Structural Aspects of Closed Characteristics on $\mathbf{S}^1$-invariant Hypersurfaces

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Abstract

Let the standard symplectic space $(\mathbf{R}^{2n}, \omega)$ be endowed with the $\mathbf{S}^1$-structure induced by the representation of the circle group $\mathbf{S}^1 \to \mathrm{Sp}(n)$ which is given by $t \mapsto \exp\left(\frac{2\pi}{\tau}\, tJ\right)$. Supposing that $\Sigma \subset \mathbf{R}^{2n} \setminus \{0\}$ is a compact connected $\mathbf{S}^1$-hypersurface of contact type with $H^1_{dR}(\Sigma) = 0$, we show that Σ carries at least n distinct symmetric closed characteristics. Moreover, if the total number of closed characteristics on Σ is finite, then all of them turn out to be symmetric. We also discuss the special case of star-shaped $\mathbf{S}^1$-hypersurfaces. In particular, there is a bijection $f \mapsto \Sigma_f$ between the functions $f : \mathbf{C}P^{n-1} \to \mathbf{R}$ and the star-shaped $\mathbf{S}^1$-hypersurfaces in $\mathbf{R}^{2n}$ which allows to reduce the investigation of symmetric closed characteristics on Σ_f to the study of the critical points of f.

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1. Introduction and results

The standard symplectic structure on the Euclidean space $(\mathbb{R}^{2n}, \langle \cdot\,,\cdot \rangle)$ is given by the 2-form $\omega(\cdot\,,\cdot) = \langle J\cdot\,,\cdot \rangle$ where J denotes the complex operator

$$(1) \qquad J \;=\; \begin{pmatrix} 0 & -\mathrm{id}_{\mathbb{R}^n} \\ \mathrm{id}_{\mathbb{R}^n} & 0 \end{pmatrix} \;\in\; \mathrm{Hom}\,(\mathbb{R}^n \times \mathbb{R}^n)\,.$$

In coordinates corresponding to a symplectic orthonormal basis $\{e_1, \ldots, e_{2n}\}$ of $\mathbb{R}^{2n}$ the symplectic form is expressed by $\omega = \sum_{j=1}^{n} dx_j \wedge dx_{n+j}$.

For every $\tau > 0$ there is a distinguished parametrization $\mathbb{S}^1_\tau := \mathbb{R}/\tau\mathbb{Z}$ of the circle group $\mathbb{S}^1 \cong \mathrm{O}(2)$. A symplectic representation of $\mathbb{S}^1$ on $\mathbb{R}^{2n}$ is then defined by the homomorphism of Lie groups $\mathbb{S}^1_\tau \to \mathrm{Sp}(n)$, $t \mapsto \exp\left(\frac{2\pi}{\tau}\, tJ\right)$. We shall write

$$(2) \qquad (t,x) \;\mapsto\; \gamma_t\, x \;:=\; \exp\left(\tfrac{2\pi}{\tau}\, tJ\right) x$$

for the $\mathbb{S}^1$-action $\mathbb{S}^1_\tau \times \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ obtained in this way. Let $\mathrm{Orb}\, x := \{\, \gamma_t\, x \mid t \in \mathbb{S}^1_\tau \,\}$ stand for the $\mathbb{S}^1$-orbit of $x \in \mathbb{R}^{2n}$. We note that $\mathrm{Fix}\,(\mathbb{R}^{2n}) := \{\, x \mid \gamma_t\, x = x \ \forall\, t \in \mathbb{S}^1_\tau \,\}$ consists exactly of the origin $0 \in \mathbb{R}^{2n}$ while on $\mathbb{R}^{2n} \setminus \{0\}$ the $\mathbb{S}^1$-action γ is free. Moreover, we observe that $\gamma_t^*\, \omega = \omega$ since $\mathbb{S}^1$ acts by symplectic transformations.

Let $\Sigma \subset \mathbb{R}^{2n} \setminus \{0\}$ be a compact connected smooth hypersurface satisfying $\gamma_t \Sigma = \Sigma$ for all $t \in \mathbb{S}^1_\tau$. For the sake of brevity, such Σ will henceforward simply be called an $\mathbb{S}^1$-hypersurface. As an additional hypothesis we require the existence of an $\mathbb{S}^1$-invariant contact structure on Σ which is compatible with the symplectic structure of the ambient space $\mathbb{R}^{2n}$. More precisely, let ω_Σ denote the pull-back of the symplectic form ω to Σ and assume that there exists a 1-form β on Σ such that

(i) $\gamma_t^*\, \beta = \beta$ for every $t \in \mathbb{S}^1_\tau$, i.e., β is $\mathbb{S}^1$-invariant,

(ii) $d\beta = \omega_\Sigma$, and

(iii) $\beta \wedge \omega_\Sigma^{n-1}$ is a volume form on Σ.

In view of the conditions (ii) and (iii) the hypersurface Σ is of contact type in the sense of A. Weinstein [17]. Taking the $\mathbb{S}^1$-symmetry of Σ into account, we shall call Σ an *$\mathbb{S}^1$-hypersurface of contact type* provided that all the conditions (i)-(iii) are satisfied.

A given hypersurface Σ having the just stated properties may be viewed as a regular level surface of a suitable $\mathbb{S}^1$-invariant Hamiltonian function $H : \mathbb{R}^{2n} \to \mathbb{R}$. In this note we study some of the orbit structure of the corresponding Hamiltonian flow on Σ. We are particularly interested in the relation between the $\mathbb{S}^1$-structure of the phase space $\mathbb{R}^{2n}$ and the periodic Hamiltonian trajectories on Σ. Formulating the problem with emphasis on the geometric viewpoint, we ask for those closed characteristics on Σ which coincide with orbits of the $\mathbb{S}^1$-action γ.

To make the statements precise, let us briefly recapitulate some basic terminology. Given a smooth hypersurface Σ as above, its *characteristic line bundle* $\mathfrak{L}_\Sigma$ is the 1-dimensional subbundle of $T\Sigma$ which is defined by the kernel of ω_Σ:

$$(3) \qquad \mathfrak{L}_\Sigma \;:=\; \ker\, \omega_\Sigma \;=\; \bigcup_{x \in \Sigma} \{\, X \in T_x\Sigma \mid \omega(X,Y) = 0 \ \ \forall\, Y \in T_x\Sigma \,\}\,.$$

A *characteristic vector field* on Σ is a nowhere vanishing section $X : \Sigma \to \mathfrak{L}_\Sigma$. The (unparametrized) integral curves of a characteristic vector field X are called the *characteristics* on Σ. In particular, a *closed characteristic* on Σ can be described as an embedding $u : \mathbb{S}^1 \to \Sigma$ whose tangent map is an embedding $u_* : T\mathbb{S}^1 \to \mathfrak{L}_\Sigma$. Two closed characteristics $u, v : \mathbb{S}^1 \to \Sigma$ are said to be *(geometrically) distinct* if $u(\mathbb{S}^1) \cap v(\mathbb{S}^1) = \emptyset$. With respect to the $\mathbb{S}^1$-symmetry we now define:

Definition 1 *A closed characteristic $u : \mathbb{S}^1 \to \Sigma$ on an $\mathbb{S}^1$-hypersurface Σ will be called symmetric if it admits a parametrization of the form $u(t) = \gamma_t\, x$.*

We present a proof of the following multiplicity result for symmetric closed characteristics:

Theorem 1 *Let $\Sigma \subset \mathbb{R}^{2n} \setminus \{0\}$ be a compact connected smooth $\mathbb{S}^1$-hypersurface of contact type having trivial first de Rham cohomology, $H^1_{dR}(\Sigma) = 0$. Then*

$$\#\,\{\,\text{distinct symmetric closed characteristics on } \Sigma\,\} \geqslant n\,, \tag{4}$$

where $n = \mathrm{Index}_{\mathbb{C}}\, \mathbb{S}^{2n-1}$ is the cohomological $\mathbb{S}^1$-index of the unit sphere $\mathbb{S}^{2n-1} \subset \mathbb{R}^{2n}$.

Hence, under the assumptions of the theorem, the minimal number of symmetric closed characteristics on the $\mathbb{S}^1$-hypersurface Σ in question is determined by the $\mathbb{S}^1$-equivariant homotopy invariants of the sphere $\mathbb{S}^{2n-1}$. Since the canonical projection $x \mapsto \mathrm{Orb}\, x$ defines exactly the Hopf fibration $\mathbb{S}^{2n-1} \to \mathbb{S}^{2n-1}/\mathbb{S}^1 \cong \mathbb{C}P^{n-1}$, the $\mathbb{S}^1$-homotopy invariants of $\mathbb{S}^{2n-1}$ are given by the cohomology ring of $\mathbb{C}P^{n-1}$.

From our proof of Theorem 1 it will be seen that the assumption $H^1_{dR}(\Sigma) = 0$ can be replaced as follows. Let λ denote the 1-form on $\mathbb{R}^{2n}$ defined by

$$\lambda \;:=\; \tfrac{1}{2} \sum_{j=1}^{n} \; x_j\, dx_{n+j} \;-\; x_{n+j}\, dx_j\,, \tag{5}$$

and let λ_Σ be its pull-back to Σ. We observe that $\gamma_t^*\, \lambda_\Sigma = \lambda_\Sigma$ and $d\lambda_\Sigma = \omega_\Sigma$. Recalling the contact 1-form β on Σ, our proof requires only that the 1-form $\beta - \lambda_\Sigma$ is exact. Thus the role of the distinguished 1-form λ here is similar to that of the Liouville form for analogous results in cotangent bundles, cf. [13].

A special situation occurs if $\beta = \lambda_\Sigma$. In this case the $\mathbb{S}^1$-hypersurface Σ is *star-shaped*, meaning that the map $x \mapsto x/|x|$ defines a diffeomorphism $\Sigma \to \mathbb{S}^{2n-1}$, cf. [17, Example 1]. A star-shaped $\mathbb{S}^1$-hypersurface Σ can be described by an $\mathbb{R}$-valued function on $\mathbb{C}P^{n-1}$ whose critical points correspond to the symmetric closed characteristics on Σ:

Theorem 2 *Let $\mathbb{S}^{2n-1} \subset \mathbb{R}^{2n}$ be the unit sphere and let $\mathbb{C}P^{n-1}$ be identified with the orbit space $\mathbb{S}^{2n-1}/\mathbb{S}^1 = \{\,\mathrm{Orb}\, x \mid x \in \mathbb{S}^{2n-1}\,\}$. To a smooth function $f : \mathbb{C}P^{n-1} \to \mathbb{R}$ we assign a star-shaped $\mathbb{S}^1$-hypersurface $\Sigma_f \subset \mathbb{R}^{2n}$ by setting*

$$\Sigma_f \;:=\; \{\, e^{f(\mathrm{Orb}\, x)}\, x \mid x \in \mathbb{S}^{2n-1}\,\}\,. \tag{6}$$

Then the distinct symmetric closed characteristics on Σ_f correspond uniquely to the distinct critical points of f.

Furthermore, for every star-shaped $\mathbb{S}^1$-hypersurface $\Sigma \subset \mathbb{R}^{2n}$ there exists a unique function $f : \mathbb{C}P^{n-1} \to \mathbb{R}$ such that $\Sigma = \Sigma_f$.

Star-shaped hypersurfaces appear in a natural way as regular level surfaces of homogeneous Hamiltonian functions. Conversely, to a given star-shaped $\mathbb{S}^1$-hypersurface Σ_f we may associate a distinguished homogeneous Hamiltonian H_f. Describing the symmetric closed characteristics on Σ_f by periodic integral curves of the Hamiltonian vector field $X_{H_f} = J \operatorname{grad} H_f$, we find a simple expression for their minimal periods:

Corollary 1 *Given a star-shaped $\mathbb{S}^1$-hypersurface $\Sigma_f \subset \mathbb{R}^{2n} \setminus \{0\}$, we denote by $H_f : \mathbb{R}^{2n} \setminus \{0\} \to \mathbb{R}$ the unique Hamiltonian which is homogeneous of degree 2 and satisfies $\Sigma_f = H_f^{-1}(1)$. Suppose that $\operatorname{Orb} x \in \mathbb{C}P^{n-1}$ is a critical point of f. Then for every $\varrho > 0$ the X_{H_f}-orbit starting at ϱx has the minimal period*

$$\tau(x) \;=\; \pi\, e^{2f(\operatorname{Orb} x)}\,. \tag{7}$$

This holds true, in particular, for the X_{H_f}-orbit starting at $e^{f(\operatorname{Orb} x)}\, x \in \Sigma_f$.

Aside from the symmetric closed characteristics, an $\mathbb{S}^1$-hypersurface of contact type as in Theorem 1 may carry additional non-symmetric closed characteristics. These are, however, cohomologically invisible, and Theorem 1 does not yield any hint on their existence or non-existence. Let us point out that the existence of one non-symmetric closed characteristic on Σ already guarantees the existence of uncountably many distinct non-symmetric closed characteristics:

Theorem 3 *Let $\Sigma \subset \mathbb{R}^{2n}$ be a compact connected $\mathbb{S}^1$-hypersurface of contact type that carries a non-symmetric closed characteristic $u : \mathbb{S}^1 \to \Sigma$. Then the family of closed characteristics $\{\, \gamma_t\, u : \mathbb{S}^1 \to \Sigma \mid t \in \mathbb{S}^1_\tau \,\}$ generated by u contains uncountably many geometrically distinct elements.*

Consequently, if Σ is known to carry a finite number of distinct closed characteristics in all, then each of its closed characteristics is symmetric.

Example Let us illustrate the above situation by means of the harmonic oscillator with n degrees of freedom, usually given by the system of Lagrangian equations $m_j\, \ddot{q}_j = -k_j\, q_j$ for $j = 1, \ldots, n$. Introducing the momenta $p_j = m_j \dot{q}_j$, the total energy of the system is described by the Hamiltonian function

$$\tilde{H}(p,q) \;=\; \sum_{j=1}^{n} \frac{1}{2m_j}\, p_j^2 \;+\; \frac{k_j}{2}\, q_j^2\;. \tag{8}$$

For fixed $c > 0$ the level surface $\tilde{H}^{-1}(c)$ is an ellipsoid in the (p,q)-space which, in general, fails to be an $\mathbb{S}^1$-hypersurface. Denoting by $\omega_j = \sqrt{k_j/m_j}$ the eigenfrequencies, the system is transformed to its so-called normal form by the following symplectic change of coordinates:

$$x_j \;=\; \frac{p_j}{\sqrt{m_j\, \omega_j}} \quad \text{and} \quad x_{n+j} \;=\; q_j\, \sqrt{k_j/\omega_j}\,, \quad j = 1, \ldots, n. \tag{9}$$

Then the transformed Hamiltonian function

$$H(x) \;=\; \tfrac{1}{2} \sum_{j=1}^{n} \omega_j \left(x_j^2 + x_{n+j}^2 \right) \tag{10}$$

is invariant under the $\mathbb{S}^1$-action γ and, consequently, the ellipsoid $H^{-1}(c)$ is a star-shaped $\mathbb{S}^1$-hypersurface. Now define $f : \mathbb{C}P^{n-1} \to \mathbb{R}$ by

$$f(\operatorname{Orb} x) \;:=\; \ln \sqrt{\frac{c}{H(x)}} \quad \text{for} \quad x \in \mathbb{S}^{2n-1}, \tag{11}$$

so that $H^{-1}(c) = \Sigma_f$. The Hamiltonian H_f is then given by $H_f(x) := c^{-1}H(x)$. Setting $\omega_j' := \omega_j/c$, we obtain the representation

$$H_f(x) \;=\; \tfrac{1}{2} \sum_{j=1}^{n} \omega_j' \left(x_j^2 + x_{n+j}^2 \right). \tag{12}$$

By the Lagrange multiplier rule we see that $\operatorname{Orb} x$ is a critical point of f if and only if $\operatorname{grad} H_f(x) = \mu x$, in which case μ is equal to some ω_j'. In particular, if the ω_j' are mutually distinct then Σ_f carries exactly n symmetric closed characteristics, corresponding to the so-called normal mode solutions. For the associated minimal periods, Corollary 1 recovers the familiar expression

$$T_j \;=\; 2\pi/\omega_j'. \tag{13}$$

Note that in the presence of resonances, i.e., in case $\omega_i/\omega_j \in \mathbb{Z}$ for some $i \neq j$, the level surface Σ_f carries uncountably many additional closed characteristics which are not symmetric unless $\omega_i = \omega_j$. □

Let us finally point out two phenomena which can already be observed in the above simple model where the hypersurface Σ_f is an ellipsoid:

Firstly, the appearance of non-symmetric closed characteristics on Σ_f is linked to a resonance condition which can be formulated in terms of the minimal periods T_j of the symmetric ones. The question arises whether the existence of non-symmetric periodic X_{H_f}-orbits on a star-shaped $\mathbb{S}^1$-hypersurface can be excluded by certain non-resonance conditions. This motivates the following

Problem 1 *Suppose that $f : \mathbb{C}P^{n-1} \to \mathbb{R}$ is a smooth function having finitely many critical points $x^1, \ldots, x^\ell$ such that the corresponding minimal periods $\tau_j := \tau(x^j)$ according to (7) satisfy $\tau_i/\tau_j \notin \mathbb{Z}$ for all $i \neq j$. Is it possible for the hypersurface Σ_f to carry any non-symmetric closed characteristics?*

Secondly, the $\mathbb{S}^1$-symmetry is evidently not preserved under symplectic diffeomorphisms of $\mathbb{R}^{2n}$. Hence it is a natural question to ask for conditions which guarantee an $\mathbb{S}^1$-symmetrization of hypersurfaces by such symplectic diffeomorphisms:

Problem 2 *Suppose that $\Sigma \subset \mathbb{R}^{2n}$ is a compact connected hypersurface of contact type with $H^1_{dR}(\Sigma) = 0$. Does there exist a symplectic diffeomorphism $\varphi : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ such that the hypersurface $\varphi(\Sigma)$ is $\mathbb{S}^1$-invariant?*

The property of a hypersurface Σ to be of contact type persists under symplectic diffeomorphisms. In case $\varphi(\Sigma)$ is $\mathbb{S}^1$-invariant, the contact 1-form on $\varphi(\Sigma)$ can also be chosen $\mathbb{S}^1$-invariant by averaging the given contact form over the Lie group $\mathbb{S}^1$.

Consequently, the existence of such a symplectic symmetrization φ already would imply the solution of a more general multiplicity problem for periodic Hamiltonian orbits on hypersurfaces in $\mathbb{R}^{2n}$. For the background, history, and earlier results connected with that we refer to [1, 6, 7, 11, 15].

An analogous problem can be posed for Liouville hypersurfaces in cotangent bundles of free $\mathbb{S}^1$-manifolds. The corresponding $\mathbb{S}^1$-symmetric case has been studied in [13]. Clearly, the above assumption $H^1_{dR}(\Sigma)$ then has to be replaced in a suitable manner. These problems are going to be treated in some forthcoming work.

2. Organization of the proofs

Most of the remainder of this work is concerned with the proof of Theorem 1. Here we present an outline of the strategy for this proof which is subdivided into five steps (sections 3-7). The proofs of Theorem 2 and Corollary 1 (section 8) as well as that of Theorem 3 (section 9) are more elementary and independent of the preceding paragraphs, although we shall use some of the notation introduced in section 3.

First of all, we observe that it suffices to carry out the proof of Theorem 1 under the hypothesis that the set of distinct symmetric closed characteristics on the $\mathbb{S}^1$-hypersurface Σ in question is finite, possibly empty. Of course, the existence of a closed characteristic on Σ is guaranteed by C. Viterbo's result [16], but in the situation in hand it is not a priori obvious whether there is a symmetric one.

The desired symmetric closed characteristics will be found by a variational approach that utilizes the representation of closed characteristics by periodic Hamiltonian orbits. The first ingredient of the proof consists of choosing an appropriate period $T > 0$ and constructing a suitable Hamiltonian $H_T : \mathbb{R}^{2n} \to \mathbb{R}$ such that the symmetric closed characteristics on Σ correspond to certain periodic X_{H_T}-orbits whose minimal period is T. Here $X_{H_T} = J \operatorname{grad} H_T : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ stands for the Hamiltonian vector field associated to H_T.

Our goal is to derive a characterization of these particular T-periodic X_{H_T}-orbits as the positive critical points of the action integral $\mathcal{A}_{H_T} : \Omega \to \mathbb{R}$, considered as a function on a suitable space of loops $\Omega \subset C^\infty(\mathbb{S}^1_T, \mathbb{R}^{2n})$. We recall that for a smooth loop $u : \mathbb{S}^1_T \to \mathbb{R}^{2n}$ the Hamiltonian action integral is given by

$$\mathcal{A}_{H_T}(u) \;=\; \int_0^T \Big\{ \tfrac{1}{2} \langle -J\dot{u} \,,\, u \rangle \;-\; H_T(u) \Big\} \, dt \,. \tag{14}$$

Speaking of a positive critical point, we mean a critical point $u \in \Omega$ of the action integral such that $\mathcal{A}_{H_T}(u) > 0$. A crucial aspect in the variational formulation is that any two *geometrically distinct* positive critical points u and v (i.e., $u(t)$ and $v(t)$ describe geometrically distinct T-periodic X_{H_T}-orbits in $\mathbb{R}^{2n}$) should correspond to geometrically distinct closed characteristics on Σ. If this condition is satisfied, the desired counting of closed characteristics on Σ can be done by counting the geometrically distinct positive critical points of $\mathcal{A}_{H_T}$ on Ω.

The requirements on $\mathcal{A}_{H_T}$ have to be taken into account for the designing of the Hamiltonian H_T. This task will be carried out in two steps: the local construction near Σ (section 3) and the global extension to $\mathbb{R}^{2n}$ (section 4). For a summary of the technical aspects entering in the construction of H_T see the comments concluding section 4. Whereas the local part is practically the same as for the corresponding construction in cotangent bundles [13], the extension is considerably simpler in the present situation, and the choice for the asymptotic behavior of H_T as $|x| \to 0$ and $|x| \to \infty$ is essentially borrowed from Hofer-Zehnder [12].

The set up for the variational formulation will be completed in section 5. In particular, we adapt the parametrization for the $\mathbb{S}^1$-action to the previously fixed $T > 0$ by setting $\gamma_t = \exp\left(\frac{2\pi}{T}\, tJ\right)$. Then the space of symmetric loops of minimal period T, defined by $\Omega_{\mathrm{sym}} := \{\, \gamma_t\, x \mid x \in \mathbb{R}^{2n} \,\} \subset C^\infty(\mathbb{S}^1_T, \mathbb{R}^{2n})$, is the natural choice for the loop space Ω. In view of the isomorphism $\Omega_{\mathrm{sym}} \cong \mathbb{R}^{2n}$ the variational problem for $\mathcal{A}_{H_T} : \Omega_{\mathrm{sym}} \to \mathbb{R}$ can be solved by estimating the number of positive critical $\mathbb{S}^1$-orbits of some $\mathbb{S}^1$-invariant function $\mathcal{F} : \mathbb{R}^{2n} \to \mathbb{R}$.

The remaining problem is treated by an $\mathbb{S}^1$-equivariant Lyusternik-Schnirelman approach for the $-\mathrm{grad}\, \mathcal{F}$-flow on $\mathbb{R}^{2n}$, using the concept of isolated invariant sets. As a general reference for the basic definitions and notations concerning isolated invariant sets we refer to C. Conley's and E. Zehnder's article [3, Sec. 3].

By our construction of H_T it turns out that the positive critical $\mathbb{S}^1$-orbits of $\mathcal{F}$ constitute a Morse decomposition of a distinguished isolated invariant set $\mathfrak{M}^+$, called the positive subcomplex of the free Morse complex of $\mathcal{F}$. The relevant $\mathbb{S}^1$-homotopy invariants of $\mathfrak{M}^+$ are analyzed in section 6. Our arguments presented here are somewhat simplified and more geometric compared to the corresponding passage in [13]. Note, however, that the proofs given for cotangent bundles can be adapted to the present case, and vice versa.

The proof of the theorem is finally completed in section 7 by applying the $\mathbb{S}^1$-equivariant Lyusternik-Schnirelman method to $\mathfrak{M}^+$, which yields the required estimate for the number of the positive critical $\mathbb{S}^1$-orbits of $\mathcal{F}$.

3. The Hamiltonian near the hypersurface

Let $\Sigma \subset \mathbb{R}^{2n} \setminus \{0\}$ be a compact connected smooth $\mathbb{S}^1$-hypersurface of contact type. The hypothesis that Σ be of contact type is alternatively expressed as follows: there exists a symplectic dilation X, defined on some open neighborhood V of Σ, which is transverse to Σ (cf. [17, Lemma 2]). We recall that a vector field $X : V \to \mathbb{R}^{2n}$ is said to be a *symplectic dilation* if the Lie derivative of the symplectic form ω along X satisfies $L_X\omega = \omega$.

The existence of a transverse symplectic dilation follows essentially from the fact that the contact form β can be extended to a 1-form $\tilde{\alpha}$ on some open neighborhood V of Σ such that $d\tilde{\alpha} = \omega$. Since Σ is an $\mathbb{S}^1$-set, every neighborhood of Σ contains an $\mathbb{S}^1$-invariant neighborhood, and we may therefore assume that the neighborhood V in question is $\mathbb{S}^1$-invariant. Then, by averaging $\tilde{\alpha}$ over the group $\mathbb{S}^1$, we obtain an $\mathbb{S}^1$-invariant 1-form α on V that again satisfies $d\alpha = \omega$ and whose pull-back α_Σ is equal to β. For the vector field $X_\alpha : V \to \mathbb{R}^{2n}$, defined by the identity

$$\alpha = X_\alpha \lrcorner\, \omega\,, \tag{15}$$

we find $L_{X_\alpha}\omega = X_\alpha \lrcorner\, d\omega + d(X_\alpha \lrcorner\, \omega) = d\alpha = \omega$ by Cartan's formula. Moreover, since α and ω are $\mathbb{S}^1$-invariant, it follows readily that X_α is $\mathbb{S}^1$-equivariant.

To see that X_α is transverse, we recall that the compact hypersurface Σ has a canonical positive orientation given by the *outward normal vector field* $N(x)$, $x \in \Sigma$. Then $0 \neq JN(x) \in T_x\Sigma$, and because of $\omega(X, JN) = \langle JX, JN\rangle = \langle X, N\rangle = 0$ for every $X \in T_x\Sigma$ it follows that $JN(x)$ indeed spans $\mathfrak{L}_\Sigma|_x$. Since $\alpha_\Sigma \wedge \omega_\Sigma^{n-1}$ is a (positive) volume form, we get $0 < \alpha(JN) = \omega(X_\alpha, JN) = \langle X_\alpha, N\rangle$ and, therefore, X_α is transverse to Σ.

Because the vector field X_α is $\mathbb{S}^1$-equivariant, so is its flow which will be denoted by $(s,x) \mapsto \psi_s(x)$, i.e., we have $\psi_s(\gamma_t\, x) = \gamma_t\, \psi_s(x)$. In view of the compactness of Σ there exists a constant $\hat{s} > 0$ such that for every $x \in \Sigma$ the trajectory $s \mapsto \psi_s(x)$ is defined for all $|s| < \hat{s}$. Hence $\{\, \Sigma_s := \psi_s(\Sigma) \mid |s| < \hat{s}\,\}$ constitutes a *parametrized family of* $\mathbb{S}^1$*-hypersurfaces modelled on* Σ. Subsequently let us assume that $\hat{s}$ is chosen sufficiently small such that, in addition, $\langle\, N(\psi_s(x))\,, X_\alpha(\psi_s(x))\,\rangle > 0$ for all $x \in \Sigma$, $|s| < \hat{s}$. Clearly, this latter assumption guarantees that each Σ_s is of contact type. For later convenience we also introduce the $\mathbb{S}^1$-invariant neighborhood U of Σ by setting

$$U := \{\, \psi_s(x) \mid x \in \Sigma\,,\ |s| < \hat{s}\,\} \subset V\,. \tag{16}$$

Example For a star-shaped $\mathbb{S}^1$-hypersurface Σ we may choose $V := \mathbb{R}^{2n}$ and $\alpha := \lambda$ with the 1-form λ given by (5). The corresponding transverse symplectic dilation $X_\lambda : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ and its flow $\psi_s(x)$ then are easily seen to be determined by

$$X_\lambda(x) \;=\; \tfrac{1}{2}\,x \quad \text{and} \quad \psi_s(x) \;=\; e^{s/2}x \quad \forall\, x \in \mathbb{R}^{2n},\ s \in \mathbb{R}\,. \tag{17}$$

There are no restrictions on the choice of $\hat{s} > 0$ since X_λ is transverse to each of the hypersurfaces $\Sigma_s := e^{s/2}\,\Sigma$.

Having completed these preparations, we now introduce a particular $\mathbb{S}^1$-invariant Hamiltonian function $H : U \to \mathbb{R}$. For this purpose let $0 < \delta < \hat{s}$ be fixed and let the smooth function $g : (-\hat{s}, \hat{s}) \to \mathbb{R}_0^+$ be defined by

$$g(s) \;:=\; \begin{cases} \exp\left(\frac{-\delta^2}{\delta^2 - s^2}\right) & \text{if} \quad 0 \leqslant |s| < \delta \\[2ex] \qquad 0 & \text{if} \quad \delta \leqslant |s| < \hat{s}\,. \end{cases} \tag{18}$$

Defining $h : (-\hat{s}, \hat{s}) \to \mathbb{R}_0^+$ by

$$h(s) \;:=\; \int_{-\hat{s}}^{s} g(t)\; dt\,, \tag{19}$$

the desired Hamiltonian $H : U \to \mathbb{R}$ is obtained by setting

$$H(\psi_s(x)) \;:=\; h(s) \quad \text{for} \quad x \in \Sigma\,,\ |s| < \hat{s}\,. \tag{20}$$

The gradient vector field $\operatorname{grad} H : U \to \mathbb{R}^{2n}$, defined according to $dH = \langle\, \operatorname{grad} H, \cdot\,\rangle$, is $\mathbb{S}^1$-equivariant since H is $\mathbb{S}^1$-invariant. The associated Hamiltonian vector field $X_H : U \to \mathbb{R}^{2n}$ is defined by the identity

$$-\, dH \;=\; X_H \lrcorner\, \omega\,, \tag{21}$$

and satisfies $X_H(x) = J\,\mathrm{grad}\,H(x)$. Consequently, X_H is $\mathbb{S}^1$-equivariant. We shall write the Hamiltonian equations as

$$\dot{u} \;=\; X_H(u)\,. \tag{22}$$

The following lemma states some properties of the non-constant periodic solutions of the Hamiltonian system (22):

Lemma 1 *For every $|s| < \delta$ the restriction $X_H|_{\Sigma_s}$ defines a characteristic vector field on $\Sigma_s = \psi_s(\Sigma)$. If u_0 is a periodic solution of (22) on Σ having the minimal period $\tau_0 > 0$, then there exists a corresponding periodic solution u_s on Σ_s which is given by*

$$u_s(t) \;=\; (\psi_s \circ u_0)\left(\tfrac{t}{\sigma(s)}\right) \tag{23}$$

and has the minimal period $\tau(s) = \sigma(s)\tau_0$, where

$$\sigma(s) \;=\; e^s\,\frac{h'(0)}{h'(s)} \quad , \quad |s| < \delta\,. \tag{24}$$

Furthermore, every non-constant periodic solution of $\dot{u} = X_H(u)$ has a representation of the form (23).

Proof From $dH(X_H) = \omega(X_H, X_H) = 0$ it follows that H remains constant along integral curves of X_H so that $X_H(\psi_s(x)) \in T_{\psi_s(x)}\Sigma_s$ for all $|s| < \hat{s}$. Moreover, from

$$\omega(X, X_H) \;=\; dH(X) \;=\; \langle\, \mathrm{grad}\,H, X\,\rangle \;=\; 0 \quad \forall\, X \in T_{\psi_s(x)}\Sigma_s \tag{25}$$

we see that $X_H(\psi_s(x)) \in \mathfrak{L}_{\Sigma_s}|_{\psi_s(x)}$. Hence $X_H|_{\Sigma_s}$ is a characteristic vector field if $X_H(\psi_s(x)) \neq 0$ everywhere on Σ_s. Observing that

$$\begin{aligned} \alpha(X_H)|_{\psi_s(x)} \;&=\; dH(X_\alpha)|_{\psi_s(x)} \;=\; L_{X_\alpha}H|_{\psi_s(x)} \;=\; \frac{d}{dt}\Big|_{t=0}\psi_t^* H(\psi_s(x)) \\ &=\; \frac{d}{dt}\Big|_{t=s} h(t) \;=\; h'(s)\,, \end{aligned} \tag{26}$$

we conclude by the definition of h that $X_H(\psi_s(x)) \neq 0$ if and only if $|s| < \delta$. This proves the first assertion of the lemma.

The gradient of H is orthogonal to the level surfaces Σ_s so that there is a smooth function $\mu : U \to \mathbb{R}$ with $\mathrm{grad}\,H(\psi_s(x)) = \mu(\psi_s(x))N(\psi_s(x))$, where N again denotes the outward normal vector field on Σ_s. Consequently, in view of (26), we have

$$h'(s) \;=\; L_{X_\alpha}H|_{\psi_s(x)} \;=\; \langle\, \mathrm{grad}\,H, X_\alpha\,\rangle|_{\psi_s(x)} \;=\; \mu\,\langle\, N, X_\alpha\,\rangle|_{\psi_s(x)}\,. \tag{27}$$

Recalling that $\langle\, N(\psi_s(x))\,, X_\alpha(\psi_s(x))\,\rangle > 0$ for all $\psi_s(x) \in U$, the gradient of H can be expressed by

$$\mathrm{grad}\,H(\psi_s(x)) \;=\; \frac{h'(s)}{\langle\, N(\psi_s(x))\,, X_\alpha(\psi_s(x))\,\rangle}\,N(\psi_s(x))\,. \tag{28}$$

Now let $s \in (-\delta, \delta)$ be fixed. The map ψ_s is thus defined for all $x' = \psi_{s'}(x) \in U$ that satisfy $|s+s'| < \hat{s}$. Since $L_{X_\alpha}\omega = \omega$ implies $\psi_s^*\omega = e^s\omega$, we obtain the following identity for the Jacobian $D\psi_s(x')$:

$$D\psi_s(x')^T\, J\, D\psi_s(x') \;=\; e^s\, J\,. \tag{29}$$

Consequently, the transformation ψ_s behaves like a symplectic transformation up to the additional dilation factor e^s. Hence, if $\dot{u}_0(t) = X_H(u_0(t))$ is a (periodic) solution on Σ, we infer

$$\begin{aligned} \frac{d}{dt}\,(\psi_s \circ u_0)(t) &= e^s\, X_{H\circ\psi_{-s}}((\psi_s \circ u_0)(t)) \\ &= e^s\, J\,\mathrm{grad}(H \circ \psi_{-s})((\psi_s \circ u_0)(t))\,. \end{aligned} \tag{30}$$

To reparametrize $\psi_s \circ u_0$ as an integral curve of X_H, we observe that similar to (28) the gradient of $H \circ \psi_{-s}$ at $\psi_s(x) \in \Sigma_s$ is represented by

$$\mathrm{grad}(H \circ \psi_{-s})(\psi_s(x)) \;=\; \frac{h'(0)}{\langle\, N(\psi_s(x))\,, X_\alpha(\psi_s(x))\,\rangle}\, N(\psi_s(x))\,. \tag{31}$$

Consequently, for all $|s| < \delta$ and $x \in \Sigma$ we find

$$X_{H\circ\psi_{-s}}(\psi_s(x)) \;=\; \frac{h'(0)}{h'(s)}\, X_H(\psi_s(x))\,. \tag{32}$$

Combining (30) and (32), the reparametrization $u_s(t) := (\psi_s \circ u_0)(t/\sigma(s))$ with $\sigma : (-\delta, \delta) \to \mathbb{R}^+$ given by (24) indeed yields a (periodic) integral curve of X_H on Σ_s. If u_0 is periodic, the transformation of the minimal period according to $\tau_0 \mapsto \sigma(s)\tau_0$ follows in an obvious way.

Given a non-constant periodic integral curve $u(t)$ of X_H, then u lies entirely on some level surface Σ_s of H for some $|s| < \delta$. Thus, by interchanging the role of Σ and Σ_s in the preceding arguments, u can be transformed into a periodic Hamiltonian orbit u_0 on Σ. The inverse transformation then shows that $u(t) = u_s(t)$, and the proof of the lemma is complete. □

The lemma shows that the closed characteristics on Σ can be described as the periodic orbits of $X_H|_\Sigma$ or, more generally, as the non-constant periodic solutions of the Hamiltonian system $\dot{u} = X_H(u)$. In the sequel, we shall speak of Hamiltonian orbits (trajectories, integral curves) rather than solutions, meaning that a Hamiltonian orbit is an equivalence class of solutions where $u \sim v$ if $u(t) = v(t+t')$ for some $t' \in \mathbb{R}$. Consequently, two periodic orbits of $X_H|_\Sigma$ are considered to be distinct if and only if the corresponding closed characteristics are geometrically distinct. Moreover, a periodic X_H-orbit on Σ will be called symmetric if the corresponding closed characteristic is symmetric.

We point out that the $\mathbb{S}^1$-symmetry of Σ and H has so far not been used, and Lemma 1 remains valid if the $\mathbb{S}^1$-invariance is not assumed. Even for an $\mathbb{S}^1$-hypersurface Σ the statements refer to arbitrary periodic Hamiltonian orbits, not just symmetric ones. Note, however, that the $\mathbb{S}^1$-equivariance of X_α implies that a symmetric X_H-orbit u_0 on Σ is transformed into a symmetric X_H-orbit u_s on Σ_s.

The next lemma serves as a preparation for the variational formulation of Theorem 1:

Lemma 2 *Let the $\mathbb{S}^1$-hypersurface Σ satisfy the hypotheses of Theorem 1 and let the Hamiltonian $H : U \to \mathbb{R}$ be given by (20). In addition, suppose that Σ carries only a finite number of symmetric periodic X_H-orbits. Then there exists $T_0 > 0$ such that every $T \geqslant T_0$ has the following property:*

($*$) *If u_0 is a symmetric periodic X_H-orbit on Σ then there exists a uniquely determined $s \in (-\delta, \delta)$, depending on u_0 and T, such that the corresponding X_H-orbit u_s on Σ_s satisfies the two conditions*

  (a) *The X_H-orbit u_s has the minimal period T.*

  (b) *The Hamiltonian action integral $\mathcal{A}_H(u_s)$ is positive, i.e.,*

$$\mathcal{A}_H(u_s) \;=\; \int_0^T \tfrac{1}{2}\langle -J\dot{u}_s\,,\,u_s\rangle\,dt \;-\; \int_0^T (H\circ u_s)\,dt \;>\; 0\,.$$

Proof Leet us assume that there are $\ell > 0$ symmetric periodic X_H-orbits on Σ, denoted by $u_0^1, \ldots, u_0^\ell$, and let $\tau_{0,i} > 0$ be the minimal period of u_0^i. For $|s| < \delta$ let the corresponding symmetric periodic X_H-orbits u_s^i on Σ_s be defined according to Lemma 1. The minimal period of u_s^i then is given by $\tau_i(s) = \sigma(s)\tau_{0,i}$ with $\sigma : (-\delta, \delta) \to \mathbb{R}^+$ defined in (24).

In view of the choice (19) of h, the derivative $\sigma'(s)$ has exactly one zero at s_0 which is the unique solution of the equation $h'(s) = h''(s)$ in $(-\delta, \delta)$. We note that $\sigma'(s) < 0$ for $-\delta < s < s_0$ and $\sigma'(s) > 0$ for $s_0 < s < \delta$. Moreover, we observe that $\sigma(s) \to +\infty$ as $|s| \to \delta$.

As a consequence, for every $T > \sigma(s_0)\tau_{0,i} = \tau_i(s_0)$ the equation $T = \tau_i(s)$ has exactly two solutions $s_i^+ \in (-\delta, s_0)$ and $s_i^- \in (s_0, \delta)$. These T-dependent solutions satisfy $s_i^+ \to -\delta$ and $s_i^- \to \delta$ monotonically as $T \to +\infty$. We note that the associated X_H-orbits $u^i_{s_i^\pm}$ lie on $\Sigma_{s_i^\pm}$ and have the minimal period T.

Now let us fix a period $T > \max\{\,\tau_i(s_0) \mid i = 1, \ldots, \ell\,\}$ and let us consider a T-periodic smooth loop $u : \mathbb{S}^1_T \cong \mathbb{R}/T\mathbb{Z} \to U$. Recalling the 1-form λ from (5), the pull-back of λ by u can be written as

$$\begin{aligned} u^*\lambda \;&=\; \tfrac{1}{2}\sum_{j=1}^{n}\{\,u_j\,du_{n+j} \;-\; u_{n+j}\,du_j\,\} \\ &=\; \tfrac{1}{2}\sum_{j=1}^{n}\{\,u_j\,\dot{u}_{n+j} \;-\; u_{n+j}\,\dot{u}_j\,\}\,dt \qquad (33) \\ &=\; \tfrac{1}{2}\langle -J\dot{u}\,,\,u\,\rangle\,dt\,. \end{aligned}$$

Similarly, the pull-back of the 1-form $\alpha = X_\alpha \lrcorner\, \omega$ by u has the representation

$$u^*\alpha \;=\; \omega(X_\alpha(u), \dot{u})\,dt \;=\; \langle -J\dot{u}\,, X_\alpha(u)\rangle\,dt\,. \qquad (34)$$

We observe that the homotopy $[\,0,1\,] \times U \to U$ defined by $(r, \psi_s(x)) \mapsto \psi_{(1-r)s}(x)$ retracts U to Σ so that $H^1_{dR}(U) \cong H^1_{dR}(\Sigma) = 0$. Consequently, since $d\alpha = d\lambda = \omega$, the 1-form $\alpha - \lambda$ on U is exact and we have $\alpha = \lambda + dF$ for some function $F : U \to \mathbb{R}$. By integration over the circle $\mathbb{S}^1_T$ we obtain

$$\int_{\mathbb{S}^1_T} u^*\lambda \;=\; \int_{\mathbb{S}^1_T} u^*\alpha \;-\; \int_{\mathbb{S}^1_T} d(u^*F) \;=\; \int_0^T \langle -J\dot{u}\,, X_\alpha(u)\rangle \; dt\,. \tag{35}$$

Now suppose that $u = u_s$ where u_s stands for any of the above symmetric T-periodic X_H-orbits $u^i_{s_i^\pm}$, $i = 1, \ldots, \ell$. Then we have $-J\dot{u}_s = \operatorname{grad} H(u_s)$, and, in view of the representation (28) of $\operatorname{grad} H$, the right hand integral becomes

$$\int_0^T \langle \operatorname{grad} H(u_s), X_\alpha(u_s)\rangle \; dt \;=\; \int_0^T \tfrac{h'(s)}{\langle\, N(u_s)\,,\, X_\alpha(u_s)\,\rangle} \,\langle N(u_s), X_\alpha(u_s)\rangle \; dt\,. \tag{36}$$

Thus from (33), (35), and (36) we deduce

$$\int_0^T \tfrac{1}{2}\,\langle\, -J\dot{u}_s\,,\, u_s\,\rangle \; dt \;=\; T\,h'(s)\,. \tag{37}$$

In combination with

$$\int_0^T H(u_s(t)) \; dt \;=\; T\,h(s)\,, \tag{38}$$

this yields the identity

$$\mathcal{A}_H(u_s) \;=\; T\;(h'(s) - h(s))\,, \tag{39}$$

where $u_s = u^i_{s_i^\pm}$ with $s_i^\pm$ being the solutions of $T = \tau_i(s)$ for $i = 1, \ldots, \ell$.

We claim that $\mathcal{A}_H(u^i_{s_i^+}) > 0$ and $\mathcal{A}_H(u^i_{s_i^-}) < 0$ for $i = 1, \ldots, \ell$ in case the prescribed period T is sufficiently large. For this purpose let us recall that $s_i^+ \to -\delta$ and $s_i^- \to \delta$ as $T \to +\infty$.

With regard to the fact that $h(\delta) < 0$ and $h'(\delta) = 0$ we can find some $\varepsilon_1 > 0$ such that $h'(s) - h(s) < 0$ if $s \in (\delta - \varepsilon_1, \delta)$. Moreover, choosing $0 < \varepsilon_2 < \min\{\delta, 1\}$, we see that h' is strictly increasing on $(-\delta, -\delta + \varepsilon_2)$. Hence, by the mean value theorem, it follows for $-\delta < s < -\delta + \varepsilon_2$ that

$$h(s) \;=\; \int_{-\delta}^{s} h'(r)\; dr \;=\; h'(\bar{r}) \int_{-\delta}^{s} dr \;<\; h'(s)\,\varepsilon_2 \;<\; h'(s) \tag{40}$$

with some $\bar{r} \in (-\delta, s)$. Consequently, for $\varepsilon := \min\{\varepsilon_1, \varepsilon_2\}$ we have

$$\begin{aligned} h'(s) - h(s) \;&>\; 0 \quad \text{if} \quad s \in (-\delta, -\delta + \varepsilon) \\ h'(s) - h(s) \;&<\; 0 \quad \text{if} \quad s \in (\delta - \varepsilon, \delta)\,. \end{aligned} \tag{41}$$

Now we can choose $T_0 > \min\{\, \tau_i(s_0) \mid i = 1, \ldots, \ell\,\}$ so large that for every $T \geqslant T_0$ the corresponding solutions $s_i^\pm$ satisfy $s_i^+ \in (-\delta, -\delta + \varepsilon)$ and $s_i^- \in (\delta - \varepsilon, \delta)$ for every $i = 1, \ldots, \ell$, and the claim follows.

This proves the assertion of the lemma if the assumed number of symmetric periodic X_H-orbits on Σ is $\ell > 0$. In the case $\ell = 0$ the condition $(*)$ is trivially satisfied for every choice of $T_0 > 0$, and the proof is hereby complete. $\square$

4. The global extension of the Hamiltonian

Observing that the neighborhood U of Σ introduced in (16) is relatively compact, we can find $R_1 > 0$ sufficiently large such that

$$U \subset \{ x \in \mathbb{R}^{2n} \mid |x| \leqslant R_1 \} . \tag{42}$$

If T_0 is again the constant from Lemma 2, then we can choose $T > T_0$ and $R_2 > R_1$ so large that

$$\tfrac{3\pi}{2T} R_1^2 < h(\hat{s}) < \tfrac{3\pi}{2T} R_2^2 , \tag{43}$$

where $h(\hat{s})$ is the maximal value of the Hamiltonian H defined in (20). Now we pick a smooth function $h_T : \mathbb{R}_0^+ \to \mathbb{R}^+$ that satisfies

$$\begin{cases} h_T(r) \equiv h(\hat{s}) & \text{if } r \leqslant R_1 \\ h_T(r) = \tfrac{3\pi}{2T} r^2 & \text{if } r \geqslant R_2 \\ h_T(r) > \tfrac{3\pi}{2T} r^2 & \text{if } R_1 < r < R_2 \\ 0 < h_T'(r) < \tfrac{3\pi}{T} r & \text{if } R_1 < r < R_2 . \end{cases} \tag{44}$$

Moreover, we shall assume that h_T is chosen in such a way that there exists only one solution $r_{\text{crit}} \in (R_1, R_2)$ of the equation

$$h_T'(r) = \tfrac{2\pi}{T} r . \tag{45}$$

Note that, by the Jordan-Brouwer separation theorem, $\mathbb{R}^{2n} \backslash \Sigma$ consists of a bounded connected component U_0 and an unbounded connected component U_∞. Now we extend the Hamiltonian $H : U \to \mathbb{R}$ defined in (20) to a smooth function on $\mathbb{R}^{2n}$ as follows:

$$H_T(x) := \begin{cases} 0 & \text{if } x \in U_0 \\ H(x) & \text{if } x \in U \\ h_T(|x|) & \text{if } x \in U_\infty \backslash U . \end{cases} \tag{46}$$

If $x \in U_\infty \backslash U$ we have: $\operatorname{grad} H_T(x) = \frac{h_T'(|x|)}{|x|} x$. Consequently, for the Hamiltonian system $\dot{u} = X_{H_T}(u)$ on $\mathbb{R}^{2n}$ we find two types of symmetric periodic X_{H_T}-orbits that have the minimal period T:

(i) The *significant* periodic X_{H_T}-orbits u_s in U which have been described in the previous section.

(ii) The *technical* periodic X_{H_T}-orbits generated by the above extension. These are of the form $u(t) = \exp\left(\frac{2\pi}{T} tJ\right) x$ where $|x| = r_{\text{crit}}$ solves the equation (45).

The choice of the function h_T implies that the corresponding Hamiltonian action $\mathcal{A}_{H_T}(u)$ of a technical periodic X_{H_T}-orbit is negative:

Lemma 3 *Suppose that u is a symmetric periodic X_{H_T}-orbit that has the minimal period T and satisfies $\mathcal{A}_{H_T}(u) > 0$. Then u is a significant periodic X_{H_T}-orbit.*

Proof For a technical X_{H_T}-orbit $u(t) = \exp\left(\frac{2\pi}{T}\, tJ\right) x$ with $|x| = r_{\text{crit}}$ we obtain

$$
\begin{aligned}
\tfrac{1}{T}\,\mathcal{A}_{H_T}(u) \;&=\; \tfrac{1}{T}\int_0^T \{\tfrac{1}{2}\,\langle -J\dot{u}\,,\,u\,\rangle \;-\; H_T(u)\}\;dt \\
&=\; \tfrac{1}{2}\,h'_T(|x|)\,|x| - h_T(|x|) \qquad (47)\\
&<\; \tfrac{3\pi}{2T}\,r_{\text{crit}}^2 - h_T(r_{\text{crit}}) \;<\; 0
\end{aligned}
$$

since $r_{\text{crit}} \in (R_1, R_2)$, and the statement thus follows. $\square$

To summarize the contents of the last two sections, we note that Lemma 1 allows to associate to each symmetric closed characteristics on Σ a so-called 'tower' $\{\, u_s \mid |s| < \delta \,\}$ of symmetric periodic X_H-orbits where u_s lies on the hypersurface Σ_s and has the minimal period $\tau(s)$. If, in addition, the number of these towers is assumed to be finite, then Lemma 2 guarantees that for every sufficiently large $T > 0$ each of these towers contains a unique X_H-orbit u_s that has the prescribed minimal period T and satisfies $\mathcal{A}_H(u_s) > 0$. Consequently, the counting of the geometrically distinct closed characteristics on Σ can be done by counting these particular T-periodic X_H-orbits.

Fixing T so large that the condition (43) is satisfied, we have extended the Hamiltonian $H : U \to \mathbb{R}$ to $H_T : \mathbb{R}^{2n} \to \mathbb{R}$ such that the additional symmetric periodic X_{H_T}-orbits of minimal period T outside U have a negative Hamiltonian action and do not affect the above orbit counting. Therefore, to prove Theorem 1 it suffices to show:

Proposition 1 *Let $\Sigma \subset \mathbb{R}^{2n}$ be an $\mathbb{S}^1$-hypersurface satisfying the hypotheses of Theorem 1 such that, in addition, the number of symmetric closed characteristics on Σ is finite. Suppose $T > 0$ is fixed in accordance with* (43) *so that the distinct symmetric periodic X_{H_T}-orbits u of minimal period T and $\mathcal{A}_{H_T}(u) > 0$ correspond uniquely to the distinct symmetric closed characteristics on Σ. Then the number of these X_{H_T}-orbits u is at least n.*

5. The variational approach and the free Morse complex

The choice of the period T in the hypotheses of Proposition 1 provides a distinguished parametrization $\mathbb{S}^1_T = \mathbb{R}/\mathbb{Z}$ of the circle group. To every $x \in \mathbb{R}^{2n}$ we may now assign the parametrized $\mathbb{S}^1$-orbit $u_x : \mathbb{S}^1_T \to \mathbb{R}^{2n}$ by setting $u_x(t) := \gamma_t\, x$. The $\mathbb{S}^1$-orbit u_x describes a symmetric T-periodic X_{H_T}-orbit if it solves the equation

$$\dot{u}_x \;=\; X_{H_T}(u_x)\,, \quad x \in \mathbb{R}^{2n}\,, \tag{48}$$

and T is the minimal period if $x \neq 0$. Hence, setting $\Omega_{\text{sym}} := \{\, u_x \mid x \in \mathbb{R}^{2n} \,\}$, the desired periodic X_{H_T}-orbits of Proposition 1 can be found as those critical points u_x of the restricted action integral $\mathcal{A}_{H_T}|_{\Omega_{\text{sym}}} : \Omega_{\text{sym}} \to \mathbb{R}$ for which

$$\mathcal{A}_{H_T}(u_x) \;=\; \int_0^T \left\{\tfrac{1}{2}\,\langle -J\dot{u}_x\,, u_x\,\rangle \;-\; H_T(u_x)\right\}\;dt \tag{49}$$

attains a positive value. However, the bijection $\mathbb{R}^{2n} \to \Omega_{\mathrm{sym}}$, $x \mapsto u_x$, suggests to study the corresponding variational problem on $\mathbb{R}^{2n}$:

Lemma 4 *The Hamiltonian action integral $\mathcal{A}_{H_T}|_{\Omega_{\mathrm{sym}}} : \Omega_{\mathrm{sym}} \to \mathbb{R}$ gives rise to a smooth $\mathbb{S}^1$-invariant function $\mathcal{F} : \mathbb{R}^{2n} \to \mathbb{R}$ by setting*

$$\mathcal{F}(x) \;:=\; \tfrac{1}{T}\, \mathcal{A}_{H_T}(u_x)\,. \tag{50}$$

In particular, $x \in \mathbb{R}^{2n}$ is a critical point of $\mathcal{F}$ if and only if $u_x \in \Omega_{\mathrm{sym}}$ solves the Hamiltonian equation $\dot{u}_x = X_{H_T}(u_x)$.

Proof Evaluating $\mathcal{A}_{H_T}(u_x)$, we get the expression $\mathcal{F}(x) = \frac{\pi}{T}\,|x|^2 - H_T(x)$. Then, clearly, $\operatorname{grad}\mathcal{F}(x) = 0$ is equivalent to $\frac{2\pi}{T}\,\gamma_t\, x = \operatorname{grad} H_T(\gamma_t\, x)$. On the other hand, $-J\dot{u}_x(t) = \frac{2\pi}{T}\, u_x(t) = \frac{2\pi}{T}\,\gamma_t\, x$, and the statement follows. □

The assertion of Proposition 1 can now be restated as follows:

Proposition 2 *Suppose that the hypotheses of Proposition 1 are satisfied. Then there exist $\ell \geqslant n$ critical points $x^1, \ldots, x^\ell$ of $\mathcal{F}$ such that*

(i) $\operatorname{Orb} x^i \cap \operatorname{Orb} x^j = \emptyset$ *if* $i \neq j$.

(ii) $\mathcal{F}(x^i) > 0$ *for* $i = 1, \ldots, \ell$.

The proof will be based on $\mathbb{S}^1$-equivariant Lyusternik-Schnirelman theory for flows, involving C. Conley's concept of isolated invariant sets. The flow in question is the $-\operatorname{grad}\mathcal{F}$-flow on $\mathbb{R}^{2n}$, i.e., the map $\mathbb{R}^{2n} \times \mathbb{R} \to \mathbb{R}^{2n}$, $(x, \tau) \mapsto x\cdot\tau$, solving the initial value problem

$$\begin{cases} \frac{d}{d\tau}\,(x\cdot\tau) &=& -\operatorname{grad}\mathcal{F}(x\cdot\tau) \\ \qquad x\cdot 0 &=& x \quad \forall\, x \in \mathbb{R}^{2n}\,. \end{cases} \tag{51}$$

Note that the trajectory $\tau \mapsto x\cdot\tau$ of each $x \in \mathbb{R}^{2n}$ is indeed defined for all $\tau \in \mathbb{R}$ because $\operatorname{grad}\mathcal{F}$ is linearly bounded. The $\mathbb{S}^1$-equivariance of $\operatorname{grad}\mathcal{F}$ implies that $(\gamma_t\, x)\cdot\tau = \gamma_t\,(x\cdot\tau)$ for all $(x, \tau) \in \mathbb{R}^{2n} \times \mathbb{R}$.

Given $r_2 > r_1 > 0$, we define the compact $\mathbb{S}^1$-set $Q_{r_1,r_2} \subset \mathbb{R}^{2n}\backslash\{0\}$ by setting

$$Q_{r_1,r_2} \;:=\; \{\, x \in \mathbb{R}^{2n} \mid r_1 \leqslant |x| \leqslant r_2 \,\}\,, \tag{52}$$

which has the boundary

$$\partial Q_{r_1,r_2} \;=\; \mathbb{S}^{2n-1}_{r_1} \cup \mathbb{S}^{2n-1}_{r_2}\,, \tag{53}$$

where $\mathbb{S}^{2n-1}_r$ denotes the Euclidean $(2n-1)$-sphere of radius r.

Lemma 5 *There exists a choice of $r_2 > \varrho > r_1 > 0$ such that:*

(i) *$Q := Q_{r_1,r_2}$ is an isolating block for the $-\operatorname{grad}\mathcal{F}$-flow, containing all critical points of $\mathcal{F}$ except for the origin $0 \in \mathbb{R}^{2n}$ and having the exit set $Q^- = \partial Q_{r_1,r_2}$.*

(ii) *$\mathcal{F}(x) = \frac{\pi}{T}\varrho^2$ if $x \in \mathbb{S}^{2n-1}_\varrho$. In particular, $\max_{x\in\mathbb{R}^{2n}} \mathcal{F}(x) > 0$ is attained in Q.*

The isolated invariant set $\mathfrak{M} \subset \operatorname{int} Q$ will be called the free Morse complex of $\mathcal{F}$.

Proof Let $B_r(0)$ denote the closed $2n$-ball of radius r centered at the origin. From the definition (16) of U we infer $B_\varrho(0) \cap \overline{U} = \emptyset$ if $\varrho > 0$ is chosen small enough. Then $H_T|_{B_\varrho(0)} = 0$ and, consequently, $\mathcal{F}|_{B_\varrho(0)} = \frac{\pi}{T}\,\varrho^2$.

Let us now fix $r_1 < \varrho$ and $r_2 > R_2$ with the constant R_2 introduced in (43). Observing that $\mathcal{F}(x) = \frac{\pi}{T}\,|x|^2$ if $|x| \leqslant \varrho$ and $\mathcal{F}(x) = \frac{\pi}{T}\,|x|^2 - \frac{3\pi}{2T}\,|x|^2 = -\frac{\pi}{2T}\,|x|^2$ if $|x| \geqslant R_2$, we infer that $\mathcal{F}$ attains its global maximum inside Q on a positive level and, moreover,

$$(54) \qquad -\operatorname{grad}\mathcal{F}(x) \;=\; \begin{cases} -\frac{2\pi}{T}\,x & \text{if} \quad x \in \mathbb{S}^{2n-1}_{r_1} \\[2ex] \frac{\pi}{T}\,x & \text{if} \quad x \in \mathbb{S}^{2n-1}_{r_2}\,. \end{cases}$$

Hence Q is an isolating block for the $-\operatorname{grad}\mathcal{F}$-flow that has the exit set $Q^- = \partial Q$ (cf. [2, p. 5]). In addition, we see that $\operatorname{grad}\mathcal{F}(x) = 0$ and $x \in \mathbb{R}^{2n} \setminus Q$ imply $x = 0$, which completes the proof. $\square$

An *admissibly ordered Morse decomposition of* $\mathfrak{M}$ is a tupel $(M_1, \ldots, M_k)$ of compact non empty subsets of $\mathfrak{M}$ such that for every $x \in \mathfrak{M} \setminus (M_1 \cup \ldots \cup M_k)$ there exist unique indices $i < j$ with $x\cdot\tau \to M_i$ as $\tau \to +\infty$ and $x\cdot\tau \to M_j$ as $\tau \to -\infty$. Let us consider the subsets

$$(55) \qquad \mathfrak{M}^+ \;:=\; \{\, x \in \mathfrak{M} \mid \mathcal{F}(x\cdot\tau) > 0 \;\; \forall \tau \in \mathbb{R} \,\}$$

and

$$(56) \qquad \mathfrak{M}^- \;:=\; \{\, x \in \mathfrak{M} \mid \mathcal{F}(x\cdot\tau) < 0 \;\; \forall \tau \in \mathbb{R} \,\}\,,$$

which we call the positive and the negative subcomplex of $\mathfrak{M}$, respectively. Note that $\mathfrak{M}^+$ contains the critical points x of $\mathcal{F}$ that correspond to the significant periodic X_{H_T}-orbits u_x with $\mathcal{A}_{H_T}(u_x) > 0$, whereas $\mathfrak{M}^-$ contains their negative counterparts together with the sphere $\mathbb{S}^{2n-1}_{r_{\mathrm{crit}}}$ which gives rise to the technical periodic X_{H_T}-orbits.

As a consequence of Lemma 5 we have $\mathfrak{M}^+ \neq \emptyset$, and it follows readily that $(\mathfrak{M}^-, \mathfrak{M}^+)$ is indeed an admissibly ordered Morse decomposition of $\mathfrak{M}$. Setting $Q^0 := \{\, x \in Q \mid \mathcal{F}(x) \leqslant 0 \,\}$, we define a filtration of index pairs $N_0 \subset N_1 \subset N_2$ associated to $(\mathfrak{M}^-, \mathfrak{M}^+)$ by

$$(57) \qquad N_0 := Q^- \,, \quad N_1 := Q^- \cup Q^0 \,, \quad N_2 := Q\,.$$

The compact $\mathbb{S}^1$-pair (N_2, N_1) is an index pair for $\mathfrak{M}^+$, i.e.,

(a) $N := \overline{N_2 \setminus N_1} = \{\, x \in Q \mid \mathcal{F}(x) \geqslant 0 \,\}$ is an isolating neighborhood for $\mathfrak{M}^+$ having the exit set $N^- = \partial N = \mathbb{S}^{2n-1}_{r_1} \cup \{\, x \in Q \mid \mathcal{F}(x) = 0 \,\}$.

(b) N_1 is positively invariant relative to N_2.

(c) If $x \in N_2$ and $x\cdot[\,0,\infty) \not\subset N_2$ then there exists $\tau \geqslant 0$ such that $x\cdot[\,0,\tau\,] \subset N_2$ and $x\cdot\tau \in N_1$.

The proof of Poposition 2 requires an estimate for the number ℓ of distinct critical $\mathbb{S}^1$-orbits $\operatorname{Orb} x^i$, $i = 1, \dots, \ell$, contained in $\mathfrak{M}^+$. So far we only know that $\ell < +\infty$ by our assumptions on the hypersurface Σ, and that $\ell \geqslant 1$ because $\mathcal{F}$ attains its global maximum along some $\operatorname{Orb} x^i \subset \mathfrak{M}^+$. To derive the claimed inequality $\ell \geqslant n$ we have to investigate the $\mathbb{S}^1$-equivariant topology of $\mathfrak{M}^+$ which will be carried out in the subsequent section.

6. $\mathbb{S}^1$-index maps related to the positive subcomplex $\mathfrak{M}^+$

The observation that the sphere $\mathbb{S}^{2n-1}_\varrho$ introduced in Lemma 5 is contained in the isolating block N of $\mathfrak{M}^+$ turns out to be crucial. This fact causes a topological linking which allows to estimate the $\mathbb{S}^1$-category of $\mathfrak{M}^+$ in N by the cohomological $\mathbb{S}^1$-index of $\mathbb{S}^{2n-1}_\varrho$. We merely recall some relevant facts on the involved $\mathbb{S}^1$-index maps and refer to [8, 9, 10] for further details.

Given a compact $\mathbb{S}^1$-pair (X, A), we denote by $H^*_{\mathbb{S}^1}(X, A)$ the $\mathbb{S}^1$-equivariant Alexander-Spanier cohomology with real coefficients, cf. e.g. [9]. Since the $\mathbb{S}^1$-action under consideration is free, the equivariant cohomology is isomorphic to the ordinary cohomology of $(X/\mathbb{S}^1, A/\mathbb{S}^1)$, i.e., we have $H^*_{\mathbb{S}^1}(X, A) \cong H^*(X/\mathbb{S}^1, A/\mathbb{S}^1)$. Likewise, we shall denote by $H_*^{\mathbb{S}^1}(X, A) := H_*(X/\mathbb{S}^1, A/\mathbb{S}^1)$ singular $\mathbb{S}^1$-homology of (X, A) with real coefficients.

The cohomological $\mathbb{S}^1$-index is defined in terms of universal characteristic classes. The universal principal $\mathbb{S}^1$-bundle $E\mathbb{S}^1 \xrightarrow{p} B\mathbb{S}^1$ is obtained as the inductive limit of the Hopf fibration

$$\begin{array}{ccccccccc}
\mathbb{S}^3 & \subset & \mathbb{S}^5 & \subset & \dots & \subset & \mathbb{S}^{2n-1} & \subset & \dots \\
\big\downarrow & & \big\downarrow & & & & \big\downarrow & & \\
\mathbb{S}^2 & \subset & \mathbb{C}P^2 & \subset & \dots & \subset & \mathbb{C}P^{n-1} & \subset & \dots
\end{array},$$

i.e., $E\mathbb{S}^1 = \mathbb{S}^\infty$ and $B\mathbb{S}^1 = \mathbb{C}P^\infty$. The base space $\mathbb{C}P^\infty$ then is called the classifying space for $\mathbb{S}^1$. The cohomology ring $H^*(\mathbb{C}P^\infty; \mathbb{Z})$ is the polynomial ring generated by the first Chern class $c_1 \in H^2(\mathbb{C}P^\infty; \mathbb{Z})$, cf. [14, Thm. 14.5]. Introducing real coefficients, let $c \in H^2(\mathbb{C}P^\infty)$ denote the cohomology class corresponding to c_1.

Now let X be a paracompact free $\mathbb{S}^1$-space and let $f : X/\mathbb{S}^1 \to \mathbb{C}P^\infty$ denote a classifying map for the principal $\mathbb{S}^1$-bundle $X \xrightarrow{p} X/\mathbb{S}^1$. The corresponding induced map $f^* : H^*(\mathbb{C}P^\infty) \to H^*(X/\mathbb{S}^1)$ is a ring homomorphism, and the cohomological $\mathbb{S}^1$-index of $X \neq \emptyset$ is defined by

$$\operatorname{Index}_{\mathbb{C}} X \;:=\; 1 + \sup\{\, k \in \mathbb{N}_0 \mid f^*(c^k) \neq 0 \,\}. \tag{58}$$

Moreover, we set $\operatorname{Index}_{\mathbb{C}} \emptyset := 0$.

If $X = \mathbb{S}^{2n-1} \subset \mathbb{R}^{2n}$ is the Euclidean $(2n-1)$-sphere (of any radius $\varrho > 0$), then the $\mathbb{S}^1$-action γ gives rise to the Hopf map $\mathbb{S}^{2n-1} \xrightarrow{p} \mathbb{C}P^{n-1}$. The corresponding classifying map f is the inclusion $\mathbb{C}P^{n-1} \to \mathbb{C}P^\infty$, and the cohomology ring $H^*(\mathbb{C}P^{n-1})$ is the truncated polynomial ring, terminating in dimension $2(n-1)$, which is generated by $f^*(c) \in H^2(\mathbb{C}P^{n-1})$, cf. [14, Thm. 14.4]. Hence we infer

$$\operatorname{Index}_{\mathbb{C}} \mathbb{S}^{2n-1} \;=\; n\,. \tag{59}$$

Given a free $\mathbb{S}^1$-pair (X, A), then A is called $\mathbb{S}^1$-*categorical in* X if A can be deformed $\mathbb{S}^1$-equivariantly through X into a single $\mathbb{S}^1$-orbit. The $\mathbb{S}^1$-category of a non empty $\mathbb{S}^1$-set $A \subset X$ in X is defined by

$$\begin{aligned} \mathbb{S}^1\text{-cat}_X(A) \;:=\; \inf\,\{\, k \in \mathbb{N} \mid & \text{ there exist closed } \mathbb{S}^1\text{-categorical subsets} \\ & A_1, \ldots, A_k \subset X \text{ with } A \subset A_1 \cup \ldots \cup A_k \,\}\;, \end{aligned} \tag{60}$$

whereas $\mathbb{S}^1\text{-cat}_X(\emptyset) := 0$. Furthermore, we write $\mathbb{S}^1\text{-cat}(X) := \mathbb{S}^1\text{-cat}_X(X)$.

Since the $\mathbb{S}^1$-action on (X, A) is supposed to be free, the $\mathbb{S}^1$-category is related to the ordinary Lyusternik-Schnirelman category (see e.g. [4, Ch. 27.3]) by

$$\mathbb{S}^1\text{-cat}_X(A) \;=\; \operatorname{cat}_{X/\mathbb{S}^1}(A/\mathbb{S}^1)\,. \tag{61}$$

The above definition of $\mathbb{S}^1\text{-cat}_X(A)$ differs from E. Fadell's [8] in that we require the covering sets $A_1, \ldots, A_k$ to be closed. However, being concerned with free $\mathbb{S}^1$-pairs (X, A) where $X/\mathbb{S}^1$ is an absolute neighborhood retract (ANR) and the subset $A/\mathbb{S}^1$ is closed, this difference does not affect the statements below.

In terms of the $\mathbb{S}^1$-category and the cohomological $\mathbb{S}^1$-index, the desired estimate can now be expressed as follows:

Proposition 3 *Let $\mathfrak{M}^+$ be the positive subcomplex defined in* (55)*, and let N denote the isolating neighborhhood obtained from the filtration of index pairs* (57)*. Then*

$$\mathbb{S}^1\text{-cat}_N(\mathfrak{M}^+) \;\geqslant\; \operatorname{Index}_{\mathbb{C}} \mathbb{S}^{2n-1}\,. \tag{62}$$

Proof First we note that N is a smooth manifold with boundary ∂N equipped with a proper free Lie group action of the circle group $\mathbb{S}^1$. Hence the orbit space $N/\mathbb{S}^1$ has the unique structure of a smooth manifold with boundary $\partial(N/\mathbb{S}^1) = (\partial N)/\mathbb{S}^1$ such that the projection $N \to N/\mathbb{S}^1$ is a submersion, cf. [5, Prop. 5.2]. In particular, $N/\mathbb{S}^1$ is an ANR so that $\mathbb{S}^1\text{-cat}_N$ has the continuity property.

Let us recall that $\mathbb{S}^{2n-1}_\varrho \subset \operatorname{int} N$ where $\varrho > 0$ is the constant introduced in Lemma 5. The inequality (62) will now be derived in three steps:

Claim 1:

$$\mathbb{S}^1\text{-cat}_N(\mathfrak{M}^+) \;\geqslant\; \mathbb{S}^1\text{-cat}_N(\mathbb{S}^{2n-1}_\varrho)\,. \tag{63}$$

Proof: Let U_ε denote the open ε-neighborhood of $\mathfrak{M}^+$. Clearly, U_ε is $\mathbb{S}^1$-invariant and, for small ε, we have $U_\varepsilon \subset N$. Moreover, by the continuity property of $\mathbb{S}^1\text{-cat}_N$ we may assume that $\varepsilon > 0$ is chosen such that $\mathbb{S}^1\text{-cat}_N(U_\varepsilon) = \mathbb{S}^1\text{-cat}_N(\mathfrak{M}^+)$.

Since $\mathfrak{M}^+$ is compact and non empty, we find $0 < k := \mathbb{S}^1\text{-cat}_N(\mathfrak{M}^+) < +\infty$. Consequently, there exist k closed $\mathbb{S}^1$-categorical subsets $A_1, \ldots, A_k \subset N$ such that $U_\varepsilon \subset A := A_1 \cup \ldots \cup A_k$. For every $x \in \overline{N \backslash A}$ there exists a unique $\tau(x) \geqslant 0$ such that $x \cdot \tau(x) \in \partial N$, i.e., the integral curve $\tau \mapsto x \cdot \tau$ of the $-\operatorname{grad} \mathcal{F}$-flow exits N at time $\tau(x)$. The compactness of $\overline{N \backslash A}$ then implies

$$\tau^- \;:=\; \sup\,\{\, \tau(x) \mid x \in \overline{N \backslash A}\,\} \;<\; +\infty\,. \tag{64}$$

If $\tau \geqslant \tau^-$ is fixed, it follows that $N \subset A_1{\cdot}\tau \cup \ldots A_k{\cdot}\tau$. Setting $B_\nu := (A_\nu{\cdot}\tau) \cap \mathbb{S}^{2n-1}_\varrho$ for $\nu = 1,\ldots,k$, we thus have found k closed $\mathbb{S}^1$-categorical subsets of N such that $\mathbb{S}^{2n-1}_\varrho \subset B_1 \cup \ldots \cup B_k$. As a consequence, $\mathbb{S}^1\text{-cat}_N(\mathbb{S}^{2n-1}_\varrho) \leqslant k$, which proves claim 1.

Claim 2:

$$\mathbb{S}^1\text{-cat}_N(\mathbb{S}^{2n-1}_\varrho) \;=\; \mathbb{S}^1\text{-cat}(\mathbb{S}^{2n-1}_\varrho)\,. \tag{65}$$

Proof: Clearly, $\mathbb{S}^1\text{-cat}_N(\mathbb{S}^{2n-1}_\varrho) \leqslant \mathbb{S}^1\text{-cat}(\mathbb{S}^{2n-1}_\varrho)$, and it remains to show the opposite inequality. For this purpose we observe that the homotopy $\bar g : Q \times [\,0,1\,] \to Q$, defined by $\bar g(x,s) := (1-s)x + \frac{s\varrho}{|x|}\,x$, deforms Q equivariantly into $\mathbb{S}^{2n-1}_\varrho$. Hence the $\mathbb{S}^1$-map $g := \bar g(\cdot,1)|_N : N \to \mathbb{S}^{2n-1}_\varrho$ is an $\mathbb{S}^1$-retraction of N to $\mathbb{S}^{2n-1}_\varrho$.

Now suppose that $k = \mathbb{S}^1\text{-cat}_N(\mathbb{S}^{2n-1}_\varrho)$ and let $A_1,\ldots,A_k$ be closed $\mathbb{S}^1$-categorical subsets of N with $\mathbb{S}^{2n-1}_\varrho \subset A_1 \cup \ldots \cup A_k$. Moreover, let $h_\nu : A_\nu \times [\,0,1\,] \to N$ denote the corresponding $\mathbb{S}^1$-homotopy that deforms A_ν into a single $\mathbb{S}^1$-orbit. We observe that $B_\nu := A_\nu \cap \mathbb{S}^{2n-1}_\varrho$ is $\mathbb{S}^1$-invariant and closed in $\mathbb{S}^{2n-1}_\varrho$ for $\nu = 1,\ldots,k$. Furthermore, by the $\mathbb{S}^1$-homotopy $g \circ h_\nu|_{B_\nu} : B_\nu \times [\,0,1\,] \to \mathbb{S}^{2n-1}_\varrho$ we see that B_ν is $\mathbb{S}^1$-categorical in $\mathbb{S}^{2n-1}_\varrho$. Hence we get $\mathbb{S}^1\text{-cat}(\mathbb{S}^{2n-1}_\varrho) \leqslant k$, and claim 2 follows.

Claim 3:

$$\mathbb{S}^1\text{-cat}(\mathbb{S}^{2n-1}) \;\geqslant\; \mathrm{Index}_{\mathbb{C}}\,\mathbb{S}^{2n-1}\,. \tag{66}$$

Proof: Arguing by contradiction, we assume that $k := \mathbb{S}^1\text{-cat}(\mathbb{S}^{2n-1}) < \mathrm{Index}_{\mathbb{C}}\,\mathbb{S}^{2n-1}$. As a consequence, there are k closed $\mathbb{S}^1$-categorical subsets $A_\nu \subset \mathbb{S}^{2n-1}$ such that $A_1 \cup \ldots \cup A_k = \mathbb{S}^{2n-1}$. Without loss of generality we may assume that $\{A_\nu, A_{\nu'}\}$ is an excisive couple for all ν,ν', i.e., $A_\nu \cup A_{\nu'} = \mathrm{int}\,A_\nu \cup \mathrm{int}\,A_{\nu'}$ in the relative topology of $A_\nu \cup A_{\nu'}$.

Denote by $i_\nu : A_\nu \to \mathbb{S}^{2n-1}$ the inclusion. Then i_ν is $\mathbb{S}^1$-homotopic to a map $i^0_\nu : A_\nu \to \mathbb{S}^{2n-1}$ such that $i^0_\nu(A_\nu) = \mathrm{Orb}\,x^\nu$ is a single $\mathbb{S}^1$-orbit. Hence we have an $\mathbb{S}^1$-map $\iota_\nu : A_\nu \to \mathrm{Orb}\,x^\nu$ and an inclusion $\kappa_\nu : \mathrm{Orb}\,x^\nu \to \mathbb{S}^{2n-1}$ such that $i^0_\nu = \kappa_\nu \circ \iota_\nu$. These maps induce the following commutative diagram in $\mathbb{S}^1$-cohomology:

$$\begin{tikzcd} H^*_{\mathbb{S}^1}(N) \arrow[r, "\kappa^*_\nu"] \arrow[dr, "i^*_\nu"] & H^*_{\mathbb{S}^1}(\mathrm{Orb}\,x^\nu) \arrow[d, "\iota^*_\nu"] \\ & H^*_{\mathbb{S}^1}(A_\nu) \end{tikzcd}$$

Consequently, $i^*_\nu = 0$ for $* \geqslant 1$. Let us now consider the exact $\mathbb{S}^1$-cohomology sequence of the pair $(\mathbb{S}^{2n-1}, A_\nu)$:

$$\longrightarrow H^{*-1}_{\mathbb{S}^1}(A_\nu) \xrightarrow{\delta^*} H^*_{\mathbb{S}^1}(\mathbb{S}^{2n-1}, A_\nu) \xrightarrow{j^*_\nu} H^*_{\mathbb{S}^1}(\mathbb{S}^{2n-1}) \xrightarrow{i^*_\nu} H^*_{\mathbb{S}^1}(A_\nu) \longrightarrow \tag{67}$$

where $j_\nu : (\mathbb{S}^{2n-1}, \emptyset) \to (\mathbb{S}^{2n-1}, A_\nu)$ is the inclusion. By the exactness of the sequence, $i^*_\nu = 0$ implies that j^*_ν is epimorphic.

Using the isomorphsm $H^*_{\mathbb{S}^1}(\mathbb{S}^{2n-1}) \cong H^*(\mathbb{S}^{2n-1}/\mathbb{S}^1) \cong H^*(\mathbb{C}P^{n-1})$, let us denote by $\alpha \in H^2_{\mathbb{S}^1}(\mathbb{S}^{2n-1})$ the generator corresponding to the class $f^*(c) \in H^2(\mathbb{C}P^{n-1})$. For $\nu = 1,\ldots,k$ we thus can find $\alpha_\nu \in H^2_{\mathbb{S}^1}(\mathbb{S}^{2n-1}, A_\nu)$ with $j^*_\nu(\alpha_\nu) = \alpha$.

Observing that $A_1 \cup \ldots \cup A_k = \mathbb{S}^{2n-1}$ and denoting by $j : (\mathbb{S}^{2n-1}, \emptyset) \to (\mathbb{S}^{2n-1}, A)$ the inclusion, we obtain the following commutative diagram

$$
\begin{array}{ccccccc}
H^2_{\mathbb{S}^1}(\mathbb{S}^{2n-1}, A_1) & \otimes & \cdots & \otimes & H^2_{\mathbb{S}^1}(\mathbb{S}^{2n-1}, A_k) & \xrightarrow{\smile} & H^{2k}_{\mathbb{S}^1}(\mathbb{S}^{2n-1}, \mathbb{S}^{2n-1}) \\
\Big\downarrow {\scriptstyle j_1^*} & & & & \Big\downarrow {\scriptstyle j_k^*} & & \Big\downarrow {\scriptstyle j^*} \\
H^2_{\mathbb{S}^1}(\mathbb{S}^{2n-1}) & \otimes & \cdots & \otimes & H^2_{\mathbb{S}^1}(\mathbb{S}^{2n-1}) & \xrightarrow{\smile} & H^{2k}_{\mathbb{S}^1}(\mathbb{S}^{2n-1})
\end{array}
\tag{68}
$$

so that $j^* = 0$. Thus, we find $\alpha^k = j^*(\alpha_1 \smile \ldots \smile \alpha_k) = 0$ in contradiction to the choice of α which implies $\alpha^k \neq 0$ for $k = 0, \ldots, n-1$. Thus the proof of claim 3 is complete.

As a side remark, we point out that indeed $\mathbb{S}^1\text{-cat}(\mathbb{S}^{2n-1}) = \mathrm{Index}_{\mathbb{C}}\, \mathbb{S}^{2n-1} = n$, cf. [10, Cor. 7.8]. The estimate (62) now follows from (63), (65), and (66). The proof of the proposition is with this complete. □

7. Lyusternik-Schnirelman theory for $\mathfrak{M}^+$ and the proof of Theorem 1

By our assumptions, $\mathfrak{M}^+$ contains only a finite number of critical $\mathbb{S}^1$-orbits. Let the corresponding critical values c_j be ordered according to $0 < c_1 < \ldots < c_k$. For $j = 1, \ldots, k$ we define the Morse sets M_j by

$$
M_j \;:=\; \{\, x \in \mathfrak{M}^+ \mid \mathrm{grad}\, \mathcal{F}(x) = 0 \;\text{ and }\; \mathcal{F}(x) = c_i \,\} \,. \tag{69}
$$

Then $(M_1, \ldots, M_k)$ is an admissibly ordered Morse decomposition of the positive subcomplex $\mathfrak{M}^+$. Consequently, we obtain

Proposition 4 *Let $(M_1, \ldots, M_k)$ denote the Morse decomposition of $\mathfrak{M}^+$ induced by the ordering of the positive critical values $c_1 < \ldots < c_k$ of $\mathcal{F}$. Then*

$$
\mathbb{S}^1\text{-cat}_N(\mathfrak{M}^+) \;\leqslant\; \sum_{j=1}^{k} \mathbb{S}^1\text{-cat}_N(M_j) \,. \tag{70}
$$

In particular, the number of the distinct critical $\mathbb{S}^1$-orbits of $\mathcal{F}$ in $\mathfrak{M}^+$ is greater or equal to $\mathbb{S}^1\text{-cat}_N(\mathfrak{M}^+)$.

Proof Let us first consider an admissibly ordered Morse decomposition $(\mathfrak{M}_1, \mathfrak{M}_2)$ of $\mathfrak{M}^+$ consisting of only two Morse sets. If $\mathbb{S}^1\text{-cat}_N(\mathfrak{M}_j) = \ell_j$, $j = 1, 2$, then there exist $\ell_1 + \ell_2$ closed $\mathbb{S}^1$-categorical subsets $A_{j,i} \subset N$ with $\mathfrak{M}_j \subset \mathrm{int}\,(A_{j,1} \cup \ldots \cup A_{j,\ell_j})$. In view of the compactness of $\mathfrak{M}^+$ there exists $\tau_0 \in [\,0, +\infty)$ such that for $\tau \geqslant \tau_0$ we have

$$
\mathfrak{M}^+ \;\subset\; \bigcup_{i=1}^{\ell_1} A_{1,i} \;\cup\; \bigcup_{i=1}^{\ell_2} \left((A_{2,i} \cdot \tau) \cap N\right) \;. \tag{71}
$$

Thus $\mathfrak{M}^+$ can be covered by $\ell_1 + \ell_2$ closed $\mathbb{S}^1$-categorical subsets of N and, therefore, $\mathbb{S}^1\text{-cat}_N(\mathfrak{M}^+) \;\leqslant\; \mathbb{S}^1\text{-cat}_N(\mathfrak{M}_1) \;+\; \mathbb{S}^1\text{-cat}_N(\mathfrak{M}_2)$.

Because the Morse sets $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are again isolated invariant sets, the same argument can now be applied to Morse decompositions of $\mathfrak{M}_1$ and $\mathfrak{M}_2$. Continuing this process, the desired Morse decomposition $(M_1, \ldots, M_k)$ of $\mathfrak{M}^+$ is reached after a finite number of repetitions. This proves the inequality (70).

Suppose that $\mathbb{S}^1\text{-cat}_N(M_j) = \ell$ and that $A_1, \ldots, A_\ell \subset N$ are closed $\mathbb{S}^1$-categorical subsets covering M_j. Then M_j contains at least ℓ critical $\mathbb{S}^1$-orbits $\operatorname{Orb} x_i \subset A_i$, $i = 1, \ldots, \ell$, which belong to mutually distinct path components of the orbit space $N/\mathbb{S}^1$. This follows immediately from the observation that a finite set of $\mathbb{S}^1$-orbits which belongs to the same path component of $N/\mathbb{S}^1$ must have the $\mathbb{S}^1$-category 1. The assertion of the proposition is thereby proven. □

Proposition 4 combined with Proposition 3 shows that $\mathfrak{M}^+$ contains at least n distinct $\mathbb{S}^1$-orbits $\operatorname{Orb} x^i$ such that $\operatorname{grad} \mathcal{F}(x^i) = 0$, which proves Proposition 2. Consequently, the proof of Theorem 1 is complete. □

8. The proofs of Theorem 2 and Corollary 1

Let $f : \mathbb{C}P^{n-1} \to \mathbb{R}$ be a smooth function and let $\bar{f} : \mathbb{S}^{2n-1} \to \mathbb{R}$ denote a lift of f, according to $\bar{f}(x) := f(\operatorname{Orb} x)$. The associated $\mathbb{S}^1$-hypersurface Σ_f is then given by

$$\Sigma_f \;=\; \{\, e^{\bar{f}(x)}\, x \mid x \in \mathbb{S}^{2n-1} \} \;\subset\; \mathbb{R}^{2n}\,, \tag{72}$$

where $\mathbb{S}^{2n-1}$ stands for the Euclidean unit sphere in $\mathbb{R}^{2n}$. Then Σ_f is star-shaped since every radius $\{\, \mu x \mid \mu \geqslant 0,\ x \in \mathbb{S}^{2n-1} \}$ intersects Σ_f transversally. The transversality is seen as follows: if $x \in \mathbb{S}^{2n-1}$ and $b_1, \ldots, b_{2n-1} \in T_x\mathbb{S}^{2n-1}$ is a basis, then a corresponding basis of $T_{e^{\bar{f}(x)}x}\Sigma_f$ is obtained via tangent map of the transformation $\mathbb{S}^{2n-1} \to \Sigma_f$, i.e., by the map

$$b_i \;\mapsto\; e^{\bar{f}(x)} \Big(b_i + \langle\, \operatorname{grad} \bar{f}(x), b_i \,\rangle\, x \Big)\,. \tag{73}$$

Thus the flow $(x, s) \mapsto e^{s/s}\, x$ of the symplectic dilation $X_\lambda(x) = \frac{1}{2}\, x$ can be used to introduce the smooth Hamiltonian function $H_f : \mathbb{R}^{2n} \setminus \{0\} \to \mathbb{R}$ by setting

$$H_f(e^{s/2}\, x) \;:=\; e^s \quad \forall\, x \in \Sigma_f,\ s \in \mathbb{R}\,, \tag{74}$$

Note that H_f is positively homogeneous of degree 2 and its gradient is positively homogeneous of degree 1, i.e., $H_f(\mu x) = \mu^2 H_f(x)$ and $\operatorname{grad} H_f(\mu x) = \mu \operatorname{grad} H_f(x)$ for all $x \in \mathbb{R}^{2n-1} \setminus \{0\}$ and $\mu > 0$. In particular, we deduce

$$H_f(e^{\bar{f}(x)}\, x) \;=\; e^{2\bar{f}(x)}\, H_f(x) \;=\; 1 \quad \forall\, x \in \mathbb{S}^{2n-1}\,, \tag{75}$$

so that $\bar{f}$ can be expressed by

$$\bar{f}(x) \;=\; -\tfrac{1}{2}\, \ln H_f(x)\,, \quad x \in \mathbb{S}^{2n-1}\,. \tag{76}$$

Then the gradient of $\bar{f}$ satisfies the identity

$$\langle\, \operatorname{grad} \bar{f}(x), X \,\rangle \;=\; \frac{-1}{2H_f(x)}\, \langle\, \operatorname{grad} H_f(x), X \,\rangle \quad \forall\, X \in T_x\mathbb{S}^{2n-1}\,. \tag{77}$$

Hence it follows that $\operatorname{grad} \bar{f}(x) = 0$ if and only if x and $\operatorname{grad} H_f(x)$ are collinear. In this case we have $\dot{\gamma}_t\, x = \frac{2\pi}{\tau}\, J\gamma_t\, x = \mu\, X_{H_f}(\gamma_t\, x)$ for some $\mu > 0$ and, consequently, the X_{H_f}-orbit starting at x can be reparametrized as an $\mathbb{S}^1_\tau$-orbit. Finally, by the homogeneity of $\operatorname{grad} H_f$, it follows that the map $\mathbb{S}^1_\tau \to \Sigma_f$ given by $t \mapsto e^{f(\operatorname{Orb} x)}\, \gamma_t\, x$ is a symmetric closed characteristic on Σ_f.

Thus the distinct symmetric closed characteristics on Σ_f correspond uniquely to the distinct $\mathbb{S}^1$-orbits $\operatorname{Orb} e^{\bar{f}(x)}\, x$ such that $\operatorname{grad} \bar{f}(x) = 0$. On the other hand, the associated distinct $\mathbb{S}^1$-orbits $\operatorname{Orb} x$ on $\mathbb{S}^{2n-1}$ represent precisely the distinct critical points of the function $f : \mathbb{C}P^{n-1} \to \mathbb{R}$.

Let now $\Sigma \subset \mathbb{R}^{2n}$ be any star-shaped $\mathbb{S}^1$-hypersurface. Setting $H(e^{s/2}\, x) := e^s$ for all $(x, s) \in \Sigma \times \mathbb{R}$, we define $f : \mathbb{C}P^{n-1} \to \mathbb{R}$ by $f(\operatorname{Orb} x) := -\frac{1}{2}\, \ln H(x)$ for all $x \in \mathbb{S}^{2n-1}$. Then we have $\Sigma_f = \Sigma$, and the proof of Theorem 1 is complete. □

For the proof of the corollary we use the above notation and we assume that $\operatorname{Orb} x$ is a critical point of f, i.e., $x \in \mathbb{S}^{2n-1}$ is a critical point of $\bar{f}$. Then for every $s \in \mathbb{R}$ the outward normal vector field at $e^{s/2} e^{\bar{f}(x)}\, x \in H_f^{-1}(e^s)$ satisfies

$$N(e^{s/2} e^{\bar{f}(x)}\, x) \;=\; x\,. \tag{78}$$

Using the symplectic dilation X_λ, the gradient of H_f at this point is determined by the formula (28). Thus we obtain the expression

$$\operatorname{grad} H_f(e^{s/2} e^{\bar{f}(x)}\, x) \;=\; \frac{2\, e^{s/2}}{e^{\bar{f}(x)}}\, x\,. \tag{79}$$

Now let the corresponding $\mathbb{S}^1$-orbit $u(t) := e^{s/2} e^{\bar{f}(x)} \exp\left(\frac{2\pi}{\tau(x)}\, tJ\right) x$ be parametrized as an integral curve of X_{H_f}, i.e., let $-J\dot{u} = \operatorname{grad} H_f(u)$. This is equivalent to

$$\frac{2\pi}{\tau(x)}\, e^{s/2} e^{\bar{f}(x)} x \;=\; \frac{2\, e^{s/2}}{e^{\bar{f}(x)}}\, x\,. \tag{80}$$

Hence the minimal period is determined by

$$\tau(x) \;=\; \pi\, e^{2\bar{f}(x)}\,, \tag{81}$$

such that the corollary follows with $\varrho = e^{s/2}$. □

9. Proof of Theorem 3

Suppose that $\Sigma \subset \mathbb{R}^{2n} \setminus \{0\}$ is a compact connected $\mathbb{S}^1$-hypersurface which carries a non-symmetric closed characteristic embedding $u : \mathbb{S}^1_\tau \to \Sigma$. We may consider u as a periodic X_H-orbit having the minimal period $\tau > 0$ where H is again the Hamiltonian function from (20). Then the period τ yields a natural choice for the parametrization of the $\mathbb{S}^1$-action γ, i.e., we shall assume $\gamma_t\, x = \exp\left(\frac{2\pi}{\tau}\, tJ\right) x$.

Denoting the flow of X_H by φ_t and setting $x_0 = u(0)$, we obtain the representation $u(t) = \varphi_t(x_0)$. Moreover, since X_H is $\mathbb{S}^1$-equivariant, it follows that

$\varphi_t(\gamma_s\, x_0) = \gamma_s\, \varphi_t(x_0)$ for all $s, t \in \mathbb{S}^1_\tau$. Thus the τ-periodic integral curve $u(t)$ of X_H gives rise to an entire family $\{\, \gamma_s\, \varphi_t(x_0) \mid s \in \mathbb{S}^1_\tau \,\}$ of periodic integral curves having the minimal period τ.

Two of these integral curves will be called equivalent if they describe the same geometric X_H-orbit on Σ, i.e., if they are related to each other by reparametrization. More precisely, we define $\varphi_t(\gamma_s\, x_0) \sim \varphi_t(\gamma_{s'}\, x_0)$ in case there exists $t' \in \mathbb{S}^1_\tau$ such that $\varphi_t(\gamma_{s'}\, x_0) = \varphi_{t+t'}(\gamma_s\, x_0)$. We show that, for u non-symmetric, an equivalence class consists of finitely many elements only. Hence there exist uncountably many distinct equivalence classes and, therefore, uncountably many geometrically distinct periodic X_H-orbits of the form $\gamma_s\, \varphi_t(x_0)$ as asserted.

Since u is supposed to be non-symmetric, the trajectory $t \mapsto \varphi_t(\gamma_s\, x_0)$ intersects the $\mathbb{S}^1$-orbit $\operatorname{Orb} x_0$ tranversally at $\gamma_s\, x_0$. Hence we infer

$$t_1 := \min\{\, t \in (0, \tau\,] \mid \varphi_t(\gamma_s\, x_0) \in \operatorname{Orb} x_0 \,\} \;>\; 0\,. \tag{82}$$

The $\mathbb{S}^1$-equivariance of the flow φ implies that t_1 is independent of the particular choice of s and hence we may fix $s = 0$ for the moment. Consequently, there exists $\vartheta \in (0, \tau]$ such that $\varphi_{t_1}(x_0) = \gamma_\vartheta\, x_0$. By induction it follows now that $\varphi_{mt_1}(x_0) = \gamma_{m\vartheta}\, x_0$ for every $m \in \mathbb{N}$.

We claim that $\tau = mt_1$ for some $m \in \mathbb{N}$. Else there exists a natural number m such that $(m-1)\, t_1 < \tau < mt_1 < \tau + t_1$. Setting $t_1' := mt_1 - \tau$, we get $0 < t_1' < t_1$ and

$$\varphi_{t_1'}(x_0) \;=\; \varphi_{mt_1}(\varphi_{-\tau}(x_0)) \;=\; \varphi_{mt_1}(x_0) \;=\; \gamma_{mt_1} x_0 \;\in\; \operatorname{Orb} x_0\,, \tag{83}$$

in contradiction to the minimality of t_1. Thus we have $mt_1 = \tau$ as claimed. Now we conclude

$$\gamma_{m\vartheta}\, x_0 \;=\; \varphi_{mt_1}(x_0) \;=\; \varphi_\tau(x_0) \;=\; x_0\,, \tag{84}$$

which implies $\vartheta = t_1 = \tau/m$. Recalling that x_0 may be replaced by $\gamma_s\, x_0$ for $s \in \mathbb{S}^1_\tau$, we finally infer that

$$\varphi_{t+t'}(\gamma_s\, x_0) \;=\; \varphi_t(\gamma_{s'}\, x_0) \tag{85}$$

holds true for $t', s'-s$ in the fundamental domain $[\,0, \tau)$ of $\mathbb{S}^1_\tau = \mathbb{R}/\tau\mathbb{Z}$ if and only if $t' = s - s' = kt_0$ for some $k = 0, \ldots, m-1$. In other words, the equivalence class of the integral curve $\varphi_t(\gamma_s\, x_0)$ is given by $\{\, \varphi_t(\gamma_{s+kt_0} x_0) \mid k = 0, \ldots, m-1 \,\}$. The proof is herewith complete. □

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