Contracting Spacelike Hypersurfaces by their Inverse Mean Curvature

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Abstract

In this work we study the behaviour of compact, smooth, orientable, spacelike hypersurfaces without boundary, which are immersed in cosmological spacetimes and move under the *inverse mean curvature flow.* We prove longtime existence and regularity of a solution to the corresponding nonlinear parabolic system of partial differential equations.

1 Introduction

A cosmological spacetimes is a time-oriented connected Lorentzian manifold $\mathcal{V}$ containing a global Cauchy surface, i.e. we can write $\mathcal{V} = \Sigma^n \times I$ with Σ^n being a Cauchy surface. In [G1] Gerhardt obtained general existence and regularity results for prescribed mean curvature surfaces in cosmological spacetimes (in [B1] Bartnik settled the corresponding problem for asymptotically flat spacetimes). In particular it was proved that in cosmological spacetimes which satisfy the timelike convergence condition and admit a *big bang* and a *big crunch* (this means suitable barriers in the future and past are known) there exists a foliation of $\mathcal{V}$ by hypersurfaces of constant mean curvature. Ecker and Huisken [EH1] studied the corresponding mean curvature flow and constructed hypersurfaces of prescribed mean curvature $\mathcal{H}$ in cosmological spacetimes satisfying the timelike convergence condition as an asymptotic limit of the geometric evolution equation $\frac{d}{ds}F = (H - \mathcal{H})\nu$. So they gave an alternative proof for the existence of maximal and constant mean curvature surfaces. However they needed as well the assumption of future and past barriers.

The aim of this work is to derive a foliation in a cosmological spacetime satisfying the timelike convergence condition without assuming the existence of, any barriers in future or past. To prove this we look at the geometric evolution equation $\frac{d}{ds}F = -\frac{1}{H}\nu$. In other words, we look at a flow which deforms hypersurfaces away from the maximal surface in the direction of the *"big bang"*. Notice that the mean curvature flow deforms hypersurfaces in the direction of a maximal surface.

Let M_0 be a smooth, orientable, compact, spacelike hypersurface without boundary in a cosmological spacetime $\mathcal{V}$ given by an immersion

$$F_0 : M^n \to \mathcal{V}.$$

1.1. Definition. *Let $F_s = F(\cdot, s) : M^n \to \mathcal{V}, s \in [0, Z)$ be a family of smooth immersions. F_s is said to move under the inverse mean curvature flow, if*

$$\begin{aligned} \frac{d}{ds}F(p,s) &= -\Big(\frac{1}{H}\nu\Big)(p,s), \\ &\qquad\qquad p \in M^n, \quad s \geq 0, \\ F(p,0) &= F_0(p). \end{aligned} \tag{1}$$

Here $\nu(p,s)$ denotes the future directed unit normal to $M_s := F(\cdot,s)(M^n)$ at $F(p,s)$ and $H > 0$ is the mean curvature on M_s.

The evolution of hypersurfaces by mean curvature has been extensively studied in Riemannian ambient spaces. In [Hu1,Hu2] Huisken looked at the so-called mean curvature flow: $\frac{d}{ds}F = -H\nu$. This flow contracts compact initial surfaces and develops singularities as the flow decreases area. In the case of spacelike hypersurfaces in Lorentzian manifolds the mean curvature flow [EH1] increases rather than decreases area which leads to a more regular behaviour. A flow related to Equation (1), has been studied in the Euclidean case by Gerhardt [G2]. For compact star-shaped $C^{2,\alpha}$ surfaces he proved longtime existence and exponential convergence of the rescaled surfaces to round spheres. Independently Urbas [U1,U2] obtained the same result using a slightly different method.

To derive the existence result for Equation (1) we have to establish a priori estimates for the height, the slope and the curvature of M_s. In Section 2 we fix the notation and recall some fundamental identities. Then we calculate the relevant evolution equations in Section 3 before we obtain the a priori estimates in Section 4. The main result is then proved in Section 5. In Section 6 we consider the case where the cosmological spacetime is a Robertson-Walker spacetime.

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2 Preliminaries

We consider $\mathcal{V}$ to be an $(n+1)$-dimensional smooth spacetime with a Lorentzian metric $\overline{g} = \{\overline{g}_{\alpha\beta}\}$ with signature $(-,+,+,...,+)$. The canonical connection will be denoteed by $\overline{\nabla}$, the metric pairing by $\langle\cdot,\cdot\rangle$ and the curvature tensor by $\overline{R}m = \{\overline{R}_{\alpha\beta\gamma\delta}\}$. Greek indices range from 0 to n. As in [B1] we shall assume the existence of a global *time function* $t \in C^\infty(\mathcal{V})$ with nonzero, past-directed timelike vector field $\overline{\nabla}t$. The reference slices $\mathcal{S}_{\mathrm{t}} := \{p \in \mathcal{V} \mid t(p) = \mathrm{t}\}$ have future-directed unit normal vector

$$T = -\psi\overline{\nabla}t,$$

where the *lapse function* $\psi \in C^\infty(\mathcal{V})$ is defined by

$$\psi^{\;-2} := -\langle\overline{\nabla}t, \overline{\nabla}t\rangle.$$

An adapted orthonormal frame on $\mathcal{S}_{\mathrm{t}}$ will be denoted by $e_0, e_1, \ldots, e_n$ such that $e_0 = T$.

Let M be a smooth spacelike hypersurface in $\mathcal{V}$, which is embedded by some map

$$F : M^n \to \mathcal{V}.$$

Let ν be the future directed timelike unit normal and choose locally an adapted orthonormal frame $\tau_0, \tau_1, \ldots, \tau_n$ in $\mathcal{V}$ such that restricted to M we have $\tau_0 = \nu$. We will denote by $g = \{g_{ij}\}$ and $Rm = \{R_{ijkl}\}$ the induced metric and the Riemann curvature tensor on M respectively, where Latin indices run from 1 to n. The covariant derivative on M will be denoted by ∇ and the Laplace-Beltrami operator by Δ. The second fundamental form $A = \{h_{ij}\}$ on M is given by

$$h_{ij} := \langle\overline{\nabla}_{\tau_i}\nu, \tau_j\rangle = -\langle\nu, \overline{\nabla}_{\tau_i}\tau_j\rangle.$$

As in [Hu1,EH1] we sum over repeated indices

$$\begin{aligned} H &= h_{ii} \\ |A|^2 &= h_{ij}h_{ij} \end{aligned}$$

The curvature, Ricci curvature $Ric = \{R_{ij}\}$ and the scalar curvature R on M are given by the Gauss' equation

$$\begin{aligned} R_{ijkl} &= \overline{R}_{ijkl} - h_{ik}h_{jl} + h_{il}h_{jk}, \\ R_{ik} &= \overline{R}_{ik} - Hh_{ik} + h_{il}h_{lk} + \overline{R}_{i0k0}, \\ R &= \overline{R} - H^2 + |A|^2 + 2\overline{R}ic(\nu,\nu). \end{aligned}$$

The Codazzi equations state that:

$$\nabla_k h_{ij} - \nabla_j h_{ik} = \overline{R}_{0ijk}.$$

We have the following rule for changing covariant derivatives:

$$\nabla_i \nabla_j Y_k - \nabla_j \nabla_i Y_k = -R_{ijkl} Y_l.$$

From these equations we can get the following fundamental identity, see e.g. [SSY, CY, N].

2.1. Lemma. *We have:*

$$\begin{aligned}(i)\Delta h_{ij} &= \nabla_i \nabla_j H + \overline{\nabla}_j \overline{R}_{0kik} + \overline{\nabla}_k \overline{R}_{0ijk} + H\overline{R}_{0ij0} \\ &\quad + h_{ij}(\overline{R}ic(\nu,\nu) + |A|^2) + h_{il}\overline{R}_{lkjk} + h_{lj}\overline{R}_{lkik} \\ &\quad + 2h_{lk}\overline{R}_{lijk} - Hh_{ik}h_{kj},\end{aligned}$$

It is essential for the a priori estimates that we take the *height function* u as

$$u(p) := t(F(p))$$

and the *gradient function* v, which measures the angel between M and the reference slices $\mathcal{S}_{\mathrm{t}}$ as

$$v := -\langle \nu, T \rangle.$$

Like in [B1,2.8] we get the identity

$$\Delta u = div\overline{\nabla}t + \psi^{-1} v H. \tag{2}$$

Furthermore we have the inequality:

$$\langle \tau_\alpha, e_\beta \rangle \leq v$$

for all $0 \leq \alpha, \beta \leq n$. This gives an estimate for the restriction of any p-tensor $B \in T^p(\mathcal{V})$ on M:

$$\|B|_{TM}\|_\infty \leq v^p \|B\|_\infty. \tag{3}$$

For a tensor B we denote by $\|B\|_{l,G}$ the C^l-norm of B in the area $G \subset \mathcal{V}$.

For the convenience of the reader let us recall the first variation formula for H (cf. [B1,CB,FM]) with respect to a deformation of the surrounding space generated by an arbitrary transverse vector field X:

$$X(H_X) = -\Delta\langle X, \nu\rangle + \langle X, \nu\rangle(|A^2| + \overline{R}ic(\nu,\nu))\langle X, \nabla H\rangle, \tag{4}$$

or in terms of the *Killing tensor* $\mathcal{L}_T g(X,Y) = \langle \overline{\nabla}_X T, Y\rangle + \langle \overline{\nabla}_Y T, X\rangle$:

$$\begin{aligned} T(H_T) &= \frac{1}{2}(\overline{\nabla}_\nu \mathcal{L}_T g)(e_i, e_i) - (\overline{\nabla}_{e_i} \mathcal{L}_T g)(\nu, e_i) - \frac{1}{2} H \mathcal{L}_T g(\nu,\nu) \\ &\quad - \mathcal{L}_T g(e_i, e_j) A(e_i, e_j).\end{aligned}$$

3 Evolution Equations

Starting from the evolution equation

$$\frac{d}{ds}F = -\frac{1}{H}\nu,$$

we will derive evolution equations for the height, the gradient, the curvature and other relevant quantities on M_s.

Proceeding exactly as in [Hu1], we first compute the derivative of the metric g, volume element μ and the unit normal ν on M:

3.1. Lemma. *We have the equations:*

$$(i) \qquad \frac{d}{ds}g_{ij} = -2\frac{1}{H}h_{ij},$$

$$(ii) \qquad \frac{d}{ds}\nu = \nabla\Big(-\frac{1}{H}\Big),$$

$$(iii) \qquad \frac{d}{ds}\mu = H\cdot\Big(-\frac{1}{H}\Big)\mu = -\mu.$$

We now notice that

$$\frac{d}{ds}u \quad = \quad = -\frac{1}{H}\psi^{-1}v,$$

and

$$\frac{d}{ds}v \quad = \quad -\langle T, \frac{1}{H^2}\nabla H\rangle + \frac{1}{H}\langle \overline{\nabla}_\nu T, \nu\rangle,$$

such that from (2) and (4) we obtain

3.2. Lemma.

$$\begin{aligned} (i) \quad \Big(\frac{d}{ds} - \frac{1}{H^2}\Delta\Big)u &= -\frac{1}{H^2}div\overline{\nabla}t - \frac{2}{H}\psi^{-1}v, \\ (ii) \quad \Big(\frac{d}{ds} - \frac{1}{H^2}\Delta\Big)v &= \frac{1}{H}\langle\overline{\nabla}_\nu T, \nu\rangle - \frac{1}{H^2}T(H_T) - \frac{1}{H^2}v(|A^2| + \overline{R}ic(\nu,\nu)). \end{aligned}$$

Finally we derive the evolution equation for the second fundamental form.

3.3. Lemma. *We have the equations*

$$\begin{aligned} (i) \quad \Big(\frac{d}{ds} - \frac{1}{H^2}\Delta\Big)h_{ij} &= -2\frac{1}{H^3}\nabla_i H\nabla_j H - 2\frac{1}{H}\overline{R}_{i00j} - 2\frac{1}{H^2}h_{kl}\overline{R}_{lijk} \\ &\quad - \frac{1}{H^2}h_{jl}\overline{R}_{lkik} - \frac{1}{H^2}h_{li}\overline{R}_{lkjk} - \frac{1}{H^2}\overline{\nabla}_k\overline{R}_{0ijk} \end{aligned}$$

$$-\frac{1}{H^2}\overline{\nabla}_j\,\overline{R}_{0kik}-\frac{1}{H^2}h_{ij}(|A^2|+\overline{R}ic(\nu,\nu)),$$

$$(ii)\qquad \Big(\frac{d}{ds}-\frac{1}{H^2}\Delta\Big)H \;=\; -2\frac{1}{H^3}\nabla_i H\nabla_i H+\frac{1}{H}(|A^2|+\overline{R}ic(\nu,\nu)).$$

Proof: Like in [Hu2] we get

$$\frac{d}{ds}h_{ij} \;=\; \nabla_i\nabla_j\Big(-\frac{1}{H}\Big)-\frac{1}{H}(h_{il}h_{lj}+\overline{R}_{i00j}).$$

and with Lemma 2.1, (i) follows. The second identity is an immediate consequence of 4 and can also be derived by taking the trace of (i). □

4 A Priori Estimates

In this and in the following section we assume that $\mathcal{V}$ is a cosmological spacetime, i.e. it is connected, globally hyperbolic and admits a compact Cauchy surface. This implies in particular the existence of a global time function $t\in C^\infty(\mathcal{V})$ as in Section 2. Moreover we will assume that the timelike convergence condition is satisfied, i.e.

$$\overline{R}ic(X,X)\geq 0$$

holds for all timelike vector fields X.

We are now going to prove the a priori estimates, which are necessary to proof Theorem 5.1, the main result. With Lemma 4.1 we will a lower a priori bound on the mean curvature H. Then we get the crucial a priori bound on the gradient function v. To prove this estimate we proceed in the same way as R. Bartnik did in [B1, Theorem 3.1].Then we get an upper a priori estimate on the mean curvature H and, using a maximum principle for tensors as Hamilton formulated it in [Ha1], an a priori bound for the full second fundamental form. So we have a a priori C^2 bound for the height function u which combined with estimates due to Krylov yields to an a priori $C^{2,\alpha}$ bound of u and thus to full regularity.

4.1. Lemma. *Let $g=g(t,y)$ be a smooth function with $t\in\mathbb{R}, y\in\mathbb{R}^k$. Let*

$$f(t)=\inf\Big\{g(t,y):y\in Y\Big\},$$

where Y is a compact set. Then $f(t)$ is Lipschitz and

$$\frac{\partial}{\partial t}f(t)\geq\inf\Big\{\frac{\partial}{\partial t}g(t,y):y\in Y(t)\Big\},$$

with $Y(t)=\Big\{y:g(t,y)=f(t)\Big\}$. For a *proof* see [Ha2]

From now on let M_s be a smooth solution of (1) on the interval $0 \leq s < s_0$, such that M_s is contained in a smooth compact subset $G \subset \mathcal{V}$ for all $0 \leq s < s_0$ and we assume that $H \geq H_0 > 0$ on M_0

4.2. Proposition. *Since $H \geq H_0 > 0$ on M_0, we have the following a priori bound for the mean curvature on M_s:*

$$H \geq e^{s\frac{1}{n}} H_0.$$

Proof: We only need to show that the function $f(s) := \min\Big\{He^{-s\frac{1}{n}}(p,s) : p \in M^n\Big\}$ is growing, i.e.

$$\frac{d}{ds} f \geq 0.$$

define $Y(s) := \Big\{p : He^{-s\frac{1}{n}}(p,s) = f(s)\Big\}$ and use Lemma 4.1:

$$\begin{aligned}
\frac{d}{ds} f(s) &\geq \min\Big\{\frac{d}{ds}(He^{-s\frac{1}{n}})(p,s) | p \in Y(s)\Big\} \\
&= \min\bigg\{\bigg(e^{-s\frac{1}{n}}\Big(\frac{1}{H^2}\Delta H - \frac{2}{H^3}\nabla_i H \nabla_i H \\
&\qquad + \frac{1}{H}(|A|^2 + \overline{R}ic(\nu,\nu) - \frac{1}{n}H^2)\Big)\bigg)(p,s) | p \in Y(s)\bigg\} \\
&\geq \min\bigg\{\bigg(e^{-s\frac{1}{n}}\Big(0 - 0 + \frac{1}{n}H + 0 - \frac{1}{n}H\Big)\bigg)\bigg\} = 0.
\end{aligned}$$

$\square$

Remark: Since $H > 0$ we get direct from Lemma 4.1 ($\frac{d}{ds}u < 0$) an upper bound for the height function: $u \leq u_0 (= sup_{M_0} u)$.

4.3. Proposition. *The gradient function v satisfies on M_s:*

$$v(p,s) \leq \Big(1 + \sup_{M_0} v\Big) \sup_{(q,r) \in M^n \times [0,s_0]} \exp\left(\lambda[u(q,r) - u(p,s)]\right) \cdot \exp(\hat{K}s),$$

where

$$\begin{aligned}
\lambda &= \lambda\Big(n, \|\mathcal{L}_T g\|_{1,G}, \|T\|_{1,G}, \|\psi^{-1}\|_{1,G}, \|\overline{R}m\|_{0,G}\Big), \\
\hat{K} &= 2\frac{2n}{4n+1} + \frac{1}{2}\lambda.
\end{aligned}$$

Proof: Following the argument in [B1, Theorem 3.1] let $\lambda, \hat{K} > 0$ be constants chosen later and define

$$C_\lambda := \Big(1 + \sup_{M_0} v\Big) \sup_{M^n \times [0,s_0]} e^{\lambda u}.$$

Suppose $ve^{\lambda u-\hat{K}s}$ reaches C_λ for the first time at $(p_1,s_1)\in M^n\times(0,s_0]$. Then at this point

$$\begin{aligned} 0 &\leq \Big(\frac{d}{ds}-\frac{1}{H^2}\Delta\Big)ve^{\lambda u-\hat{K}s},\\ 0 &= \nabla(ve^{\lambda u-\hat{K}s}). \end{aligned}$$

From Lemma 3.2 and [B1, Proof of Theorem 3.1] we infer

$$\begin{aligned} \Big(\frac{d}{ds}-\frac{1}{H^2}\Delta\Big)u &\leq |\Big(\frac{d}{ds}-\frac{1}{H^2}\Delta\Big)u|\leq \frac{1}{H^2}C_0v^2+\frac{2}{H}\psi^{-1}v\\ &\leq \frac{1}{H^2}C_1(C_0)v^2+\frac{1}{2}. \end{aligned}$$

as well as

$$\Big(\frac{d}{ds}-\frac{1}{H^2}\Delta\Big)v \leq -\frac{1}{H^2}v|A|^2+\frac{1}{H^2}C_2(v^3+|A|v^2).$$

where

$$C_0=C_0(\|\psi^{-1}\|_{1,G})\quad\text{and}\quad C_2=C_2(\|\mathcal{L}_Tg\|_{1,G},\|T\|_{1,G},n,\|\overline{R}m\|_{0,G})$$

Here we also use (3) and the timelike convergence condition. Hence we conclude

$$\frac{1}{H^2}v|A|^2\leq\frac{1}{H^2}C_2|A|v^2+\frac{1}{H^2}(C_2+C_0\lambda)v^3+\frac{1}{2}v\lambda+\frac{1}{H^2}\lambda^2|\nabla u|^2v-\hat{K}v.$$

Proceeding as in [B1, Theorem 3.1] we obtain

$$v|A|^2 \geq \Big(1+\frac{1}{2n}\Big)\lambda^2v|\nabla u|^2-C_3(n,\|T\|_{1,G})v^3-H^2v.$$

Set $\hat{K}=2\frac{2n}{4n+1}+\frac{1}{2}\lambda$ and since $|\nabla u|^2=\psi^{-2}(v^2-1)$ we finally arrive at

$$\frac{1}{2n}\lambda^2\psi^{-2}(v^2-1) \leq C_4v^2(\lambda+1).$$

at (p_1,s_1) where C_4 depends on C_0,C_1,C_2 and C_3. Since by definition of C_λ we have $v(p_1,s_1)\geq 2$ and we derive a contradiction for large enough λ. Repeating the argument with u replaced by $-u$ yields the result. □

4.4. Proposition. *We have the following a priori upper bound for the mean curvature H on M_s:*

$$H\leq C_5,$$

where

$$C_5=\max\Big\{\sup_{M_0}Hv^q,\ \sup_{M^n\times[0,s_0]}\Big(\frac{1}{H}C_6v^{4+\varepsilon}\Big)\Big\}$$

with $q := 2+\varepsilon$, $C_6 = C_6(n, \varepsilon^{-1}, \|\mathcal{L}_T g\|_{1,G}, \|T\|_{1,G})$ *and* $\varepsilon > 0$ *some fixed constant.*

Proof: Suppose Hv^q reaches $C_5 + \delta$ for the first time for any $\delta > 0$, at $(p_1, s_1) \in M^n \times (0, s_0]$. Then at this point

$$\nabla(Hv^q) = 0,$$
$$\Big(\frac{d}{ds} - \frac{1}{H^2}\Delta\Big)Hv^q \geq 0.$$

From Lemma 3.3 (ii) and the proof of Proposition 4.3 we have

$$\Big(\frac{d}{ds} - \frac{1}{H^2}\Delta\Big)H = -2\frac{1}{H^3}\nabla_i H \nabla_i H + \frac{1}{H}(|A^2| + \overline{R}ic(\nu,\nu))$$

and

$$\Big(\frac{d}{ds} - \frac{1}{H^2}\Delta\Big)v \leq -\frac{1}{H^2}v|A|^2 + \frac{1}{H^2}C_2(v^3 + |A|v^2).$$

Using again the fact that $\overline{R}ic(\nu,\nu) \geq 0$ we conclude

$$0 \leq \frac{1}{H}C_2 v^{4+\varepsilon_1} - \frac{1}{H}|A|^2 v^{2+\varepsilon_1}.$$

Since $|A|^2 \geq \frac{1}{n}H^2$ we finally get at the point (p_1, s_1)

$$C_5 + \delta = Hv^q \leq \frac{1}{H}C_6 v^{q+2} \leq C_5$$

which is a contradiction and therefor $Hv^q \leq C_5$. The Proposition follows because $H \leq Hv^q$. □

4.5. Proposition. *For the second fundamental form* h_{ij} *satisfies the following a priori estimate on* M_s*:*

$$h_{ij} \leq \bar{C}_0 g_{ij},$$

where

$$\bar{C}_0 = \max\Big\{\sup_{M^n\times[0,s_0]}\Big(\frac{1}{2} + 2H + 4\bar{C}_1\Big), \sup_{M_0}|A|\Big\},$$

with $\bar{C}_1 = \bar{C}_1\Big(H_0, \|\overline{R}m\|_{1,G}\Big)$ *and* H_0 *like in Proposition 4.2.*

Remark: It's enough to show an upper bound on h_{ij}, because H is strictly positive.

Proof: We are going to show that the eigenvalues of

$$M_{ij} = (\bar{C}_0 g_{ij} - h_{ij})$$

remain non-negative. First of all we need an evolution equation for M_{ij}. Using Proposition 3.3 (i) and Lemma 3.1 (i) we get

$$\Big(\frac{d}{ds} - \frac{1}{H^2}\Delta\Big)M_{ij} = N_{ij}$$

where

$$\begin{aligned} N_{ij} &= -2\bar{C}_0\frac{1}{H}h_{ij} + 2\frac{1}{H^3}\nabla_i H \nabla_j H + 2\frac{1}{H}\overline{R}_{i00j} + 2\frac{1}{H^2}h_{kl}\overline{R}_{lijk} \\ &\quad + \frac{1}{H^2}h_{jl}\overline{R}_{lkik} + \frac{1}{H^2}h_{li}\overline{R}_{lkjk} + \frac{1}{H^2}\overline{\nabla}_k \overline{R}_{0ijk} \\ &\quad + \frac{1}{H^2}\overline{\nabla}_j \overline{R}_{0kik} + \frac{1}{H^2}h_{ij}(|A^2| + \overline{R}ic(\nu,\nu)) \end{aligned}$$

In [Ha1, Theorem 9.1] a maximum principle for such an evolution equation was proved under the assumption that the absolute term N_{ij} is a polynomial of M_{ij} and g_{ij}. Since $\overline{R}m$ is smooth and H is bounded, it is easy to see that the argument is valid in our case as well. We have then only to consider the first time s_0, where at some point $p \in M_{s_0}$ a zero eigenvector $v = \{v_i\}$ of M_{ij} occurs, and Proposition 4.5 is proved if we can show that $N_{ij}v^iv^j$ is non-negative. For that purpose we choose an orthonormal basis $(e_1, \dots, e_n)$ for $T_pM_{s_0}$ such that h_{ij} (and thus M_{ij}) becomes diagonal. Let us assume that $v = e_1$ and that $\lambda_1, \dots, \lambda_n$ are the eigenvalues of h_{ij} at p. Then from $M_{11} = 0$ it follows that $\lambda_1 = \bar{C}_0$ at p and we obtain

$$\begin{aligned} N_{ij}v^iv^j &= N_{11} \geq -2\bar{C}_0\frac{1}{H}\lambda_1 + 2\frac{1}{H^3}\nabla_{e_1} H \nabla_{e_1} H - \frac{1}{2}\frac{1}{H^2}|A|^2 \\ &\qquad\qquad - \frac{1}{H^2}\bar{C}_1 + \frac{1}{H^2}\lambda_1|A|^2 \\ &\geq -2\bar{C}_0^2\frac{1}{H} - \frac{1}{H^2}\bar{C}_1 + \frac{1}{H^2}(\bar{C}_0 - \frac{1}{2})\bar{C}_0^2 \\ &\geq \frac{1}{H^2}(\bar{C}_0 - \frac{1}{2} - 2H)\bar{C}_0^2 - \frac{1}{H^2}\bar{C}_1 \\ &\geq \frac{1}{H^2}(\bar{C}_0 - \frac{1}{2} - 2H - 4\bar{C}_1)\bar{C}_0^2 \geq 0 \qquad \square. \end{aligned}$$

It is well known that one can get $C^{0,\alpha}$ a priori estimates on $\nabla^2 u$ in G from the a priori estimates on v, H and h_{ij} (like in the work of Gerhardt [G2: Theorem 4.2] and Urbas [U1: (3.55)]).

4.7. Proposition. *the second derivatives of the height function u are uniformly bounded in $C^{0,\alpha}(G)$. The estimate only depends on n, $\|u(\cdot,0)\|_{2,\alpha}$ and the ellipticity constant of* (1).

For the *proof* we refer to [K; Chapter 5].

4.8. Remark: With Proposition 4.6 and standard results from regularity theory, we get the full regularity of u.

5 Longtime Existence

With the help of the a priori estimates derived in the last section, we now obtain the main longtime existence result:

5.1. Theorem. *Let $M^n, \mathcal{V}$ as in Section 1 and suppose that $M_0 = F_0(M^n)$ is a smooth, compact, spacelike hypersurface in $\mathcal{V}$. Then there exists a unique family $M_s = F(\cdot, s)(M^n)$ of smooth compact spacelike hypersurfaces satisfying the initial value problem* (1) *on an interval* $0 \le s < s_0$. *Moreover, if M_s remains in a smooth compact region of $\mathcal{V}$ as $s \to s_0$, then the solution can be extended beyond s_0.*

Proof: It's easy to see that for short times the nonlinear system (1) reduces to a nonlinear scalar equation in u which is strictly parabolic (cf. [EH2]). Then, if the initial surface M_0 is smooth, short time existence of a unique smooth solution of (1) follows from standard parabolic theory, see e.g. [Ha3]. Now let s_0 be the maximal time of existence. Since by assumption M_s stays in a compact smooth region of $\mathcal{V}$ and therefor u is uniformly bounded, we infer from the last section that M_s converges to a smooth surface M_{s_0} as $s \to s_0$. Thus the solution M_s can be extended beyond s_0 by means of the local existence result.

5.2. Corollary. *Let $[b, t_0]$, $b < t_0$ be a time interval, i.e. b is in the past of t_0, with $\Sigma^n \times [b, t_0] \subset \mathcal{V}$ smooth and $\Sigma^n \times \{t_0\} \cap D_-(M_0) \neq \emptyset$. If M_s, $s \in [0, S)$, is a maximal solution of* (1) *then for each $\tau \in [b, t_0]$ there exists a $s_\tau \in [0, S)$ with $\Sigma^n \times \{\tau\} \cap M_{s_\tau} \neq \emptyset$.*

With $D_-(M_0)$ we denote the *past domain of dependence* of M_0. These are all points $p \in \mathcal{V}$ such that every future-inextendable non spacelike curve through p intersects M_0.

5.3. Corollary. *If the "inverse mean curvature flow"* (1) *exists for all times then the area of (M_s) tends to zero as $s \to \infty$ and the foliation exhausts $\mathcal{V}$ up to the singularity*

Proof of 5.2 and 5.3: The volume decreases exponentially, because it follows from Lemma 4.1 (*iii*) that $\frac{d}{ds}\int d\mu = -\int d\mu$. $\square$

5.4. Remarks. (i) We proved the existence of a foliation without using any *barrier surfaces* as in [G1,EH1].
(ii) Corollary 5.3 implies the existence of a singularity in a smooth cosmological spacetime which satisfies the timelike convergence condition and has a spacelike hypersurface with strictly positive mean curvature, which was first proved by *Hawking* and *Penrose*, see [HE].

6 Example

Finally we are interested in the special case where the cosmological spacetime is a Friedmann cosmological model, which is a special Robertson-Walker spacetime.

A Robertson-Walker spacetime is a warped product $M(k, f) = I \times_f S$. Here S is a connected, often three-dimensional Riemannian manifold of constant curvature $k = -1, 0$ or 1 and $f > 0$ is a smooth function on an open interval I in the one-dimensional Minkowski space $\mathbb{R}^1_1$.

We are talking about a Friedmann cosmological model, if the *scale function* f in a Robertson-Walker spacetime satisfies the Friedman equation $f'^2 + k = A/f$, where A is a constant depending only on mass. This is the model for *dust*.

6.1. Proposition. *Let M_0 be a smooth, orientable, compact, spacelike hypersurface without boundary in a $(n+1)$-dimensional Friedmann cosmological model $\mathcal{V} := I \times_f S$ given by an immersion*

$$F_0 : M^n \to \mathcal{V}.$$

where $F_0(M^n) = t_0 \times_f S$ with $t_0 \in I$. Then a solution to (1) exists for all time. Furthermore the hypersurfaces M_s approach the beginning singularity ($f = 0$) as s tends to infinity.

Proof: Notice that $F(p, s) = (t(s), p)$, $\nu(p, s) = T(p, s)$ and from general results on warped products (see [ON, Ch.7]) we get $H(p, s) = -\frac{f(t(s))}{nf'(t(s))}$. So that we can rewrite (1) as

$$\frac{d}{ds}t(s) = -\frac{f(t(s))}{n \cdot f'(t(s))}, \qquad t(0) = t_0. \tag{5}$$

with $f(t_0) > 0, f'(t_0) > 0$ and $f''(t_0) \leq 0$.

Let the beginning singularity be at $t_* = 0$ ($\Rightarrow f(t_*) = 0$), then we can solve the Friedmann equation explicitly in all three cases:

(α) $\quad k = 0 : \quad f(t) = Ct^{\frac{2}{3}}$, *with* $4C^3 = 9A$;

$$(\beta) \qquad k=1: \quad t=\frac{1}{2}A(\theta-\sin\theta), \quad f=\frac{1}{2}A(1-\cos\theta) \qquad \theta\in(0,2\pi);$$
$$(\gamma) \qquad k=-1: \quad t=\frac{1}{2}A(\sinh\eta-\eta), \quad f=\frac{1}{2}A(\cosh\eta-1) \qquad \eta>0.$$

In the case (α) we get the following solution to (5):

(α): $t(s)=t_0e^{-\frac{3}{2n}s}$ and the value $t(s)=t_*=0$ is attained for $s\to\infty$.

In the case (β) and (γ) we get solutions to: $\frac{d}{ds}\theta(t(s))=-f/(n\frac{df}{d\theta})$, $\theta(t_0)=\theta_0$ (which is a direct consequence from (5)). These solutions are:

(β): $\theta(s)=\arcsin(\exp[(-s+n\ln[\sin(\theta_0/2)^2])/2n])$ and the value $t(s)=t_*=0$ is attained for $s\to\infty$.

(γ): $\eta(s)=\operatorname{arcsinh}(\exp[(-s+n\ln[\sinh(\eta_0/2)^2])/2n])$ and the value $t(s)=t_*=0$ is attained for $s\to\infty$.

6.2. *Remark:* The qualitative behaviour near the singularity of (α), (β) and (γ) is the same (see e.g. [HE]).

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