On the case of Goryachev-Chaplygin and new examples of integrable conservative systems on S^2

Elena N. Selivanova*

Department of Geometry, Nizhny Novgorod State Pedagogical University
603000 Russia, Nizhny Novgorod, ul. Ulyanova 1

Mathematisches Institut, Universität Tübingen
Auf der Morgenstelle 10, 72076 Tübingen, Germany
e-mail: lena@moebius.mathematik.uni-tuebingen.de
Fax: (07071) 294322

Abstract. The aim of this paper is to describe a class of conservative systems on S^2 possessing an integral cubic in momenta. We prove that this class of systems consists off the case of Goryachev-Chaplygin, the one-parameter family of systems which has been found by the author in the previous paper and a new two-parameter family of conservative systems on S^2 possessing an integral cubic in momenta.

1 Introduction

Let M be an n-dimensional Riemannian manifold, and $U : M \to \mathbf{R}$ be a smooth function on M. For the Lagrangian $L : TM \to \mathbf{R}$ we consider

$$L(\eta) = \frac{|\eta|^2}{2} + (U \circ \pi)(\eta)$$

where $\pi : TM \to M$ is the canonical projection, see [1]. In local coordinates $q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n$ on TM we have

$$L(\eta) = \frac{1}{2} \sum g_{ij} \dot{q}^i \dot{q}^j + U(q).$$

Identifying TM and T^*M by means of the Riemannian metric, we get a Hamiltonian system with the Hamiltonian $H : T^*M \to \mathbf{R}$ which in local coordinates $q_1, \ldots, q_n$,

*Supported by DAAD.

$p_1, \ \dots\ , p_n$ on T^*M has the form

$$H = \frac{1}{2}\sum g^{ij}p_ip_j + U(q) = K + U.$$

We will call these Hamiltonian systems *conservative systems* on M.

A smooth function $F : T^*M \to \mathbf{R}$ which is an integral of the Hamiltonian system with the Hamiltonian H and which is independent of H we will call an *integral* of this system of *degree* m in momenta if in local coordinates F has the form

$$F = \sum_{k_1 + \ \dots\ + k_n \le m} a_{k_1 \ \dots\ k_n}(q)p_1^{k_1} \ \dots\ p_n^{k_n}.$$

We will say that two Hamiltonians $H_1 = K_1 + U_1$ and $H_2 = K_2 + U_2$ are *equivalent* if there exists a diffeomorphism ϕ of M and a diffeomorphism Φ of T^*M such that the diagram

$$\begin{array}{ccc} \Phi : T^*M & \to & T^*M \\ \pi' \downarrow & & \downarrow \pi' \\ \phi : M & \to & M \end{array}$$

is commutative, where Φ is linear for $p \in M$ fixed, and if there are some nonzero constants κ_1, κ_2 such that $\Phi^*(K_1) = \kappa_1 K_2$, $\phi^*(U_1) = \kappa_2 U_2$.

Clearly, if the Hamiltonians H_1, H_2 are equivalent and one of the corresponding systems possesses an integral of degree m in momenta, then the other system has the same property.

In this paper we consider integrable conservative systems on S^2 with the Hamiltonians of the form

$$H = \lambda(r^2)(r^2 d\varphi^2 + dr^2) + \rho(r^2) r\cos\varphi = K + U \tag{1}$$

in polar coordinates r, φ where λ, $\rho : \mathbf{R}^+ \to \mathbf{R}$ are some smooth functions.

There is a known example of an integrable conservative system on S^2, the case of Goryachev-Chaplygin in the dynamics of a rigid body, possessing an integral cubic in momenta. The total energy (the Hamiltonian) of this system has the following form

$$H = \frac{du_1^2 + du_2^2 + 4du_3^2}{4u_1^2 + 4u_2^2 + u_3^2} - u_1, \tag{2}$$

where S^2 is given by $u_1^2 + u_2^2 + u_3^2 = 1$, (see [2]).

The aim of this paper is to classify (up to equivalence, see above) the Hamiltonians of the form (1) of conservative systems on S^2, possessing an integral

$$F = p_\varphi^3 + \kappa p_\varphi K + A \tag{3}$$

where A is a linear polynomial in momenta and κ is an arbitrary constant. We prove that these systems include the case of Goryachev-Chaplygin, a one-parameter family from [5] and a *new* two-parameter family of conservative systems on S^2 possessing an integral cubic in momenta. It turns out that all these systems are due to smooth solutions of the following differential equation

$$x'x''' = xx'' - 2x''^2 + x'^2 + x^2. \tag{4}$$

For this reason we start with investigations of this equation. We show that there is a positive constant T such that any smooth solution of (4) which exists everywhere can be obtained from the solutions $x = x(t) : \mathbf{R} \to \mathbf{R}$ of the initial value problem

$$x'x''' = xx'' - 2x''^2 + x'^2 + x^2, \quad x(0) = 0, \ x'(0) = 1, \ x''(0) = \tau, \tag{5}$$

where $\tau \in [0, T]$, or from $x(t) = \exp t$, $x(t) = \cosh t$ by a scaling $x \to \alpha_1 x$, $\alpha_1 - const$ or a linear translation of time $t \to \pm t + t_0$, $t_0 - const$.

In the next section we obtain a criterion for integrability for geodesic flows of Riemannian metrics as a partial differential equation. In the last section we consider a class of solutions of this equation and reduce the problem of classification of conservative systems on S^2 to the classification (up to a scaling or linear translations) of smooth solutions of (4). We will show that the known case of Goryachev-Chaplygin is due to the solution of (5) if $\tau = T$ (if $\tau = 0$ in (5) we get the metric of constant positive curvature). Then we prove the main theorem of classification and find new integrable cases on S^2.

2 Smooth solutions

Theorem 2.1 *There is a positive constant T such that any smooth solution of (4) which exists everywhere can be obtained from $x(t) = \exp t$, $x(t) = \cosh t$ or from the solutions of the initial value problem (5), where $\tau \in [0, T]$, by a scaling $x \to \alpha_1 x$, $\alpha_1 - const$ or a linear translation of time $t \to \pm t + \alpha_0$, $\alpha_0 - const$.*

For any solution x_τ of (5) with $\tau \in [0, T]$ there are smooth functions ξ_τ, ζ_τ, μ_τ, ν_τ such that

$$x'_\tau(t) = (\exp(-t))\xi_\tau(\exp(2t)) = (\exp t)\zeta_\tau(\exp(-2t)),$$

$$(x''_\tau - x_\tau)x'^2_\tau(t) = (\exp t)\mu_\tau(\exp(2t)) = (\exp(-t))\nu_\tau(\exp(-2t))$$

where $x'_\tau(t)$ is positive everywhere if and only if $\tau \in [0, T)$ and there is one value of the parameter $t = t_0$ such that $x'_T(t_0) = 0$.

Proof. The initial value problem (4) has a unique solution $x(t) = \Theta_\tau(t)$ which is positive on $(0, \varepsilon)$ and negative on $(-\varepsilon, 0)$ for a sufficiently small ε.

Let us consider the case $t > 0$.

The differential equation from (4) can be replaced by the following differential equation of the second order

$$\ddot{q} = \frac{1}{q}\left(1 + 2q^2 - 3q^4 + \dot{q} - 7q^2\dot{q} - 2\dot{q}^2\right)$$

with $q(t) = R'(t)$ where $R(t) = \log x(t)$. We may rewrite this equation as a system of differential equations of the first order:

$$\dot{q} = p, \quad \dot{p} = \frac{1}{q}\left(1 + 2q^2 - 3q^4 + p - 7q^2 p - 2p^2\right). \tag{6}$$

Since system (6) is symmetric with respect to $q \mapsto -q$, $t \mapsto -t$, it suffices to consider the case $q > 0$.

In order to obtain the phase portrait of (6) we may consider the following smooth system

$$\dot{q} = qp, \quad \dot{p} = 1 + 2q^2 - 3q^4 + p - 7q^2 p - 2p^2. \tag{7}$$

The solutions of (6) are obtained from the solutions of (7) by a reparametrization. The system (7) has four singular points: two saddle points $p = 1, q = 0$, $p = -\frac{1}{2}, q = 0$ and two nodes $p = 0, q = \pm 1$.

So, if a solution of (4) is smooth everywhere, then it corresponds to a smooth solution of (6) and, therefore, it corresponds to $p = 0, q = \pm 1$ or an orbit $p = -q^2 + 1, |q| < 1$ or maybe two orbits of (6) where $q \to -\infty$ as $t \to 0-$ and $q \to +\infty$ as $t \to 0+$. In the cases $p = 0, q = \pm 1$ and $p = -q^2 + 1, |q| < 1$ a solution of (4) is obtained from $x(t) = \exp t$ and from $x(t) = \cosh t$ correspondingly by a scaling or a linear translation of time.

The aim of our further investigations is to show that the orbits of (6), corresponding to the solutions of (4), when τ belongs to a certain interval, converge to the singular point $q = 1, p = 0$.

For any solution $x(t) = \Theta_\tau(t)$, $t \in (0, \varepsilon)$ of (4) there is an orbit $\{\Gamma_\tau : p = p_\tau(q)\}$ in the phase spase of (6).

For any solution of (4) it holds $R(t) \to -\infty$ as $t \to 0+$ and, therefore, $q(t) = R'(t) \to +\infty$, $p(t) = q'(t) \to -\infty$ as $t \to 0+$. Thus $p_\tau(q) \to -\infty$ as $q \to +\infty$.

We may now consider only the orbits of (6) where $p \to -\infty$ as $q \to +\infty$.

Show that if $\tau_1 > \tau_2$, then $p_{\tau_1}(q) > p_{\tau_2}(q)$. Indeed, for a solution $x(t) = \Theta_{\tau_1}(t)$ of (4) it holds

$$\tau_1 = \lim_{t \to 0+} \frac{\Theta''_{\tau_1}(t)}{\Theta'_{\tau_1}(t)} = \lim_{q \to +\infty} \frac{q^2 + p_{\tau_1}(q)}{q}. \tag{8}$$

Note that the function $x(t) = \sinh t$ satisfies (4) when $\tau = 0$. The related orbit of (6) has then the following form $\{\Gamma_0 : p = p_0(q) = -q^2 + 1\}$.

So, the orbits of (6), corresponding to the solutions of (4), for $\tau \geq 0$ converge to the singular point $q = 1, p = 0$.

We prove below that orbit $(*)$, where $p \to -\frac{1}{2}$ as $q \to 0+$, corresponds to the solution of (4) when τ is equal to a negative constant $-T$ and all orbits of (6), lying between $(*)$ and Γ_0 correspond to the solutions of (4) for $\tau \in (-T, 0)$.

Assume first that there exists a constant $\tau_0 < 0$ such that orbit Γ_{τ_0} does not converge to the point $q = 1, p = 0$.

Consider the set W of the orbits of (6), lying between Γ_{τ_0} and Γ_0. Show that for any orbit in W the value q becomes infinite in a finite time interval.

In fact, for any solution of (6) it holds

$$\int^{q(t)} \frac{dq}{p} = t + const. \tag{9}$$

For orbits in W we have $p < -q^2 + 1$. Hence, the left hand side of (9) is bounded for any orbit in W as $q \to +\infty$. Without loss of generality we assume that for these solutions of (6) t vanishes as q becomes infinite.

We conclude that for any orbit from W it holds by (8)

$$(\tau_0 + o(1))q < p + q^2 < 1$$

and for a corresponding solution of (4) we get for $t \to 0+$

$$\tau_0 + o(1) < \frac{x''(t)}{x'(t)} < 1.$$

Thus, for a corresponding solution of (4) the function $(\log x'(t))'$ is bounded in an interval, containing $0+$. Hence, $\lim_{t\to 0+} x'(t)$ is bounded for a solution $x(t)$ of (4), corresponding to an orbit in W and, therefore, $x(t) = \frac{x'(t)}{q} \to 0$ as $t \to 0+$.

Let us consider the orbit $(*)$ where $p = p^*(q)$. If $(*)$ is the same as Γ_{τ_0} then $T = -\tau_0$.

If $(*)$ is not the same as Γ_{τ_0} then it belongs to W and, hence, corresponds to the solution of (4) for $\tau = -T$, where T is equal to a positive constant and, moreover, $T < -\tau_0$.

Thus we have shown that for any $\tau \in (-T, T)$ where T is a positive constant the system (4) has a solution for all $t \geq 0$.

Let us assume now that there is no such orbit Γ_{τ_0} and, so, all orbits of (6), corresponding to the solutions of (4), converge to the singular point $q = 1, p = 0$.

Clearly, the orbit $(*)$ corresponds to a solution $x(t) = \eta(t)$ of the differential equation in (4). As mentioned above, the function $q(t) = \frac{\eta'(t)}{\eta(t)}$ becomes infinite in a finite time interval of t, say, as $t \to 0+$. Since $p^*(q) < -q^2 - kq$ for any k and for sufficiently large q, for $\eta(t)$ it holds

$$\lim_{t\to 0+} \frac{\eta''(t)}{\eta'(t)} = -\infty. \tag{10}$$

Note that $\eta(0)$ is finite because $R(0) - R(t) = \int_t^0 q(t)dt < 0$ for $t > 0$ and $\eta(t) = \exp R(t)$. So, there are two cases: $\eta(0) = 0$ or $\eta(0) \neq 0$.

Assume that $\eta(0) \neq 0$. It follows that $\eta'(0+) = +\infty$ and from (10) we get $\eta''(0+) = -\infty$.

Rewrite now the differential equation in (4) in the following form

$$\frac{x'''}{x} = -3\frac{x'' - x}{x'} - 2\frac{(x'' - x)^2}{xx'} + \frac{x'}{x}.$$

Therefore, it holds

$$\frac{\eta'''}{\eta} = -3\frac{p^*(q)+q^2-1}{q} - 2\frac{(p^*(q)+q^2-1)^2}{q} + q.$$

We obtain $\eta'''(0+) = -\infty$. This is a contradiction $\eta''(0+) = -\infty$.

Assume that $\eta(0) = 0$ and $\eta'(0) \neq 0$. From (10) we get $\eta''(0+) = -\infty$. Rewriting the differential equation in (4) in the following form

$$x''' = x' - 3\frac{(x''-x)x}{x'} - 2\frac{(x''-x)^2}{x'},$$

we obtain again $\eta'''(0+) = -\infty$.

Therefore, we only have to consider one case $\eta(0) = 0$ and $\eta'(0) = 0$. Taking into account that $\eta'(t) = \eta(t)q(t)$ and $q(t) > 0$ as $t \to 0+$, we conclude that $\eta'(t) \to 0+$ as $t \to 0+$ but on the other hand from (10) it follows that $\eta''(t) < 0$.

These contradictions finally show that there is an orbit Γ_{τ_0} which does not converge to the point $q = 1, p = 0$ and, therefore, for any $\tau \in (-T, T)$, where

$$T = -\lim_{q\to+\infty} \frac{p^*(q)+q^2}{q} < \infty, \tag{11}$$

solutions of (4) exist on $[0, +\infty)$.

Since a solution $x(t) = \Theta_\tau(t)$ of (4) equals $-\Theta_{-\tau}(-t)$ if $t \leq 0$, solutions of (4) exist on $(-\infty, +\infty)$.

We show now the asymptotic behaviour of the solutions of (5) with $\tau \in [0, T]$ where T is defined by (11).

Put $s = \exp(-2y)$ and

$$g(s) = \sqrt{s}x\left(-\frac{1}{2}\log s\right).$$

Then the initial value problem (4) can be rewritten as

$$g''' = 3\frac{g''g'}{g-2g's} + s\frac{g''^2}{g-2g's},\ g(1) = 0, g'(1) = -\frac{1}{2}, g''(1) = \tau \tag{12}$$

We will consider $\tau \in (-T, T]$.

Compute now for a solution of (4):

$$x'(t) = (\exp t)(g - 2g's), \tag{13}$$

$$x''(t) - x(t) = 4\exp(-3t)g''(s). \tag{14}$$

Let us consider the system (6). Since the eigenvalues of the Jacobian of (6) at $(q, p) = (1, 0)$ are equal to -2, -4, there exist functions P and Q of class C^1 such that

$$q = Q(s) = 1 + Cs + o_1(s),$$

$$p = P(s) = -2Cs + o_2(s),$$

where C is a constant, see [4].

We may write:

$$R(t) = \int^t q = t + \psi_1(s).$$

Let us show that ψ_1 is of class C^1. By differentiation we obtain

$$\frac{d\psi_1(s)}{ds} = -\frac{q-1}{2s} = -\frac{Cs + o_1(s)}{2s} \in C^0.$$

Write now for a solution of (4)

$$x(t) = \exp R(t) = (\exp t)\exp\psi_1(s) = (\exp t)g(s),$$

where we have used that $g = r^{-1}u = \exp(-t)u = \exp(-t)x(t)$, and, therefore, $g(0) = \exp\psi_1(0) \neq 0$.

For a solution of (4) which can be extended into infinity it holds

$$x'(t) = x(t)q(t) = (\exp t)(\exp\psi_1(s))(1 + Cs + o_1(s)) =$$

$$= (\exp t)g(s)(1 + Cs + o_1(s)), s = \exp(-2t).$$

On the other hand, from (13) we obtain

$$g(s)(1 + Cs + o_1(s)) = g(s) - 2sg'(s).$$

Thus, for any solution of (12) where $\tau \in (-T, T]$ it holds $g'(0) < \infty$.

As mentioned above, the solution of (4) for $\tau = 0$ has the form $x(t) = \sinh t$.

Let us show that for any solution of (12) where $\tau \in (-T, T]$ and $\tau \neq 0$ it holds $0 < |g''(0)| < \infty$.

We compute

$$x''(t) - x(t) = x(t)(p + q^2 - 1) = (\exp t)(\exp\psi_1(s))(P(s) + Q^2(s) - 1) =$$

$$= (\exp t)(\exp\psi_1(s))(-2Cs + o_2(s) + (1 + Cs + o_1(s))^2 - 1).$$

Thus,

$$x''(t) - x(t) = (\exp t)\psi_2(s),$$

where $\psi_2 \in C^1$, $\psi_2(s) = o_3(s)$. Rewrite the differential equation in (4) in the following form

$$(x''' - x')x' = -3(x'' - x)x - 2(x'' - x)^2.$$

We obtain either $x''(t) - x(t) \equiv 0$ (then $\tau = 0$) or

$$(\log|x''(t) - x(t)|)' = -3\frac{1}{q} - 2|x''(t) - x(t)|(\exp t \exp\psi_1(s)(1 + Cs + o_1(s)))^{-1} =$$

$$= -3\frac{1}{q} - 2|\psi_2(s)|(\exp\psi_1(s)(1 + Cs + o_1(s)))^{-1}.$$

So,

$$(\log|x''(t)-x(t)|)' = -3(1-Cs)+o_4(s).$$

Then it follows

$$(\log((\exp t)|\psi_2(s)|))' = -3(1-Cs)+o_4(s).$$

Thus,

$$(t+\log|\psi_2(s)|)' = -3(1-Cs)+o_4(s).$$

By integrating we get

$$(\log|\psi_2(s)|) = -4t - \frac{3C}{2}s + \psi_3(s),$$

where $\psi_3 \in C^1$.

Thus,

$$|\psi_2(s)| = (\exp(-4t))\exp(-\frac{3C}{2}s+\psi_3(s)) = s^2\psi_4(s),$$

since $s = \exp(-2t)$. So, $\psi_4 \in C^1$ and $\psi_4(0) = const \neq 0$. Taking into account (14), we obtain $|g''(0)| = \frac{1}{4}\psi_4(0)$ and therefore $0 < |g''(0)| < \infty$.

So, we conclude that the solutions of (12) for $\tau \in (-T, T]$ are of class C^∞ in zero.

Now the functions ζ_τ, ν_τ can be calculated in terms of g. From (13) $\zeta_\tau(0) > 0$ in view of $g(0) > 0$. Therefore $\zeta_\tau > 0$ everywhere on $[0, +\infty)$.

Since a solution $x(t) = x_\tau(t)$ of (4) equals $-x_{-\tau}(-t)$ for $\tau \in (-T, T)$ and $x_T(t + t_0) = x_T(-t + t_0)$, the theorem follows.

□

3 A criterion for integrability

Consider a metric $ds^2 = \Theta(u,v)(du^2+dv^2)$ in conformal coordinates u, v. It can also be written as

$$ds^2 = \theta(w,\bar{w})dwd\bar{w} \tag{15}$$

where $w = u+iv$. The geodesic flow of ds^2 is a Hamiltonian system with Hamiltonian

$$H = \frac{p_w p_{\bar{w}}}{4\theta(w,\bar{w})}. \tag{16}$$

A polynomial F in momenta p_u, p_v can be also written as

$$F = \sum_{k=0}^{n} b_k(w,\bar{w}) p_w^k p_{\bar{w}}^{n-k}$$

where

$$b_k = \overline{b_{n-k}}, \quad k = 0, ..., n.$$

If the polynomial F is an additional integral of the geodesic flow with the Hamiltonian (16), then $\{F, H\} = 0$ and the following holds

$$\theta\frac{\partial b_{k-1}}{\partial w} + (n-(k-1))b_{k-1}\frac{\partial \theta}{\partial w} + \theta\frac{\partial b_k}{\partial \bar{w}} + kb_k\frac{\partial \theta}{\partial \bar{w}} = 0, \tag{17}$$

where $k = 0, ..., n+1$ and $b_{-1} = b_{n+1} = 0$. Substituting $k = 0$ and $k = n+1$ in (17) we get immediately

$$\frac{\partial}{\partial \bar{w}} b_0 \equiv 0$$

and

$$\frac{\partial}{\partial w} b_{n+1} \equiv 0.$$

One may show that if a polynomial in momenta integral F of the geodesic flow of (15) is independent of the Hamiltonian and an integral of smaller degree, then there is a conformal coordinate system $z = z(w)$ of this metric such that the coefficients of F for p_z and $p_{\bar{z}}$ are equal to 1 identically.

Theorem 3.1 *Let $ds^2 = \lambda(z, \bar{z})dzd\bar{z}$ ($z = \varphi + iy$) be a metric such that there exists a function $f : \mathbf{R}^2 \mapsto \mathbf{R}$, satisfying the following conditions*

$$\lambda = \frac{1}{4}\left(\frac{\partial^2 f}{\partial \varphi^2} + \frac{\partial^2 f}{\partial y^2}\right),$$

and

$$\frac{\partial}{\partial \varphi}\left(\left(\frac{\partial^2 f}{\partial \varphi^2} - \frac{\partial^2 f}{\partial y^2}\right)\left(\frac{\partial^2 f}{\partial \varphi^2} + \frac{\partial^2 f}{\partial y^2}\right)\right) = 2\frac{\partial}{\partial y}\left(\left(\frac{\partial^2 f}{\partial \varphi \partial y}\right)\left(\frac{\partial^2 f}{\partial \varphi^2} + \frac{\partial^2 f}{\partial y^2}\right)\right) \tag{18}$$

Then the geodesic flow of ds^2 possesses an integral cubic in momenta.

If the geodesic flow of a metric ds^2 possesses an integral which is cubic in momenta and it does not depend on the Hamiltonian and an integral of smaller degree then there exist conformal coordinates φ, y and a function $f : \mathbf{R}^2 \mapsto \mathbf{R}$ such that $ds^2 = \lambda(z, \bar{z})dzd\bar{z}$ where $z = \varphi + iy$ and (18) holds.

Proof. We will consider integrals

$$F = \sum_{k=0}^{n} a_k(z, \bar{z})p_z^k p_{\bar{z}}^{n-k}$$

of the geodesic flow of ds^2 where $a_0 = a_n \equiv 1$.

The system (17) has then the following form

$$3\frac{\partial \lambda}{\partial z} + \frac{\partial(a_1\lambda)}{\partial \bar{z}} = 0, \tag{19}$$

$$\frac{\partial(a_1\lambda^2)}{\partial z} + \frac{\partial(a_2\lambda)}{\partial \bar{z}} = 0. \tag{20}$$

(19) holds if and only if there exists a function h such that

$$\lambda = \frac{\partial h}{\partial \bar z} \text{ and } \ a_1\lambda = -3\frac{\partial h}{\partial z}.$$

(20) can be also rewritten in the following form

$$\frac{\partial(a_1\lambda^2)}{\partial z} = -\frac{\partial(\overline{a_1}\lambda^2)}{\partial \bar z} = -\overline{\frac{\partial(a_1\lambda^2)}{\partial z}}.$$

Therefore, the system (19), (20) is equivalent to the following condition

$$\text{Re } \frac{\partial(a_1\lambda^2)}{\partial z} = \text{Re } \frac{\partial}{\partial z}\left(\frac{\partial h}{\partial z}\frac{\partial h}{\partial \bar z}\right) = 0$$

where

$$\text{Im } \frac{\partial h}{\partial \bar z} = \text{Im } \lambda(z, \bar z) = 0,$$

i.e. there is a real function f such that $\frac{\partial f}{\partial z} = h$ and (18) holds.

□

The equation (18) has been obtained first in [3]. We need to give hier another proof because further we need the following corollaries which are due to the above proof of Theorem 3.1.

Corollary 3.2 *If in a conform coordinate system φ, y of $ds^2 = \lambda(\varphi, y)(d\varphi^2 + dy^2)$ there is an integral of the geodesic flow of ds^2 which has the form*

$$F = \alpha_3(p_z^3 + a_1 p_z^2 p_{\bar z} + \overline{a_1} p_z p_{\bar z}^2 + p_{\bar z}^3), \ \ z = \varphi + iy, \ \ \alpha_3 - const \neq 0, \qquad (21)$$

then there is a function f such that (18) holds.

Corollary 3.3 *If in a conform coordinate system φ, y of $ds^2 = \lambda(\varphi, y)(d\varphi^2 + dy^2)$ there is a function f such that (18) holds, then there is an integral F which has the form*

$$F = p_z^3 + a_1 p_z^2 p_{\bar z} + \overline{a_1} p_z p_{\bar z}^2 + p_{\bar z}^3, \ \ z = \varphi + iy.$$

4 The Classification

Theorem 4.1 *Let*

$$ds^2 = (\Psi_1(r^2) r\cos\varphi + \Psi_2(r^2))(r^2 d\varphi^2 + dr^2) \qquad (22)$$

be a metric on S^2 in polar coordinates. Then the geodesic flow of ds^2 possesses an integral of the form

$$F = iw^3 p_w^3 + b_1 p_w^2 p_{\bar w} + \overline{b_1} p_w p_{\bar w}^2 - i\bar w^3 p_{\bar w}^3, \quad w = r\cos\varphi + ir\sin\varphi \qquad (23)$$

if and only if
$$\Psi_1(r^2)r = ((\psi'' - \psi)(\log r))r^{-2}$$
and
$$\Psi_2(r^2) = cr^{-2}\psi'^{-2}(\log r)$$
or
$$\Psi_2(r^2) = ar^{-2}\frac{\psi'^2 - \psi^2 + b}{\psi'^2}(\log r)$$
where ψ is a smooth solution of (4) which exist everywhere on $(-\infty, +\infty)$ and a, c, p are appropriate constants.

Proof. Assume that the geodesic flow of ds^2 possesses an integral of the form (23). Let us consider $z = \varphi + iy = -i\ln w$. Then (22) can be rewritten in the following form

$$ds^2 = (\Psi_1(\exp(2y))(\exp y)\cos\varphi + \Psi_2(\exp(2y)))(\exp(2y))(d\varphi^2 + dy^2).$$

Thus,
$$ds^2 = (\Psi_3(y)\cos\varphi + \Psi_4(y))(d\varphi^2 + dy^2) = \lambda(\varphi, y)(d\varphi^2 + dy^2). \tag{24}$$
for some functions Ψ_3, Ψ_4. An integral of the form (23) has then the form (21). Now we can apply Corollary 3.2. So, there is a function f such that (18) holds.

Let us consider a function $h(\varphi, y) = \psi(y)\cos\varphi + \xi(y)$ where $\psi'' - \psi = \Psi_3$ and $\xi'' = \Psi_4$. Then a function f from (18) has the form $f = h + \delta$ where $\delta_{\varphi\varphi} + \delta_{yy} = 0$. Then for the function a_1 from (21) we get

$$a_1 = -3\frac{f_{\varphi\varphi} - f_{yy}}{f_{\varphi\varphi} + f_{yy}} + 6i\frac{f_{\varphi y}}{f_{\varphi\varphi} + f_{yy}} = -3\frac{h_{\varphi\varphi} - h_{yy} + \delta_{\varphi\varphi} - \delta_{yy}}{\lambda(\varphi, y)} + 6i\frac{h_{\varphi y} + \delta_{\varphi y}}{\lambda(\varphi, y)}. \tag{25}$$

Thus, the functions $\delta_{\varphi y}$ and $\delta_{\varphi\varphi} - \delta_{yy}$ are periodic in φ. Therefore,

$$\delta(\varphi, y) = P_1(\varphi, y) + a(\varphi^2 - y^2) + b\varphi y + \sum_{k\in\mathbf{N}}(C_k(\exp(-ky)) + D_k(\exp(ky)))\sin k\varphi$$
$$+\sum_{k\in\mathbf{N}}(E_k(\exp(-ky)) + G_k(\exp(ky)))\cos k\varphi \tag{26}$$

where a, b, C_k, D_k, E_k, G_k are some constant and $P_1(\varphi, y)$ is a linear polynomial in φ, y.

For the function b_1 from (23) we have $b_1(w, \bar{w}) = a_1|w|^2 w$ is bounded if $|w| = 0$, since ds^2 is a metric on S^2. Then from (25) and (26) we obtain C_k, D_k, E_k, G_k are equal to zero when $k \geq 2$.

Thus, we have to consider functions f of the form

$$f(\varphi, y) = \psi(y)\cos\varphi + \xi(y) + a(\varphi^2 - y^2) + (C_1(\exp(-y)) + D_1(\exp y))\sin\varphi \tag{27}$$

satisfying (18). Taking into account that ds^2 is a metric on S^2 from (18) for (27) we obtain $b = 0$ and $C_1 = D_1 = 0$.

So, we must consider solutions of (18) of the following form

$$f(\varphi, y) = \psi(y)\cos\varphi + \xi(y) + a(\varphi^2 - y^2),\ a - const.$$

Then (18) is equivalent to the following conditions

$$\lambda = (\psi'' - \psi)\cos x + \xi''$$

where

$$\xi''\psi'' + (\xi''\psi')' = 2a(\psi'' - \psi) \tag{28}$$

and

$$\psi'\psi''' = \psi\psi'' - 2\psi''^2 + \psi'^2 + \psi^2. \tag{29}$$

Multiply (28) by ψ':

$$2\xi''\psi'\psi'' + \xi'''\psi'^2 = 2a(\psi'' - \psi)\psi'$$

and integrate

$$\xi''(y) = a\frac{\psi'^2(y) - \psi^2(y) + b}{\psi'^2(y)},\ b - const \ \text{ if } \ a \neq 0$$

or

$$\xi''(y) = c\psi'^{-2}(y),\ c - const \ \text{ if } \ a = 0.$$

Conversely integrability of the corresponding geodesic flows follows immediately from Corollary 3.3 and above calculations.

□

Thus, due to the Maupertuis's principle we obtain the following corollary from Theorem 4.1.

Corollary 4.2 *Hamiltonian systems with the Hamiltonians of the form*

$$H = \frac{d\varphi^2 + dy^2}{\psi'^2(y)} - (\psi''(y) - \psi(y))\psi'^2(y)\cos\varphi \tag{30}$$

and

$$H_b = \frac{\psi'^2(y) - \psi^2(y) + b}{\psi'^2(y)}\left(d\varphi^2 + dy^2\right) - \frac{\psi'^2(y)(\psi''(y) - \psi(y))}{\psi'^2(y) - \psi^2(y) + b}\cos\varphi, \tag{31}$$

where ψ is a solution of (4), possessing an integral cubic in momenta of the form (3).

Now we prove the main theorem.

Theorem 4.3 *A Hamiltonian system with the Hamiltonian of the form (1) is a conservative system on S^2, possessing an integral of the form (3) and not possessing an integral quadratic or linear in momenta if and only if the corresponding Hamiltonian is equivalent to one of the following*

1. (30) where ψ is a solution of (5) for $\tau \in (0,T)$.

2. (31) where ψ is a solution of (5) for $\tau \in (0,T)$ and

$$b > b_*(\tau) = \max_{y\in\mathbf{R}} (\psi^2 - \psi'^2) \quad or \quad b < b^*(\tau) = \min_{y\in\mathbf{R}} (\psi^2 - \psi'^2)$$

where $b_*(\tau) < +\infty$ and $b^*(\tau) > -\infty$ for $\tau \in (0,T)$.

3. (31) where ψ is a solution of (5) for $\tau = T$ and $p = \psi^2(y_0)$, when $\psi'(y_0) = 0$, which is in fact the case of Goryachev-Chaplygin in the dynamics of a rigid body.

Proof. From Theorem 4.1 and the Maupertuis's principle we know that if a Hamiltonian system with the Hamiltonian of the form (1) defines a conservative system on S^2, possessing an integral of the form (3), then (1) is equivalent to (30) or (31) where ψ is a smooth solution of (4) which exists everywhere on $(-\infty,+\infty)$. So, we must only describe when in this case (30), (31) define a conservative system on S^2. From Theorem 2.1 it follows that we must consider only solutions of (5) for $\tau \in [0,T]$ and $\psi(y) = \exp y$, $\psi(y) = \cosh y$. We note that if ψ is a solution of (5) for $\tau = 0$ (then $\psi(y) = \sinh y$) or $\psi(y) = \exp y$, $\psi(y) = \cosh y$, then the Hamiltonians (30), (31) define simply a metric of constant curvature where there are two independent linear integrals.

So, we must only consider solutions ψ of (5) when $\tau \in (0,T]$.

Let us write then (30), (31) in polar coordinates φ, $r = \log y$ and $\tilde\varphi = -\varphi$, $\tilde r = -\log y$. Using Theorem 2.1 we compute

$$H = \frac{1}{\xi_\tau^2(r^2)}(r^2 d\varphi^2 + dr^2) - \mu_\tau(r^2) r \cos\varphi$$

$$= \frac{1}{\zeta_\tau^2(\tilde r^2)}(\tilde r^2 d\tilde\varphi^2 + d\tilde r^2) - \nu_\tau(\tilde r^2)\tilde r \cos\tilde\varphi$$

and

$$H_b = \frac{\Phi_\tau(r^2) + b + 1}{\xi_\tau^2(r^2)}(r^2 d\varphi^2 + dr^2) - \frac{\mu_\tau(r^2) r\cos\varphi}{\Phi_\tau(r^2) + b + 1}$$

$$= \frac{-\tilde\Phi_\tau(\tilde r^2) + b + 1}{\zeta_\tau^2(\tilde r^2)}(\tilde r^2 d\tilde\varphi^2 + d\tilde r^2) - \frac{\nu_\tau(\tilde r^2)\tilde r\cos\tilde\varphi}{-\tilde\Phi_\tau(\tilde r^2) + b + 1}$$

where

$$\Phi_\tau(t) = \int_1^t \mu_\tau(s)\xi_\tau^{-1}(s)ds,$$

$$\tilde\Phi_\tau(t) = \int_1^t \nu_\tau(s)\zeta_\tau^{-1}(s)ds.$$

Consider $\tau = T$. In Theorem 2.1 it has been proved that for the corresponding solution ψ of (5) there is $y = y_0$ such that $\psi'(y_0) = 0$, and therefore there is

$r = r_0 > 0$ such that $\xi_T(r_0^2) = 0$. Thus, in the case $\tau = T$ (30) does not define a system on S^2 and (31) in this case defines a system on S^2 if and only if $p = \psi^2(y_0)$.

Consider $0 < \tau < T$. Then (30) defines a family of conservative systems on S^2 which has been found in [5]. In [5] it has been proved that for $\tau_1 \neq \tau_2$ the corresponding Hamiltonians are not equivalent, the systems possess nontrivial cubic integrals (there is no quadratic or linear integral) and no Hamiltonian from this family is equivalent to the case of Goryachev-Chaplygin.

Let us find the admissible values of the parameter b in this case. From the above expressions for H_b in polar coordinates we have

$$b + 1 > \max_{[0,1]} -\Phi_\tau = M_1(\tau) \quad \text{and} \quad b + 1 > \max_{[0,1]} \tilde{\Phi}_\tau = \tilde{M}_1(\tau)$$

or

$$b + 1 < \min_{[0,1]} -\Phi_\tau = M_2(\tau) \quad \text{and} \quad b + 1 < \min_{[0,1]} \tilde{\Phi}_\tau = \tilde{M}_2(\tau).$$

So, $b > b_*(\tau) = \max\{M_1(\tau), \tilde{M}_1(\tau)\} - 1$ or $b < b^*(\tau) = \min\{M_2(\tau), \tilde{M}_2(\tau)\} - 1$ where $b_*(\tau) < +\infty$ and $b^*(\tau) > -\infty$ for $\tau \in (0, T)$.

With the same arguments as in [5] we can prove that the corresponding systems do not possess integrals linear or quadratic in momenta.

Let us show that the Hamiltonians (31) where ψ is a solution of (5) for $\tau \in (0, T)$ are not equivalent for different values of the parameters τ and p.

Assume that the Hamiltoians of the form (31) for τ_1, b_1 and τ_2, b_2 are equivalent.

We will use the following lemma which is in fact a particular case of some results of Kolokol'tsov, published in his Ph.D. Dissertation, (Moscow State University, 1984), another proof one can find in [5].

Lemma 4.4 *Let*

$$ds^2 = \lambda(r^2)(r^2 d\varphi^2 + dr^2) \tag{32}$$

be a metric on S^2 in polar coordinates r, φ. Then ds^2 can be written in the form (32) in polar coordinates $\tilde{r}, \tilde{\varphi}$ if and only if $\tilde{r} = Dr^{\pm 1}$, $D -$ const and $\tilde{\varphi} = \pm\varphi + \varphi_0$, $\varphi_0 -$ const.

Thus, from Lemma 4.4 it follows that there are some constants $C_0 \neq 0$, $C_3 \neq 0$, y_1 such that

$$\psi_1''(y) - \psi_1(y) = C_0(\psi_2''(y + y_1) - \psi_2(y + y_1)), \tag{33}$$

$$\frac{\psi_1'^2(y) - \psi_1^2(y) + b_1}{\psi_1'^2(y)} = C_3 \frac{\psi_2'^2(y + y_1) - \psi_2^2(y + y_1) + b_2}{\psi_2'^2(y + y_1)} \tag{34}$$

where ψ_1, ψ_2 are solutions of (5) for $\tau = \tau_1$ and $\tau = \tau_2$ correspondingly. So, we get from (33)

$$\psi_1(y) = C_0\psi_2(y + y_1) + C_1 \exp y + C_2 \exp(-y)$$

for some constants C_1, C_2. As in the proof of Theorem 2.1 let us write for a solution ψ of (4): $\psi(y) = (\exp y)g(\exp(-2y))$. It has been shown that g satisfies then the differential equation from (12). So, from our assupmtion it follows that there are two solutions of the differential equation from (12) g_1 and g_2 such that

$g_1(s) = g_2(s) + C_1 + C_2 s$. Then by substituting g_1 in the differential equation from (12) we obtain $C_1 = C_2 = 0$. Thus,

$$\psi_1(y) = C_0 \psi_2(y + y_1).$$

From the initial conditions of (5) we get $0 = \psi_1(0) = C_0\psi_2(y_1)$ and therefore $\psi_2(y_1) = 0$ but $\psi_2(0) = 0$ and $\psi_2'(y)$ is positive everywhere. So, $y_1 = 0$ and from $\psi_1'(0) = \psi_2'(0) = 1$ we obtain $C_0 = 1$. Thus, $\tau_1 = \tau_2$.

Prove now that $b_1 = b_2$. So, we have from (34)

$$\psi_1'^2(y) - \psi_1^2(y) + b_1 = C_3(\psi_1'^2(y) - \psi_1^2(y) + b_2),$$

and therefore, either $C_3 = 1$ and $b_1 = b_2$ or $\psi_1'^2(y) - \psi_1^2(y) \equiv const$ that is not true for $\tau \neq 0$. So we get $b_1 = b_2$.

Assume that a Hamiltonian from (1.) is equivalent to a Hamiltonian from (2.). Then we obtain in the same way as above that for a solution ψ_τ of (5) it holds $\psi_\tau'^2(y) - \psi^2(y) \equiv const$ and therefore $\tau = 0$. Thus, no Hamiltonian from (1.) is equivalent to a Hamiltonian from (2.).

Let us show that no Hamiltonian from this family is equivalent to the case of Gorychev-Chaplygin. There are polar coordinates r, φ such that (2) can be rewritten as

$$H = \gamma_1(r^2)(r^2 d\varphi^2 + dr^2) - \gamma_2(r^2) r \cos\varphi = \gamma_1(\tilde{r}^2)(\tilde{r}^2 d\varphi^2 + d\tilde{r}^2) - \gamma_2(\tilde{r}^2)\tilde{r}\cos\varphi \quad (35)$$

where $\tilde{r} = r^{-1}$.

Assume that the Hamiltonian (31) where ψ is a solution of (5) for $\tau = \tau_0 \in (0, T)$ and $b = b_1$ is equivalent to the Hamiltonian of the case of Gorychev-Chaplygin. This means also from the symmetry of (35) that (33) and (34) hold for $b_1 = b_2$ and $\tau_1 = \tau_0$, $\tau_2 = -\tau_0$. In the same way as above we obtain $\tau_0 = 0$ but then the Hamiltonian of the case of Gorychev-Chaplygin is equivalent to the Hamiltonian of a metric of constant positive curvature.

So, we described all conservative systems on S^2 with the Hamiltonians of the form (1) possessing an integral of the form (3) and not possessing an integral quadratic or linear in momenta. We proved also that no Hamiltonian from (1.) or (2.) is equivalent to the Hamiltonian of the case of Goryachev-Chaplygin. On the other hand the case of Goryachev-Chaplygin belongs to the class of conservative systems on S^2 with the Hamiltonians of the form (1) possessing an integral of the form (3) and not possessing an integral quadratic or linear in momenta, see [5]. So, the Hamiltonian of the case of Goryachev-Chaplygin is equivalent to the Hamiltonian (31) where ψ is a solution of (5) for $\tau = T$ and $b = \psi^2(y_0)$, when $\psi'(y_0) = 0$.

$\square$

References

[1] Arnold V.I. *Mathematical methods in Classical Mechanics.* Berlin Heidelberg, Springer-Verlag, 1978

[2] Bolsinov A.V., Kozlov V.V., Fomenko, A.T. *The Maupertuis principle and geodesic flows on S^2 arising from integrable cases in the dynamics of a rigid body,* Russ. Math. Surv. 50, n.3, 473-501 (1995)

[3] Hall, L.S. *A theory of exact and approximate configurational invariants,* Phys 8D, 90-116 (1983)

[4] Hartman, P. *Ordinary differential equations.* New York-London-Sydney, John Wiley and Sons, 1964

[5] Selivanova E.N. *New examples of conservative systems on S^2 possessing an integral cubic in momenta,* Preprint dg-ga/9711005 (1997)