Multiple Closed Characteristics in the Cotangent Bundle of $\mathbf{S}^1$-Manifolds

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Abstract

Let P be a compact smooth manifold equipped with a free $\mathbf{S}^1$-structure and let $\Sigma \subset T^*P$ be a compact hypersurface which is invariant under the induced $\mathbf{S}^1$-action on T^*P and bounds a neighborhood of the 0-section. In addition, Σ is assumed to be of contact type where the corresponding contact 1-form is related to the canonical Liouville form. Then Σ is shown to carry at least $1+\mathrm{cuplength}(P/\mathbf{S}^1)$ geometrically distinct closed characteristics that coincide with $\mathbf{S}^1$-orbits on T^*P. If, moreover, all these particular closed characteristics on Σ are non-degenerate then their number is greater or equal to the sum of the Betti numbers of $P/\mathbf{S}^1$.

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1. Introduction and preliminaries

Let P denote a compact connected oriented smooth manifold carrying the structure of a free smooth $\mathbb{S}^1$-space. That is, we assume that on P there is given a free smooth Lie group action $\mathbb{S}^1 \times P \to P$ where the circle group $\mathbb{S}^1$ is parametrized by $\mathbb{R}/\tau\mathbb{Z}$ for some $\tau > 0$. The group structure on $\mathbb{S}^1$ is induced from $(\mathbb{R}, +)$. We shall use the notation $\mathbb{S}^1_\tau := \mathbb{R}/\tau\mathbb{Z}$ when we refer to a distinguished period τ.

Denoting the $\mathbb{S}^1$-action on P by $(t,x) \mapsto c_t(x)$, or $(t,x) \mapsto t\cdot x$ for brevity, we write $\mathbb{S}^1\cdot x = \{\, t\cdot x \mid t \in \mathbb{S}^1_\tau \,\}$ for the $\mathbb{S}^1$-orbit of $x \in P$ and $P/\mathbb{S}^1 = \{\, \mathbb{S}^1\cdot x \mid x \in P \,\}$ for the corresponding orbit space.

We are interested in the study of closed characteristics on suitable hypersurfaces Σ in the cotangent bundle T^*P. The closed characteristics we have in mind are special in the sense that they are related to the $\mathbb{S}^1$-orbit structure of P. We shall derive lower bounds for the number of these particular closed characteristics on Σ employing homotopy invariants of the orbit space $P/\mathbb{S}^1$. Before we can state the precise formulation of our results we need some preparations.

Using the canonical projection $p : P \to P/\mathbb{S}^1$ given by $p(x) := \mathbb{S}^1\cdot x$, we may consider P as a (locally trivial) principal $\mathbb{S}^1$-bundle over $P/\mathbb{S}^1$, cf. [7]. A local trivialization of P at $p(x)$ consists of an open neighborhood $U \subset P/\mathbb{S}^1$ of $p(x)$ and an $\mathbb{S}^1$-equivariant diffeomorphism $\varphi_U : p^{-1}(U) \to U \times \mathbb{S}^1$ such that the following diagram commutes

$$\begin{tikzcd} p^{-1}(U) \arrow[rr, "\varphi_U"] \arrow[dr, "p"'] & & U \times \mathbb{S}^1 \arrow[dl, "p_U"] \\ & U & \end{tikzcd}$$

where $p_U : U \times \mathbb{S}^1 \to U$ is the projection onto the first component. In the context of principal G-bundles one usually refers to a right action of the structure group G rather than a left group action $(t,x) \mapsto t\cdot x$ as above. However, since $\mathbb{S}^1$ is commutative, the left $\mathbb{S}^1$-manifold P can be identified with a right $\mathbb{S}^1$-space by setting $x\cdot t := t\cdot x$.

We point out that the $\mathbb{S}^1$-action on P is proper, i.e., the map $\mathbb{S}^1 \times P \to P \times P$ defined by $(t,x) \mapsto (x, t\cdot x)$ is proper. As a consequence, the orbit space $P/\mathbb{S}^1$ can be endowed with a unique structure of a smooth manifold such that the canonical projection p is a submersion, cf. [4, Prop. 5.2].

The free $\mathbb{S}^1$-action c_t on P induces a free $\mathbb{S}^1$-action γ_t on T^*P. We note that for each fixed $t \in \mathbb{S}^1$ the map $x \mapsto c_t(x)$ is a diffeomorphism of P, having the inverse $c_t^{-1} = c_{-t}$. The induced $\mathbb{S}^1$-action on T^*P is given by the push-forward of cotangent vectors, i.e., the map

$$\gamma_t : T^*_x P \to T^*_{t\cdot x} P \,, \tag{1}$$

is given by

$$(\gamma_t\, \xi_x)\,(v_{t\cdot x}) \;=\; \xi_x(Tc_{-t}\, v_{t\cdot x}) \tag{2}$$

where $\xi_x \in T^*_x P$ and $v_{t\cdot x} \in T_{t\cdot x}P$. In particular, for each fixed $t \in \mathbb{S}^1$ the map $\xi \mapsto \gamma_t\, \xi$ is a point transformation and, therefore, leaves the canonical forms on T^*P

invariant. The $\mathbb{S}^1$-symmetry thus gives rise to a distinguished class of loops in T^*P:

Definition 1 *A smooth loop $u : \mathbb{S}^1_\tau \to T^*P$ will be called a principal loop in T^*P if it is of the form $u(t) = \gamma_t \xi$ for some $\xi \in T^*P$.*

Clearly, for $\xi \in T^*_x P$ we get $\pi^*(\gamma_t \xi) = c_t(x)$ by the projection $\pi^* : T^*P \to P$ so that the principal loops in T^*P cover the fibres of the principal $\mathbb{S}^1$-bundle $P \xrightarrow{p} P/\mathbb{S}^1$.

Given a smooth hypersurface $\Sigma \subset T^*P$ that is invariant under the $\mathbb{S}^1$-action γ_t, one may ask for the existence and multiplicity of special closed characteristics on Σ which can be represented by principal loops. This question is motivated by the study of the orbit structure of symmetric Hamiltonian systems at a fixed energy level, i.e., Σ is a regular level surface of a Hamiltonian function $H : T^*P \to \mathbb{R}$ which satisfies $H(\gamma_t \xi) = H(\xi)$ for all $t \in \mathbb{S}^1$, $\xi \in T^*P$. The closed characteristics on Σ correspond to the periodic trajectories of the Hamiltonian flow on the energy surface Σ. Thus we deal with the problem of finding symmetric periodic solutions of an $\mathbb{S}^1$-symmetric Hamiltonian system at a prescribed energy level.

As an example we mention the n-dimensional torus $\mathbb{T}^n$, represented by

$$\mathbb{T}^n = \{ (z_1, \ldots, z_n) \in \mathbb{C}^n \mid |z_i| = 1 \ \text{ for } i = 1, \ldots, n \} . \tag{3}$$

If $j = (j_1, \ldots, j_n) \in \mathbb{Z}^n$ is any relatively prime integer vector, a free action of $\mathbb{S}^1_\tau$ on $\mathbb{T}^n$ is defined by

$$t\cdot(z_1, \ldots, z_n) := \left(\exp\left(\tfrac{2\pi i}{\tau} j_1 t\right) z_1 , \ldots , \exp\left(\tfrac{2\pi i}{\tau} j_n t\right) z_n\right) . \tag{4}$$

The orbit space $\mathbb{T}^n/\mathbb{S}^1$ is diffeomorphic to $\mathbb{T}^{n-1}$. Note that there exist as many different free $\mathbb{S}^1$-structures on $\mathbb{T}^n$ as there are relatively prime integer vectors j, where, however, the $\mathbb{S}^1$-actions corresponding to j and $-j$ coincide up to the orientation of the fibres of the principal $\mathbb{S}^1$-bundle.

Multiplicity of closed characteristics on hypersurfaces Σ in $T^*\mathbb{T}^n$ has been studied in [8] and [10]. This present work can be understood as a step towards a generalization of the results in [10] where the $\mathbb{S}^1$-manifold P replaces $\mathbb{T}^n$. However, we point out that by the above definition of principal loops, which is purely geometrical in nature, we have to restrict ourselves to the symmetric case. This is in contrast to the situation in [10], where the $\mathbb{S}^1$-action on $\mathbb{T}^n$ determines a homotopy class of loops in $T^*\mathbb{T}^n$ and, therefore, provides a topological characterization of the loops related to the given $\mathbb{S}^1$-structure.

2. Statement of the results

By λ we denote the canonical 1-form on the cotangent bundle T^*P, also referred to as the *Liouville form.* Its exterior derivative $\omega = d\lambda$ is called the canonical symplectic form on T^*P. Locally, in the domain of a chart, we may introduce coordinates $(x_1, \ldots, x_n)$ on P. In associated local coordinates $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ of T^*P the canonical forms can be represented by

$$\lambda = \sum_{i=1}^{n} y_i \, dx_i \quad , \quad \omega = \sum_{i=1}^{n} dy_i \wedge dx_i \, . \tag{5}$$

Let $D \subset T^*P$ be a relatively compact neighborhood of the 0-section with smooth boundary $\Sigma := \partial D$. We shall assume that the hypersurface Σ is $\mathbb{S}^1$-invariant and satisfies an additional hypothesis, requiring that Σ admits a contact structure whose contact form is compatible with the Liouville form. For this purpose we consider the pull-back of the forms λ and ω to Σ, denoted by λ_Σ and $\omega_\Sigma = d\lambda_\Sigma$, respectively.

Definition 2 *A smooth $\mathbb{S}^1$-invariant hypersurface $\Sigma \subset T^*P$ will be called an invariant Liouville hypersurface if there exists a 1-form β on Σ such that the following conditions are satisfied:*

(i) $\gamma_t^* \beta = \beta$, *i.e., β is $\mathbb{S}^1$-invariant.*

(ii) $\beta - \lambda_\Sigma$ *is exact.*

(iii) $\beta \wedge \omega_\Sigma^{n-1}$ *is a volume form on Σ.*

Furthermore, we call Σ an invariant strict Liouville hypersurface if it is an invariant Liouville hypersurface with $\beta = \lambda_\Sigma$.

Let us point out that condition (ii) implies $d\beta = \omega_\Sigma$. Recall that Σ is said to be a *hypersurface of contact type* if this latter identity and condition (iii) both hold true, cf. A. Weinstein [14]. The 1-form β defines a contact structure on Σ in the sense that ω_Σ is non-degenerate on the null spaces of β.

If $\beta - \lambda_\Sigma = df$ with a smooth function f on Σ, we can extend f to an $\mathbb{S}^1$-invariant function F on some open $\mathbb{S}^1$-invariant neighborhood $U \subset T^*P$ of Σ. Then $\alpha := \lambda + dF$ is an $\mathbb{S}^1$-invariant extension of β to U. Therefore, we shall from now on refer to the pull-back α_Σ as the contact form on Σ.

Associated to the 1-form α there is a non-vanishing smooth $\mathbb{S}^1$-equivariant vector field X_α on U, defined by the identity

$$\alpha = X_\alpha \lrcorner\, \omega \,. \tag{6}$$

Using Cartan's formula, we infer $L_{X_\alpha}\omega = \omega$ for the Lie derivative of ω along X_α, i.e., X_α is a so-called *symplectic dilation*. Moreover, X_α is transverse to Σ. To see this, let us introduce the *characteristic line bundle* $\mathfrak{L}_\Sigma \subset T\Sigma$ which is defined to be the kernel of ω_Σ,

$$\mathfrak{L}_\Sigma \;:=\; \ker\, \omega_\Sigma \;=\; \bigcup_{\xi \in \Sigma} \{\, X \in T_\xi\Sigma \mid \omega(X, Y) = 0 \quad \forall\, Y \in T_\xi\Sigma \,\} \,. \tag{7}$$

A nowhere vanishing section $X : \Sigma \to \mathfrak{L}_\Sigma$ is called a *characteristic vector field* on Σ. The condition that $\alpha_\Sigma \wedge \omega_\Sigma^{n-1}$ be a volume form on Σ implies that $\alpha_\Sigma(X) \neq 0$ for every characteristic vector field X on Σ. Then

$$0 \neq \alpha(X) \;=\; (X_\alpha \lrcorner\, \omega)(X) \;=\; \omega(X_\alpha, X) \;=\; -\omega(X, X_\alpha) \,. \tag{8}$$

Consequently, the symplectic dilation X_α is transverse to Σ.

Clearly, an invariant hypersurface $\Sigma \subset T^*P$ is Liouville if there exists a transverse $\mathbb{S}^1$-equivariant symplectic dilation X_* defined on some neighborhood of Σ such that the 1-form $(X_* \lrcorner\, \omega) - \lambda$ is exact. Then the pull-back $(X_* \lrcorner\, \omega)_\Sigma$ is the desired contact form on Σ.

A *closed characteristic* on Σ can now be defined as an embedding $u : \mathbb{S}^1_\tau \to \Sigma$ whose tangent map is an embedding $u_* : T\mathbb{S}^1_\tau \to \mathfrak{L}_\Sigma$. Any two closed characteristics u, v on Σ are said to be *(geometrically) distinct* if $u(\mathbb{S}^1_\tau) \cap v(\mathbb{S}^1_\tau) = \emptyset$. Considering symmetric hypersurfaces, we are particularly interested in the study of symmetric characteristic embeddings of the circle:

Definition 3 *A closed characteristic $u : \mathbb{S}^1_\tau \to \Sigma$ will be called a principal closed characteristic on Σ if $i \circ u : \mathbb{S}^1_\tau \to T^*P$ is a principal loop according to Definition 1, where $i : \Sigma \to T^*P$ is the inclusion.*

Our first result is the following

Theorem 1 *Let P be a compact connected free $\mathbb{S}^1$-manifold and suppose $\Sigma \subset T^*P$ is an invariant Liouville hypersurface bounding a relatively compact neighborhood of the 0-section in T^*P. Then*

$$\#\{\,\text{distinct principal closed characteristics on } \Sigma\,\} \;\geqslant\; \mathrm{CL}\,(P/\mathbb{S}^1) \tag{9}$$

where $\mathrm{CL}\,(P/\mathbb{S}^1) := 1 + \mathrm{cuplength}\,(P/\mathbb{S}^1)$.

To speak of closed characteristics means to speak of periodic Hamiltonian orbits without referring to a particular parametrization. However, for a compact Liouville hypersurface Σ, we shall introduce appropriate *admissible Hamiltonian functions*, defined on some neighborhood of Σ with the aid of the flow ψ_s of the symplectic dilation X_α. Postponing the details to subsequent sections, let us call the Hamiltonian H admissible if it is of the form $H(\psi_s(\xi)) = h(s)$ for $\xi \in \Sigma$, where $h : (-\delta, \delta) \to \mathbb{R}$ is a smooth function satisfying $h'(s) \neq 0$. If X_α is $\mathbb{S}^1$-equivariant, so is its flow ψ_s, and consequently the admissible Hamiltonian H is $\mathbb{S}^1$-invariant. The associated Hamiltonian vector field X_H, defined by the identity

$$-\,dH \;=\; X_H \lrcorner\, \omega\,, \tag{10}$$

gives rise to the characteristic vector field $X_H|_\Sigma$ on Σ. Note that the choice of H induces an orientation of $\mathfrak{L}_\Sigma$, according to $\alpha(X_H) > 0$ or $\alpha(X_H) < 0$. This orientation corresponds to the sign of $h'(s)$.

Let $\chi : \Sigma \times \mathbb{R} \to \Sigma$ denote the flow of $X_H|_\Sigma$ and suppose that $\chi(\xi, t)$ is a τ-periodic integral curve. Then 1 is obviously a *relative Floquet multiplier* of $\chi(\xi, t)$, i.e., 1 is an eigenvalue of $T\chi(\xi, \tau) \in \mathrm{Hom}\,(T_\xi\Sigma)$. We say $\chi(\xi, t)$ is *non-degenerate* mod $\mathbb{S}^1$ if it has only one simple relative Floquet multiplier equal to 1. This definition is independent of the particular admissible Hamiltonian H and, therefore, we may speak of *non-degeneracy of closed characteristics* on Σ.

Theorem 2 *Let Σ satisfy the hypotheses of Theorem 1 and assume, in addition, that all the principal closed characteristics on Σ are non-degenerate. Then*

$$\#\{\,\text{distinct principal closed characteristics on } \Sigma\,\} \;\geqslant\; \mathrm{SB}\,(P/\mathbb{S}^1) \tag{11}$$

where $\mathrm{SB}\,(P/\mathbb{S}^1)$ stands for the sum of the Betti numbers of $P/\mathbb{S}^1$.

A special situation is given if P is endowed with a Riemannian metric g. Recall that g defines an isomorphism of vector bundles $TP \to T^*P$ by $\xi_* \mapsto \xi := g(\xi_*, \cdot)$. Correspondingly, there is a scalar product g^* on the fibres of T^*P defined by $g^*(\xi, \eta) := g(\xi_*, \eta_*)$. The hypersurface $\Sigma^* = \{\, \xi \in T^*P \mid g^*(\xi, \xi) = 1 \,\}$ is called the unit cosphere bundle of (P, g).

In particular, let us assume that g^* is $\mathbb{S}^1$-invariant, i.e., $g^*(\gamma_t \xi, \gamma_t \eta) = g^*(\xi, \eta)$ for all $\xi, \eta \in T^*P$. This is equivalent to saying that g is an $\mathbb{S}^1$-invariant metric, meaning that $g(Tc_t \xi_*, Tc_t \eta_*) = g(\xi_*, \eta_*)$. If g^* is $\mathbb{S}^1$-invariant then Σ^* is a strict Liouville hypersurface that meets the assumptions of Theorem 1, see Lemma 4 below.

It is well known that the closed characteristics on Σ^* correspond uniquely to closed geodesics on (P, g), cf. [11, Ch. 3.1]. Hence we may define the *principal closed geodesics* on P as the projections of the principal closed characteristics on an $\mathbb{S}^1$-invariant cosphere bundle Σ^*. Moreover, non-degenerate closed characteristics on Σ^* correspond to non-degenerate closed geodesics on P. For the unit cosphere bundle the above results thus can be reformulated in terms of closed geodesics:

Corollary 1 *Let g be an $\mathbb{S}^1$-invariant Riemannian metric on P. Then*

$$\#\{\text{ distinct principal closed geodesics on } P\,\} \;\geqslant\; \mathrm{CL}\,(P/\mathbb{S}^1)\,. \tag{12}$$

If all the principal closed geodesics on P are non-degenerate then

$$\#\{\text{ distinct principal closed geodesics on } P\,\} \;\geqslant\; \mathrm{SB}\,(P/\mathbb{S}^1)\,. \tag{13}$$

3. The strategy of the proof

Evidently it is sufficient to carry out the proofs under the assumption that there is only a finite number of distinct principal closed characteristics on Σ. This assumed number is allowed to be zero, i.e., the existence of a principal closed characteristic on Σ is not presupposed. We emphasize that additional closed characteristics which are *not* principal with respect to the $\mathbb{S}^1$-action in question may exist on Σ. In particular, by the above finiteness assumption we do not impose any restrictions on the possible number of these additional closed characteristics.

As already mentioned in the previous section, the closed characteristics on Σ can be represented as periodic trajectories of a Hamiltonian flow. The choice of a Hamiltonian function $H : T^*P \to \mathbb{R}$ with $X_H|_\Sigma \neq 0$ provides a parametrization of each integral curve of X_H and, therefore, assigns a distinguished minimal period to every periodic orbit. Let us introduce the following denotation:

Definition 4 *Suppose $u(t)$ is a periodic integral curve of X_H having the minimal period $\mathcal{T} > 0$. We call u a principal periodic Hamiltonian orbit if the map $\mathbb{S}^1_\tau \to T^*P$, defined by $t \mapsto u(\frac{\mathcal{T}}{\tau} t)$, is a principal loop according to Definition 1.*

In general, the distinct principal periodic X_H-orbits on Σ have distinct minimal periods. However, using the symplectic dilation X_α, we can achieve an $\mathbb{S}^1$-equivariant blow up of Σ which allows us to transform the problem of counting principal periodic X_H-orbits on Σ into the problem of counting principal Hamiltonian orbits which have a fixed minimal period $\mathcal{T} > 0$. This latter problem will be treated by a variational approach on a suitable space of $\mathcal{T}$-periodic loops $u : \mathbb{S}^1_{\mathcal{T}} \to T^*P$.

The blow up of Σ along the flow of X_α is described in section 4. Recall that the symplectic dilation X_α is solely defined on some neighborhood of Σ. Together with the blow up we introduce an admissible $\mathbb{S}^1$-invariant Hamiltonian $H : U \to \mathbb{R}$. Thus we use the flow of X_α to generate a tower of periodic X_H-orbits over each closed characteristic on Σ. Consequently, the required counting of the distinct principal closed characteristics can be done by counting the corresponding principal towers.

If $\mathcal{T} > 0$ is sufficiently large, we can derive a priori estimates which guarantee that each principal tower contains exactly two periodic trajectories of X_H of minimal period $\mathcal{T}$. Furthermore, in each principal tower the orbits of minimal period $\mathcal{T}$ can be distinguished by means of the Hamiltonian action. More precisely, the i-th principal tower contains a unique Hamiltonian orbit, represented by $u_i(t)$, which has both the minimal period $\mathcal{T}$ *and* a positive Hamiltonian action integral, i.e.,

$$\mathcal{A}_H(u_i) \;:=\; \int_{\mathbb{S}^1_{\mathcal{T}}} u_i^*\lambda \;-\; \int_0^{\mathcal{T}} H(u_i(t))\, dt \;>\; 0\,. \tag{14}$$

Corresponding to the choice of $\mathcal{T} > 0$ we fix an adapted parametrization for the $\mathbb{S}^1$-action γ on T^*P. Hence u_i is represented by $u_i(t) = \gamma_t\,\xi_i$ for some $\xi_i \in U$. Loosely speaking, by the variational formulation the problem is reduced to counting those distinct critical $\mathbb{S}^1$-orbits of $\mathcal{A}_H$ in U which satisfy the inequality (14).

We point out that from our construction of the Hamiltonian $H : U \to \mathbb{R}$ the intended multiplicity result for critical $\mathbb{S}^1$-orbits does not follow immediately. The situation is similar to that of the Poincaré-Birkhoff fixed point theorem on an annulus where the twist of the boundaries is well known to be an indispensable hypothesis. So far we have included no twist condition, and there is indeed no topological evidence for the existence of critical $\mathbb{S}^1$-orbits. For this reason we extend the Hamiltonian suitably to a function $H : T^*P \to \mathbb{R}$ such that H is a metric outside a compact neighborhood of the 0-section. For a more detailed discussion of the role of this asymptotic boundary condition we refer to the recent paper by C. Golé [9].

The asymptotic twist condition allows a global variational characterization of critical $\mathbb{S}^1$-orbits, but it does not guarantee that these are found on a positive level of $\mathcal{A}_H$. Moreover, the extension engenders additional critical $\mathbb{S}^1$-orbits describing principal Hamiltonian orbits outside U. These will be called *technical critical* $\mathbb{S}^1$*-orbits* and have to be distinguished from the *significant critical* $\mathbb{S}^1$*-orbits* in the towers over closed characteristics on Σ. As a consequence, our demands on the extension of H are two-fold: firstly, the technical critical $\mathbb{S}^1$-orbits have to lie on a negative $\mathcal{A}_H$-level and, secondly, the extended variational problem has to allow an analysis of the significant critical set on a positive $\mathcal{A}_H$-level.

The latter requirement does not depend on the asymptotic twist and will be settled by defining $H \equiv 0$ in a small neighborhood of the 0-section. By this construction we can establish a sort of topological linking which leads to the claimed multiplicity of critical $\mathbb{S}^1$-orbits.

In sections 5 and 6 we describe the global extension of the Hamiltonian function as indicated above. The variational setup for principal $\mathcal{T}$-periodic $\mathbb{S}^1$-orbits and the associated free Morse complex are described in section 7. We emphasize that the structure of the problem allows to separate a finite dimensional manifold Ω_{prin} of principal periodic Hamiltonian orbits on which $\mathbb{S}^1$ acts freely. In particular, the iterate orbits, i.e., the principal periodic Hamiltonian orbits having a minimal period $\mathcal{T}/m$ with $m = 2, 3, \dots$ do not contribute to the critical set.

Section 8 contains the equivariant cuplength estimate for the positive part of the free Morse complex. This is the essential topological ingredient in the proof of Theorem 1, which is completed in section 9 by the use of Lyusternik-Schnirelman theory for flows. The proof of the non-degenerate case of Theorem 2, given in section 10, is based on an $\mathbb{S}^1$-equivariant version of Morse theory for flows in the spirit of Conley-Zehnder.

4. The Hamiltonian near the hypersurface

We begin by introducing an almost complex structure on (T^*P, ω), i.e., we fix a complex operator $J \in \operatorname{Hom}(TT^*P)$, $J(\xi)^2 = -\mathrm{id}_{T_\xi T^*P}$, which is adapted to ω in the sense that

$$\langle\, X, Y \,\rangle := \omega(X, JY) \tag{15}$$

defines a scalar product with corresponding norm $\|\cdot\|$ on the fibres of TT^*P, cf. [13]. The operator J will be specified with regard to our particular purposes in section 5 below. Here we only use the fact that J can be chosen in such a way that the metric is $\mathbb{S}^1$-invariant, i.e., $\langle\, T\gamma_t X, T\gamma_t Y \,\rangle = \langle\, X, Y \,\rangle$.

The *outward unit normal vector* $N(\xi)$ at $\xi \in \Sigma$ with respect to $\langle\,\cdot\,,\,\cdot\,\rangle$ is characterized by the following properties: $\|N(\xi)\| = 1$, $\langle N(\xi), X(\xi)\rangle = 0$ for all $X(\xi) \in T_\xi\Sigma$, and $\langle N(\xi), X_\alpha(\xi)\rangle > 0$ where X_α is the $\mathbb{S}^1$-equivariant transverse symplectic dilation defined in (6). We note that JN is a characteristic vector field on Σ and that the choice of N determines a distinguished orientation of $\mathfrak{L}_\Sigma$ in view of $\alpha(JN) = \omega(X_\alpha, JN) = \langle\, X_\alpha, N \,\rangle > 0$.

By ψ_s we denote the flow of X_α. Since Σ is compact there exists $\hat{s} > 0$ such that $\psi_s(\xi)$ is defined for all $\xi \in \Sigma$ and $|s| \leqslant \hat{s}$. The flow yields a parametrized family of $\mathbb{S}^1$-invariant hypersurfaces $\Sigma_s = \psi_s(\Sigma)$ modelled on Σ, and we set

$$U := \bigcup_{|s| < \hat{s}} \Sigma_s \ \subset\ T^*P\,. \tag{16}$$

Each Σ_s is diffeomorphic to Σ and has the outward unit normal vectors $N(\psi_s(\xi))$. Choosing $\hat{s} > 0$ sufficiently small, we can assume that $\langle N(\psi_s(\xi)), X_\alpha(\psi_s(\xi))\rangle > 0$ for all $\xi \in \Sigma$ and $|s| \leqslant \hat{s}$ so that, in particular, each Σ_s is an invariant Liouville hypersurface. Moreover, we can assume that $\hat{s}$ is chosen small enough such that for all $|s| < \hat{s}$ the hypersurface Σ_s bounds a neighborhood of the 0-section in T^*P.

For a Hamiltonian function $H : T^*P \to \mathbb{R}$ the gradient of H with respect to the metric (15) is given by $dH = \langle\, \operatorname{grad} H\,, \cdot \,\rangle$ and satisfies $X_H(\xi) = J(\xi)\operatorname{grad} H(\xi)$, where X_H is the Hamiltonian vector field introduced in (10). The Hamiltonian equations thus have the form

$$(17) \qquad \dot{u} \;=\; X_H(u) \;=\; J(u)\,\mathrm{grad}\,H(u)\;.$$

If H is $\mathbb{S}^1$-invariant, $H(\gamma_t\,\xi) = H(\xi)$, then the gradient of H is $\mathbb{S}^1$-equivariant, that is, $\mathrm{grad}\,H(\gamma_t\,\xi) = T\gamma_t\,\mathrm{grad}\,H(\xi)$.

Now we introduce a special Hamiltonian on the neighborhood U of Σ:

Proposition 1 *Let the invariant Liouville hypersurface $\Sigma \subset T^*P$ satisfy the hypotheses of Theorem 1 and, moreover, assume that Σ carries at most finitely many distinct principal closed characteristics. Then there exist $\delta \in (0, \hat{s})$, $\mathcal{T} > 0$, and a smooth Hamiltonian $H : U \to \mathbb{R}$ such that the following conditions are satisfied:*

(i) *Σ_s is a regular level surface of H if $|s| < \delta$.*

(ii) *Assume that u_0 is a principal periodic orbit of X_H on Σ. Then there exist $s\in(-\delta,\delta)$ and a principal $\mathcal{T}$-periodic orbit u_s of X_H on Σ_s such that s and u_s are uniquely determined by the following properties:*

- (a) *u_s has the minimal period $\mathcal{T}$.*
- (b) *$u_s(0) \;=\; (\psi_s \circ u_0)(0)$.*
- (c) *The Hamiltonian action integral of u_s is positive, i.e.,*

$$\mathcal{A}_H(u_s) \;=\; \int_{\mathbb{S}^1_{\mathcal{T}}} u_s^*\lambda \;-\; \int_0^{\mathcal{T}} (H \circ u_s)\; dt \;>\; 0$$

where λ is the Liouville form.

Proof We fix $0 < \delta < \hat{s}$ and introduce the smooth function $g : (-\hat{s}, \hat{s}) \to \mathbb{R}_0^+$ by setting

$$(18) \qquad g(s) \;:=\; \begin{cases} \exp\left(\frac{-\delta^2}{\delta^2 - s^2}\right) & \text{if} \quad 0 \leqslant |s| < \delta \\ 0 & \text{if} \quad \delta \leqslant |s| < \hat{s}\;. \end{cases}$$

Let $h : (-\hat{s}, \hat{s}) \to \mathbb{R}$ be given by

$$(19) \qquad h(s) \;:=\; \int_{-\hat{s}}^{s} g(t)\; dt\;.$$

Now we define the Hamiltonian $H|_U : U \to \mathbb{R}$ by

$$(20) \qquad H|_U(\psi_s(\xi)) \;=\; H(\psi_s(\xi)) \;:=\; h(s) \quad \text{if} \quad \xi \in \Sigma\;,\; |s| < \hat{s}\;.$$

The condition (i) means exactly that X_H is a characteristic vector field on Σ_s for $|s| < \delta$. Clearly, since the value of H remains constant along integral curves of X_H it follows that $X_H(\psi_s(\xi)) \in T_{\psi_s(\xi)}\Sigma_s$ for all $|s| < \hat{s}$.

We claim that $X_H(\psi_s(\xi)) \in \ker \omega_{\Sigma_s}$. Let $X \in T_{\psi_s(\xi)}\Sigma_s$ be a tangent vector and let $w : (-a, a) \to \Sigma_s$ be a smooth path in Σ_s with $w(0) = \psi_s(\xi)$ and $\dot{w}(0) = X$. Because H is constant along w we find

$$(21) \qquad \omega(X_H, X) \;=\; -dH(X) \;=\; -\frac{d}{dt}\Big|_{t=0} H(w(t)) \;=\; 0\;.$$

Moreover, we claim that $X_H(\psi_s(\xi)) \neq 0$ if $|s| < \delta$. Indeed,

$$
(22) \qquad \begin{aligned} \alpha(X_H)|_{\psi_s(\xi)} &= dH(X_\alpha)|_{\psi_s(\xi)} = L_{X_\alpha} H|_{\psi_s(\xi)} = \frac{d}{dt}\Big|_{t=0} \psi_t^* H(\psi_s(\xi)) \\ &= \frac{d}{dt}\Big|_{t=s} h(t) = h'(s) \,. \end{aligned}
$$

Consequently, X_H is a characteristic vector field on Σ_s if $|s| < \delta$, and (i) of the proposition is herewith verified.

The equations (21) and (22) can be interpreted in terms of $\operatorname{grad} H$. From (21) it follows that $\operatorname{grad} H$ is parallel to the normal vector N, so that

$$
(23) \qquad \langle \operatorname{grad} H(\psi_s(\xi), X_\alpha(\psi_s(\xi)) \rangle = \gamma(\psi_s(\xi)) \langle N(\psi_s(\xi)), X_\alpha(\psi_s(\xi)) \rangle
$$

for some function $\gamma : U \to \mathbb{R}$. In view of (22) the left hand side is equal to $h'(s)$. Therefore, the gradient of H at $\psi_s(\xi)$ with $\xi \in \Sigma$, $|s| < \hat{s}$, can be represented by

$$
(24) \qquad \operatorname{grad} H(\psi_s(\xi)) = \frac{h'(s)}{\langle N(\psi_s(\xi)), X_\alpha(\psi_s(\xi)) \rangle} N(\psi_s(\xi)) \,.
$$

The next lemma shows how the Hamiltonian orbits on Σ_s are related to the Hamiltonian orbits on Σ_0. We merely state the assertion and refer to [10] for a proof, which is literally the same for the situation at hand.

Lemma 1 *Let $u_0 : \mathbb{R} \to \Sigma$ be a principal periodic Hamiltonian orbit on Σ, having the minimal period $\tau_0 > 0$. If $|s| < \delta$ then there is an associated principal periodic Hamiltonian orbit u_s on Σ_s given by*

$$
(25) \qquad u_s(t) = (\psi_s \circ u_0)\left(\tfrac{t}{\sigma(s)}\right) ,
$$

where

$$
(26) \qquad \sigma(s) = e^s \frac{h'(0)}{h'(s)} \quad , \quad |s| < \delta \,.
$$

The minimal period of u_s is given by

$$
(27) \qquad \tau(s) = \sigma(s)\,\tau_0 \,.
$$

Furthermore, there exists $\mathcal{T}_0 > 0$ depending on u_0 such that for every $\mathcal{T} > \mathcal{T}_0$ the equation $\mathcal{T} = \tau(s)$ has exactly two solutions $s^+ < s^-$ in $(-\delta, \delta)$. These solutions $s^\pm$ depend smoothly on $\mathcal{T}$ and satisfy $s^+ \to -\delta$, $s^- \to +\delta$ as $\mathcal{T} \to \infty$. Indeed, one has $\mathcal{T}_0 = \sigma(s_0)\tau_0$ where s_0 is uniquely determined by $\sigma'(s_0) = 0$.

If u_0 is any periodic orbit of X_H on Σ_0, the collection $\{ u_s \,|\, |s| < \delta \}$ constitutes a tower of periodic Hamiltonian orbits over u_0, where u_s has the minimal period $\tau(s)$. Each of these towers corresponds to a closed characteristic on Σ, while distinct towers represent distinct closed characteristics. We have presupposed that Σ carries only finitely many distinct principal closed characteristics which, subsequently, are represented by the periodic integral curves $u_0^1, \dots, u_0^\ell$ of X_H on Σ_0.

Considering the i-th principal tower, we denote by $\tau_{0,i} > 0$ the minimal period of u_0^i. Let $\tau_i(s) := \sigma(s)\tau_{0,i}$ be the minimal period of the corresponding orbit u_s^i on Σ_s. From Lemma 1 we infer that for every $\mathcal{T} > \tau_i(s_0)$ there are exactly two periodic orbits $u^i_{s_i^+}$ and $u^i_{s_i^-}$ of minimal period $\mathcal{T}$, where s_i^+, s_i^- are uniquely determined by the condition $\mathcal{T} = \tau_i(s_i^{\pm})$. In order to fulfill (ii) of the proposition it has to be shown that for sufficiently large $\mathcal{T}$ we can distinguish between $u^i_{s_i^+}$ and $u^i_{s_i^-}$ by their respective values of the Hamiltonian action integral $\mathcal{A}_H(u^i_{s_i^{\pm}})$.

A periodic Hamiltonian orbit u_s on Σ_s, having the minimal period $\mathcal{T} = \tau(s)$, can be seen as an embedding $u_s : \mathbb{S}^1_{\mathcal{T}} \to \Sigma_s \subset U$. Note that for an arbitrary smooth loop $u : \mathbb{S}^1_{\mathcal{T}} \to U$ we have

$$u^*\alpha \;=\; u^*(X_\alpha \lrcorner\, \omega) \;=\; \omega(X_\alpha(u), \dot{u})\, dt \;. \tag{28}$$

For the $\mathcal{T}$-periodic Hamiltonian orbit u_s we have $\dot{u}_s = X_H(u_s)$, and consequently, in view of (22),

$$u_s^*\alpha \;=\; \omega(X_\alpha(u_s), X_H(u_s))\, dt \;=\; \alpha(X_H)|_{u_s(t)}\, dt \;=\; h'(s)\, dt \;. \tag{29}$$

Recalling that $\alpha - \lambda$ (with λ the Liouville form) is exact, we obtain

$$\int_{\mathbb{S}^1_{\mathcal{T}}} u_s^*\lambda \;=\; \int_{\mathbb{S}^1_{\mathcal{T}}} u_s^*\alpha \;=\; \mathcal{T}\, h'(s) \;. \tag{30}$$

Therefore, the Hamiltonian action of $u_s : \mathbb{S}^1_{\mathcal{T}} \to \Sigma_s$ is given by

$$\mathcal{A}_H(u_s) \;=\; \int_{\mathbb{S}^1_{\mathcal{T}}} u_s^*\lambda \;-\; \int_0^{\mathcal{T}} (H \circ u_s)\, dt \;=\; \mathcal{T}\,(h'(s) - h(s)) \;. \tag{31}$$

In the notation of Lemma 1, we have either $s = s^+$ or $s = s^-$, and we have $s^+ \to -\delta$, $s^- \to \delta$ as $\mathcal{T} \to \infty$. Then from (31) we conclude (cf. [10]):

Lemma 2 *For each periodic orbit u_0 of X_H on Σ there exists $\mathcal{T}_1 > 0$, depending on u_0, such that for every $\mathcal{T} > \mathcal{T}_1$ the Hamiltonian action of the associated $\mathcal{T}$-periodic orbits u_{s^+}, u_{s^-} satisfies*

$$\mathcal{A}_H(u_{s^+}) \;>\; 0 \quad \text{and} \quad \mathcal{A}_H(u_{s^-}) \;<\; 0 \;. \tag{32}$$

Applying this result to the finitely many principal periodic Hamiltonian orbits $u_0^1, \ldots, u_0^\ell$ on Σ, we observe that the $\mathcal{T}_1 > 0$ can be chosen so large that the statement of Lemma 2 holds for all u_0^i simultaneously. Hence we obtain

Lemma 3 *Let $u_0^1, \ldots, u_0^\ell$ be the principal periodic Hamiltonian orbits on Σ_0. Then there exists $\mathcal{T}_1 > 0$ such that for every $\mathcal{T} \geqslant \mathcal{T}_1$ the periodic orbits $u^i_{s_i^+}$ and $u^i_{s_i^-}$ of minimal period $\mathcal{T}$ satisfy*

$$\mathcal{A}_H(u^i_{s_i^+}) \;>\; 0 \quad \text{and} \quad \mathcal{A}_H(u^i_{s_i^-}) \;<\; 0 \;, \tag{33}$$

for $i = 1, \ldots, \ell$.

To summarize the above arguments, we note that the assumption of at most finitely many principal closed characteristics on Σ implies that a principal periodic Hamiltonian orbit on Σ_0 is of the form u_0^i. If $\mathcal{T} \geqslant \mathcal{T}_1$ as in Lemma 3, the associated orbit $u^i_{s_i^+}$ is the unique orbit satisfying the conditions of (ii) of the proposition, and the proof is complete. $\square$

We end this section with a few remarks on the case of non-degenerate closed characteristics on Σ. Using the denotation from [10], the Hamiltonian function $H : U \to \mathbb{R}$ introduced in (20) is *admissible* for the pair (Σ, X_α) in the sense that it is of the form $H(\psi_s(\xi)) = h(s)$ with $h'(s) > 0$ if $|s| < \delta$. Non-degeneracy $\operatorname{mod} \mathbb{S}^1$ relative to Σ of periodic X_H-orbits is then understood with respect to the flow of the characteristic vector field $X_H|_\Sigma$. If the principal periodic X_H-orbit u_0^i is non-degenerate $\operatorname{mod} \mathbb{S}^1$ relative to Σ_0 then, for $|s| < \delta$, the corresponding orbit u_s^i is non-degenerate $\operatorname{mod} \mathbb{S}^1$ relative to Σ_s. The periodic X_H-orbit u_s^i is non-degenerate $\operatorname{mod} \mathbb{S}^1$ relative to U if, in addition, $h'(s) \neq h''(s)$, cf. [10, Lemma 4].

5. Hamiltonian functions defined by Riemannian structures

As indicated in section 3, we shall extend the Hamiltonian $H : U \to \mathbb{R}$ from Proposition 1 by the use of a metric on $T^*P \xrightarrow{\pi^*} P$. By this process we generate principal periodic Hamiltonian orbits outside U which we call technical periodic orbits. In this section we study their properties which are important for the actual choice of the extension of H described in the next paragraph.

If g is any Riemannian metric on $TP \xrightarrow{\pi} P$ then we denote by $i_g : TP \to T^*P$ the induced isomorphism $i_g(\xi_*) = g(\xi_*, \cdot)$. The corresponding tangent map is again an isomorphism $Ti_g : TTP \to TT^*P$.

Let $K : TTP \to TP$ be the Levi-Cività connection associated to g. There is an associated Riemannian connection $K^* : TT^*P \to T^*P$ such that the following diagram commutes:

$$\begin{tikzcd} TTP \arrow[r, "K"] \arrow[d, "Ti_g"] & TP \arrow[d, "i_g"] \\ TT^*P \arrow[r, "K^*"] & T^*P \end{tikzcd}$$

The connection K^* defines a decomposition $TT^*P = T_hT^*P \oplus T_vT^*P$ into the horizontal and the vertical subbundle. By $K^*|_{T_vT^*P} : T_vT^*P \to T^*P$ the vertical subbundle can be identified with T^*P. Similarly, by $T\pi^*|_{T_hT^*P} : T_hT^*P \to TP$ the horizontal subbundle is isomorphic to TP.

Let g^* be the scalar product on T^*P dual to g, i.e., $g^*(i_g(\xi_*), i_g(\eta_*)) = g(\xi_*, \eta_*)$. Writing now $X \in TT^*P$ as $(X_h, X_v) \in TP \oplus T^*P \cong T_hT^*P \oplus T_vT^*P$, the symplectic form ω can be expressed by

$$\omega(X, Y) \;=\; X_v(Y_h) - Y_v(X_h) \;=\; g^*(X_v, i_g(Y_h)) - g^*(Y_v, i_g(X_h)) \; . \tag{34}$$

This follows readily from a representation $\omega = \sum dy_i \wedge dx_i$ in symplectic local coordinates (see also [11, Prop. 3.1.3]). Furthermore, applying the Liouville form λ to

$X \in T_\xi T^*P$, we find

$$\lambda(X) \;=\; \xi(T\pi^* X) \;=\; \xi(X_h)\;. \tag{35}$$

We observe that the identification $T_hT^*P \oplus T_vT^*P \cong TP \oplus T^*P$ gives rise to an inner product on the fibres of TT^*P by setting

$$\langle X, Y\rangle \;:=\; g(X_h, Y_h) + g^*(X_v, Y_v) \tag{36}$$

so that T_hT^*P and T_vT^*P are orthogonal with respect to $\langle\cdot,\cdot\rangle$. An adapted complex operator $J(\xi)$ on $T_{\xi h}T^*P \oplus T_{\xi v}T^*P \cong T_{\pi^*(\xi)}P \oplus T^*_{\pi^*(\xi)}P$ is given by

$$J(\xi)(X_h, X_v) \;:=\; (i_g^{-1}(X_v), -i_g(X_h))\;. \tag{37}$$

The identity $\omega(X, JY) = \langle X, Y\rangle$ is verified by direct computation.

Lemma 4 *Let $\varrho > 0$ and let $\Sigma^*(\varrho) := \{\, \xi \in T^*P \mid g^*(\xi,\xi) = \varrho^2 \,\}$ denote the cosphere bundle over P of radius ϱ. Then $\Sigma^*(\varrho) \subset T^*P$ is a strict Liouville hypersurface satisfying the assumptions of Theorem 1. In particular, the symplectic dilation X_λ is normal to $\Sigma^*(\varrho)$ with respect to $\langle\,\cdot\,,\,\cdot\,\rangle$.*

Proof Because P is compact and $\Sigma^*(\varrho)$ bounds the compact set $\{\,\xi \mid |\xi|_{g^*} \leqslant \varrho\,\}$, it remains to be shown that the symplectic dilation X_λ associated to the Liouville form is transverse to $\Sigma^*(\varrho)$. For $\xi \in T^*P$ and $Y \in T_\xi T^*P$ we infer from

$$\xi(Y_h) \;=\; \lambda(Y) \;=\; (X_\lambda \lrcorner\, \omega)(Y) \;=\; g^*(X_{\lambda v}, i_g(Y_h)) - g^*(Y_v, i_g(X_{\lambda h})) \tag{38}$$

the representation

$$X_\lambda(\xi) \;=\; (0, \xi) \in T_{\xi h}T^*P \oplus T_{\xi v}T^*P\;. \tag{39}$$

On the other hand, with $G(\xi) := \frac{1}{2}\, g^*(\xi,\xi)$, we have $dG(Y)|_\xi = \langle\, \mathrm{grad}\, G, Y\rangle_\xi = g^*(\xi, Y_v)$ for $Y \in T_\xi T^*P$, so that $\langle X_\lambda, Y\rangle_\xi = \langle\, \mathrm{grad}\, G, Y\rangle_\xi$.

Given $\xi \in \Sigma^*(\varrho)$, a tangent vector $Y \in T_\xi T^*P$ is contained in $T_\xi\Sigma^*(\varrho)$ if and only if $dG(Y)|_\xi = 0$. This means that $X_\lambda(\xi)$ is normal to $\Sigma^*(\varrho)$ at ξ with respect to $\langle\,\cdot\,,\,\cdot\,\rangle$, and the proof is complete. $\square$

The preceeding observations hold true for any metric g on $TP \xrightarrow{\pi} P$. The metric we choose to extend the Hamiltonian in the fibres of T^*P is, however, required to be special because we shall need an explicit knowledge of the principal closed characteristics on a cosphere bundle $\Sigma^*(\varrho)$.

For this reason let us consider a connection κ on the principal $\mathbb{S}^1$-bundle P. Identifying the Lie algebra $T_0\mathbb{S}^1_\tau$ of $\mathbb{S}^1_\tau$ with $\mathbb{R}$, we denote the canonical basis vector by $\frac{d}{dt}|_{t=0}$. Recall that a connection κ on P is given by a 1-form $\kappa(x) : T_xP \to T_0\mathbb{S}^1_\tau$ such that

(i) $\kappa(x)\,\dot c_0(x) = \frac{d}{dt}|_{t=0}$ where $\dot c_t(x) := \frac{d}{dt}\, c_t(x)$.

(ii) $\kappa(c_t(x)) \circ Tc_t = \kappa(x)$.

The connection κ provides a decomposition $TP = T_hP \oplus T_vP$ into the horizontal subbundle $T_hP = \ker \kappa$ and the vertical subbundle $T_vP = \ker Tp$. Hence we may identify $T_hP \xrightarrow{Tp} T(P/\mathbb{S}^1)$ and $T_vP \xrightarrow{\kappa} T\mathbb{S}^1_\tau$ (see e.g. [3, Ch. 16]).

Let g_0 denote the flat metric on $\mathbb{S}^1_\tau = \mathbb{R}/\tau\mathbb{Z}$ inherited from the standard metric on $\mathbb{R}$, which is normalized by setting

$$(40) \qquad g_0(\tfrac{d}{dt}|_{t=0}, \tfrac{d}{dt}|_{t=0}) \;=\; 1/\tau^2 \quad \text{for} \quad \tfrac{d}{dt}|_{t=0} \in T_0\mathbb{S}^1_\tau .$$

Moreover, let g_1 be a fixed Riemannian metric on $P/\mathbb{S}^1$. Then we define the Riemannian metric g on P by setting

$$(41) \quad g(\xi_*, \eta_*) \;:=\; g_1(\xi_{*h}, \eta_{*v}) + g_0(\xi_{*v}, \eta_{*v}) \;=\; g_1(Tp\,\xi_*, Tp\,\eta_*) + g_0(\kappa\,\xi_*, \kappa\,\eta_*)$$

where $\xi_* = (\xi_{*h}, \xi_{*v}) \in T_hP \oplus T_vP \cong T(P/\mathbb{S}^1) \oplus T\mathbb{S}^1$. We note that $|\dot c_0(x)|_g = 1/\tau$ by the above normalization.

If $\xi_* \in T_xP$ then the vertical component has the form $\xi_{*v} = r\,\dot c_0(x)$ for some $r \in \mathbb{R}$, and we obtain

$$(42) \qquad Tc_t\,\xi_* \;=\; Tc_t\,(\xi_{*h}, r\,\dot c_0(x)) \;=\; (\xi_{*h}, r\,\dot c_t(x)) \;\in\; T_{t\cdot x}P\,.$$

Consequently, it follows readily that $Tc_t\,\xi_*$ is parallel along $c_t(x)$ with respect to the Levi-Cività connection associated to g.

Lemma 5 *Let g be the metric from* (41)*. Then the corresponding dual metric g^* is $\mathbb{S}^1$-invariant, i.e., $g^*(\gamma_t\,\xi, \gamma_t\,\eta) = g^*(\xi, \eta)$.*

Proof We note that $p \circ c_t = p$ implies $Tp \circ Tc_t = Tp$. Hence, together with $\kappa \circ Tc_t = \kappa$, it follows from (41) that $g(Tc_t\,\xi_*, Tc_t\,\eta_*) = g(\xi_*, \eta_*)$. By the definition of γ_t we have

$$(43) \quad \gamma_t\,\xi \;=\; g(i_g^{-1}(\xi), Tc_{-t}\,\cdot\,) \;=\; g(Tc_{-t}(Tc_t\,i_g^{-1}(\xi)), Tc_{-t}\,\cdot\,) \;=\; g(Tc_t\,i_g^{-1}(\xi), \,\cdot\,)\,,$$

so that $\gamma_t = i_g \circ Tc_t \circ i_g^{-1}$. Consequently, we obtain

$$(44) \quad g^*(\gamma_t\,\xi, \gamma_t\,\eta) \;=\; g(Tc_t\,i_g^{-1}(\xi), Tc_t\,i_g^{-1}(\eta)) \;=\; g(i_g^{-1}(\xi), i_g^{-1}(\eta)) \;=\; g^*(\xi, \eta)\,.$$

$\square$

As a consequence, the corresponding cosphere bundle $\Sigma^*(\varrho)$ is invariant under the $\mathbb{S}^1$-action γ_t. Moreover, the principal closed characteristic on $\Sigma^*(\varrho)$ can be described as follows:

Lemma 6 *Let g be the metric from* (41) *and let g^* be the corresponding dual metric. Given $\varrho > 0$ and a parametrization of the circle by $\mathbb{S}^1_\tau = \mathbb{R}/\tau\mathbb{Z}$, we have:*

(a) *Every $\mathbb{S}^1$-orbit $\gamma_t\,\chi_x$ with $\chi_x := i_g(\varrho\,\tau\,\dot c_0(x))$ is a closed characteristic on $\Sigma^*(\varrho)$.*

(b) *Consider the Hamiltonian function $G(\xi) = \frac{1}{2}\,g^*(\xi, \xi)$ on T^*P. Every principal periodic orbit $u(t)$ of X_G on $\Sigma^*(\varrho)$ is of the form $u(t) = \gamma_{\varrho\tau t}\,\chi_x$ and has the minimal period $1/\varrho$.*

Proof Suppose that $\xi \in \Sigma^*(\varrho)$. Then $v(t) = \gamma_t \xi$ represents a closed characteristic on $\Sigma^*(\varrho)$ if $\dot v(t) = \mu(t)\, X_G(v(t))$ with a scalar function $\mu(t) > 0$.

Let us introduce the co-geodesic spray $S^* : T^*P \to TT^*P$ which is defined by $S^*(\xi) = (i_g^{-1}(\xi), 0) \in TP \oplus T^*P \cong T_hT^*P \oplus T_vT^*P$. Then one has $S^*(\xi) = X_G(\xi)$, cf. [11, Ch. 3.1]. Hence, if $\xi \in T_x^*P$, we may write

$$X_G(\gamma_t \xi) \;=\; S^*(\gamma_t \xi) \;=\; (i_g^{-1}(\gamma_t \xi), 0) \;\in\; T_{t\cdot x}P \oplus T^*_{t\cdot x}P\,. \tag{45}$$

Associated to $\xi \in T_x^*P$ there is a section $\mathbb{S}^1 \cdot x \to T^*P$, defined by $t \cdot x \mapsto \gamma_t \xi$, whose tangent map $T\gamma_t \xi$ maps $\dot c_t(x)$ to $\dot\gamma_t \xi := \frac{d}{dt}\gamma_t \xi$. Denoting by ∇ the covariant derivative associated to the Levi-Cività connection K, we obtain

$$\begin{aligned} K^*(\dot\gamma_t \xi) &= (K^* \circ T\gamma_t \xi)(\dot c_t(x)) \\ &= (i_g \circ K \circ T i_g^{-1} \circ T\gamma_t \xi)(\dot c_t(x)) \\ &= (i_g \circ K \circ T(i_g^{-1}(\gamma_t \xi)))(\dot c_t(x)) \\ &= i_g(\nabla_{\dot c_t(x)} i_g^{-1}(\gamma_t \xi)) \\ &= i_g(\nabla_{\dot c_t(x)}(Tc_t\, i_g^{-1}(\xi)))\,. \end{aligned} \tag{46}$$

Recalling that Tc_t is the parallel displacement, we get $\nabla_{\dot c_t(x)}(Tc_t\, i_g^{-1}(\xi)) = 0$ and, therefore, $\dot\gamma_t \xi \in \ker K^*$, i.e., $\dot\gamma_t \xi \in T_hT^*P \oplus \{0\}$. From $\pi^*(\gamma_t \xi) = c_t(x)$ we finally obtain $T\pi^*(\dot\gamma_t \xi) = \dot c_t(x)$, implying the representation

$$\dot\gamma_t \xi \;=\; (\dot c_t(x), 0) \;\in\; T_{t\cdot x}P \oplus T^*_{t\cdot x}P\,. \tag{47}$$

Using (45) and (47), we deduce that the equation $\dot\gamma_t \xi = \mu(t)\, X_G(\gamma_t \xi)$ is equivalent to $\mu(t)\, \gamma_t \xi = i_g(\dot c_t(x))$. Because $|\gamma_t \xi|_{g^*} = \varrho$ and $|\dot c_t(x)|_g = 1/\tau$ it follows that $\mu(t) \equiv 1/\varrho\tau$. Hence $v(t) = \gamma_t \xi$ describes a periodic X_G-orbit on $\Sigma^*(\varrho)$ if and only if $\xi = i_g(\varrho\tau\, \dot c_0(x)) = \chi_x$. The assertion (b) then follows by the reparametrization $u(t) := v(\varrho\tau t)$, and the proof is complete. $\square$

Finally let us point out the following fact: if g is the metric from (41) and g^* its dual metric, then the corresponding scalar product $\langle\,\cdot\,,\,\cdot\,\rangle$ is $\mathbb{S}^1$-invariant, $\langle T\gamma_t X, T\gamma_t Y\rangle = \langle X, Y\rangle$. We recall that the map $\xi \mapsto \gamma_t \xi$ is a point transformation so that $\gamma_t^*\omega = \omega$. Hence it follows that $J(\gamma_t \xi) \circ T\gamma_t = T\gamma_t \circ J(\xi)$ for all $\xi \in T^*P$ and $t \in \mathbb{S}^1$.

6. Global extension of the Hamiltonian

From now on let g always be the $\mathbb{S}^1$-invariant metric defined in (41), with the corresponding dual metric g^*. Then there exists $\varrho_1 > 0$ such that

$$\overline{U} \;\subset\; \operatorname{int}\{\, \xi \in T^*P \mid |\xi|_{g^*} \leqslant \varrho_1 \,\}\,. \tag{48}$$

Furthermore, we can choose $a > 0$ small and $\varrho_2 > 0$ large such that

$$\frac{a\,\varrho_1^2}{2} \;<\; h(\hat s) \;<\; \frac{a\,\varrho_2^2}{2}\,, \tag{49}$$

where $h(\hat{s})$ is the maximal value of the Hamiltonian H on U, see (19).

We shall denote by φ_s the flow of the symplectic dilation X_λ. From the representation $X_\lambda(\xi) = (0, \xi) \in T_{\xi h}T^*P \oplus T_{\xi v}T^*P$ it follows that

$$\varphi_s(\xi) = e^s \xi \quad \text{for} \quad s \in \mathbb{R},\ \xi \in T^*P\,. \tag{50}$$

Using the flow φ_s, the complement of the 0-section in T^*P can be represented by a parametrized family of hypersurfaces modelled over a fixed cosphere bundle, say $\Sigma^*(\varrho_2)$. It is convenient to label the cosphere bundles by the flow parameter s rather than the radius ϱ. Hence we introduce the notation

$$\Sigma^*_s := \varphi_s(\Sigma^*(\varrho_2)) = \Sigma^*(\varrho_2\, e^s)\,, \quad s \in \mathbb{R}\,. \tag{51}$$

Now we set $s_1 := \ln(\varrho_1/\varrho_2) < 0$ and we choose a smooth function $h_\infty : \mathbb{R} \to \mathbb{R}$ according to the following conditions:

$$\begin{cases} h_\infty(s) \equiv h(\hat{s}) & \text{if} \quad s \leqslant s_1 \\ h_\infty(s) = \frac{a\,\varrho_2^2}{2}\, e^{2s} & \text{if} \quad s \geqslant 0 \\ 0 < h'_\infty(s) < h''_\infty(s) & \text{if} \quad s > s_1\,. \end{cases} \tag{52}$$

Recall that $\Sigma = \partial D$ with D a relatively compact neighborhood of the 0-section 0_{T^*P}, and the neighborhood U of Σ was defined such that $0_{T^*P} \subset D \backslash U$. The Hamiltonian $H|_U : U \to \mathbb{R}$ defined in (20) is now extended to a smooth Hamiltonian $H : T^*P \to \mathbb{R}$ by setting

$$H(\xi) := \begin{cases} 0 & \text{if} \quad \xi \in D\backslash U \\ H|_U(\xi) & \text{if} \quad \xi \in U \\ h_\infty(s) & \text{if} \quad \xi \in T^*P\backslash(D \cup U) \ \text{ with } \ e^{-s}\xi \in \Sigma^*_0\,. \end{cases} \tag{53}$$

In particular, we note that $H \equiv 0$ in a neighborhood of 0_{T^*P}, and

$$g^*(\xi, \xi) = \varrho_2^2\, e^{2s} \quad \text{for} \quad e^{-s}\xi \in \Sigma^*_0\,,\ s \in \mathbb{R}\,, \tag{54}$$

so that $H(\xi) = \frac{a}{2}\, g^*(\xi, \xi) = \frac{a}{2}\, |\xi|^2_{g^*}$ if $|\xi|_{g^*} \geqslant \varrho_2$.

Lemma 7 *The principal periodic orbits of X_H on Σ^*_0 are given by*

$$v_0(t; x) := \gamma_{a\varrho_2\tau t}\chi_x \quad \textit{for} \quad x \in P\,, \tag{55}$$

where $\chi_x = i_g(\varrho_2\tau\, \dot{c}_0(x)) \in \Sigma^*_0$, and where the circle is parametrized by $\mathbb{S}^1_\tau = \mathbb{R}/\tau\mathbb{Z}$. Each orbit $v_0(t; x)$ has the minimal period $1/a\varrho_2$.

Proof We point out that $X_H(\xi) = a\, X_G(\xi)$ if $\xi \in \Sigma^*_0$, where G is the Hamiltonian from Lemma 6. The assertion thus is an immediate consequence of Lemma 6. □

The flow φ_s provides a transfer of the orbits $v_0(t; x)$ from Σ^*_0 to Σ^*_s. The transferred orbits can be parametrized as integral curves of X_H by a procedure analogous to Lemma 1 in section 4:

Lemma 8 *Let $v_0(t)$ stand for any of the principal periodic orbits $v_0(t;x)$ of X_H on Σ_0^*, having the minimal period $\tau_0^* := 1/a\varrho_2$. If $s > s_1$ then an associated principal periodic orbit v_s of X_H on Σ_s^* is defined by*

$$v_s(t) \ := \ (\,\varphi_s \circ v_0\,)\left(\tfrac{t}{\sigma^*(s)}\right)\ , \tag{56}$$

where

$$\sigma^*(s) \ := \ e^s\,\frac{h'_\infty(0)}{h'_\infty(s)}\ , \quad s > s_1\,. \tag{57}$$

The orbit v_s has the minimal period

$$\tau^*(s) \ := \ \sigma^*(s)\,\tau_0^*\ . \tag{58}$$

For every $\mathcal{T} > 0$ the equation $\mathcal{T} = \tau^*(s)$ has exactly one solution $s(\mathcal{T}) \in (s_1, +\infty)$ which depends smoothly on $\mathcal{T}$ and satisfies $s(\mathcal{T}) \to s_1$ as $\mathcal{T} \to \infty$.

Proof It follows immediately from our definitions that $(\varphi_s \circ v_0)(t)$ is a principal closed characteristic on Σ_s^*. Using that $\varphi_s^*\omega = e^s\omega$, we readily obtain for $s > s_1$

$$\frac{d}{dt}\,(\varphi_s\circ v_0)(t) \ = \ e^s\,J((\varphi_s\circ v_0)(t))\,\mathrm{grad}(H\circ\varphi_{-s})((\varphi_s\circ v_0)(t))\,, \tag{59}$$

where $J(\xi) \in \mathrm{Hom}(T_\xi T^*P)$ is the complex operator according to (37). Furthermore, let $\langle\,\cdot\,,\,\cdot\,\rangle = \omega(\,\cdot\,, J\,\cdot\,)$ denote the associated scalar product on $T_\xi T^*P$, having the corresponding norm $\|\cdot\|$.

For $\xi \in \Sigma_0^*$ and $s \in \mathbb{R}$ let $N^*(\varphi_s(\xi)) := X_\lambda(\varphi_s(\xi))/\|X_\lambda(\varphi_s(\xi))\|$ denote the outward unit normal vector on Σ_s^* at $\varphi_s(\xi)$. If, in addition, $s > s_1$ then we can represent the gradient of H at $\varphi_s(\xi)$ by

$$\mathrm{grad}\,H(\varphi_s(\xi)) \ = \ \frac{h'_\infty(s)}{\langle\, N^*(\varphi_s(\xi))\,, X_\lambda(\varphi_s(\xi))\,\rangle}\,N^*(\varphi_s(\xi))\,. \tag{60}$$

This expression is derived analogously to (24) in the proof of Proposition 1. Using a similar representation for the gradient of $H\circ\varphi_{-s}$, it is readily seen that

$$\mathrm{grad}(H\circ\varphi_{-s})(\varphi_s(\xi)) \ = \ \frac{h'_\infty(0)}{h'_\infty(s)}\,\mathrm{grad}\,H(\varphi_s(\xi))\,. \tag{61}$$

Combined with (59) this yields $\dot v_s(t) = X_H(v_s(t))$.

Evidently, $\tau^*(s)$ is the minimal period of v_s provided that τ_0^* is the minimal period of v_0. The condition $h'_\infty(s) < h''_\infty(s)$ for $s > s_1$ in (52) implies that $\tau^{*\prime}(s) < 0$. Hence $\tau^*(s)$ is strictly decreasing. Moreover, we have $\tau^*(s) \to +\infty$ as $s \to s_1$ because of $h'_\infty(s) \to 0$, and $\tau^*(s) \to 0$ as $s \to +\infty$ because of $\sigma^*(s) = e^{-s}$ for $s > 0$. Consequently, $\tau^* : (s_1, +\infty) \to (0, +\infty)$ is invertible, and the lemma follows. $\square$

In the next section it will turn out to be a crucial observation that the Hamiltonian action of the orbit $v_{s(\mathcal{T})}$ is negative if the minimal period $\mathcal{T}$ is sufficiently large:

Lemma 9 *Given $\mathcal{T} > 0$, let $s = s(\mathcal{T})$ and v_s be defined according to Lemma 8. The Hamiltonian action of v_s is then given by*

$$\mathcal{A}_H(v_s) \;=\; \mathcal{T}\,(h'_\infty(s) - h_\infty(s))\,. \tag{62}$$

In particular, there exists $\mathcal{T}_2 > 0$ such that $\mathcal{A}_H(v_{s(\mathcal{T})}) < 0$ for all $\mathcal{T} \geqslant \mathcal{T}_2$.

Proof In respect of $\dot{v}_s = X_H(v_s)$ we have

$$v_s^*\lambda \;=\; \omega(X_\lambda, X_H)|_{v_s(t)}\,dt \;=\; \lambda(X_H)|_{v_s(t)}\,dt \;=\; h'_\infty(s)\,dt\,. \tag{63}$$

Since $H(v_s(t)) = h_\infty(s)$ we conclude

$$\mathcal{A}_H(v_s) \;=\; \int_{\mathbb{S}^1_{\mathcal{T}}} v_s^*\lambda \;-\; \int_0^{\mathcal{T}} H(v_s(t))\,dt \;=\; \mathcal{T}\,(h'_\infty(s) - h_\infty(s))\,. \tag{64}$$

Furthermore, we have $s(\mathcal{T}) \to s_1$ as $\mathcal{T} \to +\infty$, which implies $h'_\infty(s(\mathcal{T})) \to 0$ and $h_\infty(s(\mathcal{T})) \to h(\hat{s}) > 0$. Hence there exists $\mathcal{T}_2 > 0$ such that $h'_\infty(s(\mathcal{T})) < h_\infty(s(\mathcal{T}))$ if $\mathcal{T} \geqslant \mathcal{T}_2$, so that $\mathcal{A}_H(v_{s(\mathcal{T})}) < 0$. $\square$

Let us summarize the aspects of the Hamiltonian description which are meaningful for the subsequent variational setup:

Proposition 2 *Let $\Sigma \subset T^*P$ be a Liouville hypersurface satisfying the hypotheses of Theorem 1 and carrying at most finitely many principal closed characteristics. Let the Hamiltonian $H : T^*P \to \mathbb{R}$ be defined by (53), and let $\mathcal{T}_1, \mathcal{T}_2 > 0$ be chosen in accordance with Lemma 3 and Lemma 9, respectively. If $\mathcal{T} \geqslant \max\{\mathcal{T}_1, \mathcal{T}_2\}$ then the set of principal periodic X_H-orbits of minimal period $\mathcal{T}$ is described by*

$$\{\, u^i_{s_i^\pm}(t) \mid i = 1, \ldots, \ell \,\} \;\cup\; \{\, v_s(t; x) \mid s = s(\mathcal{T}),\, x \in P \,\}\,. \tag{65}$$

Furthermore, if u is any principal periodic X_H-orbit of minimal period $\mathcal{T}$ with $\mathcal{A}_H(u) > 0$, then there exist $i \in \{1, \ldots, \ell\}$ and $\vartheta \in \mathbb{S}^1_{\mathcal{T}}$ such that $u(t) = u^i_{s_i^+}(t + \vartheta)$.

7. The variational formulation and the associated Morse complex

Let $\mathcal{T} > 0$ be fixed such that Proposition 2 holds true. Correspondingly we fix the parametrization of the circle by $\mathbb{S}^1_{\mathcal{T}} = \mathbb{R}/\mathcal{T}\mathbb{Z}$. To every $\xi \in T^*P$ we assign the principal loop $u_\xi : \mathbb{S}^1_{\mathcal{T}} \to T^*P$, defined by $u_\xi(t) := \gamma_t\,\xi$, which has the minimal period $\mathcal{T}$. The principal loop $u_\xi(t)$ is a principal periodic X_H-orbit if it solves the equation

$$\dot{u}_\xi(t) \;=\; X_H(u_\xi(t))\,, \quad \xi \in T^*P\,. \tag{66}$$

We have to derive lower bounds for the number of geometrically distinct solutions of (66) which, moreover, have a positive Hamiltonian action. Recall that the Hamiltonian action of u_ξ is given by

$$\mathcal{A}_H(u_\xi) \;=\; \int_{\mathbb{S}^1_{\mathcal{T}}} u_\xi^*\lambda \;-\; \int_0^{\mathcal{T}} H(u_\xi(t))\,dt\,. \tag{67}$$

Let $\Omega_{\text{prin}} := \{\, u_\xi \mid \xi \in T^*P \,\}$ denote the manifold of $\mathcal{T}$-periodic principal loops, which can be seen as a finite dimensional submanifold of $C^\infty(\mathbb{S}^1_{\mathcal{T}}, T^*P)$. Indeed, by $\xi \mapsto u_\xi$ we have a diffeomorphism $T^*P \to \Omega_{\text{prin}}$. Considering $\mathcal{A}_H$ as a function $\Omega_{\text{prin}} \to \mathbb{R}$, then the critical points of $\mathcal{A}_H$ on Ω_{prin} are precisely the principal periodic Hamiltonian orbits from (66).

Instead of formulating the variational set up for $\mathcal{A}_H$ on Ω_{prin} we study the corresponding problem on T^*P:

Lemma 10 (a) *The Hamiltonian action integral $\mathcal{A}_H : \Omega_{\text{prin}} \to \mathbb{R}$ gives rise to a smooth $\mathbb{S}^1$-invariant function $\mathcal{F} : T^*P \to \mathbb{R}$ by setting*

$$\mathcal{F}(\xi) \;:=\; \tfrac{1}{\mathcal{T}}\, \mathcal{A}_H(u_\xi)\,. \tag{68}$$

(b) *The gradient of $\mathcal{F}$ is given by*

$$\operatorname{grad}\mathcal{F}(\xi) \;=\; -J(\xi)\,\dot\gamma_0\,\xi \;-\; \operatorname{grad} H(\xi)\,. \tag{69}$$

In particular, $\operatorname{grad}\mathcal{F}(\xi) = 0$ *if and only if $u_\xi(t)$ satisfies the equation* (66).

(c) *Let $\tilde\nabla$ denote the Levi-Cività covariant derivative associated to the metric $\langle\,\cdot\,,\,\cdot\,\rangle$ on T^*P so that the Hessean form of $\mathcal{F}$ is represented by*

$$D^2\mathcal{F}(\xi)(X,Y) \;=\; \langle\, \tilde\nabla_X \operatorname{grad}\mathcal{F}(\xi)\,, Y\,\rangle \quad \textit{for} \quad X, Y \in T_\xi T^*P\,. \tag{70}$$

Then $\{\, \gamma_t\,\xi \mid t \in \mathbb{S}^1_{\mathcal{T}} \,\}$ is a non-degenerate critical manifold of $\mathcal{F}$ if and only if $u_\xi(t)$ is non-degenerate $\operatorname{mod}\mathbb{S}^1$ *as a periodic orbit of X_H, i.e., $u_\xi(t)$ has only one simple Floquet multiplier equal to* 1.

Proof (a) Let $\xi \in T^*P$ with $\pi^*(\xi) = x \in P$. We observe that

$$\begin{aligned} u_\xi^*\lambda \;&=\; \lambda(\dot u_\xi)|_{u_\xi(t)}\, dt \;=\; \omega(X_\lambda, \dot u_\xi)|_{u_\xi(t)}\, dt \\ &=\; g^*(\gamma_t\,\xi, i_g(\dot c_t(x)))\, dt \;=\; g^*(\gamma_t\,\xi, \gamma_t\, i_g(\dot c_0(x)))\, dt \\ &=\; g^*(\xi, i_g(\dot c_0(x)))\, dt\,. \end{aligned} \tag{71}$$

Since, moreover, $H(u_\xi(t)) = H(\gamma_t\,\xi) = H(\xi)$, we deduce

$$\mathcal{F}(\xi) \;=\; g^*(\xi, i_g(\dot c_0(x))) \;-\; H(\xi)\,. \tag{72}$$

From this expression it is seen that $\mathcal{F}$ is indeed a smooth function.

(b) To derive a representation of $\operatorname{grad}\mathcal{F}(\xi)$ let us proceed as follows. Consider an $\mathbb{S}^1$-equivariant vector field X on T^*P and let χ_s denote its flow. The corresponding flow $\hat\chi_s$ on Ω_{prin} is given by $(\hat\chi_s(u_\xi))(t) := \chi_s(\gamma_t\,\xi) = \gamma_t\,\chi_s(\xi)$. Then we have $\mathcal{F}(\chi_s(\xi)) = \frac{1}{\mathcal{T}}\,\mathcal{A}_H(\hat\chi_s(u_\xi))$, and we conclude

$$\langle\, \operatorname{grad}\mathcal{F}(\xi)\,, X(\xi)\,\rangle \;=\; \frac{1}{\mathcal{T}}\, \frac{d}{ds}\Big|_{s=0} \mathcal{A}_H(\hat\chi_s(u_\xi))\,. \tag{73}$$

The directional derivative on the right hand side is readily determined: in view of Cartan's formula $L_X\lambda = X \lrcorner\, d\lambda + d(X \lrcorner\, \lambda)$ and $d\lambda = \omega$ we obtain

$$(74)\quad \frac{d}{ds}\int_{\mathbb{S}^1_{\mathcal{T}}} (\chi_s \circ u_\xi)^*\lambda \;=\; \int_{\mathbb{S}^1_{\mathcal{T}}} \tfrac{d}{ds}\, u_\xi^*(\chi_s^*\lambda) \;=\; \int_{\mathbb{S}^1_{\mathcal{T}}} u_\xi^*(L_X\lambda) \;=\; \int_{\mathbb{S}^1_{\mathcal{T}}} u_\xi^*(X \lrcorner\, \omega)\,.$$

Consequently,

$$(75)\qquad \frac{d}{ds}\bigg|_{s=0} \mathcal{A}_H(\hat{\chi}_s(u_\xi)) \;=\; \int_{\mathbb{S}^1_{\mathcal{T}}} u_\xi^*(X \lrcorner\, \omega) \;-\; \int_0^{\mathcal{T}} dH(X)|_{\gamma_t\xi}\, dt\,.$$

Decomposing $X = (X_h, X_v) \in T_hT^*P \oplus T_vT^*P \cong TP \oplus T^*P$ into the horizontal and the vertical subbundle according to section 5, we find

$$(76)\qquad \begin{aligned} u_\xi^*(X \lrcorner\, \omega) \;&=\; \omega(X, \dot{u}_\xi)|_{\gamma_t\xi}\, dt \;=\; g^*(X_v(\gamma_t\,\xi), i_g(\dot{c}_t(x)))\, dt \\ &=\; g^*(\gamma_t\, X_v(\xi), \gamma_t\, i_g(\dot{c}_0(x)))\, dt \;=\; g^*(X_v(\xi), i_g(\dot{c}_0(x)))\, dt \\ &=\; \langle\, X(\xi), -J(\xi)\, \dot{\gamma}_0\, \xi \,\rangle\, dt\,. \end{aligned}$$

Hence it follows that

$$(77)\qquad \langle\, \operatorname{grad}\mathcal{F}(\xi), X(\xi) \,\rangle \;=\; \langle\, -J(\xi)\,\dot{\gamma}_0\,\xi \;-\; \operatorname{grad} H(\xi)\,, X(\xi) \,\rangle\,,$$

Let $\operatorname{grad}\mathcal{F}(\xi) = 0$. Then we find $T\gamma_t \operatorname{grad}\mathcal{F}(\xi) = -J(\gamma_t\,\xi)\, T\gamma_t\,\dot{\gamma}_0\,\xi - \operatorname{grad} H(\gamma_t\,\xi) = -J(u_\xi(t))\,\dot{u}_\xi(t) - \operatorname{grad} H(u_\xi(t)) = 0$, so that the critical points of $\mathcal{F}$ correspond uniquely to the solutions of (66).

(c) The $\mathbb{S}^1$-orbit $\mathbb{S}^1{\cdot}\xi = \{\, \gamma_t\,\xi \mid t \in \mathbb{S}^1_{\mathcal{T}} \,\}$ is a non-degenerate critical manifold of $\mathcal{F}$ if ξ is a critical point and the kernel of $\tilde{\nabla}\operatorname{grad}\mathcal{F}(\gamma_t\,\xi)$ is spanned by $\dot{\gamma}_t\,\xi$. We can introduce a symplectic orthonormal basis $e_1, \ldots, e_{2n}$ in $T_\xi T^*P$ by choosing $e_1, \ldots, e_n \in T_{\xi h}T^*P \oplus \{0\}$ and $e_{n+i} := -J(\xi)e_i \in \{0\} \oplus T_{\xi v}T^*P$. An orthonormal basis at $\gamma_t\,\xi$ then is given by $e_i(t) := T\gamma_t\, e_i$.

Let $a_{i,j} = \langle\, \tilde{\nabla}_{e_i(t)}\operatorname{grad} H(\gamma_t\,\xi), e_j(t) \,\rangle$ denote the components of the Hessean matrix $A := (a_{i,j})$ of H expressed in the above basis. Note that $a_{i,j} = a_{j,i}$ is indeed independent of t in view of the $\mathbb{S}^1$-invariance of H and the choice of the basis.

Using the coordinate representation, the linearized Hamiltonian equation along the $\mathcal{T}$-periodic critical orbit u_ξ takes the form $\frac{d}{dt}X = J(\xi)A\,X$ of an autonomous linear differential equation in $T_\xi T^*P \cong \mathbb{R}^{2n}$. Hence it follows that $u_\xi(t)$ is non-degenerate $\bmod\, \mathbb{S}^1$ if the non trivial $\mathcal{T}$-periodic solutions of this equation are given by $X(t) \equiv \mu\,\dot{\gamma}_0\xi$ for $\mu \in \mathbb{R} \setminus \{0\}$, and the proof is complete. □

Now we are ready to reformulate the statements of Theorems 1 and 2 in terms of critical $\mathbb{S}^1$-orbits of $\mathcal{F}$:

Proposition 3 *Suppose that the hypotheses of Proposition 2 are satisfied. Then there exist $\ell \geqslant \mathrm{CL}(P/\mathbb{S}^1)$ critical points $\xi^1, \ldots, \xi^\ell$ of $\mathcal{F}$ such that*

(i) $\mathbb{S}^1{\cdot}\xi^i \cap \mathbb{S}^1{\cdot}\xi^j = \emptyset$ *if* $i \neq j$.

(ii) $\mathcal{F}(\xi^i) > 0$ *for* $i = 1, \ldots, \ell$.

In case that all the critical $\mathbb{S}^1$*-orbits* $\mathbb{S}^1\cdot\xi^i$ *satisfying* (i) *and* (ii) *are non-degenerate critical manifolds of* $\mathcal{F}$ *then it follows that* $\ell \geqslant \mathrm{SB}(P/\mathbb{S}^1)$.

For the proof of Proposition 3 we have to study the structure of the Morse complex consisting of the critical points of $\mathcal{F}$ together with their connecting orbits under the gradient flow. For this purpose we observe

Lemma 11 *The function* $\mathcal{F} : T^*P \to \mathbb{R}$ *is bounded above, and for every* $c \in \mathbb{R}$ *the set*

$$\mathcal{F}_c \;:=\; \{\, \xi \in T^*P \mid \mathcal{F}(\xi) \geqslant c \,\} \tag{78}$$

is compact. As a consequence, the critical set $\mathrm{Crit}(\mathcal{F}) := \{\, \xi \mid \operatorname{grad}\mathcal{F}(\xi) = 0 \,\}$ *is compact.*

Proof Let $\xi \in \Sigma_0^*$ and consider $\varphi_s(\xi) = e^s\xi$ for $s \geqslant 0$. Then, by the definition of H, we have $H(\varphi_s(\xi)) = h_\infty(s)$, and consequently, with $x = \pi^*(\xi)$,

$$\mathcal{F}(e^s\,\xi) \;=\; e^s\, g^*(\xi, i_g(\dot c_0(x))) \;-\; \tfrac{a\,\varrho_2^2}{2}\, e^{2s}\,, \tag{79}$$

Since $g^*(\xi, i_g(\dot c_0(x))) \leqslant |\xi|_{g^*}\, |\dot c_0(x)|_g = \frac{\varrho_2}{\mathcal{T}}$, we infer $\mathcal{F}(e^s\,\xi) \leqslant \frac{\varrho_2}{\mathcal{T}}\, e^s - \frac{a\,\varrho_2^2}{2}\, e^{2s} \to -\infty$ as $s \to +\infty$, and the first statement of the lemma follows.

In view of Proposition 2 the structure of the critical set of $\mathcal{A}_H$ on Ω_{prin} is known by the choice of H and $\mathcal{T}$. It follows immediately that the set of critical values of $\mathcal{F}$ is bounded so that $\mathrm{Crit}(\mathcal{F}) \subset \mathcal{F}_c$ for some suitable c. This implies the second statement since $\mathcal{F}_c$ is compact and $\mathrm{Crit}(\mathcal{F})$ is closed. □

Let us now introduce the flow of $-\operatorname{grad}\mathcal{F}$, i.e., $\frac{d}{d\tau}(\xi\cdot\tau) = -\operatorname{grad}\mathcal{F}(\xi\cdot\tau)$. Because the gradient vector field of $\mathcal{F}$ is linearly bounded in the fibres of T^*P, its flow is globally defined as a map $T^*P \times \mathbb{R} \to T^*P$. By the semi-boundedness of $\mathcal{F}$ its Morse complex is given by

$$\mathfrak{M} \;:=\; \{\, \xi \in T^*P \mid \mathcal{F}(\xi\cdot\tau) > -\infty \;\;\forall\, \tau \in \mathbb{R} \,\}\,. \tag{80}$$

We recall that a Morse decomposition of $\mathfrak{M}$ consists of a finite collection of compact non-empty subsets $M_i \subset \mathfrak{M}$ which can be ordered, say $(M_1, \ldots, M_k)$. This ordering is said to be amissible if for every $\xi \in \mathfrak{M} \setminus (M_1 \cup \ldots \cup M_k)$ there exist unique subscripts $i < j$ with $\xi\cdot\tau \to M_i$ as $\tau \to +\infty$ and $\xi\cdot\tau \to M_j$ as $\tau \to -\infty$.

From Lemma 9 and Proposition 2 it follows that 0 is a regular value of $\mathcal{F}$. Consequently, $\mathfrak{M}$ admits a Morse decomposition $(\mathfrak{M}^-, \mathfrak{M}^+)$ where

$$\mathfrak{M}^+ := \;\{\, \xi \in \mathfrak{M} \mid \mathcal{F}(\xi\cdot\tau) > 0 \;\;\forall\, \tau \in \mathbb{R} \,\}\,, \tag{81}$$

and

$$\mathfrak{M}^- := \;\{\, \xi \in \mathfrak{M} \mid \mathcal{F}(\xi\cdot\tau) < 0 \;\;\forall\, \tau \in \mathbb{R} \,\}\,. \tag{82}$$

We shall refer to $\mathfrak{M}^+$ and $\mathfrak{M}^-$ as the positive and the negative Morse complex of $\mathcal{F}$, respectively. Note that $\mathfrak{M}^- \neq \emptyset$ because by Proposition 2 it contains the technical critical $\mathbb{S}^1$-orbits v_s. Moreover, $\mathfrak{M}^+ \neq \emptyset$ since $\mathcal{F}_0 \supset 0_{T^*P} \neq \emptyset$ so that in view of Lemma 11 there exists at least one critical $\mathbb{S}^1$-orbit on a positive level.

Let us fix a regular level $c<0$ of $\mathcal{F}$ such that the Morse complex $\mathfrak{M}$ is contained in $\mathcal{F}_c$. Using Lemma 11, we can choose c such that $\mathcal{F}^{-1}(\{c\})$ is a cosphere bundle.

Associated to the Morse decomposition $(\mathfrak{M}^-,\mathfrak{M}^+)$ there is a filtration of index pairs $N_0\subset N_1\subset N_2$ for the $-\mathrm{grad}\,\mathcal{F}$-flow in the sense of Conley, cf. [2]. For this purpose we set

$$N_0:=\mathcal{F}^{-1}(\{c\})\,,\qquad N_1:=\overline{\mathcal{F}_c\backslash\mathcal{F}_0}\,,\qquad N_2:=\mathcal{F}_c\,. \tag{83}$$

The pair (N_2,N_1) is an index pair for $\mathfrak{M}^+$, i.e.,

(a) $N:=\overline{N_2\backslash N_1}=\mathcal{F}_0$ is an isolating neighborhood for $\mathfrak{M}^+$ having the exit set $\partial N=\mathcal{F}^{-1}(\{0\})$.

(b) N_1 is positively invariant relative to N_2.

(c) If $\xi\in N_2$ and $\xi\cdot[\,0,\infty)\not\subset N_2$ then there exists $\tau\geqslant 0$ such that $\xi\cdot[\,0,\tau\,]\subset N_2$ and $\xi\cdot\tau\in N_1$.

Similarly, $\mathcal{F}_c$ is an isolating neighborhood for the total Morse complex $\mathfrak{M}$, having the exit set $\mathcal{F}^{-1}(\{c\})=\partial\mathcal{F}_c$.

It is a crucial observation that the interior of N contains an $\mathbb{S}^1$-invariant subset which carries the $\mathbb{S}^1$-cohomology of the base space P:

Lemma 12 *There exists $r>0$ such that*

$$Q\;:=\;\{\,r\,i_g(\dot c_0(x))\mid x\in P\,\}\;\subset\;\mathcal{F}_0\,. \tag{84}$$

Moreover, Q is $\mathbb{S}^1$-equivariantly homotopic to the zero section 0_{T^*P}.

Proof If $r>0$ is small enough then we have $H|_Q=0$ by the definition of the Hamiltonian. From (72) it follows that $\mathcal{F}|_Q=r\,|\dot c_0(x)|_g^2=r/\mathcal{T}>0$ so that $Q\subset\mathcal{F}_0$. Clearly, the map $0_{T^*P}\times[\,0,1\,]\to T^*P$, defined by $(x,\mu)\mapsto\mu\,r\,i_g(\dot c_0(x))$, deforms 0_{T^*p} equivariantly into Q. $\square$

8. Equivariant index maps and cuplength estimates for $\mathfrak{M}^+$

If (X,A) is a paracompact $\mathbb{S}^1$-pair then let $H^*_{\mathbb{S}^1}(X,A)$ denote the $\mathbb{S}^1$-equivariant Alexander-Spanier cohomology with real coefficients, cf. e.g. [6]. Since we deal with a free $\mathbb{S}^1$-action, the equivariant cohomology is isomorphic to the ordinary cohomology of $(X/\mathbb{S}^1,A/\mathbb{S}^1)$, i.e., we have $H^*_{\mathbb{S}^1}(X,A)\cong H^*(X/\mathbb{S}^1,A/\mathbb{S}^1)$. Likewise, we denote by $H^{\mathbb{S}^1}_*(X,A):=H_*(X/\mathbb{S}^1,A/\mathbb{S}^1)$ singular $\mathbb{S}^1$-homology of (X,A) with real coefficients.

Given an $\mathbb{S}^1$-space X, the $\mathbb{S}^1$-equivariant cuplength of X is defined by

$$\begin{aligned}\mathbb{S}^1\text{-cuplength}(X)\;&:=\;\sup\,\{\,k\in\mathbb{N}_0\mid\exists\,\omega_i\in H^{q_i}_{\mathbb{S}^1}(X)\,,\;q_i\geqslant 1\,,\;i=1,\dots,k\,,\\&\qquad\qquad\text{such that }\omega_1\smile\ldots\smile\omega_k\neq 0\,\}\,.\end{aligned} \tag{85}$$

If (X,A) is an $\mathbb{S}^1$-pair, then A is called $\mathbb{S}^1$-*categorical in* X if A can be deformed equivariantly through X into a single $\mathbb{S}^1$-orbit. The $\mathbb{S}^1$-category of a non empty $\mathbb{S}^1$-invariant subset $A \subset X$ in X is defined by

$$
\begin{aligned}
\mathbb{S}^1\text{-cat}_X(A) \;&:=\; \inf\,\{\, k \in \mathbb{N} \mid \text{there exist closed } \mathbb{S}^1\text{-categorical subsets} \\
&\qquad\qquad A_1, \ldots, A_k \subset X \text{ with } A \subset A_1 \cup \ldots \cup A_k \,\}\ ,
\end{aligned}
\tag{86}
$$

whereas $\mathbb{S}^1\text{-cat}_X(\emptyset) := 0$.

The following proposition relates the $\mathbb{S}^1$-homotopy invariants of the base space P to those of the positive Morse complex $\mathfrak{M}^+$:

Proposition 4 *Let* $\mathfrak{M}^+$ *be the positive Morse complex of* $\mathcal{F}$ *introduced in* (81), *and let* $\mathcal{F}_c$ *be chosen as in* (83). *Then*

$$
\mathbb{S}^1\text{-cat}_{\mathcal{F}_c}(\mathfrak{M}^+) \;\geqslant\; 1 + \mathbb{S}^1\text{-cuplength}(P)\ .
\tag{87}
$$

Proof Since $\mathcal{F}_c$ can equivariantly be retracted to 0_{T^*P} and since the zero section can equivariantly be deformed into $Q \subset \mathcal{F}_0$, there exists an $\mathbb{S}^1$-equivariant strong deformation retraction $\bar{g} : \mathcal{F}_c \times [\,0,1\,] \to \mathcal{F}_c$ deforming $\mathcal{F}_c$ into Q. The retraction map $g := \bar{g}(\cdot, 1) : \mathcal{F}_c \to Q$ induces an isomorphism $g^* : H^*_{\mathbb{S}^1}(Q) \to H^*_{\mathbb{S}^1}(\mathcal{F}_c)$. In particular, $H^*_{\mathbb{S}^1}(Q) \cong H^*_{\mathbb{S}^1}(P)$ implies $\mathbb{S}^1\text{-cuplength}(\mathcal{F}_c) = \mathbb{S}^1\text{-cuplength}(P)$.

Assuming that $\ell := \mathbb{S}^1\text{-cat}_{\mathcal{F}_c}(\mathfrak{M}^+) \leqslant \mathbb{S}^1\text{-cuplength}(\mathcal{F}_c)$, we argue by contradiction. We have to distinguish between the two cases $\ell = 0$ and $\ell > 0$.

In the latter case we can find a non-vanishing cup product $\omega_1 \smile \ldots \smile \omega_\ell \neq 0$ where $\omega_\nu \in H^{k_\nu}_{\mathbb{S}^1}(\mathcal{F}_c)$ with $k_\nu \geqslant 1$ for $\nu = 1, \ldots, \ell$. Let $\langle \cdot\,, \cdot \rangle$ denote the duality pairing between cohomology and homology. Then there exists a homology class $a_0 \in H_k^{\mathbb{S}^1}(\mathcal{F}_c)$ with $k = k_1 + \ldots + k_\ell$, such that

$$
\langle\, \omega_1 \smile \ldots \smile \omega_\ell\,,\, a_0 \,\rangle \;\neq\; 0\ .
\tag{88}
$$

We emphasize that T^*P is a proper free smooth $\mathbb{S}^1$-manifold with the $\mathbb{S}^1$-structure induced from P. Hence the orbit space carries the unique structure of a smooth manifold such that $Tp : T^*P \to (T^*P)/\mathbb{S}^1$ is a submersion, cf. [4, Prop. 5.2]. In particular, $\mathcal{F}_c/\mathbb{S}^1$ is a compact manifold with boundary $\partial\,(\mathcal{F}_c/\mathbb{S}^1) = (\partial\mathcal{F}_c)/\mathbb{S}^1$.

Set $d := \dim \mathcal{F}_c/\mathbb{S}^1$ and let $\zeta_{\mathcal{F}_c} \in H_d^{\mathbb{S}^1}(\mathcal{F}_c, \partial\mathcal{F}_c)$ be the corresponding fundamental class. By Lefschetz duality there is an isomorphism given by the cap product,

$$
H^{d-k}_{\mathbb{S}^1}(\mathcal{F}_c, \partial\mathcal{F}_c) \;\xrightarrow{\;\frown \zeta_{\mathcal{F}_c}\;}\; H_k^{\mathbb{S}^1}(\mathcal{F}_c)\ .
\tag{89}
$$

Let $\omega_0 \in H^{d-k}_{\mathbb{S}^1}(\mathcal{F}_c, \partial\mathcal{F}_c)$ be the unique class with $a_0 = \omega_0 \frown \zeta_{\mathcal{F}_c}$. Consequently, we have

$$
\langle\, \omega_1 \smile \ldots \smile \omega_\ell\,,\, \omega_0 \frown \zeta_{\mathcal{F}_c} \,\rangle \;=\; \langle\, \omega_1 \smile \ldots \smile \omega_\ell \smile \omega_0\,,\, \zeta_{\mathcal{F}_c} \,\rangle \;\neq\; 0\ .
\tag{90}
$$

The latter cup product needs to be related to the $\mathbb{S}^1$-cohomology of the isolating neighborhood $N = \mathcal{F}_0$ of $\mathfrak{M}^+$.

We have a sequence $\mathcal{F}_c \overset{g}{\to} Q \overset{i}{\to} \mathcal{F}_0 \overset{j}{\to} \mathcal{F}_c$ with $j \circ i \circ g \simeq id$, where i and j are the inclusions. From the induced sequence in $\mathbb{S}^1$-cohomology,

$$H^*_{\mathbb{S}^1}(\mathcal{F}_c) \xrightarrow{j^*} H^*_{\mathbb{S}^1}(\mathcal{F}_0) \xrightarrow{(i\circ g)^*} H^*_{\mathbb{S}^1}(\mathcal{F}_c)\ , \tag{91}$$

it follows that $(i \circ g)^* \circ j^* = id^*$. Hence $(i \circ g)^*$ is epimorphic and j^* is monomorphic. Note that $j^*(\omega_1 \smile \ldots \smile \omega_\ell) = j^*(\omega_1) \smile \ldots \smile j^*(\omega_\ell)$, i.e., j^* commutes with the cup product. We shall abbreviate $\hat{\omega}_\nu := j^*(\omega_\nu) \in H^{k\nu}_{\mathbb{S}^1}(\mathcal{F}_0)$. Consequently we get

$$\hat{\omega}_1 \smile \ldots \smile \hat{\omega}_\ell \neq 0\,. \tag{92}$$

To take the relative class ω_0 into account, we observe that there exists a homotopy $\bar{h} : \mathcal{F}_c \backslash Q \times [\,0,1\,] \to \mathcal{F}_c \backslash Q$ which deforms $\mathcal{F}_c \backslash Q$ equivariantly to $\partial \mathcal{F}_c$. Let this retraction be denoted by $h := \bar{h}(\cdot, 1) : \mathcal{F}_c \backslash Q \to \partial \mathcal{F}_c$. Thus we obtain a sequence $\partial \mathcal{F}_c \overset{j_0}{\to} \mathcal{F}_c \setminus \operatorname{int} \mathcal{F}_0 \overset{i_0}{\to} \mathcal{F}_c \setminus Q \overset{h}{\to} \partial \mathcal{F}_c$ with $h \circ i_0 \circ j_0 = id$, where i_0 and j_0 are the inclusions. Then $j_0^* \circ (h \circ i_0)^* = id^*$ for the induced homomorphisms in $\mathbb{S}^1$-cohomology. Now let us consider the following commutative diagram:

$$\begin{tikzcd}
 & & H^{*-1}_{\mathbb{S}^1}(\mathcal{F}_c \backslash \operatorname{int} \mathcal{F}_0) \arrow[d, "j_0^*"] \\
 & H^{*-1}_{\mathbb{S}^1}(\partial \mathcal{F}_c) \arrow[r, "id^*"] \arrow[ur, "(h\circ i_0)^*"] \arrow[d] & H^{*-1}_{\mathbb{S}^1}(\partial \mathcal{F}_c) \arrow[d, "\delta^*"] \\
H^*_{\mathbb{S}^1}(\mathcal{F}_c, \mathcal{F}_c \backslash \operatorname{int} \mathcal{F}_0) \arrow[r, "\hat{\jmath}^*"] & H^*_{\mathbb{S}^1}(\mathcal{F}_c, \partial \mathcal{F}_c) \arrow[r, "\hat{\imath}^*"] \arrow[d, "\iota_1^*"] & H^*_{\mathbb{S}^1}(\mathcal{F}_c \backslash \operatorname{int} \mathcal{F}_0, \partial \mathcal{F}_c) \arrow[d, "\iota_3^*"] \\
 & H^*_{\mathbb{S}^1}(\mathcal{F}_c) \arrow[r] \arrow[d, "\iota_2^*"] & H^*_{\mathbb{S}^1}(\mathcal{F}_c \backslash \operatorname{int} \mathcal{F}_0) \arrow[d, "j_0^*"] \\
 & H^*_{\mathbb{S}^1}(\partial \mathcal{F}_c) \arrow[r, "id^*"] \arrow[ur, "(h\circ i_0)^*"] & H^*_{\mathbb{S}^1}(\partial \mathcal{F}_c)
\end{tikzcd}$$

Note that the middle row is given by the exact $\mathbb{S}^1$-cohomology sequence of the triple $(\mathcal{F}_c, \mathcal{F}_c \setminus \operatorname{int} \mathcal{F}_0, \partial \mathcal{F}_c)$ while the columns are obtained by the exact $\mathbb{S}^1$-cohomology sequences of the pairs $(\mathcal{F}_c, \partial \mathcal{F}_c)$ and $(\mathcal{F}_c \backslash \operatorname{int} \mathcal{F}_0, \partial \mathcal{F}_c)$, respectively.

We claim that $\hat{\jmath}^*$ is epimorphic. In view of the exactness of the cohomology sequence this claim follows if $\hat{\imath}^* = 0$. Indeed, since j_0^* is epimorphic the exactness of the right column implies $\delta^* = 0$ and, therefore, ι_3^* is monomorphic. By the exactness of the left column we deduce

$$\iota_3^* \circ \hat{\imath}^* \ = \ (h \circ i_0)^* \circ \iota_2^* \circ \iota_1^* \ = \ 0\,, \tag{93}$$

so that $\hat{\imath}^* = 0$ in view of the injectivity of ι_3^*.

Since $\hat{\jmath}^*$ is an epimorphism it has a right inverse, i.e., there is a homomorphism $\hat{g}$ such that for the following sequence in relative $\mathbb{S}^1$-cohomology

$$H^*_{\mathbb{S}^1}(\mathcal{F}_c,\partial\mathcal{F}_c) \xrightarrow{\hat{g}} H^*_{\mathbb{S}^1}(\mathcal{F}_c,\mathcal{F}_c\backslash \operatorname{int}\mathcal{F}_0) \xrightarrow{\hat{\jmath}^*} H^*_{\mathbb{S}^1}(\mathcal{F}_c,\partial\mathcal{F}_c)\ , \tag{94}$$

we have $\hat{\jmath}^*\circ\hat{g} = id^*$. We note that the cohomology groups are finite-dimensional real vector spaces: for $H^*_{\mathbb{S}^1}(\mathcal{F}_c,\partial\mathcal{F}_c)$ this fact follows because the corresponding singular homology groups are finitely generated (cf. [12, p. 298]). For $H^*_{\mathbb{S}^1}(\mathcal{F}_c,\mathcal{F}_c\backslash \operatorname{int}\mathcal{F}_0)$ it follows similarly, using the fact that by Čech-excision there is the following isomorphism which is induced by inclusion (cf. [5, Ch.VIII, 6.17]):

$$\iota^*_{\mathcal{F}_0} : H^*_{\mathbb{S}^1}(\mathcal{F}_c,\mathcal{F}_c\backslash \operatorname{int}\mathcal{F}_0) \xrightarrow{\cong} H^*_{\mathbb{S}^1}(\mathcal{F}_0,\partial\mathcal{F}_0)\ . \tag{95}$$

Let $\hat{\omega}_0 := (\iota^*_{\mathcal{F}_0}\circ\hat{g})(\omega_0) \in H^{d-k}_{\mathbb{S}^1}(\mathcal{F}_0,\partial\mathcal{F}_0)$. Thus we have a commutative diagram

$$\begin{array}{ccccc}
H^k_{\mathbb{S}^1}(\mathcal{F}_c) & \otimes & H^{d-k}_{\mathbb{S}^1}(\mathcal{F}_c,\partial\mathcal{F}_c) & \xrightarrow{\smile} & H^d_{\mathbb{S}^1}(\mathcal{F}_c,\partial\mathcal{F}_c) \\
\downarrow{\scriptstyle j^*} & & \downarrow{\scriptstyle \iota^*_{\mathcal{F}_0}\circ\hat{g}} & & \downarrow{\scriptstyle \iota^*_{\mathcal{F}_0}\circ\hat{g}} \\
H^k_{\mathbb{S}^1}(\mathcal{F}_0) & \otimes & H^{d-k}_{\mathbb{S}^1}(\mathcal{F}_0,\partial\mathcal{F}_0) & \xrightarrow{\smile} & H^d_{\mathbb{S}^1}(\mathcal{F}_0,\partial\mathcal{F}_0)
\end{array}$$

where the vertical maps are monomorphisms. Hence, setting $k = k_1 + \ldots + k_\ell$, we deduce from (90) and (92) the following cup product:

$$\hat{\omega}_1 \smile \ldots \smile \hat{\omega}_\ell \smile \hat{\omega}_0 \neq 0\ . \tag{96}$$

If $\ell = \mathbb{S}^1\text{-cat}_{\mathcal{F}_c}(\mathfrak{M}^+) > 0$ then there exist ℓ closed $\mathbb{S}^1$-categorical subsets $A_\nu \subset \mathcal{F}_c$ with $\mathfrak{M}^+ \subset A_1\cup\ldots\cup A_\ell$. Since $\mathfrak{M}^+ \subset \operatorname{int}\mathcal{F}_0$ we may assume $A_\nu \subset \operatorname{int}\mathcal{F}_0$ and, in addition, $(A_\nu, A_{\nu'})$ is an excisive pair for any choice of $\nu,\nu' \in \{1,\ldots,\ell\}$. By $\iota_\nu : A_\nu \to \mathcal{F}_c$ and $j_\nu : A_\nu \to \mathcal{F}_0$ we denote the inclusions. Since A_ν is $\mathbb{S}^1$-categorical in $\mathcal{F}_c$ the inclusion ι_ν is equivariantly homotopic to a map $\iota^0_\nu : A_\nu \to \mathcal{F}_c$ where the image $\iota^0_\nu(A_\nu) = \mathbb{S}^1\cdot\xi_0$ is a single $\mathbb{S}^1$-orbit. Thus we have an $\mathbb{S}^1$-map $\hat{\iota}_\nu : A_\nu \to \mathbb{S}^1\cdot\xi_0$ and an inclusion $\kappa : \mathbb{S}^1\cdot\xi_0 \to \mathcal{F}_c$ with $\iota^0_\nu = \kappa\circ\hat{\iota}_\nu$. These maps induce the following commutative diagram in $\mathbb{S}^1$-cohomology:

$$\begin{tikzcd}
H^*_{\mathbb{S}^1}(\mathcal{F}_c) \arrow[r, "\kappa^*"] \arrow[d, "j^*"] \arrow[dr, "\iota^*_\nu"] & H^*_{\mathbb{S}^1}(\mathbb{S}^1\cdot\xi_0) \arrow[d, "\hat{\iota}^*_\nu"] \\
H^*_{\mathbb{S}^1}(\mathcal{F}_0) \arrow[r, "j^*_\nu"] & H^*_{\mathbb{S}^1}(A_\nu)
\end{tikzcd}$$

Then $\ker \iota^*_\nu = H^*_{\mathbb{S}^1}(\mathcal{F}_c)$ for $* \geqslant 1$, and therefore $\operatorname{im} j^* \subset \ker j^*_\nu$. In particular, we infer $\hat{\omega}_\nu = j^*(\omega_\nu) \in \ker j^*_\nu \subset H^{k_\nu}_{\mathbb{S}^1}(\mathcal{F}_0)$. For $*\geqslant 1$ we consider the exact cohomology sequence of the $\mathbb{S}^1$-pair $(\mathcal{F}_0, A_\nu)$,

$$\longrightarrow H^{*-1}_{\mathbb{S}^1}(A_\nu) \xrightarrow{\delta^*} H^*_{\mathbb{S}^1}(\mathcal{F}_0,A_\nu) \xrightarrow{i^*_\nu} H^*_{\mathbb{S}^1}(\mathcal{F}_0) \xrightarrow{j^*_\nu} H^*_{\mathbb{S}^1}(A_\nu) \longrightarrow \tag{97}$$

where $i_\nu : (\mathcal{F}_0, \emptyset) \to (\mathcal{F}_0, A_\nu)$ is the inclusion. In view of the exactness of the sequence, for $\nu = 1, \ldots, \ell$ there exists a class $\tilde{\omega}_\nu \in H^{k_\nu}_{\mathbb{S}^1}(\mathcal{F}_0, A_\nu)$ with $i^*_\nu(\tilde{\omega}_\nu) = \hat{\omega}_\nu$. Setting $A := A_1 \cup \ldots \cup A_\ell$, we thus obtain the following commutative diagram

$$\begin{array}{ccccccc} H^{k_1}_{\mathbb{S}^1}(\mathcal{F}_0, A_1) & \otimes & \cdots & \otimes & H^{k_\ell}_{\mathbb{S}^1}(\mathcal{F}_0, A_\ell) & \xrightarrow{\smile} & H^{k}_{\mathbb{S}^1}(\mathcal{F}_0, A) \\ \Big\downarrow{\scriptstyle i_1^*} & & & & \Big\downarrow{\scriptstyle i_\ell^*} & & \Big\downarrow{\scriptstyle i_A^*} \\ H^{k_1}_{\mathbb{S}^1}(\mathcal{F}_0) & \otimes & \cdots & \otimes & H^{k_\ell}_{\mathbb{S}^1}(\mathcal{F}_0) & \xrightarrow{\smile} & H^{k}_{\mathbb{S}^1}(\mathcal{F}_0) \end{array} \tag{98}$$

where $i_A : (\mathcal{F}_0, \emptyset) \to (\mathcal{F}_0, A)$ is the inclusion. We note that $\hat{\omega}_1 \smile \ldots \smile \hat{\omega}_\ell = i^*_A(\tilde{\omega}_1 \smile \ldots \smile \tilde{\omega}_\ell) \neq 0$. To derive the desired contradiction to our assumption $\ell \leqslant \mathbb{S}^1\text{-cuplength}(Q)$ we use Wazewski's principle for equivariant flows:

Lemma 13 *The exit set $\partial\mathcal{F}_0$ is a strong $\mathbb{S}^1$-deformation retract of $\mathcal{F}_0 \setminus \mathfrak{M}^+$.*

The proof is analogous to [1, Lemma 3]. Now we choose a closed $\mathbb{S}^1$-invariant subset $A_0 \subset \mathcal{F}_0$ with $\partial\mathcal{F}_0 \subset A_0 \subset \mathcal{F}_0 \setminus \mathfrak{M}^+$ such that $\overline{\mathcal{F}_0 \setminus (A_1 \cup \ldots \cup A_k)} \subset \operatorname{int} A_0$. By Lemma 13 it follows that $\partial\mathcal{F}_0$ is an $\mathbb{S}^1$-retract of A_0. Hence the inclusion of $\mathbb{S}^1$-pairs $i_0 : (A_0, \partial\mathcal{F}_0) \to (\mathcal{F}_0, \partial\mathcal{F}_0)$ induces the trivial homomorphism $i^*_0 = 0$ in $\mathbb{S}^1$-cohomology. In view of the exactness cohomology sequence of the triple $(\mathcal{F}_0, A_0, \partial\mathcal{F}_0)$,

$$\xrightarrow{\delta^*} H^*_{\mathbb{S}^1}(\mathcal{F}_0, A_0) \xrightarrow{j_0^*} H^*_{\mathbb{S}^1}(\mathcal{F}_0, \partial\mathcal{F}_0) \xrightarrow{i_0^*} H^*_{\mathbb{S}^1}(A_0, \partial\mathcal{F}_0) \xrightarrow{\delta^*} \tag{99}$$

we infer j^*_0 is onto. Hence we can find a relative class $\tilde{\omega}_0 \in H^{d-k}_{\mathbb{S}^1}(\mathcal{F}_0, A_0)$ with $j^*_0(\tilde{\omega}_0) = \hat{\omega}_0 \in H^{d-k}_{\mathbb{S}^1}(\mathcal{F}_0, \partial\mathcal{F}_0)$. Thus we obtain another commutative diagram

$$\begin{array}{ccccc} H^{k}_{\mathbb{S}^1}(\mathcal{F}_0, A) & \otimes & H^{d-k}_{\mathbb{S}^1}(\mathcal{F}_0, A_0) & \xrightarrow{\smile} & H^{d}_{\mathbb{S}^1}(\mathcal{F}_0, \mathcal{F}_0) \\ \Big\downarrow{\scriptstyle i_A^*} & & \Big\downarrow{\scriptstyle j_0^*} & & \Big\downarrow{\scriptstyle i_{\mathcal{F}_0}^*} \\ H^{k}_{\mathbb{S}^1}(\mathcal{F}_0) & \otimes & H^{d-k}_{\mathbb{S}^1}(\mathcal{F}_0, \partial\mathcal{F}_0) & \xrightarrow{\smile} & H^{d}_{\mathbb{S}^1}(\mathcal{F}_0, \partial\mathcal{F}_0) \end{array}$$

where $i_{\mathcal{F}_0} : (\mathcal{F}_0, \partial\mathcal{F}_0) \to (\mathcal{F}_0, \mathcal{F}_0)$ is the inclusion. Now we are led to

$$\begin{aligned} \hat{\omega}_1 \smile \ldots \smile \hat{\omega}_\ell \smile \hat{\omega}_0 &= i^*_A(\tilde{\omega}_1 \smile \ldots \smile \tilde{\omega}_\ell) \smile j^*_0(\tilde{\omega}_0) \\ &= i^*_{\mathcal{F}_0}(\tilde{\omega}_1 \smile \ldots \smile \tilde{\omega}_\ell \smile \tilde{\omega}_0) \;=\; 0\,, \end{aligned} \tag{100}$$

in contradiction to (96).

It remains to consider the case $\ell = \mathbb{S}^1\text{-cat}_{\mathcal{F}_0}(\mathfrak{M}^+) = 0$, i.e., we assume $\mathfrak{M}^+ = \emptyset$. Again by Lefschetz duality we find $\omega_0 \in H^d_{\mathbb{S}^1}(\mathcal{F}_c, \partial\mathcal{F}_c)$ having the property that $\langle 1, \omega_0 \frown \zeta_{\mathcal{F}_c} \rangle = \langle \omega_0, \zeta_{\mathcal{F}_c} \rangle \neq 0$, where $1 \in H^0_{\mathbb{S}^1}(\mathcal{F}_c)$ is the unit element of the $\mathbb{S}^1$-cohomology ring of $\mathcal{F}_c$. Then $0 \neq \hat{\omega}_0 = (\iota^*_{\mathcal{F}_0} \circ \hat{g})(\omega_0) \in H^d_{\mathbb{S}^1}(\mathcal{F}_0, \partial\mathcal{F}_0)$. On the other hand, since $\mathfrak{M}^+ = \emptyset$ we may choose $A_0 := \mathcal{F}_0$. Similar as above, we conclude

$$\hat{\omega}_0 \;=\; j^*_0(\tilde{\omega}_0) \;=\; 0\,, \tag{101}$$

which yields a contradiction. Thus the proof of the proposition is complete. □

9. Lyusternik-Schnirelman theory and the proof of Theorem 1

We are now prepared to apply Lyusternik-Schnirelman theory for flows to the positive Morse complex $\mathfrak{M}^+$. By our assumptions, $\mathfrak{M}^+$ contains only a finite number of critical $\mathbb{S}^1$-orbits. Let the corresponding critical values c_j be ordered according to $0 < c_1 < \ldots < c_k$. For $j = 1, \ldots, k$, we define the Morse sets M_j by

$$(102) \qquad M_j \;:=\; \{\, \xi \in \mathfrak{M}^+ \mid \operatorname{grad} \mathcal{F}(\xi) = 0 \text{ and } \mathcal{F}(\xi) = c_i \,\} \;.$$

Then $(M_1, \ldots, M_k)$ is an admissible ordering of the Morse decomposition of $\mathfrak{M}^+$. Consequently, we obtain

Proposition 5 *Let $(M_1, \ldots, M_k)$ denote the Morse decomposition of $\mathfrak{M}^+$ induced by the ordering of the positive critical values $c_1 < \ldots < c_k$ of $\mathcal{F}$. Then*

$$(103) \qquad \mathbb{S}^1\text{-}\mathrm{cat}_{\mathcal{F}_c}(\mathfrak{M}^+) \;\leqslant\; \sum_{j=1}^{k} \mathbb{S}^1\text{-}\mathrm{cat}_{\mathcal{F}_c}(M_j) \;.$$

In particular, the number of the distinct critical $\mathbb{S}^1$-orbits of $\mathcal{F}$ in $\mathfrak{M}^+$ is greater or equal $\mathbb{S}^1\text{-}\mathrm{cat}_{\mathcal{F}_c}(\mathfrak{M}^+)$.

We omit the proof which can be taken over from [10, Prop. 5].

Combining this result with Proposition 4, we have shown the existence of $\ell \geqslant 1 + \mathbb{S}^1\text{-cuplength}(P)$ distinct critical $\mathbb{S}^1$-orbits $\mathbb{S}^1{\cdot}\xi^1, \ldots, \mathbb{S}^1{\cdot}\xi^\ell$ of $\mathcal{F}$ such that $\mathcal{F}(\xi^i) > 0$. Thus we have verified the first part of Proposition 3 in section 7, which is the variational reformulation of Theorem 1. The proof of Theorem 1 is therefore complete.

10. Morse theory and the proof in the non-degenerate case

Suppose now that all the critical $\mathbb{S}^1$-orbits contained in $\mathfrak{M}^+$ are non-degenerate critical manifolds of $\mathcal{F}$. In this case their number can be estimated by the sum of the equivariant Betti numbers of P.

Let $\mathbb{S}^1{\cdot}\xi^1, \ldots, \mathbb{S}^1{\cdot}\xi^\ell \in T^*P$ denote the distinct critical orbits in $\mathfrak{M}^+$ and let $c_i > 0$ be the critical level corresponding to $M_i := \mathbb{S}^1{\cdot}\xi^i$. Here we assume that the critical sets M_i are indexed according to the ordering $c_1 \leqslant \ldots \leqslant c_\ell$ of the critical values. Then $(M_1, \ldots, M_\ell)$ constitutes an admissibly ordered Morse decomposition of $\mathfrak{M}^+$.

In the case of free $\mathbb{S}^1$-spaces the equivariant Conley index $h_{\mathbb{S}^1}(A)$ of an isolated invariant set A is defined to be the ordinary Conley index of the orbit set $A/\mathbb{S}^1$:

$$(104) \qquad h_{\mathbb{S}^1}(A) \;:=\; h(A/\mathbb{S}^1) \;.$$

Now consider the Morse equation (cf. [2, Thm. 3.3]),

$$(105) \qquad \sum_{i=1}^{\ell} \mathcal{P}(t, h_{\mathbb{S}^1}(\mathbb{S}^1{\cdot}\xi^i)) \;=\; \mathcal{P}(t, h_{\mathbb{S}^1}(\mathfrak{M}^+)) \;+\; (1{+}t)\,\mathcal{Q}(t) \;,$$

where $\mathcal{P}(t,\cdot)$ denotes the Poincaré polynomial and $\mathcal{Q}(t)$ is a polynomial with non-negative integer coefficients. From (105) we deduce the following result:

Proposition 6 *Suppose that the critical* $\mathbb{S}^1$*-orbits* $\mathbb{S}^1 \cdot \xi^i$ *in* $\mathfrak{M}^+$ *are non-degenerate* $\operatorname{mod} \mathbb{S}^1$ *and let* $(\mathbb{S}^1{\cdot}\xi^1, \ldots, \mathbb{S}^1{\cdot}\xi^\ell)$ *constitute an admissibly ordered Morse decomposition of* $\mathfrak{M}^+$*. Then we have*

$$\ell \geqslant \mathrm{SB}\,(P/\mathbb{S}^1)\,, \tag{106}$$

where $\mathrm{SB}(P/\mathbb{S}^1)$ *denotes the sum of the Betti numbers of the orbit space* $P/\mathbb{S}^1$*.*

Proof Consider the orbit space $(T^*P)/\mathbb{S}^1$ endowed with its unique differentiable structure such that the projection $Tp : T^*P \to (T^*P)/\mathbb{S}^1$ is a submersion. Then $Tp(\mathbb{S}^1 \cdot \xi^i)$ is a non-degenerate critical point of the function $\mathcal{F}^* : (T^*P)/\mathbb{S}^1 \to \mathbb{R}$ which is defined by the identity $\mathcal{F} = \mathcal{F}^* \circ Tp$.

The Conley index of $Tp(\mathbb{S}^1{\cdot}\xi^i)$ is the homotopy type of a pointed sphere of dimension ν_i, where ν_i is equal to the Morse index of $Tp(\mathbb{S}^1{\cdot}\xi^i)$:

$$h_{\mathbb{S}^1}(\mathbb{S}^1{\cdot}\xi^i) \;=\; h(Tp(\mathbb{S}^1{\cdot}\xi^i)) \;=\; [\,\dot{\mathbb{S}}^{\nu_i}\,]\,. \tag{107}$$

Clearly, ν_i is the number of the negative eigenvalues of $D^2\mathcal{F}(\xi) \in \mathrm{Hom}(T_\xi T^*P)$ at every point $\xi \in \mathbb{S}^1{\cdot}\xi^i$. From the Poincaré polynomial $\mathcal{P}(t, [\,\dot{\mathbb{S}}^{\nu_i}\,]) = t^{\nu_i}$ we get

$$\sum_{i=1}^{\ell} \mathcal{P}(t, h_{\mathbb{S}^1}(\mathbb{S}^1{\cdot}\xi^i)) \;=\; \sum_{i=1}^{\ell} t^{\nu_i}\,. \tag{108}$$

Considering now the right hand side of (105), we observe that the equivariant Conley index of $\mathfrak{M}^+$ is given by

$$h_{\mathbb{S}^1}(\mathfrak{M}^+) \;=\; [\,((\mathcal{F}_0/\mathbb{S}^1)/(\partial\mathcal{F}_0/\mathbb{S}^1), \{\partial\mathcal{F}_0/\mathbb{S}^1\})\,]\,. \tag{109}$$

In Alexander-Spanier cohomology (with real coefficients) we have the following isomorphisms (see also [5, Ch.VIII, 6.20]):

$$\begin{aligned} H^*(\,(\mathcal{F}_0/\mathbb{S}^1)/(\partial\mathcal{F}_0/\mathbb{S}^1), \{\partial\mathcal{F}_0/\mathbb{S}^1\}\,) \;&\cong\; H^*(\mathcal{F}_0/\mathbb{S}^1, \partial\mathcal{F}_0/\mathbb{S}^1) \\ &\cong\; H^*_{\mathbb{S}^1}(\mathcal{F}_0, \partial\mathcal{F}_0)\,. \end{aligned} \tag{110}$$

Consequently, the Poincaré polynomial of $h_{\mathbb{S}^1}(\mathfrak{M}^+)$ is given by

$$\mathcal{P}(t, h_{\mathbb{S}^1}(\mathfrak{M}^+)) \;=\; \sum_{j \geqslant 0} t^j \dim H^j_{\mathbb{S}^1}(\mathcal{F}_0, \partial\mathcal{F}_0)\,. \tag{111}$$

In the proof of Proposition 4 we have already used the monomorphism

$$\iota^*_{\mathcal{F}_0} \circ \hat{g} : H^*_{\mathbb{S}^1}(\mathcal{F}_c, \partial\mathcal{F}_c) \to H^*_{\mathbb{S}^1}(\mathcal{F}_0, \partial\mathcal{F}_0)\,, \tag{112}$$

which implies the inequality

$$\dim H^*_{\mathbb{S}^1}(\mathcal{F}_0, \partial\mathcal{F}_0) \;\geqslant\; \dim H^*_{\mathbb{S}^1}(\mathcal{F}_c, \partial\mathcal{F}_c)\,. \tag{113}$$

Using Lefschetz duality and recalling that the set Q from Lemma 12 is a strong $\mathbb{S}^1$-deformation retract of $\mathcal{F}_c$, we get

$$H^{d-k}_{\mathbb{S}^1}(\mathcal{F}_c, \partial\mathcal{F}_c) \;\cong\; H^{\mathbb{S}^1}_k(\mathcal{F}_c) \;\cong\; H^{\mathbb{S}^1}_k(Q) \;\cong\; H_k(Q/\mathbb{S}^1)\,, \tag{114}$$

where $d = \dim \mathcal{F}_c/\mathbb{S}^1$. Hence we conclude $\dim H^{d-k}_{\mathbb{S}^1}(\mathcal{F}_c, \partial\mathcal{F}_c) \geqslant \dim H_k(P/\mathbb{S}^1)$. Summing up the terms on the right hand of (111) which are labelled by $j = d-k$, we derive for $t \geqslant 0$ the following estimate from the Morse equation (105):

$$\sum_{i=1}^{\ell} t^{\nu_i} \ \geqslant \ \sum_{k=0}^{d} t^{d-k} \dim H_k(P/\mathbb{S}^1)\,. \tag{115}$$

Setting now $t=1$, we find $\ell \geqslant \mathrm{SB}\,(P/\mathbb{S}^1)$ as claimed. □

This proves the second part of Proposition 3 in section 7, and the proof of Theorem 2 is complete.

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