Bohmian Trajectories as the Foundation of Quantum Mechanics and Quantum Field Theory

Roderich Tumulka

Eberhard Karls University, Tübingen (Germany)

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Happy 100-th birthday, David Bohm!
In 1952, David Bohm solved the biggest of all problems in quantum mechanics,
which is to provide an explanation of quantum mechanics.
His theory is known as Bohmian mechanics, pilot-wave theory, de Broglie–Bohm theory, or the ontological interpretation.
This theory makes a proposal for how the our world might work.
It agrees with all empirical observations of quantum mechanics.
It is widely under-appreciated.
It achieves what was often (before and even after 1952) claimed impossible: To explain the rules of quantum mechanics through a coherent picture of microscopic reality.
It is remarkably simple and elegant.
It is probably the true theory of quantum reality.
Compared to Bohmian mechanics, orthodox quantum mechanics appears quite “unprofessional” (John Bell) and “incoherent” (Albert Einstein).

In fact, orthodox quantum mechanics appears like the narrative of a dream whose logic doesn’t make sense any more once you are awake although it seemed completely natural while you were dreaming.

According to Bohmian mechanics, electrons and other elementary particles are particles in the literal sense, i.e., they have a well-defined position $Q_j(t) \in \mathbb{R}^3$ at all times $t$. They have trajectories.

These trajectories are governed by Bohm’s equation of motion (next slide).

Given the claim that it was impossible to explain quantum mechanics, it is remarkable that something as simple as particle trajectories does the job.

What went wrong in orthodox QM? Some variables were left out of consideration: the particle positions!
Laws of Bohmian mechanics

1. Bohm’s equation of motion

\[ \frac{d Q_j}{dt} = \frac{\hbar}{m_j} \text{Im} \frac{\psi^* \nabla_j \psi}{\psi^\ast \psi} (Q_1, \ldots, Q_N) \]

2. The Schrödinger equation for \( \psi \),

\[ i\hbar \frac{\partial \psi}{\partial t} = -\sum_j \frac{\hbar^2}{2m_j} \nabla^2_j \psi + V \psi \]

3. The initial configuration \( Q(0) = (Q_1(0), \ldots, Q_N(0)) \) is random with probability density

\[ \rho = |\psi_0|^2. \]

It follows that at any time \( t \in \mathbb{R} \), \( Q(t) \) is random with density \( \rho_t = |\psi_t|^2 \) ("equivariance theorem").
“This idea seems to me so natural and simple, to resolve the wave–particle dilemma in such a clear and ordinary way, that it is a great mystery to me that it was so generally ignored.”

John S. Bell
(1928–1990)
Bohmian mechanics is clearly non-local.

Bohmian mechanics avoids the problematical idea that the world consists only of wave function.

It provides precision, clarity, and a clear ontology in space-time.

It allows for an analysis of quantum measurements, thus replacing postulates of orthodox QM by theorems.
Extensions of Bohmian mechanics

- Particle creation
- Relativistic space-time
Natural extension of Bohmian mechanics to particle creation:

\[ \Psi \in \text{Fock space } \mathcal{F} = \bigoplus_{n=0}^{\infty} \mathcal{H}_n, \]

configuration space of a variable number of particles

\[ Q = \bigcup_{n=0}^{\infty} \mathbb{R}^{3n} \]

jumps (e.g., \( n \)-sector \( \rightarrow \) \((n + 1)\)-sector) occur in a stochastic way, with rates governed by a further equation of the theory.
Jump rate formula

- Jump rate from $q'$ to $q \in Q$:

$$\sigma^\psi(q' \rightarrow q) = \max\left\{0, \frac{2}{\hbar} \text{Im} \langle \psi | P(q) H_I P(q') | \psi \rangle \right\}$$

- Here, $H_I = \text{interaction Hamiltonian}$, $H = H_0 + H_I$, and
- $P(q)$ the configuration operators
  - e.g., $P(q) = |q\rangle \langle q|$
  - or generally, a POVM (positive-operator-valued measure) on configuration space
- Between jumps, Bohm’s equation of motion applies.
- $|\psi|^2$ distribution $= \langle \psi | P(q) | \psi \rangle$ holds at every time $t$.

Essentially, if you have a Hilbert space $\mathcal{H}$, a state vector $\psi \in \mathcal{H}$, a Hamiltonian $H$, a configuration space $Q$, and configuration operators $P(q)$, then we know how to set up Bohmian trajectories $Q(t)$. 
An UV divergence problem

For example, consider a simplified model QFT:
- x-particles can emit and absorb y-particles.
- There is only 1 x-particle, and it is fixed at the origin. \( \mathcal{H} = \mathcal{F}_y^{\text{bosonic}} \)
- configuration space \( \mathcal{Q} = \bigcup_{n=0}^{\infty} \mathbb{R}^{3n} \)

Original Hamiltonian in the particle-position representation:

\[
(H_{\text{orig}} \psi)(y_1 \ldots y_n) = -\frac{\hbar^2}{2m_y} \sum_{j=1}^{n} \nabla^2_{y_j} \psi(y_1 \ldots y_n) \\
+ g \sqrt{n+1} \psi(y_1 \ldots y_n, 0) \\
+ \frac{g}{\sqrt{n}} \sum_{j=1}^{n} \delta^3(y_j) \psi(y_1 \ldots \hat{y}_j \ldots y_n),
\]

is UV divergent. (\( \hat{\;} \) = omit)
Well-defined, regularized version of $H$:

UV cut-off $\varphi \in L^2(\mathbb{R}^3)$:

$$(H_{\text{cutoff}} \psi)(y_1 \cdots y_n) = -\frac{\hbar^2}{2m_y} \sum_{j=1}^{n} \nabla^2_{y_j} \psi(y_1 \cdots y_n) +$$

$$+ g \sqrt{n+1} \sum_{i=1}^{m} \int_{\mathbb{R}^3} d^3y \varphi^*(y) \psi(y_1 \cdots y_n, y) +$$

$$+ \frac{g}{\sqrt{n}} \sum_{i=1}^{m} \sum_{j=1}^{n} \varphi(y_j) \psi(y_1 \cdots \hat{y}_j \cdots y_n)$$

“smearing out” the $x$-particle with “charge distribution” $\varphi(\cdot)$
But then . . .

... emission and absorption occurs anywhere in a ball around the \( x \)-particle (= in the support of \( \varphi \)).

This UV problem can be solved!

[Teufel and Tumulka 2015; Lampart, Schmidt, Teufel, and Tumulka 2017]
Novel idea: Interior-boundary condition

Here: boundary config = where y-particle meets x-particle; interior config = one y-particle removed

Interior–boundary condition (IBC)

\[ \psi^{(n+1)}(\text{bdy}) = (\text{const.}) \psi^{(n)} \]

links two configurations connected by the creation or annihilation of a particle. For example, with an x-particle at 0,

\[ \psi(y^n, 0) = \frac{g m_y}{2\pi \hbar^2 \sqrt{n + 1}} \psi(y^n). \]

with \( y^n = (y_1, \ldots, y_n) \).
Self-adjoint Hamiltonian, rigorously

- IBC \( \lim_{r \searrow 0} r \psi(y^n, r\omega) = \frac{g m_y}{2\pi \hbar^2 \sqrt{n + 1}} \psi(y^n) \) \hspace{1cm} (1)

- \( H_{IBC}\psi = H_{\text{free}}\psi + \frac{g \sqrt{n + 1}}{4\pi} \int_{S^2} d^2\omega \lim_{r \searrow 0} \frac{\partial}{\partial r} \left( r\psi(y^n, r\omega) \right) \)

  \( + \frac{g}{\sqrt{n}} \sum_{j=1}^{n} \delta^3(y_j) \psi(y^n \setminus y_j) \) \hspace{1cm} (2)

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**Theorem [Lampart, Schmidt, Teufel, Tumulka 2017]**

On a suitable dense domain \( \mathcal{D}_{IBC} \) of \( \psi \)s in \( \mathcal{H} \) satisfying the IBC (1), \( H_{IBC} \) is well defined, self-adjoint, and positive. No UV divergence!

**Bohmian particles:**

- when \( Q(t) \in Q^{(n)} \) reaches \( y_j = 0 \), it jumps to \( (y^n \setminus y_j) \in Q^{(n-1)} \)
- emission of new \( y \)-particle at \( 0 \) at random time with random direction with a rate dictated by time reversal invariance.
Extensions of Bohmian mechanics: Relativistic space-time
If a preferred foliation (= slicing) of space-time into spacelike hypersurfaces ("time foliation" $\mathcal{F}$) is permitted, then there is a simple, convincing analog of Bohmian mechanics, $\text{BM}_\mathcal{F}$.

[Bohm and Hiley 1993 for flat foliations; Dürr, Goldstein, Münch-Berndl, Zanghì 1999 for curved foliations; Tumulka 2001 for curved space-time]

Without a time foliation, no version of Bohmian mechanics is known that would make predictions anywhere near quantum mechanics. (And I have no hope that such a version can be found in the future.)
To grant a time foliation (\(\equiv\) preferred foliation) is against the spirit of relativity.

But it is a real possibility that our world is like that.

It doesn’t mean relativity would be irrelevant:

- There is still a metric \(g_{\mu\nu}\).
- The free Hamiltonian is still the Dirac operator.
- Formulas are still expressed with 4-vector indices \((j^\mu\ etc.)\),
- Just there is also the vector \(n_\mu\) normal to the time foliation.

Still no superluminal signaling.

The hypothesis of a time foliation provides a very simple explanation of the non-locality required by Bell’s theorem.
A preferred foliation may be provided anyhow by the metric:

**Simplest choice of time foliation** \( \mathcal{F} \)

Let \( \mathcal{F} \) be the level sets of the function

\[ T : \text{space-time} \rightarrow \mathbb{R}, \quad T(x) = \text{timelike-distance}(x, \text{big bang}). \]

E.g., \( T(\text{here-now}) = 13.7 \) billion years

Alternatively, \( \mathcal{F} \) might be defined in terms of the quantum state vector \( \psi \), \( \mathcal{F} = \mathcal{F}(\psi) \) [Dürr, Goldstein, Norsen, Struyve, Zanghì 2014]

Or, \( \mathcal{F} \) might be determined by an evolution law (possibly involving \( \psi \)) from an initial time leaf.
Consider $N$ particles. Suppose that, for every $\Sigma \in \mathcal{F}$, we have $\psi_\Sigma$ on $\Sigma^N$. $Q(\Sigma) = (Q_1 \cap \Sigma, \ldots, Q_N \cap \Sigma) =$ configuration on $\Sigma$.

Equation of motion:

$$\frac{dQ^\mu_k}{d\tau} = \text{expression} \left[ \psi(Q(\Sigma)) \right]$$

Example for $N$ Dirac particles

$\psi_\Sigma : \Sigma^N \rightarrow (\mathbb{C}^4)^\otimes N$. Equation of motion:

$$\frac{dQ^\mu_k}{d\tau} \propto j^\mu_k(Q(\Sigma)), $$

$$j^{\mu_1 \cdots \mu_N} = \overline{\psi} [\gamma^{\mu_1} \otimes \cdots \otimes \gamma^{\mu_N}] \psi, $$

$$j^\mu_k(q_1 \ldots q_N) = j^{\mu_1 \cdots \mu_N}(q_1 \ldots q_N) \, n_{\mu_1}(q_1) \cdots (k\text{-th omitted}) \cdots n_{\mu_N}(q_N) $$

with $n_\mu(x) =$ unit normal vector to $\Sigma$ at $x \in \Sigma$. 
Equivariance

Suppose initial configuration is $|\psi|^2$-distributed. Then the configuration of crossing points $Q(\Sigma) = (Q_1 \cap \Sigma, \ldots, Q_N \cap \Sigma)$ is $|\psi_\Sigma|^2$-distributed (in the appropriate sense) on every $\Sigma \in \mathcal{F}$.

Predictions

The detected configuration is $|\psi_\Sigma|^2$-distributed on every spacelike $\Sigma$.

As a consequence,

$\mathcal{F}$ is invisible, i.e., experimental results reveal no information about $\mathcal{F}$.

All empirical predictions of BM$_{\mathcal{F}}$

agree with the standard quantum formalism and the empirical facts.
Theorem [Lienert and Tumulka 2017]

If detectors are placed along any spacelike surface $\Sigma$, the joint distribution of detection events is $|\psi_\Sigma|^2$.

BM$_{\mathcal{F}}$ is very robust:

- works for arbitrary foliation $\mathcal{F}$
- works even if the foliation has kinks [Struyve and Tumulka 2014]
- works even if the leaves of $\mathcal{F}$ overlap [Struyve and Tumulka 2015]
- can be combined with the stochastic jumps for particle creation
- works also in curved space-time [Tumulka 2001]
- works still if space-time has singularities [Tumulka 2010]
Multi-time wave function $\phi(t_1, x_1, \ldots, t_N, x_N)$

as a generalization of the $N$-particle wave function $\psi(t, x_1, \ldots, x_N)$ of non-relativistic quantum mechanics:

$$\psi(t, x_1, \ldots, x_N) = \phi(t, x_1, \ldots, t, x_N)$$

$\psi_{\Sigma}(x_1, \ldots, x_N) = \phi(x_1, \ldots, x_N)$.

Intended: if detectors along $\Sigma$ then prob distribution of outcomes $= |\psi_{\Sigma}|^2$

$$i \frac{\partial \psi}{\partial t} = H \psi \quad \text{ vs. } \quad i \frac{\partial \phi}{\partial t_i} = H_i \phi \quad \forall i = 1, \ldots, N$$

$$\sum_{i=1}^{N} H_i = H$$

It’s the covariant particle-position representation of the state vector. Closely related to the Tomonaga-Schwinger wave function, but simpler.
Multi-time wave function $\phi(t_1, x_1, \ldots, t_N, x_N)$

Consistency condition

$$\left[ i \frac{\partial}{\partial t_i} - H_i, i \frac{\partial}{\partial t_j} - H_j \right] = 0 \quad \forall i \neq j$$

[Dirac, Fock, Podolsky 1932; F. Bloch 1934]

- trivially satisfied for non-interacting particles
- interaction is a challenge, potentials violate consistency
- zero-range interactions possess consistent multi-time equations [Lienert 2015]
- interaction through emission and absorption of bosons possesses consistent multi-time equations [Petrat and Tumulka 2014]
Thank you for your attention