Introduction to Combinatorial Birational Geometry

Algebraic geometry studies solution sets to systems of algebraic equations in many variables and considers them as geometric objects called *algebraic varieties.* It turns out that the best understanding of algebraic varieties arises only after they are reduced by means of certain transformations to the most convenient geometric models for their study. Our goal is to familiarize students with the most accessible transformations of algebraic varieties which are called birational. Using birational transformations one comes to definition of birational equivalence classes of algebraic varieties, and one of the main tasks of birational geometry is to identify "the best" representatives which are called *minimal models.* In general, finding minimal birational models is a rather difficult task. But in these lectures we will restrict ourselves to the case of *non-degenerate toric hypersurfaces.* For these it turns out to be possible to completely solve this problem in arbitrary dimension using methods of combinatorial birational geometry based on the elementary convex geometry of *Newton polyhedra* of toric hypersurfaces. To maintain maximum accessibility of the presented methods of combinatorial birational geometry, we will devote the bulk of the lectures to the classical case of algebraic curves on algebraic surfaces.

Content

- Basic notions from algebraic geometry: affine and projective spaces, Zariski topology, dimension of algebraic varieties. The field of rational functions of an irreducible variety.

- The language of commutative algebra. The local ring of a subvariety. Unique factorisation domains. Normal varieties. The divisor class group. Examples of explicit computations.

- Smooth projective toric surfaces $\mathbb{P}_\Delta$ associated with lattice polytopes Δ. The moment map for surfaces over $\mathbb{C}$. The rational equivalence on boundary divisors. The divisor class group $Cl(X)$ of a toric surface X, the bilinear intersection pairing on $Cl(X)$.

- Cartier divisors, line bundles, invertible sheaves. The canonical divisor of a normal variety. Computations for toric surfaces. Ample and very ample divisors.

- Nondegenerate algebraic curves in toric surfaces. Blow ups of toric surfaces. Birational modification of surfaces via blow ups and blow downs.

- The cone of curves of a surface. The Zariski decomposition. Birational Cremona transformations.

- Desingularization of nondegenerate curves D on smooth toric surfaces X via blow ups. Combinatorial constructing minimal models of pairs (X, D) for normal toric surfaces X.

- Cyclic quotients surface singularities and their combinatorial minimal desingularization.

- Finite subgroups of $SU(2)$ and Du Val singularities and their minimal desingularization.

- Birational classification of nondegenerate surfaces in 3-dimensional toric varieties via the Fine interior $F(\Delta)$ of their Newton polytopes Δ.

- The Kodaira dimension of algebraic varieties. Combinatorial constructing minimal models of nondegenerate surfaces.

- Combinatorial formulas for the Hodge numbers of minimal models.

Literature

- Igor R. Schafarewitsch: Grundzüge der algebraischen Geometrie. Springer 1972.

- Klaus Hulek: Elementare Algebraische Geometrie. Springer 2012.

- David Mumford: Lectures on curves on an algebraic surface. Princeton 1966.

- Robin Hartshorne: Algebraic Geometry. Springer 1977.

- Tadao Oda: Convex Bodies and Algebraic Geometry. Springer 1988.