Since (13) is a Riemannian metric on S^2, from (12) it follows that $b_0(w)$ is a polynomial of degree n where $n \leq 4$. If F does not depend on the Hamiltonian and linear integrals, then $b_0(w)$ does not equal zero identically.

Then from (11) when $k = 1$ we obtain

$$\lambda \frac{\partial b_0}{\partial w} + 2 b_0 \frac{\partial \lambda}{\partial w} + \lambda \frac{\partial b_1}{\partial \bar{w}} + b_1 \frac{\partial \lambda}{\partial \bar{w}} = 0.$$

This is equivalent to the following condition

$$\mathrm{Im}\, \frac{\partial}{\partial w} \left(\lambda \frac{\partial b_0}{\partial w} + 2 b_0 \frac{\partial \lambda}{\partial w} \right) = 0$$

where $b_0(w) = B_0 w^4 + B_1 w^3 + B_2 w^2 + B_3 w + B_4$ and B_0, B_1, B_2, B_3, B_4 are some constants such that at least one of them does not equal zero.

Denote $w\bar{w}$ as t. So, we get the following conditions for $\lambda(t)$

$$\mathrm{Im}\, (w^2(12B_0\lambda + 12B_0\lambda' t + 2B_0\lambda'' t^2) + \bar{w}^2(2B_4\lambda''))+$$

$$\mathrm{Im}\, (w(6B_1\lambda + 9B_1\lambda' t + 2B_1\lambda'' t^2) + \bar{w}(3B_3\lambda' + 2B_3\lambda'' t))+$$

$$\mathrm{Im}\ (B_2(2\lambda + 6\lambda' t + 2\lambda'' t^2)) \equiv 0.$$

So, we have to consider the following linear differential equations

$$\epsilon_1(6\lambda + 6\lambda' t + \lambda'' t^2) = \epsilon_2(\lambda''), \tag{14}$$

$$\epsilon_1(6\lambda + 9\lambda' t + 2\lambda'' t^2) = \epsilon_2(3\lambda' + \lambda'' t), \tag{15}$$

$$\lambda + 3\lambda' t + 2\lambda'' t^2 = 0 \tag{16}$$

where ϵ_1, ϵ_2 are constants equal to -1, 0 or 1. (In another case consider the rescaling $t \mapsto Dt$, $D - const$, of a solution of (14) or (15).)

It is easy to see that for any smooth solution of (16) $\lambda(0) = 0$ and then ds^2 is not a Riemannian metric on S^2. We get $\mathrm{Im}\ B_2 = 0$.

It is also easy to see that any solution of the equations (14), (15) if $\epsilon_1 = 0$ or $\epsilon_2 = 0$ does not define by (13) a Riemannian metric on S^2.

Solutions of (14) if $\epsilon_1 = 1$ and $\epsilon_2 = 1$ have the following form

$$\lambda(t) = C_1 \frac{1+t^2}{(-1+t^2)^2} + C_2 \frac{t}{(-1+t^2)^2}, C_1, C_2 - const.$$

Solutions of (14) if $\epsilon_1 = 1$ and $\epsilon_2 = -1$ have the following form

$$\lambda(t) = C_1 \frac{1-t^2}{(1+t^2)^2} + C_2 \frac{t}{(1+t^2)^2}, C_1, C_2 - const.$$

We get it noting that the solutions of (14) if $\epsilon_1 = 1$ and $\epsilon_2 = 1$ can be obtained from the solutions of (14) if $\epsilon_1 = 1$ and $\epsilon_2 = -1$ by complex transformation $t \to it$.

It is easy to note that if $\lambda(0) = 1$ there is only one smooth solution of (15) if $\epsilon_1 = 1$ or $\epsilon_2 = \pm 1$. These solutions correspond to the metrics of curvature -1 if $\epsilon_2 = 1$ and to the metrics of curvature -1 if $\epsilon_2 = -1$. So, if $\epsilon_1 = 1$ and $\epsilon_2 = 1$ any solution of (15) which is smooth in zero has the following form

$$\lambda(t) = \frac{C_1}{(-1+t)^2}, C_1 - const$$

and if $\epsilon_1 = 1$ and $\epsilon_2 = -1$ any solution of (15) which is smooth in zero has the following form

$$\lambda(t) = \frac{C_1}{(1+t)^2}, C_1 - const.$$

Thus, we may see that either $\mathrm{Im}\ B_2 = 0$ and $B_i = 0, i \neq 2$ or

$$\lambda(t) = \frac{C_1}{(1+Dt)^2}, C_1, D - const$$

and, therefore,

$$ds^2 = \frac{C_1}{(1+Dr^2)^2}\left(r^2 d\varphi^2 + dr^2\right), \tag{17}$$

where $C_1, D - const$, i.e. ds^2 is a metric of constant positive curvature.

Remember that the geodesic flow of a metric (13) possesses the linear integral $F_1 = iwp_w - i\bar{w}p_{\bar{w}}$. If Im $B_2 = 0$ and $B_i = 0, i \neq 2$, we will consider the integral $F_2 = F + B_2 F_1^2$ which is equal, obviously, to $\hat{C}H$ where $\hat{C}$ is a constant. Therefore, F depends on H and F_1.

So, if the geodesic flow of a metric (13) on S^2 possesses a nontrivial quadratic integral, it can be only a metric of constant positive curvature. But it is well known that the geodesic flows of metrics of constant curvature possess two independent linear integrals. (Indeed, the geodesic flows with the Hamiltonians $H = l'^2(y)(p_\varphi^2 + p_y^2)$ where $l''(y) = l(y)$ possess the integrals $F_1 = p_\varphi$ and $\hat{F}_1 = l(y)\cos\varphi p_\varphi - l'(y)\sin\varphi p_y$.) So, in this case any quadratic integral is also trivial, i.e. depends on the Hamiltonian and linear integrals. □

Proposition 3.2 *The geodesic flow of a metric (13) on S^2 possesses two independent linear integrals if and only if it has the form (17) in polar coordinates, i.e. if it is a metric of constant positive curvature.*

Proof. As mentioned above the geodesic flow of a metric (13) on S^2 possesses the linear integral $F_1 = p_\varphi$.

In the coordinates $w, \bar{w}, p_w, p_{\bar{w}}$ where $w = r\cos\varphi + ir\sin\varphi$ it can be written as follows $F_1 = iwp_w - i\bar{w}p_{\bar{w}}$. (Indeed, in the coordinates $z = \varphi + iy$, $y = \log r$ we have $F_1 = p_z + p_{\bar{z}}$ and, clearly, $w = \exp iz$.)

Assume that a metric ds^2 of the form (13) on S^2 possesses another linear integral $F_2 = d_0(w)p_w + \bar{d}_0(w)p_{\bar{w}}$ which is independent of F_1 and, therefore, the function $d_0(w)$ does not equal iwC_3 where C_3 is a real constant.

Let us consider then the function $F_2^2 = d_0^2(w)P_w + 2|d_0|^2 p_w p_{\bar{w}} + \bar{d}_0(w)p_{\bar{w}}^2$ which is also an integral of the geodesic flow of ds^2.

Compare now $d_0^2(w)$ with $b_0(w)$ from Proposition 3.1. Since $d_0(w) \neq iwC_3$ in our case, then at least one of the coefficients B_i, $i \neq 2$ (see Proposition 3.1) does not equal zero. In the same way as in Proposition 3.1 we may show that ds^2 has the form (17) in polar coordinates, i.e. ds^2 is a metric of constant positive curvature.

□

Remember that conform coordinates x, y of a metric ds^2 are Liouville if $ds^2 = (f(x) + h(y))(dx^2 + dy^2)$ for some functions f, h. Due to Darboux [6] Liouville coordinates exist if and only if the geodesic flow of ds^2 possesses an integral quadratic in momenta and moreover if one of the function f or h is constant then the geodesic flow possesses an integral linear in momenta. So, we may formulate the following

Corollary 3.3 *Liouville coordinates $\varphi, y = \log r$ related to polar coordinates φ, r of a metric (13) on S^2 are unique up to shifts and the transform $y \to -y$.*

Proof. Assume that a metric ds^2 on S^2 can be written in two different forms (13)

$$ds^2 = \lambda_1(r^2)(r^2 d\varphi^2 + dr^2)$$

and

$$ds^2 = \lambda_2(\tilde{r}^2)(\tilde{r}^2 d\tilde{\varphi}^2 + d\tilde{r}^2)$$

where $\tilde{r} \neq Dr^{\pm 1}$, $D - const$. This means that the geodesic flow of ds^2 possesses two independent linear integrals. So, from Proposition 3.2 it follows that ds^2 has the form (17), and therefore, $\tilde{r} = Dr^{\pm 1}$, $D - const$. Thus, the Liouville coordinates $\varphi, y = \log r$ related to polar coordinates φ, r of ds^2 on S^2 are unique up to shifts and the transform $y \to -y$.

$\square$

We must note that Proposition 3.1 follows also from the results of Kolokol'tsov published in his Ph.D. thesis (in Russian) but we give here a complete another proof.

One may show that if a polynomial in momenta integral F of the geodesic flow of (9) is independent of the Hamiltonian and an integral of smaller degree, then there is a conformal coordinate system $z = z(w)$ of this metric such that the coefficients of F for p_z and $p_{\bar{z}}$ are equal to 1 identically.

Theorem 3.4 *Let $ds^2 = \lambda(z, \bar{z}) dz d\bar{z}$ be a metric such that there exists a function $f : \mathbf{R}^2 \mapsto \mathbf{R}$, satisfying the following conditions*

$$\lambda(z, \bar{z}) = \frac{\partial^2 f}{\partial z \partial \bar{z}}, \quad Im \left(\frac{\partial^4 f}{\partial z^4} \frac{\partial^2 f}{\partial z \partial \bar{z}} + 3 \frac{\partial^3 f}{\partial z^3} \frac{\partial^3 f}{\partial z^2 \partial \bar{z}} + 2 \frac{\partial^2 f}{\partial z^2} \frac{\partial^4 f}{\partial z^3 \partial \bar{z}} \right) = 0. \qquad (18)$$

Then the geodesic flow of ds^2 possesses an integral of fourth degree in momenta.

If the geodesic flow of a metric ds^2 possesses an integral which is a polynomial of fourth degree in momenta and it does not depend on the Hamiltonian and an integral of smaller degree then there exist conformal coordinates x, y and a function $f : \mathbf{R}^2 \mapsto \mathbf{R}$ such that $ds^2 = \lambda(z, \bar{z}) dz d\bar{z}$ where $z = x + iy$ and (18) holds.

Proof. Suppose that (18) holds. We will now construct an integral

$$F = \sum_{k=0}^{n} a_k(z, \bar{z}) p_z^k p_{\bar{z}}^{n-k}$$

of the geodesic flow of ds^2 where $a_0 = a_n \equiv 1$.

The system (11) has then the following form

$$4 \frac{\partial \lambda}{\partial z} = - \frac{\partial (a_1 \lambda)}{\partial \bar{z}}, \qquad (19)$$

$$\frac{\partial (a_1 \lambda^3)}{\partial z} = -\lambda \frac{\partial (a_2 \lambda^2)}{\partial \bar{z}}, \qquad (20)$$

$$a_2 = \overline{a_2}. \qquad (21)$$

Consider

$$a_1 = -4 \frac{\partial^2 f}{\partial z^2} \left(\frac{\partial^2 f}{\partial z \partial \bar{z}} \right)^{-1}.$$

Then (19) holds.

Show that for

$$g = \frac{\partial (a_1 \lambda^3)}{\partial z} \lambda^{-1}$$

it holds

$$\mathrm{Im}\ \frac{\partial g}{\partial z} = 0.$$

Indeed, by computation we obtain

$$\mathrm{Im}\ \frac{\partial g}{\partial z} = -4\ \mathrm{Im}\left(\frac{\partial^4 f}{\partial z^4}\frac{\partial^2 f}{\partial z \partial \bar{z}} + 3\frac{\partial^3 f}{\partial z^3}\frac{\partial^3 f}{\partial z^2 \partial \bar{z}} + 2\frac{\partial^2 f}{\partial z^2}\frac{\partial^4 f}{\partial z^3 \partial \bar{z}}\right) = 0.$$

So, it follows that there is a function $h : \mathbf{R}^2 \mapsto \mathbf{R}$ such that

$$\frac{\partial h}{\partial \bar{z}} = g.$$

Consider $a_2 = -\frac{h}{\lambda^2}$. Then $a_2 : \mathbf{R}^2 \mapsto \mathbf{R}$ and, moreover,

$$\frac{\partial (a_1 \lambda^3)}{\partial z}\lambda^{-1} = -\frac{\partial (a_2 \lambda^2)}{\partial \bar{z}},$$

i.e. conditions (20) and (21) hold also.

Put then $a_3 = \bar{a}_1$. So, we have an integral of fourth degree in momenta of the geodesic flow of ds^2.

Prove now the second statement of the theorem. As mentioned above, in this case there exists a conformal coordinate system of ds^2 where $a_0 = a_4 \equiv 1$. Note that this system is unique up to a shift. So, in this coordinate system (19)-(21) hold. From (19) it follows that there exists a function V where

$$\frac{\partial V}{\partial \bar{z}} = \lambda$$

and

$$a_1 \lambda = -4\frac{\partial V}{\partial z}.$$

From (20) and (21) it follows immediately that

$$\mathrm{Im}\ \frac{\partial}{\partial z}\left(\frac{\partial (a_1 \lambda^3)}{\partial z}\lambda^{-1}\right) = 0.$$

On the other hand

$$\mathrm{Im}\ \frac{\partial}{\partial z}\left(\frac{\partial (a_1 \lambda^3)}{\partial z}\lambda^{-1}\right) = -4\ \mathrm{Im}\ \frac{\partial}{\partial z}\left(\frac{\partial^2 V}{\partial z^2}\frac{\partial V}{\partial \bar{z}} + 2\frac{\partial^2 V}{\partial z \partial \bar{z}}\frac{\partial V}{\partial z}\right) = 0.$$

Taking into account that $\mathrm{Im}\ \frac{\partial V}{\partial \bar{z}} = \mathrm{Im}\ \lambda = 0$, we conclude that there exists a function $f : \mathbf{R}^2 \mapsto \mathbf{R}$ such that

$$\frac{\partial f}{\partial z} = V.$$

Thus,

$$\lambda = \frac{\partial^2 f}{\partial z \partial \bar{z}}$$

and

$$0 = \mathrm{Im}\ \frac{\partial}{\partial z}\left(\frac{\partial^3 f}{\partial z^3}\frac{\partial^2 f}{\partial z\partial\bar z} + 2\frac{\partial^3 f}{\partial z^2\partial\bar z}\frac{\partial^2 f}{\partial z^2}\right) =$$

$$= \mathrm{Im}\ \left(\frac{\partial^4 f}{\partial z^4}\frac{\partial^2 f}{\partial z\partial\bar z} + 3\frac{\partial^3 f}{\partial z^3}\frac{\partial^3 f}{\partial z^2\partial\bar z} + 2\frac{\partial^2 f}{\partial z^2}\frac{\partial^4 f}{\partial z^3\partial\bar z}\right).$$

□

Equation (18) in some other form has been obtained in [7] but in order to show when the corresponding integral is nontrivial we needed to give here a complete proof.

Using this criterion we may find families of metrics with integrable geodesic flows.

We will look for a class of solutions of (18) and show that in this class of solutions the problem of integrability can be reduced to an ordinary differential equation.

Denote

$$\frac{\partial^2 f}{\partial z^2} = (A(x,y) + iB(x,y)).$$

By computation we obtain

$$\mathrm{Im}\ \frac{\partial^4 f}{\partial z^4} = \frac{1}{4}\left(A_{xx} - A_{yy} - 2A_{xy}\right),$$

$$\mathrm{Im}\ \frac{\partial^3 f}{\partial z^3}\frac{\partial^3 f}{\partial z^2\partial\bar z} = \frac{1}{2}\left(A_x A_x + A_y A_y\right),$$

$$\mathrm{Im}\ \frac{\partial^2 f}{\partial z^2}\frac{\partial^4 f}{\partial z^3\partial\bar z} = \frac{1}{4}\left(B(A_{xx} + A_{yy}) + A(A_{xx} + A_{yy})\right).$$

Thus, (18) can be written as follows

$$6(A_yA_y + A_xA_x) + 2(B(A_{xx} + A_{yy}) + A(A_{xx} + A_{yy})) + (A_{xx} - A_{yy} - 2A_{xy})\lambda = 0. \quad (22)$$

Consider now the solutions of (18) of the following form:

$$f(x,y) = \psi(y)\cos x + \xi(y) + a(x^2 - y^2) \quad (23)$$

where ψ, ξ are some smooth functions and a is a constant.

Then we obtain

$$\lambda = \frac{1}{4}\left((\psi''(y) - \psi(y))\cos x + \xi''(y)\right)$$

and

$$A = -\frac{1}{4}\left((\psi''(y) + \psi(y))\cos x + \xi''(y) - 4a\right), B = \frac{1}{2}\left(\psi'(y)\sin x\right). \quad (24)$$

Substituting (24) in (22) we get the following condition of integrability

$$6(((\psi''' + \psi')\cos x + \xi''')2\psi''\sin x - (\psi'' + \psi)2\psi'\sin x\cos x +$$

$$2(2\psi'\sin x((\psi^{(4)} - \psi)\cos x + \xi^{(4)}) + ((\psi'' + \psi)\cos x + \xi'' - 4a)2(\psi''' - \psi')\sin x)) +$$

$$4(\psi''' + \psi')\sin x((\psi'' - \psi)\cos x + \xi'') \equiv 0.$$

Therefore,

$$3\psi''\xi''' + \psi'\xi^{(4)} + 2\psi'''\xi'' = 4a(\psi''' - \psi'), \tag{25}$$

$$5\psi'''\psi'' - 6\psi'\psi + \psi'\psi^{(4)} = 0. \tag{26}$$

Equation (25) can be written as

$$(\psi'\xi''')' + 2(\psi''\xi'')' = 4a(\psi'' - \psi)'.$$

By integrating we get

$$\psi'\xi''' + 2\psi''\xi'' = 4a(\psi'' - \psi) + a_1$$

where a_1 is a constant. So, if $a = 0$

$$\xi'' = \frac{a_1\psi(y) + c}{(\psi'(y))^2}. \tag{27}$$

or if $a \neq 0$

$$\xi'' = 2a\frac{\psi'^2(y) - \psi^2(y) + a_1(2d)^{-1}\psi(y) + p}{\psi'(y))^2}. \tag{28}$$

where p, c are some constants.

On the other hand there is an integral of (26):

$$2\psi''^2 - 3\psi^2 + \psi'\psi''' = const. \tag{29}$$

So, we may now formulate the following proposition.

Proposition 3.5 *For any solution ψ of (29) there are two families of metrics $ds_1^2 = \Lambda(x, y)(dx^2 + dy^2)$ where*

$$\Lambda(x, y) = (\psi''(y) - \psi(y))\cos x + \frac{a_1\psi(y) + c}{\psi'^2(y)} \tag{30}$$

or

$$\Lambda(x, y) = (\psi''(y) - \psi(y))\cos x + c\frac{\psi'^2(y) - \psi^2(y) + a_1\psi(y) + p}{\psi'^2(y)} \tag{31}$$

where c, a_1, p are constants and the corresponding geodesic flows possess an integral of fourth degree in momenta. If $\frac{\partial^2\Lambda}{\partial x\partial y}$ and $\frac{\partial^2\Lambda}{\partial x^2} - \frac{\partial^2\Lambda}{\partial y^2}$ are not equal to zero identically, then this integral is nontrivial, i.e. it does not depend on the Hamiltonian and an integral of smaller degree.

Proof. As we have shown, for every metric in one of these families there exists a function f of the form (23) satisfying (18). In fact, compare (30) with (27) and (31) with (28) (in the last case we denote $4d$ in (28) as c).

Since every metric in one of these families satisfies the conditions of theorem 3.4, the integrability follows.

So, we have only to prove that the integrals of the geodesic flows of these geodesic flows are nontrivial.

According to our construction the integral F of the geodesic flow with Hamiltonian

$$H = \frac{p_z p_{\bar{z}}}{4\Lambda}$$

where Λ is given by (30), (31) has the following form

$$F = p_z^4 + D(x,y)p_z^3 p_{\bar{z}} + E(x,y)p_z^2 p_{\bar{z}}^2 + \bar{D}(x,y)p_z p_{\bar{z}}^3 + p_{\bar{z}}^4$$

where $D(x,y) : \mathbf{R^2} \mapsto \mathbf{C}$, $E(x,y) : \mathbf{R^2} \mapsto \mathbf{R}$.

Thus, if F depends on H and another integral polynomial of smaller degree, then there is an integral $\tilde{F}$ quadratic in momenta which has the following form

$$\tilde{F} = p_z^2 + \tilde{E}_1(x,y)p_z^2 p_{\bar{z}}^2 + p_{\bar{z}}^2$$

or

$$\tilde{F} = ip_z^2 + \tilde{E}_2(x,y)p_z^2 p_{\bar{z}}^2 - ip_{\bar{z}}^2$$

where $\tilde{E}_1(x,y), \tilde{E}_2(x,y) : \mathbf{R^2} \mapsto \mathbf{R}$.

Therefore, due to Darboux [6] the following holds

$$\frac{\partial^2 \Lambda}{\partial x \partial y} \equiv 0$$

or

$$\frac{\partial^2 \Lambda}{\partial x^2} - \frac{\partial^2 \Lambda}{\partial y^2} \equiv 0.$$

$\square$

4 Smooth metrics on S^2

In this chapter we will prove Theorem 1.1 and Theorem 1.2.

Proof of Theorem 1.1. Prove that the Hamiltonian system with the Hamiltonian (4) is a smooth system on S^2. Using theorem 2.1, we may write

$$H = \frac{1}{\nu^2(r^2)}(r^2 d\varphi^2 + dr^2) + \mu(r^2) r\cos\varphi =$$

$$\frac{1}{\nu^2(\tilde{r}^2)}(\tilde{r}^2 d\tilde{\varphi}^2 + d\tilde{r}^2) - \mu(\tilde{r}^2)\tilde{r}\cos\tilde{\varphi}$$

where $\tilde{r} = \frac{1}{r}$, $\tilde{\varphi} = -\varphi$. Since ν, μ are of class C^∞ and $\nu \neq 0$, see Theorem 2.1, then this system is a smooth system on S^2.

Prove that (4) possesses an integral of fourth degree in momenta. Consider the metrics (30) from Proposition 3.5 and put in (30) $\psi(y) = \psi_0(y)$ and $d_1 = 0$. Then we obtain the following one-parameter family of metrics:

$$\left((\psi_0''(y) - \psi_0(y))\cos x + c\frac{1}{\psi_0'^2(y)} \right)(dx^2 + dy^2)$$

or in polar coordinates $\varphi = x$, $r = \exp y$:

$$\left((\Psi_2(r) - \Psi_0(r)) \cos\varphi + c\frac{1}{\Psi_1^2(r)} \right) (d\varphi^2 + \frac{1}{r^2} dr^2)$$

where c is an arbitrary constant. From Proposition 3.5 we know that any metric in this family possesses an integral of fourth degree in momenta. Using the well-known Maupertuis's principle, see [1], we conclude that (4) possesses an integral of fourth degree in momenta, see also [3], [4].

So, we have to prove only that this integral nontrivial, i.e. depends on the Hamiltonian and an integral of smaller degree. Denote the kinetik energy in (4) as $K = \lambda(y)(d\varphi^2 + dy^2)$ and the potential as V. We will denote the Hamiltonian of the geodesic flow of the metric K as K also.

Let us assume that a system of this family has an integral which is independent of the Hamiltonian (total energy (4)) and which is a polynomial of second degree in momenta (clearly, this assumption includes the case of linear integrals). So, there is an integral $\tilde{F}$ of (4) which is quadratic in momenta p_φ, p_y. Thus, $\tilde{F} = A(p_\varphi, p_y, \varphi, y) + B(\varphi, y)$ where $A(p_\varphi, p_y, \varphi, y)$ is a polynomial of second degree in momenta. We may write $\{\tilde{F}, H\} = \{A(p_\varphi, p_y, \varphi, y) + B(\varphi, y), K + V\} \equiv 0$ and, therefore, $\{A(p_\varphi, p_y, \varphi, y), K\} \equiv 0$.

Thus, the geodesic flow of K has an integral which is a polynomial of second degree in momenta.

Since a function ψ such that $\psi'(y) = c_1 \exp(-y)(1 + \exp(2y)), c_1 - const$ does not satisfy (3) and, therefore, $\Psi_1(r) \neq r^{-1}(1 + Dr^2)$, D is a positive constant, then K is not a metric of constant positive curvature. So, the quadratic integral A of the geodesic flow of K depends on the Hamiltonian K and the integral p_φ. Since F does not depend on H, w.l.o.g. we may put $A = p_\varphi^2$.

So, we may write

$$\{A, V\} + \{B, K\} = \{p_\varphi^2, V\} + \{B, K\} \equiv 0.$$

By computation we obtain

$$\frac{\partial B}{\partial y} \equiv 0,$$

and

$$\frac{\partial B}{\partial \varphi} = \frac{\partial V}{\partial \varphi} \lambda(y).$$

So, we get

$$V = \frac{B(\varphi) + \alpha(y)}{\lambda(y)}$$

for a smooth function $\alpha(y)$.

In our case

$$\lambda(y) = \frac{1}{\psi_0'^2(y)}$$

and

$$V(x, y) = (\psi_0''(y) - \psi_0(y))\psi_0'^2(y) \cos\varphi$$

and we get then $\psi_0''(y) - \psi_0(y) \equiv const$, that is not true. So, there is no quadratic integral of the system given by (4).

In oder to prove that the system given by (4) is really a new example of the integrability by a polynomial of fourth degree in momenta we have to compare this case with the known cases of Kovalevskaya and Goryachev. First of all, we note that from Corollary 3.3 it follows that in the family (2) we have to consider only the case $B_1 = B_2 = 0$. As mentioned above, for $B_1 = B_2 = 0$ the system given by (2) is simply the case of Kovalevskaya. So, we may consider only the case of Kovalevskaya. Let us write the Hamiltonian (1) of the case of Kovalevskaya in polar coordinates. We obtain

$$H = \gamma_1(r^2)(r^2 d\varphi^2 + dr^2) - \gamma_2(r^2)\cos\varphi \tag{32}$$

where $\gamma_2(r^2) \neq 0$ for all $0 < r < +\infty$.

From Corollary 3.3 we have that if the Hamiltonians (32) (or (1)) and (4) are equivalent then the potentials U in (32) and in (4) are proportional, see the Introduction. Then comparing the potential in (32) with the potential in (4) we see that they cannot be proportional, because $\gamma_2(r) > 0$, $r \in (0, +\infty)$ in (32) but $\Psi_2(1) - \Psi_0(1) = 0$.

□

Proof of Theorem 1.2. Using theorem 2.1, we may rewrite H_p in (5) as

$$H_p = \frac{-\Phi(r^2) + p + 1}{\nu^2(r^2)}(r^2 d\varphi^2 + dr^2) + \frac{\mu(r^2)}{-\Phi(r^2) + p + 1} r\cos\varphi =$$

$$\frac{-\Phi(\tilde{r}^2) + p + 1}{\nu^2(\tilde{r}^2)}(\tilde{r}^2 d\tilde{\varphi}^2 + d\tilde{r}^2) - \frac{\mu(\tilde{r}^2)}{-\Phi(\tilde{r}^2) + p + 1}\tilde{r}\cos\tilde{\varphi}$$

where $\tilde{r} = \frac{1}{r}$, $\tilde{\varphi} = -\varphi$ and

$$\Phi(t) = \int_1^t \mu(s)\nu^{-1}(s)ds.$$

Since ν, μ are of class C^∞ and $\nu \neq 0$, see Theorem 2.1, there are the constants

$$M_1 = \max_{[0,1]} \Phi(t) \quad \text{and} \quad M_2 = \min_{[0,1]} \Phi(t),$$

where $M_2 < M_1$. Thus, for $p > p_1 = M_1 - 1$ or $p < p_2 = M_2 - 1$ a system with the Hamiltonian H_p or $-H_p$ correspondingly, where H_p has the form (5), is a conservative system on S^2.

Prove that the system with energy (5) possesses an integral of fourth degree in momenta. Consider the metrics (31) from Proposition 3.5 and put in (31) $\psi(y) = \psi_0(y)$ and $a_1 = 0$. We get the following two-parameter family of metrics:

$$\left((\psi_0''(y) - \psi_0(y))\cos x + c\frac{\psi_0'^2(y) - \psi_0^2(y) + p}{\psi_0'^2(y)}\right)(dx^2 + dy^2)$$

which can be written also in polar coordinates $\varphi = x$, $r = \exp y$ as

$$\left((\Psi_2(r) - \Psi_0(r))\cos\varphi + c\frac{\Psi_1^2(r) - \Psi_0^2(r) + p}{\Psi_1^2(r)}\right)(d\varphi^2 + \frac{1}{r^2}dr^2)$$

where c is an arbitrary constant. It was shown in Proposition 3.5 that any metric in this family possesses an integral of fourth degree in momenta. From the well-known Maupertuis's principle, see [1], it follows that a system with the Hamiltonian (5) possesses an integral of fourth degree in momenta.

In order to prove that for any $p > p_0$ the system with the Hamiltonian (5) does not possess an integral quadratic or linear in momenta and in oder to compare these cases with the above cases of Kovalevskaya and Goryachev we can apply the same arguments as for the system with the Hamiltonian (4), see the proof of Theorem 1.1.

We must prove that for different values of parameter p the Hamiltonians (5) are not equivalent.

Let us introduce the coordinates $\varphi, y = \log r$. From Corollary 3.3 it follows that we need to consider only the transformations $y \mapsto \pm y + \kappa_1$ where κ_1 is a constant.

Let us rewrite H_p in the coordinates φ, y:

$$H_p = \frac{\psi_0'^2(y) - \psi_0^2(y) + p}{\psi_0'^2(y)}\left(d\varphi^2 + dy^2\right) -$$

$$(\psi_0''(y) - \psi_0(y))\left(\frac{\psi_0'^2(y) - \psi_0^2(y) + p}{\psi_0'^2(y)}\right)^{-1}\cos\varphi \tag{33}$$

From Corollary 3.3 it follows that if the Hamiltonians (33) for $p = p_1$ and $p = p_2$ are equivalent, then there are some constants $K_1 \neq 0$, $K_2 \neq 0$ and κ_1 such that the following holds

$$\frac{\psi_0'^2(y) - \psi_0^2(y) + p}{\psi_0'^2(y)} =$$

$$K_1\frac{\psi_0'^2(\pm y + \kappa_1) - \psi_0^2(\pm y + \kappa_1) + p}{\psi_0'^2(\pm y + \kappa_1)} \tag{34}$$

and

$$(\psi_0''(y) - \psi_0(y))\left(\frac{\psi_0'^2(y) - \psi_0^2(y) + p}{\psi_0'^2(y)}\right)^{-1} =$$

$$K_2(\psi_0''(\pm y + \kappa_1) - \psi_0(\pm y + \kappa_1))\left(\frac{\psi_0'^2(\pm y + \kappa_1) - \psi_0^2(\pm y + \kappa_1) + p}{\psi_0'^2(\pm y + \kappa_1)}\right)^{-1}. \tag{35}$$

Thus, we get

$$\psi_0''(y) - \psi_0(y) = K_1K_2(\psi_0''(\pm y + \kappa_1) - \psi_0(\pm y + \kappa_1)). \tag{36}$$

Put $y = 0$ in (36) and we get $\psi_0''(\kappa_1) - \psi_0(\kappa_1) = 0$. W.l.o.g. we assume that $\kappa_1 \geq 0$. From the proof of Theorem 2.1 we know that the function $\psi_0''(y) - \psi_0(y)$, $y > 0$ can be written as follows

$$\psi_0''(y) - \psi_0(y) = (\exp R(y))(p(q) + q^2 - q), y > 0$$

where $p(q) = -q^2 + \frac{1}{q^2}$ and $q > 1$.

So, we obtain $\psi_0''(y) - \psi_0(y) = \exp R(y)(q^{-2} - q)$ and, therefore, if $\psi_0''(\kappa_1) - \psi_0(\kappa_1) = 0$ and $\kappa_1 \neq 0$, then $q(\kappa_1) = 1$ but as mentioned above $q(y) > 1$ for finite values of y. Thus, $\kappa_1 = 0$. Then taking into account that $\psi_0(y) = -\psi_0(-y)$ from (34) we get $K_1 = 1$ and $p_1 = p_2$ and then from (35) we get $K_2 = 1$.

Thus, for $p_1 \neq p_2$ the Hamiltonians H_{p_1} and H_{p_2} of the form (5) are not equivalent.

□

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After the paper was finished, K. P. Hadeler informed the author that the differential equation in (3) can be solved by successive integration and the inverse function of the solution $\psi_0(y)$ to the initial value problem (3) can be given as:

$$y = y(\psi_0) = \frac{1}{4} \log \frac{(\psi_0^4 + 1)^{\frac{1}{4}} + \psi_0}{(\psi_0^4 + 1)^{\frac{1}{4}} - \psi_0} - \frac{1}{2} \arctan \frac{(\psi_0^4 + 1)^{\frac{1}{4}}}{\psi_0}.$$

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