MODULI SPACE OF HOLOMORPHIC

DISCS WITH BOUNDARY

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1. Introduction. Let (X, J) be a complex surface, i.e., a two complex dimensional manifold with holomorphic coordinate changes, and J the associated complex structure on X. In case that X is compact, the moduli space of rational curves in X is an important tool in classification theory. For the study of open complex surfaces it is natural to consider holomophically embedded open discs with boundary. To obtain a well posed problem for such one needs to impose a boundary condition. If it is restricted to lie in a real two-dimensional surface that is totally real - that is infinitesimally transverse to its image under the complex structure J, rather than invariant, as are holomorphic curves - then one obtains a classical Riemann-Hilbert problem [5], an elliptic boundary value problem. Here we study the corresponding moduli space and examine its dependence on the boundary condition. The study of holomorphic discs with boundary was initiated by Bishop [2] and the arguments given here follow [4], chapter 2.1. The ellipticity and the index of the

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problem for compact curves is studied in detail in [1], 6.22 and [6], 2.3.4. The index formula in Theorem 1c) occurs in [4] and Theorem 1d) is proven similarly as in [6].

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2. The analytic framework. Let $\mathfrak{M}$ be the infinite dimensional manifold of C^∞ - smooth embedded totally real surfaces in X. Here a real surface $M \hookrightarrow X$ is called totally real if $T_m M \oplus J(m) T_m M = T_m X$ for all $m \in M$ or equivalently since $\dim_{\mathbb{R}} M = 4$, $T_m M \cap J(m) T_m M = 0 \subseteq T_m X$. Let now $\Delta \subset \mathbb{C}$ denote the set $\{z \in \mathbb{C} : |z| < 1\}$. Since $W^{1,m}(\bar{\Delta}, \mathbb{R}^N)$, the Sobolev space of functions whose derivative is in L^p, embeds compactly into $C^0(\bar{\Delta}, \mathbb{R}^N)$ for $p > 2$, we have a well-defined space $W^{1,p}(\Delta, \partial\Delta; X, M)$ if $X \hookrightarrow \mathbb{R}^N$ for some $N \in \mathbb{N}$. Let

$$\mathfrak{F} = \{(f, M) \in W^{1,p} \times \mathfrak{M} : f : (\Delta, \partial\Delta) \to (X, M)\}.$$

Together with the natural map $\pi : \mathfrak{F} \to \mathfrak{M}$, this is a locally trivial fibre bundle over $\mathfrak{M}$, and a complete Banach manifold. For differentiable structures on $\mathfrak{M}$ and $\mathfrak{F}$ one may consult [3]. The moduli space of holomorphic discs in X with boundary in elements of $M \in \mathfrak{M}$ is now given by the points in $\mathfrak{F}$ that satisty the Cauchy-Riemann equation:

$$\mathcal{H} = \{(f, M) \in \mathfrak{F} : df \circ i = J(f) \cdot df\}.$$

For $(f, M) \in \mathfrak{F}$ denote by $c_1(f^\star TX, f^\star TM) \in \mathbb{Z}$ the first relative Chern class of the bundle $(f^\star TX, f^\star TM)$ over $(\Delta, \partial\Delta)$ with distinguished totally real subbundle over $\partial\Delta$. It is also called the Maslov class and equals the first Chern class of the relative to $f^\star TM$ doubled bundle over $\mathbb{C}P(1)$.

Theorem 1. *Let $(f, M) \in \mathcal{H}$ and assume that $c_1(f^\star TX, f^\star TM) \geq 1$. Then there exists a neighborhood $\mathfrak{U} \subset \mathfrak{F}$ of (f, M) with:*

a) $\mathcal{H} \cap \mathfrak{U}$ is an infinite dimensional submanifold of $\mathfrak{F}$

b) $\pi : \mathcal{H} \cap \mathfrak{U} \to \mathfrak{M}$ is a smooth Fredholm map

c) The index of $\pi | \mathcal{H} \cap \mathfrak{U}$ equals $2 + c_1(f^\star TX, f^\star TM)$

d) the cokernel of π vanishes at $(f, M) \in \mathcal{H}$, i.e.

$$\dim \pi^{-1}(M_1) = 2 + c_1(f^\star TX, f^\star TM_1) \quad \text{for } M_1 \in \pi(\mathfrak{U}) \subset \mathfrak{M}.$$

3. An Elliptic Boundary Value Operator. To study the space $\mathcal{H}$ it is convenient to view it as the zero set of a section of a vector bundle $\mathfrak{G}$ over $\mathfrak{F}$. The fibre at $(f, M) \in \mathfrak{F}$ of $\mathfrak{G}$ is given by $\Gamma(\Delta, \mathrm{Hom}_{\bar{\mathbb{C}}}(T\Delta, f^\star TX))$, the vector space of sections of the bundle over Δ whose fibre in turn at $\delta \in \Delta$ is given by the $(i, -J)$-linear maps Λ from $(T_\delta\Delta, i) \approx (\mathbb{C}, i)$ to $\left(T_{f(\delta)}X, J(f(\delta))\right) : \Lambda(iv) + J(f(\delta))\Lambda v = 0$ for all $v \in T_\delta\Delta$. We denote by $\mathfrak{G} \to \mathfrak{F}$ the resulting bundle and the section $\bar{\partial} : \mathfrak{F} \to \mathfrak{G}$ we are looking at is given by

$$\bar{\partial}(f, M) = \frac{1}{2}(df + J(f) \circ df \circ i).$$

Then we have $\mathcal{H} = (\bar{\partial})^{-1}0_{\mathfrak{G}} \subset \mathfrak{F}$, if $0_{\mathfrak{G}}$ denotes the zero section of $\mathfrak{G} \to \mathfrak{F}$. We next consider the linearization of the section $\bar{\partial}$, using a connection ∇ on the bundle $\mathfrak{G}$. The result is a linear operator

$$\nabla\bar{\partial} : T_f\mathfrak{F} \approx \Gamma(\Delta, f^\star TX) \to \Gamma(\Delta, \mathrm{Hom}(T\Delta, f^\star TX)).$$

For a not necessarily integrable complex structure J, the value of $\nabla\bar{\partial}$ at a section $s \in T_f\mathfrak{F}$ is given by the following expression.

$$\begin{aligned} \nabla_s\bar{\partial} =& \frac{1}{2}[f^\star(\nabla)_{(\cdot)}s + J(f) \circ f^\star(\nabla)_{(\cdot)}s \circ i] \\ &+ \frac{1}{2}[\mathrm{Tor}^{f^\star\nabla}(s, df(\cdot)) + J(f) \cdot \mathrm{Tor}^{f^\star\nabla}(s, df \circ i(\cdot))] \\ &+ \frac{1}{2}(f^\star(\nabla)_s(f^\star J) \circ df \circ i \end{aligned} \tag{1}$$

One checks that this is an element of $\Gamma(\Delta, \mathrm{Hom}_{\bar{\mathbb{C}}}(T\Delta, f^\star TX))$ for $f \in \ker\bar{\partial}$.

To verify this expression, let $\varphi : \Delta \times (-1, 1) \to X; \quad (\delta, t) \to \varphi(\delta, t)$ be a curve in $\mathfrak{F}$ tangent to $s \in T_f\mathfrak{F} : \varphi(\cdot, 0) = f$, $\partial/\partial t\Big|_{t=0} \varphi = s$, and v a section of $T\Delta$, which can be trivially extended to a section of $T(\Delta \times (-1, 1))$, as can the vector field $\partial/\partial t \in T(-1, 1)$. We are now in a position to compute $(\nabla_s d)(f) \cdot v$. It equals

$$\nabla^{\varphi^\star TX}_{\partial/\partial t} \; d\varphi \circ v \;\Big|_{t=0}.$$

On the other hand, the torsion of ∇ on $\varphi^\star T$ is defined to be

$$\mathrm{Tor}\ (d\varphi \cdot \partial/\partial t, d\varphi \cdot v) = \nabla^{\varphi^\star TX}_{\partial/\partial t} \; d\varphi \cdot v - \nabla^{\varphi^\star TX}_{v} d\varphi \cdot \partial/\partial t - d\varphi\,[\partial/\partial t, v]\,.$$

The expression $\nabla_s(J(f)\circ df\circ i)$ splits up into $(\nabla_s J(f)df\circ i + J(f)(\nabla_s df)\circ i$ and collecting terms, one arrives at (1).

To keep track of the boundary we need to extend the operator as follows. One needs to understand the tangent space of $\mathfrak{M}$, the manifold of totally real surfaces in X. If we fix such a surface $M \in \mathfrak{M}$, the tangent space to $\mathfrak{M}$ at M is given by $\Gamma(M, TX|_M/TM)$. The full linearization of our operator is thus

$$\nabla\bar{\partial}\oplus b : \Gamma(f^\star TX) \to \mathrm{Hom}_{\bar{\mathbb{C}}}(T\Delta, f^\star TM) \oplus \Gamma(\partial\Delta; f|^\star_{\partial\Delta}TX|_M/TM),$$

where $b : \Gamma(f^\star TX) \to \Gamma(\partial\Delta; f|^\star_{\partial\Delta}TX|_M/TM)$ is given by $b(s) = s|_{\partial\Delta}/TM$. Regarding this operator, we have the following.

Proposition 2. The linearized operator $\nabla\bar{\partial}\oplus b$ is an elliptic boundary value operator.

Proof. There are two parts to this dicussion. First we shall show that the symbol of our differential operator $\nabla\bar{\partial}$ is invertible. Then the boundary condition is shown to be elliptic. Let $\delta \in \Delta, v \in T^\star_\delta\Delta \setminus 0$, then the symbol is a linear map

$$\sigma_\delta(v) : f^\star TM|_\delta \to \mathrm{Hom}_{\bar{\mathbb{C}}}(T_\delta\Delta, f^\star TM|_\delta)$$

Since the domain and range of σ are of the same dimension to check invertibility it suffices to show that $\sigma_\delta(v)e \neq 0$ for all $e \in f^\star TM|_\delta\backslash\{0\}$.

But the value of $\sigma_\delta(v)\cdot e$ is

$$\frac{1}{2}\left(v\otimes e + v\cdot i\otimes J(f(\delta))\cdot e\right) \tag{2}$$

which is nonzero.

To check the boundary conditions we fix $\delta \in \partial\Delta$ and also require the following data. Near $\partial\Delta$ we fix a collar neighborhood $C = [0,\epsilon]\times\partial\Delta$. Let ξ denote a unit covector along $\partial\Delta$ in the neighborhood and fix $n = \xi\circ i$ a conormal vector. Using this notation we consider the ordinary differential equation

$$\sigma\left(\xi + n\left(-i\frac{\partial}{\partial t}\right)\right)e(t) = 0 \tag{3}$$

for $e(t)$ a curve in $f^\star TX|_\delta\otimes\mathbb{C}$. Let $B \subset f^\star TX|_\delta\otimes\mathbb{C}$ denote the space of initial conditions at $t = 0$ giving rise to solutions of equation (3) which are bounded for $t > 0$. Let the map

$R : f^\star TX|_\delta \otimes \mathbb{C} \to f^\star TX/f^\star TM|_\delta \otimes \mathbb{C}$ denote the quotient map. To check ellipticity it suffices to show that the induced map $R : B \to f^\star TX/f^\star TM|_\delta \otimes \mathbb{C}$ is an isomorphism. To this end we write equation (2) more explicitly. Using equation (1) we have

$$n \otimes -i\frac{\partial e(t)}{\partial t} + n \circ i \otimes J \cdot \left(-i\frac{\partial e(t)}{\partial t}\right) + \xi \otimes e(t) + \xi \circ i \otimes Je(t) = 0.$$

Let $e(t) = e_+(t) + e_-(t)$ denote the decomposition along the $\pm i$ eigenspaces of J. The previous equation then splits into the follwing pair of equations.

$$\pm ie_\pm - i\frac{\partial e_\pm}{\partial t} = 0.$$

The solutions to these equations are

$$e_\pm(t) = e_\pm(0)e^{\pm t}.$$

Thus B is the $-i$ eigenspace of J. Thus B is naturally isomorphic to $f^\star TX|_\delta$ by the map sending $f \in f^\star TX|_\delta$ to $\frac{1}{2}(f + iJf) \in B$. Since M is totally real there is a natural isomorphism of $f^\star TX/f^\star TM|_\delta \otimes \mathbb{C}$ with $f^\star TX|_\delta$ induced from the map sending $\xi + i\eta \in f^\star TX|_\delta \otimes \mathbb{C}$ to $\xi + J\eta \in f^\star TX|_\delta$. Since the corresponding composition is the identity it follows directly that R is an isomorphism. q.e.d.

4. Calculation of the Index. Let $E \to \Delta$ be a complex vector bundle and $R \to \partial\Delta$ a totally real subbundle of $E|\partial\Delta$. We begin by showing that there is a well defined integer valued relative first Chern class also called the Maslov index of (E, R).

By taking determinants we may reduce to the case of a complex line bundle. If the (necessarily totally) real line subbundle is orientable, it provides a preferred choice of nonvanishing section of E along the boundary of Δ and hence there is a well defined relative Chern class. If the totally real line subbundle is not orientable we take the complex tensor square of E. The induced totally real line subbundle is now orientable and we can argue as before to define twice the relative Chern class.

One may also use the short exact sheaf sequence

$$\mathcal{O} \to \mathbb{Z} \to \mathcal{O}(\Delta, \partial\Delta; E, R) \xrightarrow{\exp} \mathcal{O}^\star(\Delta, \partial\Delta; E, R) \to \mathcal{O}$$

of holomorphic sections of E which take values in the subbundle R on $\partial\Delta$ and define the Chern class by the connecting homomorphism in homology

$$H^1(\Delta, \partial\Delta; \mathcal{O}^\star(E), \mathbb{R}) \xrightarrow{c_1} H^2(\Delta, \partial\Delta; \mathbb{Z}),$$

the latter being isomorphic to $\mathbb{Z}$ via Poincaré duality. Note that we have here $\mathbb{R}$ instead of the real bundle R.

A third way that one may go is to double the vector bundle $(E, R) \to (\Delta, \partial\Delta)$ with respect to R to a bundle $E_1 \to \mathbb{C}P(1)$ over the rational sphere by identifying points on the boundary of the two copies of Δ and glueing the fibres along the oriented prescribed subbundle R. By Schwarz reflection principle, holomorphic sections of (E, R) may be continued to sections of E_1. This operation defines an involution on the space $\mathcal{O}(E_1)$ and the eigenspaces to the eigenvalues ± 1 are subspaces of equal dimension, span the entire spaces, and $H^0(\Delta, \partial\Delta; \mathcal{O}(E), R)$ is isomorphic to either. The same holds for $\mathcal{O}(E_1^\star)$ and $H^1(\Delta, \partial\Delta; \mathcal{O}(E), \mathbb{R})$.

In this situation the theorem of Riemann-Roch states that

$$h^0(\Delta, \partial\Delta; \mathcal{O}(E), R) - h^1(\Delta, \partial\Delta; \mathcal{O}(E), R) = r(1-g) + c_1(E, R).$$

Here h^0, h^1 are the real dimensions of the real vector spaces of holomorphic sections of E with boundary values in R and corresponding holomorphic one-forms, and r is the rank over $\mathbb{C}$ of the vector bundle E, and g the genus of $\mathbb{C}P(1)$, namely zero. The left hand side of the above formula is the real index of the usual $\bar\partial$-operator on the relative bundle (E, R). Note that in (1) the zero order terms vanish if we choose the connection ∇ to be torsion free and since for an integrable complex structure, $f^\star(\nabla)f^\star J$ vanishes. We conclude in this case that $\nabla\bar\partial$ is identical to the usual $\bar\partial$-operator and proceed to show that the cokernel of the latter vanishes in the vector bundle $E_1 \to \mathbb{C}P^1$ described above as the doubling of $(f^\star TX, f^\star TM)$. This is the case precisely if the $(0,1)$-Dolbeault cohomology group $H^{0,1}_{\bar\partial}(\mathbb{C}P(1), E_1)$ vanishes. Now E_1 splits into the direct sum of the tangent line bundle L_1 and the normal line bundle L_2. By Dolbeault isomorphy and Kodaira-Serre duality

$$H^{0,1}_{\bar\partial}(\mathbb{C}P(1), L_j) \cong H^1\mathbb{C}P(1), \mathcal{O}(L_j)) \cong H^0(\mathbb{C}P(1), \mathcal{O}(L_j^\star \otimes K)).$$

Here K is the canonical bundle of $\mathbb{C}P(1)$. Now the last group is trivial if the first Chern class of the bundle $L_j^\star \otimes K$ is negative. Since $c_1(L_j^\star \otimes K) = -c_1(L_\imath) - 2$, the condition is $c_1(L_\imath) \geq -1$. We have $c_1(L_1) = 2$, so the condition is satisfied. For L_2 this condition is satisfied if $c_1(E_1) \equiv c_1(L_1) + c_1(L_2) \equiv 2 + c_1(L_2) \geq 1$. This is the condition mentioned in Theorem 1. We conclude that $\nabla\bar{\partial}$ is regular at $(f, M) \in \mathcal{H}$, i.e. part a) and part d) of Theorem 1. This concludes the proof of Theorem 1.

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