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A GEOMETRIC BOUNDARY VALUE PROBLEM RELATED TO THE STATIC VACUUM EQUATIONS IN GENERAL RELATIVITY

The Schwarzschild spacetime is one of if not *the* most important example of a spacetime in Mathematical General Relativity. It describes the static, vacuum exterior region of a spherically symmetric, isolated star or black hole. The $n+1$-dimensional Schwarzschild spacetime of mass $m \in \mathbb{R}$ is given by

$$\mathfrak{g} := -u_m^2 dt^2 + \frac{1}{u^2} dr^2 + r^2 \, d\Omega^2, \tag{1}$$

$$u_m := u_m(r) := \sqrt{1 - \frac{2m}{r^{n-2}}} \tag{2}$$

on the spacetime manifold $\mathbb{R} \times (r_m, \infty) \times \mathbb{S}^{n-1}$, where $d\Omega^2$ denotes the canonical metric on $\mathbb{S}^{n-1}$ and $r_m := 0$ for $m \leq 0$ and $r_m := (2m)^{\frac{1}{n-2}}$ for $m > 0$. This definition applies whenever $n \geq 3$.

The Schwarzschild spacetime is known to be rigid in various ways:

- *Birkhoff's theorem* [1] asserts that the Schwarzschild spacetime is the only spherically symmetric Lorentzian spacetime $(\mathcal{L}^{n+1}, \mathfrak{g})$ which solves the vacuum Einstein equations $\mathfrak{Ric} = 0$.
- The *static vacuum black hole uniqueness theorem* asserts that the Schwarzschild spacetime is the only asymptotically flat spacetime $(\mathcal{L}^{n+1}, \mathfrak{g})$ with "black hole inner boundary" which is "static" and solves the vacuum Einstein equations $\mathfrak{Ric} = 0$. Here, being *static* means that $(\mathcal{L}^{n+1}, \mathfrak{g}) = (\mathbb{R} \times M^n, \mathfrak{g} = -u^2 dt^2 + g)$, where (M^n, g) is an asymptotically Euclidean Riemannian manifold and $u \colon M^n \to \mathbb{R}^+$ is a function with $u \to 1$ near infinity. Static vacuum black hole uniqueness was proved by many authors under a variety of assumptions, in particular by Bunting and Masood-ul-Alam [2] for $n = 3$, using a very elegant method. Gibbons, Ida, and Shiromizu [6] and Hwang [7] generalized this method to $n \geq 3$ for spin manifolds. In this context, the definition of a *black hole inner boundary* is that ∂M consists of finitely many compact components with vanishing mean curvature, $H = 0$, such that $u = 0$ on ∂M, and such that the normal derivative $\nu(u)$ has a sign on ∂M.
- Analogously, the *static vacuum photon sphere uniqueness theorem* asserts that the Schwarzschild spacetime is the only asymptotically flat spacetime $(\mathcal{L}^{n+1}, \mathfrak{g})$ with "photon sphere inner boundary" which is static and solves the vacuum Einstein equations $\mathfrak{Ric} = 0$. Here, a *photon sphere inner boundary* is defined as a timelike umbilic hypersurface $\mathcal{P}^n \hookrightarrow (\mathcal{L}^{n+1}, \mathfrak{g})$ on which $u \equiv \text{const}$, see [4]. Static vacuum photon sphere uniqueness was proved by the author and Galloway [5] for $n = 3$, relying on the method suggested by Bunting and Masood-ul-Alam [2].

The goal of this talk was to show that the Schwarzschild spacetime is indeed rigid in a much more general way [3]. Before we discuss the main rigidity theorem, let us briefly recall the symmetry reduced Einstein vacuum equation for static spacetimes $(\mathbb{R}^n \times M^n, \mathfrak{g} = -u^2 dt^2 + g)$, the so-called *static vacuum equations*

$$u \operatorname{Ric} = \nabla^2 u, \tag{3}$$

$$\triangle u = 0 \tag{4}$$

on M^n which follow directly from plugging the special form of $\mathfrak{g}$ into the vacuum Einstein equations $\mathfrak{Ric} = 0$. Here, Ric denotes the Ricci tensor of g. A straightforward consequence obtained by tracing (3) is that the scalar curvature of (M^n, g) vanishes, which we denote as $\mathrm{R} = 0$. These equations are used in [2, 6, 7, 4, 5].

If not discussing vacuum but matter models with non-negative energy density, one finds $\mathrm{R} \geq 0$ — at least in the so-called "Riemannian" case. This condition is related to the *dominant energy condition* in General Relativity. The rigidity theorem we prove does not assume (3) and neither $\mathrm{R} = 0$, only (4) and $\mathrm{R} \geq 0$.

Theorem 1 (Rigidity of Schwarzschild manifold)**.** *Assume $n \geq 3$ and let M^n be a smooth, connected, n-dimensional manifold with non-empty, possibly disconnected, smooth, compact inner boundary $\partial M = \dot{\bigcup}_{i=1}^{I} \Sigma_i^{n-1}$. Let g be a smooth Riemannian metric on M^n. Assume that (M^n, g) has non-negative scalar curvature $\mathrm{R} \geq 0$ and that it is geodesically complete up to its inner boundary ∂M. Assume in addition that (M^n, g) is asymptotically isotropic with one end of mass $m \in \mathbb{R}$.*

Furthermore, assume that the inner boundary ∂M is umbilic in (M^n, g), and that each component Σ_i^{n-1} has constant mean curvature H_i with respect to the outward pointing unit normal ν_i. Assume that there exists a function $u \colon M^n \to \mathbb{R}$ with $u > 0$ away from ∂M which is smooth and harmonic on (M^n, g), so that $\triangle u = 0$. We ask that u is such that $u|_{\Sigma_i^{n-1}} \equiv\colon u_i$ is constant on each Σ_i^{n-1} and u is asymptotically isotropic of the same mass m.

Finally, we assume that for each $i = 1, \ldots, I$, we are either in the semi-static horizon case

$$H_i = 0, \quad u_i = 0, \quad \nu_i(u) \neq 0, \tag{5}$$

or we are in the true CMC case *with $H_i > 0$, $u_i > 0$, and such that there exist constants $c_i > \frac{n-2}{n-1}$ so that*

$$\mathrm{R}_{\sigma_i} = c_i H_i^2, \tag{6}$$

$$2\nu(u)_i = \left(c_i - \frac{n-2}{n-1}\right) H_i u_i, \tag{7}$$

where R_{σ_i} denotes the scalar curvature of Σ_i^{n-1} with respect to its induced metric σ_i and $\nu_i(u)|_{\Sigma_i^{n-1}} \equiv\colon \nu(u)_i$ denotes the normal derivative of u.

Then $m > 0$ and (M^n, g) is isometric to a suitable portion of the spatial Schwarzschild manifold of mass m $((r_m, \infty) \times \mathbb{S}^{n-1}, g_m := \frac{1}{u^2} dr^2 + r^2 \, d\Omega^2)$. Moreover, u coincides with the restriction of u_m (up to the isometry).

In Theorem 1, the asymptotic isotropy conditions are defined as follows:

Definition 2. *We say that* (M^n, g) *is* asymptotically isotropic with one end of mass $m \in \mathbb{R}$, *if* M^n *is diffeomorphic to* $\mathbb{R}^n \setminus$ *ball outside a compact set, and with respect to the coordinates* (y^i) *induced by this diffeomorphism, we have*

$$g_{ij} = \left(1 + \frac{m}{2s^{n-2}}\right)^{\frac{4}{n-2}} \delta_{ij} + \mathcal{O}_2(\frac{1}{s^{n-1}}) \tag{8}$$

as $s := \sqrt{(y^1)^2 + \cdots + (y^n)^2} \to \infty$. *We say that a function* $u\colon M^n \to \mathbb{R}$ *is* asymptotically isotropic of mass m *if, with respect to the same diffeomorphism and coordinates described above, we have*

$$u = 1 - \frac{m}{s^{n-2}} + \mathcal{O}_2(\frac{1}{s^{n-1}}) \tag{9}$$

as $s \to \infty$.

We remark that Theorem 1 recovers the static vacuum black hole uniqueness theorem in all dimensions (and dropping the spin assumption in [7, 6] and recovers and generalizes the static vacuum photon sphere uniqueness theorem to all dimenions.

Sketch of Proof of Theorem 1. In the talk, we gave a sketch of the proof of Theorem 1 by pictures. For more details, please see [3].

The **first step** is to extend (M^n, g) across each true CMC inner boundary component Σ_i^{n-1} by gluing a suitable, explicitly constructed Riemannian manifold (M_i^n, g_i) into (M^n, g) across Σ_i^{n-1} in a $C^{1,1}$ fashion. The glue-in manifolds (M_i^n, g_i) are constructed such that they give rise to new inner boundary components which are totally geodesic semi-static horizons, i.e. satisfy (5). Also by construction, (M_i^n, g_i) has vanishing scalar curvature. We will also extend the harmonic function u by gluing it to a (positive multiple of a) g_i-harmonic function $u_i\colon M_i^n \to \mathbb{R}$ with $C^{1,1}$-regularity across the gluing surface Σ_i^{n-1}, in a manner that $u_i > 0$ away from the new horizon boundary. This is possible because of the constraint conditions (6), (7). The described argument reduces Theorem 1 to the case where there are only semi-static horizon boundary components, see also Figure 1.

The glue-in manifolds (M_i^n, g_i) are defined as

$$g_i := \frac{1}{u_i(r)^2} dr^2 + \frac{r^2}{r_i^2}\sigma_i, \tag{10}$$

$$u_i(r) := \sqrt{1 - \frac{2\mu_i}{r^{n-2}}}, \tag{11}$$

on $M_i^n := ((2\mu_i)^{\frac{1}{n-2}}, r_i) \times \Sigma_i^{n-1}$, where r_i is the scalar curvature radius of $(\Sigma_i^{n-1}, \sigma_i)$ given by $\mathrm{R}_{\sigma_i} =: \frac{(n-2)(n-1)}{r_i^2}$ and $\mu_i > 0$ is a suitably chosen mass. This glue-in strategy generalizes that used in [5] to higher dimensions and possibly non-round inner boundary $(\Sigma_i^{n-1}, \sigma_i)$. See [3] for more properties of the manifolds (M_i^n, g_i).

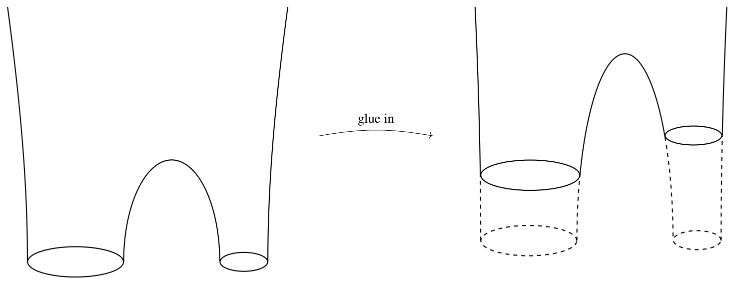

FIGURE 1. Gluing in a suitable, explicitly constructed Riemannian manifold (M_i^n, g_i) into each inner boundary component Σ_i^{n-1}. The new boundary components are totally geodesic semi-static horizons.

As a **second step**, we adapt [2, 7, 6] and double the extended manifold constructed above across its umbilic, semi-static horizon boundary (again with $C^{1,1}$-regularity across the doubling surfaces) to obtain a new Riemannian manifold $(\widetilde{M}^n, \widetilde{g})$ which is geodesically complete and has two asymptotically isotropic ends of the same ADM-mass m as (M^n, g), see Figure 2. We denote the original part $M^n \subset \widetilde{M}^n$ as $\widetilde{M}^+$ and the new copy as $\widetilde{M}^-$. At the same time, we extend the function u to $\widetilde{M}^n$ by

$$\widetilde{u} : \widetilde{M}^n \to \mathbb{R} \colon p \mapsto \begin{cases} u(p) & \text{if } p \in \widetilde{M}^+ \\ -u(p) & \text{if } p \in \widetilde{M}^- \end{cases} \tag{12}$$

and observe that $\widetilde{u}$ is smooth away from the gluing surfaces and $C^{1,1}$ across the gluing surfaces. Also, $\widetilde{u}$ is harmonic with respect to $\widetilde{g}$, $\widetilde{u}(\widetilde{M}^n) = (-1, 1)$, and $\pm\widetilde{u} \to 1$ as $r \to \infty$ in $\widetilde{M}^\pm$ is also asymptotically isotropic of mass m. This doubling construction first employed by Bunting and Masood-ul-Alam [2] works even though we do not assume the static vacuum equations (3), (4).

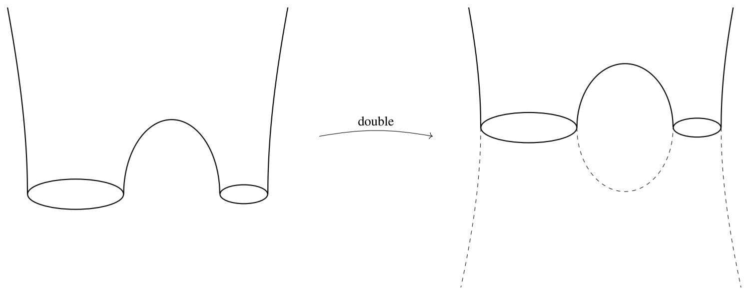

FIGURE 2. Doubling the extended manifold to a geodesically complete one across the totally geodesic boundary.

The **third step** consists in performing the conformal transformation and one point insertion method from [2, 7, 6] and ensuring that it makes no use of (3). More precisely, we conformally transform $(\widetilde{M}^n, \widetilde{g})$ to $\widehat{M}^n := \widetilde{M}^n$ via

$$\widehat{g} := \left(\frac{1+\widetilde{u}}{2}\right)^{-\frac{4}{n-2}} \widetilde{g}. \tag{13}$$

Exploiting that $\widetilde{u}$ is harmonic with respect to $\widetilde{g}$ and the fact that we chose the magical Yamabe power, we find that $\widehat{\mathrm{R}} \geq 0$. Under this conformal transform, the original asymptotically isotropic end $\widetilde{M}^+$ then transforms into an asymptotically isotropic end $\widehat{M}^n$ of vanishing ADM-mass $\widehat{m} = 0$, see Figure 3. The asymptotics of $\widetilde{u}$ and $\widetilde{g}$ of the doubled end allows to insert a point p_∞ in a $C^{1,1}$ fashion so that we obtain a geodesically complete manifold $(\widehat{M}^n_\infty := \widehat{M}^n \cup \{p_\infty\}, \widehat{g}_\infty)$. This manifold satisfies the assumptions of the rigidity case of the positive mass theorem [9, 10], except the regularity assumptions across the finitely many gluing hypersurfaces and the point p_∞. To remedy this problem, we appeal to McFeron and Székelyhidi [8]. This shows that $(\widehat{M}^n_\infty, \widehat{g}_\infty)$ is globally isometric to Euclidean space.

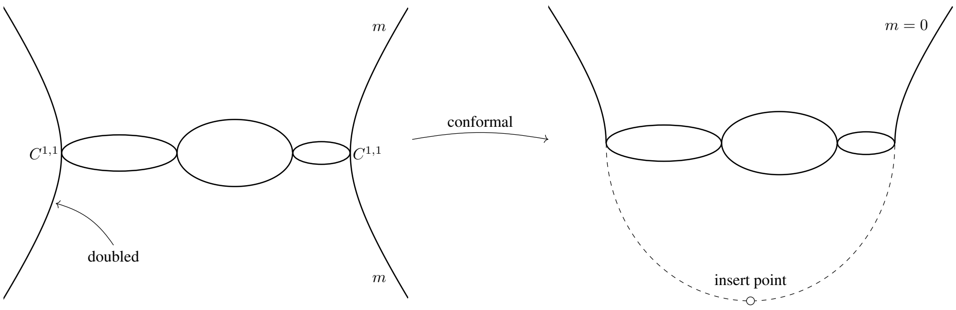

FIGURE 3. Conformal transformation and one-point insertion to a geodesically complete Riemannian manifold with vanishing ADM-mass and non-negative scalar curvature.

In order to conclude that (M^n, g) must have been isometric to a portion of Schwarzschild $(\widetilde{M}^n_m, \widetilde{g}_m)$, we proceed as follows: First, recall that each boundary component $\Sigma_i^{n-1} \hookrightarrow (M^n, g)$ is closed and umbilic. As g is conformally equivalent to $\widehat{g}$ and $\widehat{g}$ is isometric to δ, we find that the image of $\Sigma_i^{n-1} \hookrightarrow (\mathbb{R}^n, \delta)$ is a closed, totally umbilic hypersurface and thus necessarily a round sphere and thus in particular a topological sphere by standard arguments. Second, we know that $\widehat{M}^n_\infty$ is diffeomorphic to $\mathbb{R}^n$ and thus $\widehat{M}^n$ diffeomorphic to $\mathbb{R}^n \setminus \{0\}$. From topological considerations, this shows that the boundary ∂M must have been connected.

Now, let us consider the picture in $(\mathbb{R}^n, \delta)$: A standard computation shows that the conformal factor $\varphi := \left(\frac{1+\widetilde{u}}{2}\right)^{-1}$ is harmonic with respect to $\widehat{g}$ and thus with respect to δ outside the round sphere image of Σ^{n-1}. The boundary value of φ on the round sphere image is a constant by construction, and φ tends to 1 near infinity. Thus by the maximum principle and standard facts on Green's functions, we find that φ is the conformal factor of Schwarzschild of mass m. Because of the boundary data assumptions (5), (6), (7), respectively, $m > 0$.

This finishes the sketch of the proof of Theorem 1.

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