THE NULL-TIMELIKE BOUNDARY PROBLEM FOR MAXWELL'S EQUATIONS IN MINKOWSKI SPACE

Robin Balean *
Robert Bartnik †
University of New England
Armidale, NSW Australia

July 11, 2001

Abstract

We consider Maxwell's equations in an exterior region of Minkowski space, with boundary conditions specified on a timelike cylinder and on a characteristic cone. We prove gauge fixing results and uniqueness and existence results which lead us to introduce a class of gauge-reduced boundary data, which has the property that there is a one–one correspondence between boundary data in the class and the gauge-equivalence classes of solutions of Maxwell's equations in the exterior region. This class of boundary data provides a constraint-free parameterisation of the space of all solutions of Maxwell's equations up to gauge equivalence.

Keywords: *Maxwell's equations; characteristic boundary value problem; existence and uniqueness theorems; configuration space, phase space*

1 Introduction

The modelling of radiation in the hyperbolic equations of mathematical physics leads naturally to problems involving posing boundary data on an initial characteristic (null) cone and on a timelike cylinder. Data on the timelike cylinder describes the outgoing fields produced by the unspecified

*From 1/2/97: Fachbereich Mathematik, Technische Hochschule Darmstadt, Schloßgartenstrasse 7, 64289 Darmstadt, Germany

†From 1/3/97: School of Mathematics and Statistics, University of Canberra, PO Box 1, Belconnen ACT 2616, Australia

interior sources; data on the future characteristic cone represents incoming radiation generated in the exterior region. It could even be argued that such problems are more natural from the physical viewpoint than the Cauchy problems considered traditionally, where initial data is specified on a space-like hypersurface. In any case, it is important to understand the properties of null-timelike boundary problems for common hyperbolic equations and systems.

Numerical algorithms based on a characteristic coordinate have been used in recent years to study various problems arising in general relativity, including the massless scalar field (wave equation) in Minkowski space [GW92],[Betal96]; the spherically symmetric Einstein equations coupled with scalar field matter [C91], [GP87], [HS96]; and the full vacuum Einstein equations [GPW94], [Bar96]. However, numerical instabilities have been observed in some cases, arising apparently from violations of the contraint equations [GP96]. These doubts underline the need for more detailed study of hyperbolic equations in characteristic coordinates.

Particularly in situations where gauge degeneracies are present, one would like to know how much boundary data can be prescribed, firstly in order to obtain a well-posed boundary value problem (ie. so that existence and uniqueness theorems can be established), and secondly, so that *all* solutions of the equations (perhaps up to gauge degeneracy) are covered by the boundary conditions. Ideally this would lead to a classification, or parameterisation, of the space of all solutions (the "configuration space"), by the class of boundary data.

One important difference between the Cauchy surface and characteristic approaches is that characteristic formulations naturally extend to include the propagation of shocks, or discontinuities in the field, which are usually precluded in Cauchy problem formulations by regularity assumptions on the initial data. This is because hyperbolic equations in characteristic coordinates have a natural transport equation structure, which is adapted to modelling shock propagation in the direction of the characteristic coordinate, since the resulting energy estimates differentiate between derivatives transverse and tangent to the characteristic surface. The study [Bal96b] exploited the transport form of the linear wave equation to clearly delineate the differences in the regularity properties of the solution in directions along, tangent and transverse to the characteristic hypersurfaces.

Our aim in the present paper is to understand the space of free boundary data for Maxwell's equations with null–timelike boundary, by proving global existence and uniqueness theorems, firstly for the Lorentz gauge with appropriate boundary conditions, and then for Maxwell's equations. These results provide a rather complete picture of the configuration space of Maxwell fields in the exterior domain, since they lead to a bijection with the space of gauge-reduced boundary data.

Since the emphasis here is on determining the nature of the "best" boundary conditions on handling the gauge degeneracy of the equations, we work only with C^∞ data and solutions. However, the wave equation existence results of [Bal96b] which provide our main technical support, provide existence in Sobolev spaces tailored to the differing regularities in the transverse and tangential-characteristic directions, so regularity results which are more general than those given here are certainly possible.

The main results of this paper are then as follows. The boundary conditions (4.8a–4.8c) coupled with the Lorentz gauge condition, uniquely determine the gauge potential (lemma 4.3), irrespective of Maxwell's equations. Then theorem 4.4 shows that every electromagnetic potential is gauge–equivalent to a potential satisfying these gauge and boundary conditions. Solutions of Maxwell's equations are uniquely determined by their boundary data (5.11), (5.12), by theorem 5.2. Existence of solutions satisfying these boundary conditions is shown in theorem 5.4, by solving a sequence of modified scalar wave equations. The existence and uniqueness theorems combine to show that *every* solution of Maxwell's equations may be obtained by this construction, leading to the classification theorem 6.2, which collects the Lorentz gauge-fixing and the Maxwell existence and uniqueness results to deduce a parameterisation of the space of all solutions in terms of the free boundary data.

Since the conditions (5.11), (5.12) do not involve any differential constraints, they give a class of boundary data which provides a faithful parameterisation of the space of gauge-equivalence classes of solutions on Maxwell's equations in the exterior region Ω, by unconstrained fields on $S^2 \times \mathbb{R}^+$. This lack of differential constraints on the data should be compared with the necessity of requiring the divergence constraint $\operatorname{div} E = 0$ on the electric field along the spacelike initial surface in the Cauchy problem. This constraint complicates the analysis of the free data for the Cauchy problem — see for example [EM82].

There is one curious feature of our results which we wish to emphasise. In several places, the requirements of the analysis force us to first normalise the potential to have both magnetic and electric charges vanishing. This may well be related to the duality invariance of the equations, and to the existence of representations of the solution in terms of Hertz potentials [H89],[P65],[S79], but we do not further investigate this here.

2 Notation and equations

We shall work in Minkowski space $\mathbb{R}^{1,3}$, with Cartesian coordinates x^0, x^1, x^2, x^3, with the Minkowski metric

$$g(\partial_\alpha, \partial_\beta) = \eta_{\alpha\beta} = \mathrm{diag}(-1, 1, 1, 1)$$

We sometimes write $t = x^0$ for the time coordinate. Partial differentiation will be denoted with ∂, so that $\partial_\alpha = \partial/\partial x^\alpha$. Greek indices take values $0, 1, 2, 3$, Roman indices take values $1, 2, 3$, and we use the Einstein convention with repeated indices indicating summation. Indices are raised and lowered using the metric, for example

$$\partial^\alpha = \eta^{\alpha\beta}\partial_\beta = \sum_{\beta=0}^{3} \eta^{\alpha\beta}\partial_\beta$$

In particular, $-\partial^0 = \partial_0 = \partial_t$ and $\partial^i = \partial_i$.

We wish to solve Maxwell's equations with data prescribed on a null cone $\mathcal{C}$ and a timelike tube $\mathcal{T}$ given by (with $\rho_0 > 0$ a fixed constant)

$$\begin{aligned} \mathcal{T} &= \left\{(t, x) \in \mathbb{R}^+ \times \mathbb{R}^3 \;\middle|\; r = \rho_0, t \geq 0\right\} \\ \mathcal{C} &= \left\{(t, x) \in \mathbb{R}^+ \times \mathbb{R}^3 \;\middle|\; t - r + \rho_0 = 0, r \geq \rho_0\right\}, \end{aligned}$$

where $r := |x| = ((x^1)^2 + (x^2)^2 + (x^3)^2)^{1/2}$, and $\mathcal{T}$ and $\mathcal{C}$ bound the exterior domain

$$\Omega = \left\{(t, x) \in \mathbb{R}^+ \times \mathbb{R}^3 \;\middle|\; t > r - \rho_0,\ r > \rho_0\right\}$$

The corner intersection sphere $\mathcal{S} = \mathcal{C} \cap \mathcal{T}$ is thus

$$\mathcal{S} = \left\{(t, x) \in \mathbb{R}^+ \times \mathbb{R}^3 \;\middle|\; t = 0, r = \rho_0\right\}$$

It will be convenient to use the null-polar coordinate system $(z, \rho, \vartheta, \varphi)$ on $\mathbb{R}^4$, which is given in terms of the usual Cartesian coordinates by $z = t - r + \rho_0$, $\rho = r$ and (ϑ, φ) are the standard spherical polar coordinates on S^2. $\mathcal{T}$ then corresponds to the set $\{(z, \rho, \vartheta, \varphi) \in \Omega;\ \rho = \rho_0, z \geq 0\}$ and $\mathcal{C}$ to $\{(z, \rho, \vartheta, \varphi) \in \Omega;\ z = 0, \rho \geq \rho_0\}$. The coordinate tangent vectors are related by

$$\begin{aligned} \partial_z &= \partial_t \\ \partial_\rho &= \partial_t + \partial_r \end{aligned}$$

where $\partial_z \phi$ denotes the directional derivative of the function ϕ in the direction of increase of z in the (z, ρ)-coordinate system.

In terms of a 1-form $A = A_\mu dx^\mu$ (the electromagnetic potential), Maxwell's equations can be written as $d * dA = 0$, where $* : \Lambda^2(T^*\mathbb{R}^{1,3}) \to \Lambda^2(T^*\mathbb{R}^{1,3})$ is the Hodge star and d is the exterior derivative. In component form this is

$$\Box A_\mu + \partial_\mu \partial^\alpha A_\alpha = 0 \tag{2.1}$$

where $\Box u := -\eta^{\alpha\beta}\partial_\alpha \partial_\beta u \;= \partial_t^2 u - \partial^i \partial_i u$. If we now impose the Lorentz gauge,

$$L(A) := -\partial^\mu A_\mu = 0 \tag{2.2}$$

then each of the components of A will satisfy the linear wave equation

$$\Box A_\mu = 0 \tag{2.3}$$

In the (z, ρ) coordinates, the linear wave equation $\Box u = f$ takes the form

$$\frac{2}{\rho}\partial_\rho(\rho \partial_z u) - \frac{1}{\rho^2}\partial_\rho(\rho^2 \partial_\rho u) - \frac{1}{\rho^2}\Delta_{S^2} u = f \quad \text{in } \Omega \tag{2.4}$$

where Δ_{S^2} is the Laplacian on the sphere S^2.

We shall consider both Dirichlet and mixed Dirichlet–Neumann boundary conditions. The Dirichlet conditions are:

$$\begin{aligned} u\big|_{\mathcal{C}} &= u_0 \\ u\big|_{\mathcal{T}} &= u_1 \\ u_0\big|_{\mathcal{T}} &= u_1\big|_{\mathcal{C}} \end{aligned} \tag{DC}$$

Notice that the final condition, $u_0\big|_{\mathcal{T}} = u_1\big|_{\mathcal{C}}$, is required to ensure compatibility at the 'corner' $\mathcal{S} = \mathcal{C} \cap \mathcal{T}$. The Dirichlet–Neumann conditions are

$$\begin{aligned} u\big|_{\mathcal{C}} &= u_0 \\ \partial_r u\big|_{\mathcal{T}} &= u_1 \end{aligned} \tag{NC}$$

Our constructions will rely on the following simplified version of the existence results in [Bal96b]. Define the modified wave operator (cf. (2.4))

$$\tilde{\Box} u := 2\partial_\rho \partial_z u - \partial_\rho \partial_\rho u - \frac{1}{\rho^2}\Delta_{S^2} u. \tag{2.5}$$

Theorem 2.1 *Suppose* $u_0 \in C^\infty(\mathcal{C})$*,* $u_1 \in C^\infty(\mathcal{T})$ *and* $f \in C^\infty(\bar{\Omega})$*. There exist unique solutions* $u, \tilde{u} \in C^\infty(\bar{\Omega})$ *of the wave equation* $\Box u = f$*, and of the modified wave equation* $\tilde{\Box}\tilde{u} = f$*, with either Dirichlet–Neumann boundary conditions (NC) or, assuming* $u_0\big|_{\mathcal{S}} = u_1\big|_{\mathcal{S}}$*, with Dirichlet boundary conditions (DC).*

Thus for example, if $\Box u = 0$ and $u\big|_{\mathcal{C}\cup\mathcal{T}} = 0$, then $u = 0$ in Ω; unlike the Cauchy problem, information about derivatives transverse to the boundary $\mathcal{C} \cup \mathcal{T}$ is not required. Note that $u \in C^\infty(\bar{\Omega})$ means that there are uniform bounds for all derivatives on any compact subset of $\bar{\Omega}$, which may include parts of the boundary $\mathcal{C} \cup \mathcal{T}$. Note also that this result does not require decay conditions such as finiteness of energy; indeed, as shown in [Bal96a], such conditions are unnatural for this exterior characteristic boundary value problem.

3 Magnetic and Electric Charge

Maxwell's equations in terms of the curvature 2-form $F = \frac{1}{2}F_{\mu\nu}dx^\mu \wedge dx^\nu$, $F_{\mu\nu} = -F_{\nu\mu}$, are

$$dF = 0, \qquad d * F = 0$$

Since $dF = 0$ we may locally introduce a gauge potential 1-form $A = A_\mu dx^\mu$ satisfying $F = dA$, and then proceed to analyse the equations in terms of A. However this is in general not possible globally in Ω since the second cohomology $H^2(\Omega)$ is non-vanishing, and we now show how to sidestep this complication.

Since $dF = 0$, F represents a de Rham cohomology class $[F] \in H^2_{deR}(\Omega, \mathbb{R})$. Representatives of each class are given in polar coordinates by the 2-forms

$$F^{mag}_k := k \sin\vartheta \, d\vartheta \wedge d\varphi \tag{3.1}$$

where $k \in \mathbb{R}$ parameterises $H^2_{deR}(\Omega, \mathbb{R}) \simeq \mathbb{R}$. If $F \in [F^{mag}_k]$ then $\tilde{F} = F - F^{mag}_k = dA$ for some globally defined 1-form A, and it follows that both F^{mag}_k and $\tilde{F} = dA$ satisfy Maxwell's equations. Thus a general solution of Maxwell's equations may be parameterised by a globally defined 1-form A satisfying the potential form (2.1) of the equations, and a constant $k \in \mathbb{R}$.

Note that if we accept the interpretation of F as representing ($-i$ times) the curvature of some $U(1)$ (complex line) bundle over Ω, then the constant k satisfies the Dirac quantisation condition $k \in 2\pi\mathbb{Z}$, and represents the Chern class or magnetic charge of the line bundle.

For any 2-surface $\Sigma \subset \Omega$, the magnetic charge of a solution F of Maxwell's equations within Σ is given by the flux integral $\oint_\Sigma F$, and this is constant on the homology class $[\Sigma] \in H_2(\Omega)$ since $dF = 0$. Since $\mathcal{S} = \mathcal{C} \cap \mathcal{T}$ generates $H_2(\Omega)$, the total magnetic charge $k(F)$ of F in Ω is given by

$$k = k(F) = \frac{1}{4\pi} \int_{\mathcal{S}} F = \frac{1}{4\pi} \int_{\mathcal{S}} F_{\vartheta\varphi} \, d\vartheta \, d\varphi \tag{3.2}$$

Similarly, we define the total electric charge $e(F)$ of a general solution F by

$$e = e(F) = \frac{1}{4\pi} \oint_{\mathcal{S}} *F = \frac{1}{4\pi} \oint_{\mathcal{S}} F_{tr} \rho_0^2 \sin\vartheta \, d\vartheta \, d\varphi \tag{3.3}$$

Just as F_k^{mag} satisfies Maxwell's equations and has magnetic charge k, the dual form

$$F_e^{elec} = e/r^2 \, dt \wedge dr \tag{3.4}$$

satisfies Maxwell's equations and has electric charge $e \in \mathbb{R}$. Unlike F_k^{mag} however, there is a global potential representation $F_e^{elec} = dA^{elec}$, with $A^{elec} = e/r \, dt$. The electric charge coincides with the flux integral $\oint_\Sigma *F$, for any 2-surface Σ homologous to $\mathcal{S}$ in Ω, provided that the second of Maxwell's equations, $d*F = 0$, is satisfied. Note that unlike the magnetic case, where the condition $dF = 0$ is automatically satisfied as a concommitant of the topological interpretation of F as the curvature of a $U(1)$ connection, the Maxwell equation $d*F = 0$ needed to ensure that the electric charge is well-defined, is a differential rather than a topological restriction on the 2-form F.

Since Maxwell's equations are linear, we may represent the general solution as $F = F_k^{mag} + F_e^{elec} + dA$, where A is a globally defined 1-form (hence with magnetic charge zero), with zero electric charge as well. In particular, it suffices to study Maxwell's equations in terms of a global 1-form A with zero electric charge, and the remainder of this paper will be concerned solely with this "zero charge" case. The necessity of the zero electric charge condition arises in the analysis of the gauge normalisation — see the following sections.

4 Lorentz gauge conditions

We shall now prove a number of properties of the Lorentz gauge, supplemented by appropriate boundary conditions. First we show that since the Lorentz gauge condition is propagated by the wave equation, we only need to impose conditions on the boundary.

Lemma 4.1 *If A is a 1-form whose component functions A_μ in rectangular coordinates satisfy the wave equation (2.3) on the region Ω, and if*

$$L(A) := -\partial^\mu A_\mu = 0 \qquad \text{on} \quad \mathcal{C} \cup \mathcal{T},$$

then $L(A) = 0$ in Ω and A satisfies Maxwell's equations.

Proof: This follows simply from the uniqueness of solution property for the wave equation with null-timelike data. If equation (2.3) is satisfied on the whole domain, then we also have

$$-\Box L = \Box(\partial^\mu A_\mu) = \partial^\mu \Box A_\mu = 0 \tag{4.1}$$

with zero boundary conditions. Lemma 2.1 now tells us that the unique solution is $L = L(A) = 0$, so the Lorentz condition is satisfied everywhere. It follows from (2.1) that A then satisfies Maxwell's equations. □

Another property of the Lorentz gauge is that it is not too restrictive – we do not lose any solutions by imposing it.

Lemma 4.2 *If A satisfies Maxwell's equations in the region Ω then there is a function Λ on Ω such that $B = A + d\Lambda$ satisfies the Lorentz gauge condition $L(B) = 0$.*

Proof: Since A is a solution of Maxwell's equations, the rectangular components A_μ satisfy

$$\Box A_\mu = -\partial_\mu \partial^\alpha A_\alpha \tag{4.2}$$

Let Λ be the solution of the equation $\Box\Lambda = \partial^\mu A_\mu$ Dirichlet conditions $\Lambda\big|_{\mathcal{C}} = \Lambda\big|_{\mathcal{T}} = 0$ (the existence of such a solution follows from lemma 2.1). It follows easily that $B = A + d\Lambda$ satisfies

$$L(B) = -\partial^\mu B_\mu = -\partial^\mu A_\mu + \Box\Lambda = 0, \tag{4.3}$$

as required. □

A drawback of the Lorentz gauge is that it does not uniquely determine the potential within the gauge equivalence class: two distinct but gauge-equivalent potentials A and $B = A + d\Lambda$ will satisfy the Lorentz gauge condition together if $\Box\Lambda = 0$. To fix the gauge we need to impose boundary conditions which ensure that such a Λ must be constant, whilst at the same time taking care that the conditions are not so restrictive that some potentials cannot be gauge-transformed to satisfy the conditions.

To describe the conditions we use to fix the gauge, it is useful to decompose the 1-form A into $(z, \rho, \vartheta, \varphi)$ components, parameterised by

$$A = A_0 dt + A_i dx^i = A_z dz + A_\rho d\rho + A_\vartheta \rho d\vartheta + A_\varphi \rho \sin\vartheta \, d\varphi \tag{4.4}$$

Let A_r denote the radial component, so $A_r = r^{-1} x^i A_i = A_\rho - A_z$. The rectangular and null-polar coefficients are related explicitly by

$$A_z = A_0 \tag{4.5a}$$

$$A_r = \cos\varphi \sin\vartheta \, A_1 + \sin\varphi \sin\vartheta \, A_2 + \cos\vartheta \, A_3 \tag{4.5b}$$

$$A_\vartheta = \cos\varphi \cos\vartheta \, A_1 + \sin\varphi \cos\vartheta \, A_2 - \sin\vartheta \, A_3 \tag{4.5c}$$

$$A_\varphi = -\sin\varphi \, A_1 + \cos\varphi \, A_2 \tag{4.5d}$$

For ease of notation, we shall write A_{S^2} for the pair (A_ϑ, A_φ); more precisely, A_{S^2} is the 1-form $A_{S^2} = A_\vartheta d\vartheta + A_\varphi \sin\vartheta d\varphi$, so $A = A_z dz + A_\rho d\rho + \rho A_{S^2}$. The spherical divergence of A_{S^2} is defined by

$$\operatorname{div}_{S^2} A_{S^2} = \frac{\partial A_\vartheta}{\partial \vartheta} + \cot\vartheta A_\vartheta + \frac{1}{\sin\vartheta} \frac{\partial A_\varphi}{\partial \varphi} \tag{4.6}$$

and the Lorentz gauge condition in the null-polar coordinates is

$$\mathsf{L}(A) = \rho^{-2} \partial_\rho(\rho^2 A_z) - \rho^{-2} \partial_\rho(\rho^2 A_\rho) + \partial_z A_\rho - \rho^{-1} \operatorname{div}_{S^2} A_{S^2} = 0 \tag{4.7}$$

The relations between the equations satisfied by the rectangular coefficients (A_0, A_i) and the null-polar coefficients $(A_z, A_\rho, A_\vartheta, A_\varphi)$ are examined in more detail in the following section.

We say that the potential A satisfies the *Lorentz conditions* if $\mathsf{L}(A) = 0$ and the following boundary conditions hold:

$$A_\rho\big|_{\mathcal{C}} = 0 \tag{4.8a}$$

$$A_z\big|_{\mathcal{T}} = 0 \tag{4.8b}$$

$$\partial_\rho A_z\big|_{\mathcal{S}} = 0 \tag{4.8c}$$

We first show that the Lorentz conditions suffice to fix the potential uniquely.

Lemma 4.3 *If A, B are gauge-equivalent potentials which both satisfy the Lorentz conditions, then $A = B$.*

Proof: Since $B = A + d\Lambda$ for some Λ, it will suffice to show that Λ is constant on Ω. The Lorentz gauge condition satisfied by both A, B, shows that Λ automatically satisfies $\Box\Lambda = 0$. From (4.8a) and (4.8b) it follows that $\partial_\rho\Lambda = 0$ on $\mathcal{C}$ and $\partial_z\Lambda = 0$ on $\mathcal{T}$, so it will sufffice to show that Λ is constant on $\mathcal{S} = \mathcal{C}\cap\mathcal{T}$, since then the boundary conditions, and uniqueness for the wave equation (2.4), combine to show that Λ is constant in Ω.

So, consider the restriction to $\mathcal{S}$ of the expression (2.4) for the wave equation in null-polar coordinates, together with the conditions $\partial_\rho\Lambda = 0$ on $\mathcal{C}$ and $\partial_z\Lambda = 0$ on $\mathcal{T}$. These give

$$0 = \Box\Lambda = 2\partial_\rho\partial_z\Lambda - \rho^{-2}\Delta_{S^2}\Lambda \quad \text{on}\mathcal{S} \tag{4.9}$$

Since A, B both satisfy the condition (4.8c) we have $\partial_\rho\partial_z\Lambda = 0$ and thus $\Delta_{S^2}(\Lambda|_\mathcal{S}) = 0$, which implies that $\Lambda|_\mathcal{S}$ is constant, as required. $\Box$

We shall now show the existence of a gauge equivalent potential satisfying the Lorentz conditions (4.7),(4.8a)–(4.8c). Lemma 4.3 and this result provide a canonical representative in each gauge-equivalence class of 1-form potentials on Ω — in other words, a "slice" of the action of the group of gauge transformations on the space of all smooth connection 1-forms on Ω. Note that it is here that the condition that the electric charge vanish (at least, on the corner sphere $\mathcal{S} = \mathcal{C}\cap\mathcal{T}$) is used. This condition is necessary because the Lorentz boundary conditions (4.8a)–(4.8c) combine with $L(A) = 0$ to show that on $\mathcal{S}$, the *radial electric field* $E_r = F_{tr} = F_{z\rho}$ satisfies

$$E_r := \partial_z A_\rho - \partial_\rho A_z = \rho^{-1}\operatorname{div}_{S^2} A_{S^2} \tag{4.10}$$

which has vanishing integral over $\mathcal{S}$.

Theorem 4.4 *Let* B *be any potential in* Ω *satisfying*

$$\oint_\mathcal{S} *dB = 0 \tag{4.11}$$

Then there exists a unique gauge-equivalent potential $A = B + d\Lambda$ *which satisfies the Lorentz gauge condition (4.7) and the boundary conditions (4.8a)–(4.8c).*

Proof: By lemma 4.2 we may assume without loss of generality that $L(B) = 0$. Let $\alpha \in C^\infty(\mathcal{S})$ satisfy $\oint_\mathcal{S}\alpha = 0$ and

$$\Delta_{S^2}\alpha = \rho_0^2 E_r\big|_\mathcal{S} - \rho_0\operatorname{div}_{S^2} B_{S^2}\big|_\mathcal{S} \tag{4.12}$$

The condition (4.11) says precisely that E_r has mean value zero on $\mathcal{S}$, so the electric charge of B along $\mathcal{S}$ vanishes. Since both terms on the right hand side have mean value zero, by the Fredholm Alternative a solution α exists and is unique up to a constant. Let Λ be the (unique) solution of the equation

$$\Box\Lambda = 0 \qquad \text{in } \Omega \tag{4.13}$$

with Dirichlet boundary conditions

$$\Lambda\big|_{\mathcal{C}}(\rho) = \alpha - \int_{\rho_0}^{\rho} B_\rho\big|_{\mathcal{C}}(\rho')d\rho' \tag{4.14}$$

$$\Lambda\big|_{\mathcal{T}}(z) = \alpha - \int_{0}^{z} B_z\big|_{\mathcal{T}}(z')dz' \tag{4.15}$$

where the integrals are along the lines $(z, \vartheta, \varphi) = \mathsf{const.}$ and $(\rho, \vartheta, \varphi) = \mathsf{const.}$ respectively, and the (ϑ, φ) dependence of Λ is not indicated explicitly. The construction of Λ ensures that $A = B + d\Lambda$ satisfies $L(A) = 0$ and that the following relations hold on the boundary

$$\begin{aligned}
A_\rho\big|_{\mathcal{C}} &= B_\rho\big|_{\mathcal{C}} + \partial_\rho\Lambda\big|_{\mathcal{C}} = 0 \\
A_z\big|_{\mathcal{T}} &= B_z\big|_{\mathcal{T}} + \partial_z\Lambda\big|_{\mathcal{T}} = 0 \\
\operatorname{div}_{S^2} A_{S^2}\big|_{\mathcal{S}} &= \operatorname{div}_{S^2} A_{S^2}\big|_{\mathcal{S}} + \rho_0^{-1}\Delta_{S^2}\alpha \\
&= \rho_0 E_r\big|_{\mathcal{S}}
\end{aligned}$$

Substituting for $\operatorname{div}_{S^2} A_{S^2}\big|_{\mathcal{S}}$ and $E_r = \partial_z A_\rho - \partial_\rho A_z$ into the restriction of $L(A) = 0$ to $\mathcal{S}$

$$L(A)\big|_{\mathcal{S}} = \partial_\rho A_z + \partial_z A_\rho - \rho_0^{-1} \operatorname{div}_{S^2} A_{S^2}\big|_{\mathcal{S}}$$

shows that $\partial_\rho A_z\big|_{\mathcal{S}} = 0$, as required by (4.8c). Thus B satisfies the required gauge and boundary conditions; and B is the unique such potential by lemma 4.3. $\Box$

5 Uniqueness and existence for Maxwell's equations

We now consider the effect of imposing Maxwell's equations on the class of canonical gauge potentials selected in the previous section. The first step is to select additional boundary conditions, for which *uniqueness* can be shown, within the space of all solutions of Maxwell's equations. The next step is to show *existence* of solutions for general boundary data in this class. The uniqueness and existence results are then combined into the final classification theorem 6.2 in the next section.

The first difficulty is to choose boundary conditions which are sufficiently general so as to encompass all solutions of Maxwell's equations, but which at the same time, do not conflict with the differential constraints resulting from the Lorentz gauge and Maxwell's equations.

We shall need relations between the wave equations satisfied by the rectangular and polar coefficients of A, which we compute by summarising the relations (4.5b)–(4.5d) in the form

$$\begin{bmatrix} A_r \\ A_\vartheta \\ A_\varphi \end{bmatrix} = \begin{bmatrix} \cos\varphi\sin\vartheta & \sin\varphi\sin\vartheta & \cos\vartheta \\ \cos\varphi\cos\vartheta & \sin\varphi\cos\vartheta & -\sin\vartheta \\ -\sin\varphi & \cos\varphi & 0 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} = J \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} \tag{5.1}$$

where we use $A_r = A_\rho - A_z$ for simplicity, and J is an orthogonal matrix, $J^T = J^{-1}$. A direct and potentially tedious computation using the expression (2.4) for the scalar wave equation in polar coordinates, now leads to the relations

(5.2)
$$\begin{bmatrix} \Box A_r + 2r^{-2}(A_r + \operatorname{div}_{S^2} A_{S^2}) \\ \Box A_\vartheta + r^{-2}(\csc^2\vartheta(A_\vartheta + 2\cos\vartheta\,\partial_\varphi A_\varphi) - 2\partial_\vartheta A_r) \\ \Box A_\varphi + r^{-2}(\csc^2\vartheta(A_\varphi - 2\cos\vartheta\,\partial_\varphi A_\vartheta) - 2\csc\vartheta\,\partial_\varphi A_r) \end{bmatrix} = J \begin{bmatrix} \Box A_1 \\ \Box A_2 \\ \Box A_3 \end{bmatrix}$$

Note that $\Box$ represents the *scalar*, not the tensor wave operator. These formulae may of course be understood tensorially, although we shall not use this interpretation.

In order to clarify the structure of the equations for $\Box A_\vartheta$ and $\Box A_\varphi$, we use the Hodge–Helmhotz representation for 1-forms on S^2 to express

$$A_{S^2} = d_2 v + *_2 d_2 w \tag{5.3}$$

where $d_2, *_2$ are the exterior derivative and Hodge dual operators on S^2, and $v, w \in C^\infty(\bar{\Omega})$ are determined up to functions constant in the S^2 directions. We normalise v, w by requiring their spherical averages $\mathrm{Ave}_{S^2}(v), \mathrm{Ave}_{S^2}(w)$ to vanish, ie.

$$\mathrm{Ave}_{S^2}(v)(z,\rho) := \oint_{\{(z,\rho)\}\times S^2} v(z,\rho,\vartheta,\varphi)\sin\vartheta\, d\vartheta\, d\varphi = 0 \tag{5.4}$$

which determines v, w uniquely. Explicitly we have

$$\begin{aligned} A_\vartheta &= \partial_\vartheta v + \csc\vartheta\,\partial_\varphi w \\ A_\varphi &= -\partial_\vartheta w + \csc\vartheta\,\partial_\varphi v \end{aligned} \tag{5.5}$$

which may be expressed in a complexified form using the operator $D_{S^2} := \partial_\vartheta - i\csc\vartheta\,\partial_\varphi$ by

$$A_\vartheta - iA_\varphi = D_{S^2}(v + iw) = (\partial_\vartheta - i\csc\vartheta\,\partial_\varphi)(v + iw)$$

Again after some computation we find that the relations (5.2) are equivalent to

$$\begin{bmatrix} \Box(A_r) + 2r^{-2}(A_r + \Delta_{S^2} v) \\ D_{S^2}\left(\Box(v + iw) - 2\rho^{-2}A_r\right) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -i \end{bmatrix} J \begin{bmatrix} \Box A_1 \\ \Box A_2 \\ \Box A_3 \end{bmatrix} \tag{5.6}$$

The following lemma summarises the computations we shall need.

Lemma 5.1 *The equations $\Box A_\mu = 0$, $\mu = 0, \ldots, 3$ are equivalent to the system*

$$\Box A_z = 0 \tag{5.7a}$$

$$\Box A_\rho = -2\rho^{-2}\left(A_\rho - A_z + \Delta_{S^2} v\right) \tag{5.7b}$$

$$\Box v = 2\rho^{-2}\left(A_\rho - A_z - \mathrm{Ave}_{S^2}(A_\rho - A_z)\right) \tag{5.7c}$$

$$\Box w = 0 \tag{5.7d}$$

Any 1-form A on Ω satisfies

$$\Box(\rho A_\rho) = -2\rho^{-1}\,\partial_\rho(\rho A_z) + \rho\Box A_z + x^i\Box A_i + 2L(A) \tag{5.8}$$

$$\tilde{\Box}(\rho^2 A_\rho) = -2\partial_\rho(\rho A_z) + \rho^2\Box A_z + \rho x^i\Box A_i + 2\rho L(A) \tag{5.9}$$

$$\begin{aligned} \partial_\rho(\rho\partial_\rho(\rho A_z)) = {} & \tfrac{1}{2}\partial_\rho^2(\rho^2 A_\rho) - \tfrac{1}{2}\Delta_{S^2} A_\rho + \partial_\rho(\rho\,\mathrm{div}_{S^2} A_{S^2}) \\ & - \rho\left(\tfrac{1}{2}\rho\Box A_z + \tfrac{1}{2}x^i\Box A_i - \partial_\rho(\rho L(A))\right) \end{aligned} \tag{5.10}$$

Proof: The equivalence follows easily from (5.2) since $A_z = A_0$, $A_\rho - A_z = A_r$ and $D_{S^2}U = 0$ only when the function $U = U(z, \rho, \vartheta, \varphi)$ is constant on each 2-sphere.

The remaining identities may be derived from (5.2) and (4.7), or by direct computation as follows. The first identity is obtained by combining the two formulas

$$\begin{aligned} \Box(\rho A_z) &= \rho\,\Box A_z + 2\partial_z A_z - 2\rho^{-1}\partial_\rho(\rho A_z) \\ \Box(x^i A_i) &= x^i\Box A_i + 2L(A) - 2\partial_z A_z \end{aligned}$$

and the second is an immediate consequence. Equation (5.10) may be verified by direct substitution, or it may be derived from (5.8) by expanding the definition of $\Box(\rho A_\rho)$ using (2.4), eliminating all resulting z partial derivatives against $\partial_\rho(\rho^2 L(A))$ using (4.7), and then gathering terms. $\square$

We now have the equipment needed to prove uniqueness of solutions of Maxwell's equations satisfying appropriate boundary conditions.

Theorem 5.2 *Suppose* A, B *are two solutions of Maxwell's equations such that in the Lorentz gauge with the Lorentz boundary conditions (4.8a)–(4.8c),*

$$A_{S^2}\big|_{\mathcal{C}} = B_{S^2}\big|_{\mathcal{C}} \tag{5.11}$$

$$A_{S^2}\big|_{\mathcal{T}} = B_{S^2}\big|_{\mathcal{T}} \tag{5.12}$$

Then $A = B$ *everywhere in* Ω.

Proof: We will show $\tilde{A} := A - B = 0$. Maxwell's equations and the Lorentz gauge, combined with the boundary conditions (4.8a), (5.11) on $\mathcal{C}$ ensure that all terms on the right hand side of (5.10) vanish along $\mathcal{C}$, so we have

$$\partial_\rho(\rho\partial_\rho(\rho\tilde{A}_z)) = 0 \qquad \text{on } \mathcal{C}.$$

Integrating twice with respect to ρ and using the Lorentz condition (4.8c) for $\partial_\rho \tilde{A}_z\big|_{\mathcal{S}}$ and $\tilde{A}_z = 0$ on $\mathcal{C} \cap \mathcal{T}$ gives $\tilde{A}_z\big|_{\mathcal{C}} = 0$. Since $\Box\tilde{A}_z = 0$, it follows by uniqueness and the Dirichlet conditions $\tilde{A}_z\big|_{\mathcal{C}\cup\mathcal{T}} = 0$ that $\tilde{A}_z = 0$ in Ω.

From (5.9) we see now that $\tilde{\Box}(\rho^2\tilde{A}_\rho) = 0$, and from (4.7) we have the Neumann boundary condition

$$\partial_r(\rho^2\tilde{A}_\rho) = 0 \quad \text{on } \mathcal{T} \tag{5.13}$$

Since $\tilde{A}_\rho\big|_{\mathcal{C}} = 0$, the uniqueness of solutions to the Dirichlet–Neumann problem for $\tilde{\Box}$ asserted by theorem 2.1 shows that $\tilde{A}_\rho = 0$ in Ω. Alternatively, to establish uniqueness we may apply the following energy lemma.

Lemma 5.3 *Suppose* $\tilde{\Box}u = 0$, *where* $\tilde{\Box} = 2\partial_z\partial_\rho - \partial_\rho^2 - \rho^{-2}\Delta_{S^2}$. *Then for any* $T \geq 0$,

$$\begin{aligned} &\int_{\mathcal{D}_T} \tfrac{1}{2}\left((\partial_t u)^2 + (\partial_r u)^2 + r^{-2}|\nabla_{S^2}u|^2\right) \sin\vartheta \, dr \, d\vartheta \, d\varphi \\ &= \int_{\mathcal{C}_T} \tfrac{1}{2}\left((\partial_\rho u)^2 + r^{-2}|\nabla_{S^2}u|^2\right) \sin\vartheta \, d\rho \, d\vartheta \, d\varphi \\ &+ \int_{\mathcal{T}_T} \partial_t u \partial_r u \sin\vartheta \, dt \, d\vartheta \, d\varphi \end{aligned} \tag{5.14}$$

where $\mathcal{D}_T = \{(z,\rho,\vartheta,\varphi) \,\big|\, t = z+\rho = T, \rho_0 \leq \rho \leq \rho_0 + T\}$ *and* $\mathcal{C}_T = \mathcal{C} \cap \{t \leq T\}$, $\mathcal{T}_T = \mathcal{T} \cap \{t \leq T\}$.

Proof: Multiplying $\tilde{\Box}u = 0$ by $\partial_z u \sin\vartheta\, dz\, dr\, d\vartheta\, d\varphi$, we obtain the identities

$$\begin{aligned}
0 &= d\Big\{\big(\partial_z u \partial_\rho u - (\partial_z u)^2\big)\sin\vartheta\, dz\, d\vartheta\, d\varphi \\
&\qquad + \tfrac{1}{2}\big((\partial_\rho u)^2 + r^{-2}|\nabla_{S^2} u|^2\big)\sin\vartheta\, d\rho\, d\vartheta\, d\varphi \\
&\qquad + \rho^{-2}\partial_z u *_2(d_2 u)\, dz\, d\rho\Big\} \\
&= d\Big\{\partial_t u\, \partial_r u \sin\vartheta\, dt\, d\vartheta\, d\varphi \\
&\qquad + \tfrac{1}{2}\big((\partial_r u)^2 + (\partial_t u)^2 + r^{-2}|\nabla_{S^2} u|^2\big)\sin\vartheta\, dr\, d\vartheta\, d\varphi \\
&\qquad + r^{-2}\partial_t u *_2(d_2 u)\, dt\, dr\Big\}
\end{aligned}$$

Integrating over the space-time region bounded by $\mathcal{D}_T, \mathcal{T}_T$ and $\mathcal{C}_T$ and applying Stokes' theorem using the form of the identity appropriate to each boundary component, gives the required energy formula. □

Applying the lemma with $u = \rho^2\tilde{A}_\rho$ and noting that the boundary conditions for u are now $u = 0$ on $\mathcal{C}$ and $\partial_r u = 0$ on $\mathcal{T}$, we see that the integrals over $\mathcal{C}$ and $\mathcal{T}$ vanish, so

$$0 = \int_{\mathcal{D}_T} \big((\partial_r u)^2 + (\partial_t u)^2 + r^{-2}|\nabla_{S^2} u|^2\big)\sin\vartheta\, dr\, d\vartheta\, d\varphi$$

from which it follows that $\tilde{A}_\rho = 0$. Finally, since $\tilde{A}_{S^2} = 0$ on $\mathcal{C}\cup\mathcal{T}$ and the potentials v, w for $\tilde{A}_{S^2}$ both vanish on $\mathcal{C}\cup\mathcal{T}$, uniqueness for the wave equations (5.7c), (5.7d) with $\tilde{A}_z = \tilde{A}_\rho = 0$ implies that $v = w = 0$ everywhere in Ω. Hence $\tilde{A} = 0$ and we have shown the required uniqueness. □

We now turn to the problem of establishing *existence* of solutions of Maxwell's equations with the above boundary conditions. Let $\Gamma^\infty(\mathcal{C}, T^*S^2)$ denote the space of C^∞ sections of the bundle $T^*S^2\times\mathbb{R} \to \mathcal{C} \simeq S^2\times\mathbb{R}$.

Theorem 5.4 *Let* $H_{\mathcal{C}} \in \Gamma^\infty(\mathcal{C}, T^*S^2)$ *and* $H_{\mathcal{T}} \in \Gamma^\infty(\mathcal{T}, T^*S^2)$, *where* $H_{\mathcal{C}}$ *and* $H_{\mathcal{T}}$ *agree along* $\mathcal{S}$,

$$H_0 := H_{\mathcal{C}}\big|_{\mathcal{S}} = H_{\mathcal{T}}\big|_{\mathcal{S}} \tag{5.15}$$

Then there exists a smooth solution, A, *of Maxwell's equations satisfying the Lorentz gauge (4.7), the Lorentz boundary conditions (4.8a)-(4.8c) and boundary conditions:*

$$A_{S^2}\big|_{\mathcal{C}} = H_{\mathcal{C}} \tag{5.16}$$

$$A_{S^2}\big|_{\mathcal{T}} = H_{\mathcal{T}} \tag{5.17}$$

The electric and magnetic charges of the curvature $F = dA$ *both vanish.*

Proof: The method is by explicit construction of a solution. We construct A_z, $\rho^2 A_\rho$ and ν, w as the solutions of the following sequence of boundary value problems:

$$
\begin{aligned}
&(5.18a) \quad \Box A_z &&= 0\\
&(5.18b) \quad A_z\big|_{\mathcal{T}} &&= 0\\
&(5.18c) \quad A_z\big|_{\mathcal{C}} &&= \rho^{-1}\int_{\rho_0}^{\rho} \operatorname{div}_{S^2} H_{\mathcal{C}}(0,\rho')\,d\rho' - \frac{\rho_0}{\rho}\log\left(\frac{\rho}{\rho_0}\right)\operatorname{div}_{S^2} H_0
\end{aligned}
$$

where the integral is along the null rays of $\mathcal{C}$;

$$
\begin{aligned}
&(5.19a) \quad \tilde{\Box}(\rho^2 A_\rho) &&= -2\partial_\rho(\rho A_z)\\
&(5.19b) \quad \partial_r(\rho^2 A_\rho)\big|_{\mathcal{T}} &&= -\rho_0 \operatorname{div}_{S^2} H_{\mathcal{T}} + \rho_0^2(\partial_\rho A_z)\big|_{\mathcal{T}}\\
&(5.19c) \quad \rho^2 A_\rho\big|_{\mathcal{C}} &&= 0
\end{aligned}
$$

where the value of A_z from the previous step is used to provide both the source term in (5.19a), and the Neumann data in (5.19b); and

$$
\begin{aligned}
&(5.20a) \quad \Box\nu &&= 2\rho^{-2}(A_\rho - A_z - \operatorname{Ave}_{S^2}(A_\rho - A_z))\\
&(5.20b) \quad \Box w &&= 0\\
&(5.20c) \quad \nu\big|_{\mathcal{T}} &&= \nu_{\mathcal{T}}, \quad w\big|_{\mathcal{T}} = w_{\mathcal{T}}\\
&(5.20d) \quad \nu\big|_{\mathcal{C}} &&= \nu_{\mathcal{C}}, \quad w\big|_{\mathcal{C}} = w_{\mathcal{C}}
\end{aligned}
$$

where $\nu_{\mathcal{C}}, w_{\mathcal{C}}$ are constructed from $H_{\mathcal{C}}$ by the Hodge–Helmholtz representation $H_{\mathcal{C}} = d_2\nu_{\mathcal{C}} + *_2 d_2 w_{\mathcal{C}}$, and similarly for $\nu_{\mathcal{T}}, w_{\mathcal{T}}$. Note that A_{S^2} is reconstructed directly from ν, w by $A_{S^2} = d_2\nu + *_2 d_2 w$.

All these problems have unique solutions, by theorem 2.1. We now show that the resulting potential A satisfies Maxwell's equations in the Lorentz gauge.

The boundary condition (5.18c) implies that $\partial_\rho A_z\big|_{\mathcal{S}} = 0$, which gives (4.8c), and also that $\partial_\rho(\rho\partial_\rho(\rho A_z)) = \partial_\rho(\rho \operatorname{div}_{S^2} H_{\mathcal{C}})$. Comparing this with (5.10) and using the conditions (5.18a), (5.19a) and (5.19c) shows that

$$
(5.21) \qquad \partial_\rho(L(A)) = 0 \quad \text{on } \mathcal{C}
$$

¿From (5.18b) and (5.20c) the Lorentz operator (4.7) on $\mathcal{T}$ satisfies

$$
(5.22) \qquad L(A) = -\rho_0^{-2}\partial_r(\rho^2 A_\rho) + \partial_\rho A_z - \rho_0^{-1}\operatorname{div}_{S^2} H_{\mathcal{T}}
$$

which vanishes by (5.19b). Hence $L(A) = 0$ on $\mathcal{C} \cup \mathcal{T}$.

Now consider the equations satisfied by the rectangular coefficients A_i of A. The identity (5.9) combined with (5.19a) and (5.18a) shows that $x^i \Box A_i + 2L(A) = 0$. Combining this with (5.20a) and (5.20b) brings (5.6) to the form

$$\begin{bmatrix} -2\rho^{-1}L(A) \\ 0 \\ 0 \end{bmatrix} = J \begin{bmatrix} \Box A_1 \\ \Box A_2 \\ \Box A_3 \end{bmatrix} \tag{5.23}$$

Since J is just a frame rotation, this shows that the 1-form $M := M_\mu dx^\mu$, $M_\mu = \Box A_\mu$, has the null-polar coordinate representation $M = -2\rho^{-1}L(A)\, d\rho$. Setting $L = L(A)$, by (4.7) we thus have

$$\Box L = L(\Box A) = L(M) = 2\rho^{-2}(\partial_\rho - \partial_z)(\rho L) \tag{5.24}$$

which shows exactly that $\tilde{\Box}(\rho^2 L) = 0$. Since $L = 0$ on $\mathcal{C} \cup \mathcal{T}$, it follows that $L = 0$ in Ω, by the uniqueness argument for $\tilde{\Box}$ given above. We now have $\Box A_i = 0$, $i = 1, 2, 3$ by (5.23) and thus A satisfies Maxwell's equations in the Lorentz gauge, with the required boundary conditions (4.8a)-(4.8c) and (5.16), (5.17). The magnetic charge (3.2) vanishes by Stokes' theorem since $F = dA$. The Lorentz conditions (4.8a)-(4.8c) and (4.7) imply that on $\mathcal{S}$ we have

$$F_{tr} = F_{z\rho} = \partial_z A_\rho - \partial_\rho A_z = \rho_0^{-1} \operatorname{div}_{S^2} A_{S^2} \quad \text{on } \mathcal{S} \tag{5.25}$$

which shows that the electric charge (3.3) vanishes. □

6 Classification

By combining the uniqueness and existence results for the Lorentz gauge and for Maxwell's equations, we obtain a classification theorem for the gauge-equivalence classes of all solutions to Maxwell's equations in the exterior domain Ω, in terms of the unconstrained boundary data $H_\mathcal{C}, H_\mathcal{T}$.

Before presenting the classification theorem, we note that the boundary conditions (5.16), (5.17) have an invariant character, since they determine those curvature components

$$F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$$

which are tangential to the boundary in each case. On the null cone $\mathcal{C}$ these are (since $A = A_\vartheta\, \rho\, d\vartheta + A_\varphi\, \rho \sin\vartheta\, d\varphi$ on $\mathcal{C}$ in our notation, and we are assuming also $A_\rho\big|_{\mathcal{C}} = 0$)

$$\text{(6.1)} \qquad \begin{aligned} F_{\rho\vartheta} &= \partial_\rho(\rho A_\vartheta) \\ F_{\rho\varphi} &= \partial_\rho(\rho \sin\vartheta A_\varphi) \\ F_{\vartheta\varphi} &= \partial_\vartheta(\rho\sin\vartheta A_\varphi) - \partial_\varphi(\rho A_\vartheta) \end{aligned}$$

In terms of the decomposition of $F_{\alpha\beta}$ into electric $E_i = F_{0i}$ and magnetic $B_i = \frac{1}{2}\epsilon_{ijk}F_{jk}$ components, expressed with respect to the polar orthonormal frame e_0, e_r, $e_{\hat\vartheta} = 1/\rho\,\partial_\vartheta$, $e_{\hat\varphi} = 1/(\rho\sin\vartheta)\,\partial_\varphi$, these components are $F_{\rho\vartheta} = \rho(B_{\hat\varphi} + E_{\hat\vartheta})$, $F_{\rho\varphi} = \rho\sin\vartheta(-B_{\hat\vartheta} + E_{\hat\varphi})$ and $F_{\vartheta\varphi} = \rho^2\sin\vartheta\, B_r$. Similarly on the timelike boundary $\mathcal{T}$ we are specifying the curvature components

$$\text{(6.2)} \qquad \begin{aligned} E_{\hat\vartheta} &= \partial_z A_\vartheta \\ E_{\hat\varphi} &= \partial_z A_\varphi \\ B_r &= \rho^{-1}(\partial_\vartheta A_\varphi + \cot\vartheta A_\varphi - \csc\vartheta\partial_\varphi A_\vartheta), \end{aligned}$$

assuming $A_z\big|_{\mathcal{T}} = 0$.

Conversely, the boundary data $A_{S^2}\big|_{\mathcal{C}\cup\mathcal{T}}$ may be recovered from the boundary curvatures by solving the equations (6.1) and (6.2) for A_{S^2}, as follows.

Proposition 6.1 *Given $F_{\rho\vartheta}, F_{\rho\varphi}, F_{\vartheta\varphi}$ on $\mathcal{C}$ satisfying*

$$\text{(6.3)} \qquad \partial_\varphi F_{\rho\vartheta} - \partial_\vartheta F_{\rho\varphi} + \partial_\rho F_{\vartheta\varphi} = 0 \quad \text{on } \mathcal{C}$$

and $\oint_{\mathcal{S}} F_{\vartheta\varphi} d\vartheta d\varphi = 0$, and $F_{z\vartheta}, F_{z\varphi}, F_{\vartheta\varphi}$ on $\mathcal{T}$ satisfying

$$\text{(6.4)} \qquad \partial_\varphi F_{z\vartheta} - \partial_\vartheta F_{z\varphi} + \partial_z F_{\vartheta\varphi} = 0 \quad \text{on } \mathcal{T}$$

and a function $f \in C^\infty(\mathcal{S})$ with $\mathrm{Ave}_{S^2} f = 0$, there exists a unique boundary potential A_{S^2} on $\mathcal{C}\cup\mathcal{T}$ satisfying the Lorentz boundary conditions (4.8a), (4.8b), together with the curvature conditions (6.1), (6.2), and the corner condition $\mathrm{div}_{S^2} A_{S^2}\big|_{\mathcal{S}} = f$.

Proof: We first reconstruct $A_{S^2}\big|_{\mathcal{S}}$ using the Hodge-Helmholtz decomposition $A_{S^2}\big|_{\mathcal{S}} = d_2 v + {*_2} d_2 w$. Using (5.5) we see that the potentials v, w satisfy

$$\Delta_{S^2} v = \mathrm{div}_{S^2} A_{S^2}\big|_{\mathcal{S}} = f, \qquad \Delta_{S^2} w = -\frac{1}{\rho_0\sin\vartheta} F_{\vartheta\varphi}\big|_{\mathcal{S}}$$

Since we have required $\mathrm{Ave}_{S^2} f = 0$ and $\oint_{\mathcal{S}} F_{\vartheta\varphi} d\vartheta d\varphi = 0$, both equations admit solutions, determined up to irrelevant constants, and thereby $A_{S^2}\big|_{\mathcal{S}}$ is fixed. The remaining equations in (6.1) and (6.2) are ordinary differential equations for A_{S^2} along the $\rho-$ and $z-$ coordinate curves in $\mathcal{C}$ and $\mathcal{T}$ respectively, which may be integrated using the initial condition $A_{S^2}\big|_{\mathcal{S}}$ to define A_{S^2} on all of $\mathcal{C}\cup\mathcal{T}$. The resulting potential is consistent with the prescribed field $F_{\vartheta\varphi}$ on the boundary, by virtue of the integrability equations $dF\big|_{\mathcal{C}} = 0$, $dF\big|_{\mathcal{T}} = 0$, ie. (6.3) and (6.4). □

Note that this result concerns only the boundary values of A, and is therefore independent of any extension of A into Ω. If the boundary potential is extended to a global solution of Maxwell's equations in the Lorentz gauge using theorem 5.4, then the prescribed field $f \in C^\infty(\mathcal{S})$ will describe the radial electric field along $\mathcal{S}$, by (5.25) and the corner gauge condition (4.8c) (which in turn follows from the the construction in theorem 5.4).

It is also possible to recover the free data (5.16), (5.17) from a potential without making any assumptions about the gauge, by following the algorithm in theorem 4.4. Thus, if A is any potential in Ω, then we have explicit formulas for the free boundary data $H(A) = (H_{\mathcal{C}}, H_{\mathcal{T}}) := \tilde{A}_{S^2}\big|_{\mathcal{C}\cup\mathcal{T}}$ of the gauge-equivalent potential $\tilde{A}$ satisfying the Lorentz gauge and boundary conditions (4.8a–4.8c). Observe that these formulas depend only on the boundary values $A_\rho\big|_{\mathcal{C}}$, $A_z\big|_{\mathcal{T}}$, $A_{S^2}\big|_{\mathcal{C}\cup\mathcal{T}}$ and the radial electric field $E_r\big|_{\mathcal{S}}$ at the corner, and it is essential that the electric charge vanish on $\mathcal{S}$, ie. $\mathrm{Ave}_{S^2} E_r = 0$. Explicitly, if for any $f \in C^\infty(S^2)$ with $\mathrm{Ave}_{S^2} f = 0$, we denote the unique solution of $\Delta_{S^2}u = f$ with $\mathrm{Ave}_{S^2} u = 0$ by $u = \Delta_{S^2}^{-1} f$, then

$$
\begin{aligned}
(6.5)\qquad H_{\mathcal{C}}(A) &= A_{S^2}\big|_{\mathcal{C}} - \frac{\rho_0}{\rho} A_{S^2}\big|_{\mathcal{S}} \\
&\quad + \frac{\rho_0^2}{\rho} d_2\left(\Delta_{S^2}^{-1}(E_r\big|_{\mathcal{S}})\right) - \frac{1}{\rho}\int_{\rho_0}^{\rho} d_2 A_\rho(0,\rho')d\rho' \\
(6.6)\qquad H_{\mathcal{T}}(A) &= A_{S^2}\big|_{\mathcal{T}} - A_{S^2}\big|_{\mathcal{S}} \\
&\quad + \rho_0 d_2\left(\Delta_{S^2}^{-1}(E_r\big|_{\mathcal{S}})\right) - \frac{1}{\rho_0}\int_0^z d_2 A_z(z',\rho_0)dz
\end{aligned}
$$

We may now formulate the final classification theorem. Define the linear spaces

$$
\begin{aligned}
(6.7)\quad \mathcal{D} &= \big\{(e,k,H_{\mathcal{C}},H_{\mathcal{T}}) : e\in\mathbb{R}, k\in\mathbb{R}, \\
&\qquad H_{\mathcal{C}} \in \Gamma^\infty(\mathcal{C},T^*S^2), H_{\mathcal{T}} \in \Gamma^\infty(\mathcal{T},T^*S^2), H_{\mathcal{C}}\big|_{\mathcal{S}} = H_{\mathcal{T}}\big|_{\mathcal{S}}\big\} \\
(6.8)\quad \mathcal{M} &= \big\{F = \tfrac{1}{2}F_{\mu\nu}dx^\mu\wedge dx^\nu \in \Gamma^\infty(\Lambda^2 T^*\bar{\Omega}) : dF = 0, d*F = 0\big\}
\end{aligned}
$$

Theorem 6.2 *There is a linear bijection $\Phi : \mathcal{D} \leftrightarrow \mathcal{M}$. Denoting the solution of Maxwell's equations constructed in theorem 5.4 by $F_0 = F_0(H_{\mathcal{C}}, H_{\mathcal{T}})$, and* *the boundary data projection constructed in proposition 6.1 by* $\mathcal{H}_{\mathcal{C}\cup\mathcal{T}}(F) = (A_{S^2}|_{\mathcal{C}}, A_{S^2}|_{\mathcal{T}})$*, the bijection is given by*

$$\Phi(e, k, H_{\mathcal{C}}, H_{\mathcal{T}}) = F_e^{elec} + F_k^{mag} + F_0(H_{\mathcal{C}}, H_{\mathcal{T}}) \tag{6.9}$$

$$\Phi^{-1}(F) = (e(F), k(F), \mathcal{H}_{\mathcal{C}\cup\mathcal{T}}(F - F_{e(F)}^{elec} - F_{k(F)}^{mag})) \tag{6.10}$$

Proof: This is simply a combination of the results of the previous sections. □

Finally, we observe that the Hodge dual $* : \Lambda^2(\mathbb{R}^{1,3}) \to \Lambda^2(\mathbb{R}^{1,3})$ preserves solutions of Maxwell's equations, and hence extends to an invertible map $* : \mathcal{M} \to \mathcal{M}$ satisfying $*^2 = -1$. This in turn determines an invertible map on $\mathcal{D}$ by $\Phi^{-1} \circ * \circ \Phi$. Although it is easy to see the effect of duality rotations on the boundary curvature components (6.1), (6.2), the fact that the dual components are expressed in terms of transverse derivatives of potential components different from those gauged in our formulation, makes it a rather more difficult problem to explicitly present the duality map in terms of the potential boundary data.

References

[Bal96a] Robin Balean, *The Null-Timelike Boundary Problem*, thesis, University of New England 1996.

[Bal96b] ———, *The Null-Timelike Boundary Problem for the Linear Wave Equation*, submitted, 1996.

[Bar96] R. Bartnik, *Einstein equations in the null quasi-spherical gauge*, `gr-qc/9611045`

[Betal96] N. T. Bishop, R. Gomez, P. R. Holverem, R. A. Matzner, P. Papadopoulos, J. Winicour, *Cauchy-characteristic matching: a new approach to radiation boundary conditions*, Phys. Rev. Lett. **76** (1996), 4303–4306

[C91] D. Christodoulou, *The formation of black holes and singularities in spherically symmetric gravitational collapse* Commun. Pure Appl. Math. **44** (1991) 339–373.

[EM82] D. Eardley and V. Moncrief, *The global existence of Yang-Mills-Higgs fields in 4-dimensional Minkowski space, I*, Commun. Math. Phys. **83** (1982) 171–191.

[GP96] C. Gundlach and J. Pullin, *Instability of free evolution in double null coordinates*, gr-qc/9606022

[GP87] A. Goldwirth and T. Piran, *Gravitational collapse of massless scalar fields and cosmic censorship*, Phys. Rev. D **36** (1987) 3575–3581.

[GPW94] R. Gómez, P.Papadopoulos, and J Winicour, *Null Cone Evolution of Axisymmetric Vacuum Spacetimes*, J. Math. Phys. **35** (1994), 4184.

[GW92] R. Gómez and J. Winicour, *Gravitational wave forms at finite distances and at null infinity*, Phys. Rev. D **45** (1992), no. 8, 2776–2782.

[HS96] R. S. Hamade and J. M. Stewart, *The spherically symmetric collapse of a massless scalar field*, Class. Quant. Grav. **13** (1996) 497–512

[H89] H. Hertz, *Die Kräfte electrischer Schwingungen, behandelt nach der Maxwell'schen Theorie*, Ann. Physik **36** (1889) 1–???

[P65] R. Penrose, *Zero rest mass fields including gravitation: asymptotic behaviour*, Proc. Roy. Soc. London A **284** (1965) 159–203

[S79] J. M. Stewart, *Hertz-Bromwich-Debye-Whittaker-Penrose potentials in general relativity*, Proc. Roy. Soc. London A **367** (1979) 527–538

[TW66] L. Tamburino and J. Winicour, *Gravitational fields in finite and conformal Bondi frames*, Phys. Rev. **150** (1966), 1039.