

SECTIONAL CURVATURE-TYPE CONDITIONS ON METRIC SPACES

MARTIN KELL

ABSTRACT. In the first part Busemann concavity as non-negative curvature is introduced and a bi-Lipschitz splitting theorem is shown. Furthermore, if the Hausdorff measure of a Busemann concave space is non-trivial then the space is doubling and satisfies a Poincaré condition and the measure contraction property. Using a comparison geometry variant for general lower curvature bounds $k \in \mathbb{R}$, a Bonnet-Myers theorem can be proven for spaces with lower curvature bound $k > 0$.

In the second part the notion of uniform smoothness known from the theory of Banach spaces is applied to metric spaces. It is shown that Busemann functions are (quasi-)convex. This implies the existence of a weak soul. In the end properties are developed to further dissect the soul.

In order to understand the influence of curvature on the geometry of a space it helps to develop a synthetic notion. Via comparison geometry sectional curvature bounds can be obtained by demanding that triangles are thinner or fatter than the corresponding comparison triangles. The two classes are called $CAT(\kappa)$ - and resp. $CBB(\kappa)$ -spaces. We refer to the book [BH99] and the forthcoming book [AKP] (see also [BGP92, Ots97]). Note that all those notions imply a Riemannian character of the metric space. In particular, the angle between two geodesics starting at a common point is well-defined. Busemann investigated a weaker notion of non-positive curvature which also applies to normed spaces [Bus55, Section 36]. A similar idea was developed by Pedersen [Ped52] (see also [Bus55, (36.15)]). Pedersen's condition is better suited for the study of Hilbert geometries, see [KS58]. In [Oht07a] Ohta studied an even weaker convexity notion, called L -convexity which can be seen as a relaxed form of Busemann's non-positive curvature assumption.

In the recent years a synthetic notion of a lower bound on the Ricci curvature was defined by Lott–Villani [LV09] and Sturm [Stu06]. Surprisingly, their condition includes also Finsler manifolds [Oht09, Oht13]. The notion of lower curvature bound in the sense of Alexandrov, i.e. $CBB(\kappa)$ -spaces, is compatible with this Ricci bound [Pet11, GKO13]. However, by now there is no known sectional curvature analogue for Finsler manifolds which is compatible to Ohta's Ricci curvature bounds and thus the synthetic Ricci bounds.

In this note we present two approaches towards a sectional curvature-type condition. The first is the “converse” of Busemann's non-positive curvature condition. This condition implies a bi-Lipschitz splitting theorem, uniqueness of tangent cones and if the space admits a non-trivial Hausdorff measure then it satisfies doubling

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and Poincaré conditions, and even the measure contraction property. This approach rather focuses on the generalized angles formed by two geodesics. Using ideas from comparison geometry one can easily define general lower curvature bounds and prove a Bonnet-Myers theorem if the lower bound is positive.

The second approach can be seen as a dual to the theory of uniformly convex metric spaces which were studied in [Oht07a, Kuw13, Kel14]. We call this condition uniform smoothness. This rather weak condition is only powerful in the large as any compact Finsler manifold is 2-uniformly smooth, see [Oht08, Corollary 4.4]. Nevertheless, if the space is unbounded then Busemann functions associated to rays are (quasi-)convex and the space has a weak soul. In order to match the theory in the smooth setting we try to develop further assumptions which imply existence of a retractions onto the soul and a more local curvature assumption in terms of Gromov's characterization of non-negative curvature [Gro91].

1. PRELIMINARIES

Throughout this manuscript let (X, d) be a *proper geodesic metric space*, i.e. (X, d) is a complete metric space such that every bounded closed subset is compact and for every $x, y \in X$ there is a continuous map $\gamma : [0, 1] \rightarrow X$ with $\gamma(0) = x$, $\gamma(1) = y$ and

$$d(\gamma(t), \gamma(s)) = |t - s|d(x, y).$$

The map γ will be called a (constant speed) *geodesic* connecting x and y which is parametrized by $[0, 1]$. We say that a continuous curve $\gamma : I \rightarrow X$ defined on some interval $I \subset \mathbb{R}$ is *locally geodesic* if for all $t \in I$ there is an interval $I_t \subset I$ with $t \in \text{int } I_t$ such that γ restricted to I_t is a constant speed geodesic.

We say that (X, d) is *non-branching* if for all geodesics $\gamma, \eta : [0, 1] \rightarrow X$ with $\gamma|_{[0,t]} = \eta|_{[0,t]}$ for some $t \in (0, 1)$ implies $\gamma \equiv \eta$. Note that this is equivalent to the following: for all geodesics $\gamma, \eta : [0, 1] \rightarrow X$ with $\gamma(0) = \eta(0)$ and $\gamma(t) = \eta(t)$ for some $t \in (0, 1)$ it holds $\gamma(t) = \eta(t)$ for all $t \in [0, 1]$. In other words two geodesics starting at the same point and intersecting in the middle must agree.

The latter condition is easily seen to be stronger than non-branching. To prove the equivalence of the two conditions note that whenever $\gamma(0) = \eta(0)$ and $\gamma(t) = \eta(t)$ for some geodesics γ and η and some $t \in (0, 1)$ then we may construct a new curve $\tilde{\eta}$ by first traversing γ on $[0, t]$ and then traversing η on $[t, 1]$. The resulting curve $\tilde{\eta}$ is a geodesic that agrees with γ on $[0, t]$ so that non-branching implies $\eta|_{[t,1]} = \tilde{\eta}|_{[t,1]} = \gamma|_{[t,1]}$. Reversing the direction of γ and η also shows that γ and η agree on $[0, t]$.

1.1. Notions of convexity. A function $f : X \rightarrow \mathbb{R}$ is said to be (geodesically) *convex* if $t \mapsto f(\gamma(t))$ is convex for all geodesics $\gamma : [0, 1] \rightarrow X$, i.e.

$$f(\gamma(t)) \leq (1 - t)f(\gamma(0)) + tf(\gamma(1)).$$

We say that f is *p-convex* if f^p is convex. Furthermore, f is *concave* if $-f$ is convex, and if f is both convex and concave then it is said to be *affine*. The limiting notion $p \rightarrow \infty$ is usually called quasi-convex. More precisely, we say that a function $f : X \rightarrow \mathbb{R}$ is said to be *quasi-convex* if

$$f(\gamma(t)) \leq \max\{f(\gamma(0)), f(\gamma(1))\}.$$

It is said to be *strictly quasi-convex* if the inequality is strict whenever $\gamma(0) \neq \gamma(1)$. Furthermore, we say f is *properly quasi-convex* if the inequality is strict whenever

f restricted to γ is non-constant. Note that any p -convex function is automatically properly quasi-convex. As above *quasi-concavity* of f is just quasi-convexity of $-f$, and f is said to be *monotone* if it is both quasi-convex and quasi-concave.

One may obtain a variety of quasi-convex functions as follows: Let $h : \mathbb{R} \rightarrow \mathbb{R}$ be a non-decreasing function. Then $h \circ f$ is quasi-convex for any quasi-convex function f . Furthermore, if h is strictly increasing then $h \circ f$ is strictly quasi-convex if f is strictly quasi-convex.

Remark. In [BP83] it is suggested to use the terminology *peaklessness* for proper quasi-convexity and *weak peaklessness* for quasi-convexity. However, this terminology does not seem to be frequently used in the literature. Because we only obtain quasi-convex functions we stay with the better known term of quasi-convexity.

A subset C of X is said to be *convex* if for all geodesics $\gamma : [0, 1] \rightarrow X$ with $\gamma(0), \gamma(1) \in C$ it holds $\gamma(t) \in C$ for $t \in (0, 1)$. If, in addition, $\gamma(t) \in \text{int } C$ whenever γ is non-constant and $t \in (0, 1)$ then C is said to be *strictly convex*. A stronger notion, called *totally geodesic*, is obtained by requiring that C also contains all local geodesics, i.e. if $\gamma : [a, b] \rightarrow X$ is locally geodesic with $\gamma(a), \gamma(b) \in C$ then $\gamma(t) \in C$ for $t \in (a, b)$.

As is well-known a function $f : X \rightarrow \mathbb{R}$ is (strictly) convex if the epigraph $\{(x, t) \in X \times \mathbb{R} \mid t \geq f(x)\}$ is (strictly) convex in the product space $(X \times \mathbb{R}, \tilde{d})$ where $\tilde{d}((x, t), (y, s)) = \sqrt{d(x, y)^2 + |t - s|^2}$. In a similar way f is quasi-convex if the sublevel $C_s = f^{-1}((-\infty, s])$ are totally geodesic. Furthermore, if f is strictly convex then each C_s is strictly convex.

In general strict convexity of the sublevels of f is not related to strict quasi-convexity of f . However, one can always construct a quasi-convex function out of an exhaustive non-decreasing family of closed convex sets as follows. Under some additional assumptions on the family the function is also strictly quasi-convex if the sublevels are strictly convex. In the following for two closed subsets $A \subset B$ of X denote by $\text{int}_B A$ the relative interior of A in the metric space (B, d) .

Lemma 1.1. *Assume $(C_s)_{s \in I}$, $I \subset \mathbb{R} \cup \{-\infty\}$, is a non-decreasing family of closed convex sets with $\cup_{s \in I} C_s = X$. Then the function $f : X \rightarrow \mathbb{R}$ defined by*

$$f(x) = \inf\{\arctan s \in \mathbb{R} \mid x \in C_s\}$$

is quasi-convex. If, in addition, I is connected and the following properties hold:

(1) $\bigcap_{s \in I} C_s$ contains at most one element,

$$(2) \quad \bigcap_{s' > s} C_{s'} = C_s$$

and

$$(3) \quad \bigcup_{s' < s} C_{s'} = \text{int } C_s,$$

then f is continuous and strictly quasi-convex iff each C_s is strictly convex.

Proof. Choose distinct points $x, y \in X$ and let m be a midpoint of x and y . By exchanging x and y if necessary we may also assume $f(x) \leq f(y)$. If $f(y) < a$ then $x, y \in C_{\tan a}$. By convexity of $C_{\tan a}$ we must have $m \in C_{\tan a}$ and hence $f(m) < a$. This shows $f(m) \leq f(y) = \max\{f(x), f(y)\}$.

Assume $(C_s)_{s \in I}$ satisfies properties (1), (2) and (3). We make the following observations: As x and y are distinct, property (1) shows $f(y) > -\frac{\pi}{2}$. Furthermore,

property (2) shows $-\frac{\pi}{2} < f(z) \leq \arctan s$ if and only if $z \in C_s$. Thus $x, y \in C_{\tan f(y)}$.

If f strictly quasi-convex then $f(\gamma(t)) < f(y)$ for any geodesic γ connecting x any y . Thus by property (3) we see $\gamma(t) \in \text{int } C_{\tan f(y)}$. Since $C_{\tan f(y)} \subset C_s$ whenever $x, y \in C_s$ we must have $\gamma(t) \in \text{int } C_s$ implying strict convexity of C_s .

We show the converse: Assume each C_s is strictly convex. Then any midpoint m of x and y must be in the interior of $C_{\tan f(y)}$. Property (3) implies that there is an $s < \tan f(y)$ such that $m \in C_s$. Thus $f(m) \leq s < f(y)$ proving strict quasi-convexity of f .

It remains to show that f is continuous: Let $x_n \rightarrow x$. Set $t_1 = \liminf_{n \rightarrow \infty} f(x_n)$ and $t_2 = \limsup_{n \rightarrow \infty} f(x_n)$. Assume by contradiction $f(x) < t_1$ or $t_2 < f(x)$. In the first case $t_1 > -\frac{\pi}{2}$ and $x \in \partial C_{\tan t_1}$ by property (2). However, x must be in the interior of $C_{\tan t_1}$ by property (3) which shows $f(x) \leq t_1$. If $t_2 < f(x)$ then $x \notin C_{\tan t}$ for all $t \in (t_2, f(x))$ by definition of f . Note $\tan t \in I$ by connectedness of I . If, by contradiction, $f(x_n) < f(x) - \epsilon < f(x)$ then property (3) shows $x \in C_{\tan(f(x)-\epsilon)}$ which is not possible. Hence $f(x) \leq t_2$. In conclusion we see that $f(x) = t_1 = t_2$, i.e. f is continuous. \square

Note that all properties above are necessary to show that f is strictly quasi-convex. Indeed, the following family of strictly convex closed intervals in \mathbb{R} given by

$$C_s^1 = \begin{cases} [-s, s] & s < 1 \\ [-(s+1), s+1] & s \geq 1 \end{cases}$$

satisfies properties (1) and (2) but not property (3). Similarly,

$$C_s^2 = \begin{cases} [-s, s] & s \leq 1 \\ [-(s+1), s+1] & s > 1 \end{cases}$$

satisfies properties (1) and (3) but property (2). Finally, let $\phi : \mathbb{R} \rightarrow (0, \infty)$ be an increasing homeomorphism. Then

$$C_s^3 = (-\infty, \phi(s)]$$

satisfies property (2) and (3) but not property (1). In either case the induced functions f_i , $i = 1, 2, 3$, are constant on some open interval and cannot be strictly quasi-convex. In particular, they are not strictly quasi-convex. One may replace the intervals by geodesic balls resp. horoballs of the same radius if the metric space has strictly convex geodesic balls and resp. strictly convex horoballs to get more general examples.

1.2. Busemann functions. A central tool to study the structure of spaces with certain generalized curvature bounds are Busemann function associated to rays. Here a ray $\gamma : [0, \infty) \rightarrow X$ is an isometric embedding of the half line, i.e.

$$d(\gamma(t), \gamma(s)) = |t - s| \quad s, t \geq 0.$$

Definition 1.2 (Busemann function). Given a ray $\gamma : [0, \infty) \rightarrow X$ we define the *Busemann function* $b_\gamma : X \rightarrow \mathbb{R}$ as follows

$$b_\gamma(x) = \lim_{t \rightarrow \infty} t - d(\gamma(t), x).$$

Note that the right hand side is non-decreasing in t so that the limit is well-defined.

In case $\gamma : \mathbb{R} \rightarrow X$ is a line we define $\gamma^\pm : [0, \infty) \rightarrow X$ by $\gamma^\pm(t) = \gamma(\pm t)$. One can show that

$$b_{\gamma^+} + b_{\gamma^-} \leq 0.$$

We say a ray $\eta : [0, \infty) \rightarrow X$ is *asymptotic to γ* if there is sequence $t_n \rightarrow \infty$ and unit speed geodesics $\eta_n : [0, d(\eta(0), \gamma(t_n))] \rightarrow X$ from $\eta(0)$ to $\gamma(t_n)$ such that η_n converges uniformly on compact subsets to η . It is not difficult to see that

$$b_\gamma(\eta(t)) = t + b_\gamma(\eta(0)) \quad t \geq 0.$$

If (X, d) is a proper metric space then for any $x \in X$ we can select a subsequence of geodesics $(\eta_n)_{n \in \mathbb{N}}$ connecting x and $\gamma(t_n)$ such that $(\eta_{n_k})_{k \in \mathbb{N}}$ converges to a ray η starting at x which is asymptotic to γ .

A line $\eta : \mathbb{R} \rightarrow X$ is said to be *bi-asymptotic* to the line $\gamma : \mathbb{R} \rightarrow X$ if η^\pm is asymptotic to γ^\pm . We say that η is *parallel* to γ if the shifted lines $\eta_{(s)} : t \mapsto \eta(t+s)$ are bi-asymptotic to γ . It is not clear whether every bi-asymptotic line η to γ is also parallel to γ . Note, that if (X, d) is non-branching then it suffices to show that b_{γ^+} restricted to η is affine. Assuming Busemann concavity this is indeed the case, see Lemma 2.10 below.

1.3. Gromov-Hausdorff convergence. Given two subsets A and B of a metric space (Z, d_Z) the *Hausdorff distance* $d_Z^{(H)}$ of A and B is defined as

$$d_Z^{(H)}(A, B) = \inf\{\epsilon > 0 \mid A \subset B_\epsilon, B \subset A_\epsilon\}$$

where $A_\epsilon = \cup_{x \in A} B_\epsilon(x)$ and $B_\epsilon = \cup_{x \in B} B_\epsilon(x)$. Let (X, d_X) and (Y, d_Y) be two metric spaces. We say that a metric space (Z, d_Z) together with two maps $i_X : X \rightarrow Z$ and $i_Y : Y \rightarrow Z$ is a *metric coupling* of (X, d_X) and (Y, d_Y) if i_X and i_Y are isometric embeddings, i.e. for all $x_i \in X$ and $y_i \in Y$ it holds

$$\begin{aligned} d_Z(i_X(x_1), i_X(x_2)) &= d_X(x_1, x_2) \\ d_Z(i_Y(y_1), i_Y(y_2)) &= d_Y(y_1, y_2). \end{aligned}$$

Then the *Gromov-Hausdorff distance* of (X, d_X) and (Y, d_Y) is defined as

$$d_{GH}((X, d_X), (Y, d_Y)) = \inf d_Z^{(H)}(i_X(X), i_Y(Y))$$

where the infimum is taken over all metric couplings of (X, d_X) and (Y, d_Y) . Note that d_{GH} is zero iff the completions of (X, d_X) and (Y, d_Y) are isometric. Thus d_{GH} induces a metric on the equivalence classes of isometric complete metric spaces. One may restrict the metric couplings further if a certain point is supposed to be preserved. More precisely, let (X, d_X, x) and (Y, d_Y, y) be *pointed metric spaces*. Then a *metric coupling* is a triple $((Z, d_Z, z), i_X, i_Y)$ such that i_X and i_Y are isometric embeddings with $i_X(x) = i_Y(y) = z$. The Gromov-Hausdorff distance is then defined as above.

In general this convergence is rather strong in case of non-compact/unbounded spaces. A weaker notion is given by the *pointed Gromov-Hausdorff convergence*. More precisely, we say that a sequence (X_n, d_n, x_n) converges to the pointed metric space (X, d, x) in the *pointed Gromov-Hausdorff topology* if for each $r > 0$

$$d_{GH}((B_r^n(x_n), d_n), (B_r(x), d)) \rightarrow 0$$

where $B_r^n(x_n)$ and $B_r(x)$ are the usual balls of radius r with respect to d_n and resp. d .

2. BUSEMANN CONCAVITY

In this section we define a form of non-negative curvature which is similar to Busemann's notion of non-positive curvature. As it turns out this notion is not new. It appeared already in the study of Hilbert geometry as "has defined curvature" [KS58] and in a paper of Kann [Kan61] who studied two dimensional G -spaces of positive curvature which is defined via an additional quadratic term. Note that Kann's definition differs from the definition in terms of comparison geometry presented below.

Definition 2.1 (Busemann concave). A geodesic metric space (X, d) is said to be *Busemann concave* if for any three point $x, y_1, y_2 \in X$ and any geodesics γ_{x, y_i} connecting x and y_i the function

$$t \mapsto \frac{d(\gamma_{x, y_1}(t), \gamma_{x, y_2}(t))}{t}$$

is non-increasing on $[0, 1]$.

Busemann concavity implies that the space is non-branching. One readily verifies that any strictly convex Banach space is Busemann concave. Below we give further examples.

It is possible to define Busemann concavity in terms of comparison geometry. More precisely, let $\Delta(\tilde{x}\tilde{y}_1\tilde{y}_2)$ be a comparison triangle in \mathbb{R}^2 with side lengths $d(x, y_1)$, $d(x, y_2)$ and $d(y_1, y_2)$. Then Busemann concavity is equivalent to requiring

$$d(\gamma_{x, y_1}(t), \gamma_{x, y_2}(t)) \geq d_{\mathbb{R}^2}(\tilde{\gamma}_{\tilde{x}, \tilde{y}_1}(t), \tilde{\gamma}_{\tilde{x}, \tilde{y}_2}(t))$$

for all $t \in [0, 1]$. With the help of this, it is possible to define spaces with lower bound k on the curvature for general $k \in \mathbb{R}$. Note that for $k > 0$ the existence of a comparison triangle is implicitly assumed, see also Section 2.5.

Using the triangle comparison definition for Alexandrov spaces and the Toponogov comparison theorem for Riemannian manifolds we obtain the following.

Lemma 2.2. *Every Alexandrov space with sectional curvature bounded below by k has Busemann curvature bounded below by k . Furthermore, a Riemannian manifold has sectional curvature bounded below k if and only if it has Busemann curvature bounded below by k .*

Remark. Similar to the argument in [FLS07] the existence of angles implies that a metric space with Busemann curvature bounded below by $k \in \mathbb{R}$ is an Alexandrov space with the same lower curvature bound. We leave it to the reader to work out the details.

In contrast to Busemann convexity it is not clear whether it suffices to check the property above only for midpoints, i.e.

$$d(m_{x, y_1}, m_{x, y_2}) \geq \frac{1}{2}d(y_1, y_2).$$

Note that similar to Busemann convexity, Busemann concavity is not stable under Gromov-Hausdorff convergence. Nevertheless, a weaker property is preserved.

Definition 2.3. A function $\sigma : X \times X \times [0, 1]$ is called a *geodesic bicombing* if $\sigma_{xy}(0) = x$, $\sigma_{xy}(1) = y$ and $d(\sigma_{xy}(t), \sigma_{xy}(t')) = |t - t'|d(x, y)$ for all $x, y \in X$ and $t, t' \in [0, 1]$. We say the bicombing is *closed* if $(x_n, y_n) \rightarrow (x, y)$ implies $\sigma_{x_n y_n}(t) \rightarrow \sigma_{xy}(t)$ for all $t \in [0, 1]$.

Definition 2.4 (weak Busemann concavity). A metric space (X, d) is said to be *weakly Busemann concave* if there is a closed geodesic bicombing σ such that for all $x, y, z \in X$ it holds

$$t \mapsto \frac{d(\sigma_{xy}(t), \sigma_{xz}(t))}{t}$$

is non-increasing.

Remark. This property resembles Kleiner's notion of often convex spaces [KL97], resp. the notion of convex bicombings [DL14].

It is easy to see that any Banach space is weakly Busemann concave. The corresponding geodesic bicombing is given by straight lines. In a future work we try to give generalizations of Theorem 2.14 and Proposition 2.5 using only weak Busemann concavity.

Below it is shown that tangent cones of Busemann concave spaces are uniquely defined. If the space is doubling or admits a doubling measure then one can adjust the proofs of [LD11] to show that the tangent cones are (locally compact) Carnot groups away from a thin set, i.e. a set which has zero measure for every doubling measure. We refer to [LD11] for necessary definitions of Carnot groups.

The following shows that the only Busemann concave Carnot groups are Banach spaces with strictly convex norm. As Busemann concavity is not stable under Gromov-Hausdorff convergence this is not sufficient to conclude that almost all tangent cones are Banach spaces.

Proposition 2.5. *Any finite dimensional Busemann concave Carnot group equipped with Carnot-Carathéodory metric d must be a Banach space with strictly convex norm.*

Proof. Let $V_1 \in TX$ be a horizontal vector and $\eta^1, \eta^2 : (-\epsilon, \epsilon) \rightarrow X$ be tangent to V_1 at $\eta^1(0) = \eta^2(0)$. Now define the delated curves

$$\eta_\lambda^i(t) = \delta_\lambda(\eta^i(t)).$$

Then by invariance it holds $d(\eta_\lambda^1(t), \eta_\lambda^2(t)) = \lambda d(\eta^1(t), \eta^2(t))$. Together with Busemann concavity we get for $\lambda > 1$

$$d\left(\eta_\lambda^1\left(\frac{t}{\lambda}\right), \eta_\lambda^2\left(\frac{t}{\lambda}\right)\right) \geq d(\eta^1(t), \eta^2(t)).$$

As η^1 and η^2 are both tangent to V_1 , their Pansu differential is also V_1 . In particular, it holds

$$\eta_\lambda^i\left(\frac{t}{\lambda}\right) \rightarrow \exp(tV_1).$$

But this implies that $d(\eta_\lambda^1(\frac{t}{\lambda}), \eta_\lambda^2(\frac{t}{\lambda})) \rightarrow 0$ and thus $d(\eta^1(t), \eta^2(t)) = 0$.

To conclude, we just need to note that any k -step Carnot group with $k > 1$ has distinct geodesics which are tangent to the same horizontal vector. Thus X must be a rank 1 Carnot group, i.e. a Banach space. Strict convexity of the norm follows as otherwise Busemann concavity cannot hold. \square

Remark (Asymmetric metrics). In principle it is possible to define Busemann concavity also for asymmetric metrics. In order to prove Propositions 2.9 and 2.23 one needs the concavity property between any two forward resp. backward geodesics starting at a fixed point $x \in X$. The proof of the splitting theorem requires minor adjustments taking care of the asymmetry of the distance.

2.1. Constructions and examples. A whole family of Busemann concave spaces is obtained by products of Busemann concave spaces.

Lemma 2.6. *If $(X_i, d_i)_{i \in I}$ are Busemann concave spaces for some finite index set $I \subset \mathbb{N}$ and F is a strictly convex norm on $\mathbb{R}^{|I|}$ then $X = \times_{i \in I} X_i$ equipped with the metric $d((x_i)_{i \in I}, (y_i)_{i \in I}) = F((d(x_i, y_i))_{i \in I})$ is Busemann concave.*

Remark. (1) It is possible to allow for countably infinite index sets. The obtained space is then an extended metric space, i.e. the metric may be infinite.

(2) If F is not strictly convex or some factors are only weakly Busemann concave then their product is weakly Busemann concave.

Proof. If F is strictly convex then geodesics in X are obtained as “product geodesics”. Thus

$$\begin{aligned} d((\gamma_i(t))_{i \in I}, (\gamma'_i(t))_{i \in I}) &= F((d(\gamma_i(t), \gamma'_i(t))_{i \in I})) \\ &\geq tF(d(\gamma_i(1), \gamma'_i(1))_{i \in I}) \\ &= td((\gamma_i(1))_{i \in I}, (\gamma'_i(1))_{i \in I}). \end{aligned}$$

□

Remark (Berwald spaces). Using Jacobi fields it is possible to show that Berwald spaces with flag curvature bounded below by $k \in \mathbb{R}$ satisfy the corresponding Busemann comparison result locally until the conjugate radius is reached. In [KK06] it was shown that in the class Berwald spaces non-positive flag-curvature is equivalent to Busemann convexity. In the current setting we were not able to “invert” the inequalities to show the same for non-negative flag curvature and obtain global Busemann concavity. Note, however, that any simply connected non-negatively curved Berwald space which does not contain a higher rank symmetric factor can be exactly described [Sza81]. More precisely, they are metric products as above and each factor is either flat or a Riemannian manifold of non-negative curvature (see [Kel15]). Thus a deeper understanding of higher rank symmetric spaces and their Riemannian and Berwald structures would allow to characterize non-negatively curved Berwald spaces in terms of Busemann concavity.

Recall that the (Euclidean) cone $\text{Con}(X)$ of a metric space (X, d) is the set $X \times [0, \infty)$ where the points $(x, 0)$ are identified and the metric on $\text{Con}(X)$ is given by

$$d((x, r), (y, s))^2 = r^2 + s^2 - 2rs \cos(\min\{\pi, d(x, y)\}).$$

Lemma 2.7. *Assume (X, d) has diameter bounded by π . Then the Euclidean cone over $\text{Con}(X)$ is Busemann concave iff (X, d) has Busemann curvature bounded below by 1.*

Remark. (1) The bound π ensures that there is a comparison triangle. By Theorem 2.29 this bound always holds if the space (X, d) is not 1-dimensional.

(2) More generally, one can define general k -cones $\text{Con}_k(X)$, called spherical suspension for $k = 1$ and elliptical cone for $k = -1$, see [BGP92, Section 4.3]

and [BH99, Definition 5.6]. A similar proof would show that they have Busemann curvature bounded below by k iff X has Busemann curvature bounded below by 1 and diameter at most π .

Proof. Let x_1, x_2 and x_3 be points in X and m_1 and m_2 the t -midpoints of (x_1, x_3) and resp. (x_2, x_3) . The curvature bound of X translates to

$$d(m_1, m_2) \geq d_{S^2}(\tilde{m}_1, \tilde{m}_2)$$

where \tilde{m}_1 and \tilde{m}_2 are the t -midpoints in the comparison triangle whose existence is ensured by the bound on the diameter. Then $\cos(d(m_1, m_2)) \leq \cos(d_{S^2}(\tilde{m}_1, \tilde{m}_2))$.

Now let (x_i, r_i) be three point in $\text{Con}(X)$ and (m_1, s_1) and (m_2, s_2) be the corresponding t -midpoints. With the help of the comparison space we obtain

$$\begin{aligned} t^2 d((x_1, r_1), (x_2, r_2))^2 &= t^2 d_{\mathbb{R}^3}((\tilde{x}_1, r_1), (\tilde{x}_2, r_2))^2 \\ &= d_{\mathbb{R}^3}((\tilde{m}_1, s_1), (\tilde{m}_2, s_2))^2 \\ &= s_1^2 + s_2^2 - 2s_1 s_2 \cos(d_{S^2}(\tilde{m}_1, \tilde{m}_2)) \\ &\leq s_1^2 + s_2^2 - 2s_1 s_2 \cos(d(m_1, m_2)) \\ &= d((m_1, s_1), (m_2, s_2))^2. \end{aligned}$$

Assuming conversely that $\text{Con}(X)$ is Busemann concave we see that $\cos(d(m_1, m_2)) \leq \cos(d_{S^2}(\tilde{m}_1, \tilde{m}_2))$. As the diameter of X is at most π , monotonicity of cosine on $[0, \pi]$ implies the required comparison

$$d(m_1, m_2) \geq d_{S^2}(\tilde{m}_1, \tilde{m}_2).$$

□

An open problem is whether quotients via isometry actions of Busemann concave spaces are still Busemann concave. The current proofs in the Alexandrov setting rely heavily on the notion of angle which is not present in the current setting. This would also imply that any non-negatively curved Berwald space whose connection does not have a higher rank symmetric factor is Busemann concave.

2.2. Busemann functions, lines and a splitting theorem. For non-negatively curved Riemannian manifolds the existence of a line implies that the space splits, i.e. there is a metric spaces X' such that X is isometric/diffeomorphic to $X' \times \mathbb{R}$. A key point of the splitting theorem is the existence of a unique line η_x through every $x \in X$ which is parallel to a given line γ . This is usually done by showing that the Busemann functions b_{γ^\pm} are affine and the rays asymptotic to γ^\pm can be glued to lines. In this section we show more directly that the gluing property holds and that the space also splits into a product. However, it is not clear whether the Busemann functions associated to lines are affine or whether their level sets are convex.

First the following useful lemma.

Lemma 2.8. *Let $\gamma^+ : [0, \infty) \rightarrow X$ be a ray then it holds*

$$\lim_{t \rightarrow \infty} \frac{d(x, \gamma(t))}{t} = 1.$$

Proof. By definition $b_\gamma(x)$ is the limit of the non-decreasing bounded sequence $t - d(x, \gamma(t))$. Thus

$$0 = \lim_{t \rightarrow \infty} \frac{t - d(x, \gamma(t))}{t} = 1 - \lim_{t \rightarrow \infty} \frac{d(x, \gamma(t))}{t}.$$

□

The following proposition shows that moving in the ray direction induces a natural expansion. In case of a line one may also move in the opposite direction to show that this movement is an isometry, see Lemma 2.12 below.

Proposition 2.9. *Let $\gamma : [0, \infty) \rightarrow X$ be a ray, and η and ξ be rays asymptotic to γ that are generated by the same sequence $t_n \rightarrow \infty$. Then it holds*

$$d(\eta(t), \xi(s)) \leq d(\eta(t+a), \xi(s+a))$$

for all $t, s, a \geq 0$.

Proof. From the assumption there are sequences $(\eta_n)_{n \in \mathbb{N}}$ and $(\xi_n)_{n \in \mathbb{N}}$ where η_n and resp. ξ_n are geodesics from $\eta(0)$ to $\gamma(t_n)$ and resp. $\xi(0)$ to $\gamma(t_n)$ such that for any $t \geq 0$ it holds $\eta_n(t) \rightarrow \eta(t)$ and $\xi_n(t) \rightarrow \xi(t)$. Now fix $s, t, a \geq 0$ and set $b_n = d(\eta(0), \gamma(t_n))$ and $c_n = d(\xi(0), \gamma(t_n))$. Define geodesics $\bar{\eta}_n(r) = \eta_n(b_n - r)$ and $\bar{\xi}_n(r) = \xi_n(c_n - r)$. Also define $\bar{b}_n = b_n - t$ and $\bar{c}_n = c_n - s$ and note that $\bar{b}_n, \bar{c}_n > 0$ for sufficiently large n .

By Busemann concavity applied the hinge formed by $\bar{\eta}_n$ and $\bar{\xi}_n$ with contraction factor $\lambda_n = 1 - a/\bar{b}_n$ we have

$$\begin{aligned} d(\bar{\eta}_n(\lambda_n \bar{b}_n), \bar{\xi}_n(\lambda_n \bar{c}_n)) &\geq \lambda_n d(\bar{\eta}_n(\bar{b}_n), \bar{\xi}_n(\bar{c}_n)) \\ &= \lambda_n d(\eta_n(t), \xi_n(s)). \end{aligned}$$

Note that

$$\begin{aligned} \bar{\eta}_n(\lambda_n \bar{b}_n) &= \eta_n(t+a) \\ \bar{\xi}_n(\lambda_n \bar{c}_n) &= \xi_n\left(s + \frac{\bar{c}_n}{\bar{b}_n} a\right). \end{aligned}$$

By Lemma 2.8 we have $\frac{\bar{c}_n}{\bar{b}_n} \rightarrow 1$ so that $\bar{\xi}_n(\lambda_n \bar{c}_n) \rightarrow \xi(s+a)$ and hence

$$d(\eta(t), \xi(s)) \leq d(\eta(t+a), \xi(s+a)).$$

□

We will now prove the splitting theorem in a sequence of lemmas. Assume in the following that (X, d) is Busemann concave and contains a line $\gamma : \mathbb{R} \rightarrow X$. Denote by η_x^\pm the rays asymptotic to γ^\pm and let

$$\eta_x(t) = \begin{cases} \eta_x^+(t) & t \geq 0 \\ \eta_x^-(t) & t \leq 0. \end{cases}$$

Lemma 2.10. *The Busemann functions b_{γ^\pm} are affine when restricted to η_x . Furthermore, η_x is a line bi-asymptotic to γ .*

Proof. We only need to prove that

$$b_{\gamma^+}(\eta_x^-(s)) = b_{\gamma^+}(x) - s$$

for $s \geq 0$. Indeed, this will show that b_{γ^+} is affine on η_x . A similar argument also works for b_{γ^-} .

From the previous lemma we have

$$d(\eta_x^-(s), \gamma(t)) \geq d(\eta_x^-(0), \gamma(t+s)).$$

Thus

$$t - d(\eta_x^-(s), \gamma(t)) \leq t + s - d(x, \gamma(t+s)) - s.$$

Taking the limit as $t \rightarrow \infty$ we obtain

$$b_{\gamma^+}(\eta_x^-(s)) \leq b_{\gamma^+}(x) - s.$$

But then

$$s \leq b_{\gamma^+}(x) - b_{\gamma^+}(\eta_x^-(s)) \leq d(x, \eta^-(s)) = s$$

and thus $b_{\gamma^+}(\eta_x^-(s)) = b_{\gamma^+}(x) - s$.

This also implies that for $t, s \geq 0$ it holds

$$\begin{aligned} b_{\gamma^+}(\eta_x^+(t)) - b_{\gamma^+}(\eta_x^-(s)) &= s + t \\ &\leq d(\eta_x^-(s), \eta_x^+(t)) \\ &\leq d(\eta_x^-(s), x) + d(x, \eta_x^+(t)) = s + t. \end{aligned}$$

Therefore, η_x is a line bi-asymptotic to γ . \square

Lemma 2.11. *Through each point there is exactly one line parallel to γ .*

Proof. By non-branching there is at most one bi-asymptotic line through each x . Indeed, assume η is a line through x such that b_{γ^+} is affine along η and let $\tilde{\eta}$ be a ray which is asymptotic to γ^+ . Then it is easy to see that b_{γ^+} is affine on $\eta' = \eta^- \cup \tilde{\eta}$ and thus $d(\eta^-(t), \tilde{\eta}(s)) = t + s$. But then by non-branching assumption we must have $\eta = \eta'$. The same argument also show that η is the unique line bi-asymptotic to γ that starts at $\eta(t)$ for any $t \in \mathbb{R}$. Hence η is parallel to γ . \square

Lemma 2.12. *For any $x, y \in X$ and $t, s, a \in \mathbb{R}$ it holds*

$$d(\eta_x(t+a), \eta_y(s+a)) = d(\eta_x(t), \eta_y(s)).$$

Proof. Observe that uniqueness of the lines η_x and η_y through x and resp. y implies that for any $t_n \rightarrow \infty$ the sequences $(\eta_n)_{n \in \mathbb{N}}$ and $(\xi_n)_{n \in \mathbb{N}}$ connecting x and $\gamma(t_n)$ and resp. y and $\gamma(t_n)$ converge to η and resp. ξ . Thus we can apply Proposition 2.9 either with the ray γ^+ or with the ray γ^- to conclude

$$d(\eta(t), \xi(s)) = d(\eta(t+a), \xi(s+a)).$$

\square

This means that moving along the lines induces an isometry. In particular, all level sets of b_{γ^+} are isometric with isometry generated by moving along the parallel lines.

Corollary 2.13. *Assume (X, d) is a proper Busemann concave metric space. If through every point $x \in X$ there is a line connecting x with some fixed x_0 then (X, d) is homogeneous, i.e. for every $x, y \in X$ there is an isometry $g_{xy} \in \text{Isom}(X, d)$ such that $g_{xy}(x) = y$.*

We are now able to prove the bi-Lipschitz splitting theorem.

Theorem 2.14 (Splitting Theorem). *Let (X, d) a proper Busemann concave space and assume X contains a line $\gamma : \mathbb{R} \rightarrow X$. Then through every $x \in X$ there is a unique line η parallel to γ and (X, d) is bi-Lipschitz to a metric space $(X' \times \mathbb{R}, \tilde{d})$ where \tilde{d} is a product metric.*

Remark. The proof shows that X' is a subset of X but it is not clear whether it can be chosen to be convex/totally geodesic. In case X' is totally geodesic the product metric \tilde{d} on $X' \times \mathbb{R}$ can be chosen such that $(X' \times \mathbb{R}, \tilde{d})$ is Busemann concave.

Proof. We claim that (X, d) is bi-Lipschitz to $(X' \times \mathbb{R}, \tilde{d})$ where $X' = b_{\gamma^+}^{-1}(0)$ and

$$\tilde{d}((x, t), (y, s)) = d(x, y) + |t - s|.$$

This is obviously a product metric on $X' \times \mathbb{R}$. Now let $\Psi : X \rightarrow (X' \times \mathbb{R})$ assign to each $x \in X$ the unique line η_x parallel to γ such that $b_{\gamma^+}(\eta_x(0)) = 0$. We claim that the map

$$\Psi(x) = (\eta_x(0), b_{\gamma^+}(x))$$

is a bi-Lipschitz homeomorphism between (X, d) and $(X' \times \mathbb{R}, \tilde{d})$. Note that Ψ is obviously bijective, so that for simplicity of notation we identify X and $(X' \times \mathbb{R})$ set-wise and assume $(x, t), (y, s)$ are points living in X .

Using the triangle inequality of d we get

$$d((x, t), (y, s)) \leq d((x, t), (x, s)) + d((x, s), (y, s)) = |t - s| + d(x, y) = \tilde{d}((x, t), (y, s)).$$

which implies that Ψ^{-1} is 1-Lipschitz.

We claim that $d(x, y) + |t - s| \leq 3d((x, t), (y, s))$. This would imply that Ψ is 3-Lipschitz and finish the proof.

To show the claim note that b_{γ^+} is 1-Lipschitz so that

$$|t - s| = |b_{\gamma^+}(x, t) - b_{\gamma^+}(y, s)| \leq d((x, t), (y, s))$$

and from triangle inequality

$$\begin{aligned} d(x, y) &\leq d((x, t), (y, s)) + d((y, s), (y, t)) \\ &= d((x, t), (y, s)) + |t - s|. \end{aligned}$$

Combining the inequalities above we obtain

$$d(x, y) + |t - s| \leq d((x, t), (y, s)) + 2|t - s| \leq 3d((x, t), (y, s)).$$

Note that it is possible to change \tilde{d} by any metric product of (X', d) and $(\mathbb{R}, |\cdot - \cdot|)$ as any two norms on \mathbb{R}^2 are bi-Lipschitz with Lipschitz constants only depending on the two norms. In particular, if X' was convex w.r.t. d then one may choose the L^2 -product so that $X' \times_2 \mathbb{R}$ is Busemann concave. \square

If the Hausdorff measure is non-trivial and the space is ‘‘Riemannian-like’’, then it is possible to show that X' is indeed convex and (X, d) is isometric to the L^2 -product of X' and the real line.

Corollary 2.15. *Assume, in addition, that \mathcal{H}^n is non-trivial and (X, d, \mathcal{H}^n) is infinitesimally Hilbertian then (X, d) is isometric to $X' \times_2 \mathbb{R}$.*

Proof. This follows from the proof of Gigli’s splitting theorem for RCD -spaces [Gig13]. Just note that the ‘‘gradient flow’’ of the Busemann function is just the isometry induced by moving along the lines and this isometry also preserves the Hausdorff measure. Using [Gig13, Theorem 5.23] one shows that X' is totally geodesic. Furthermore, L^2 -products of infinitesimally Hilbertian Busemann concave spaces are also infinitesimally Hilbertian (compare with [Gig13, Theorem 6.1]). \square

As mentioned above if (X, d) is Busemann concave and angles are well-defined then (X, d) is an Alexandrov space. In Gigli’s proof of the splitting theorem, pointwise angles are replaced by ‘‘smoothed’’ angles. More precisely, instead of looking at two intersecting geodesic in X one can look at intersecting geodesics in the Wasserstein space $\mathcal{P}_2(X)$. If the geodesics are pointwise absolutely continuous measures

then being infinitesimally Hilbertian shows that there is a notion of angle. Thus one might ask.

Problem 2.16. Assume (X, d, \mathcal{H}^n) is an infinitesimally Hilbertian Busemann concave metric measure space and \mathcal{H}^n is non-trivial. Is (X, d) a non-negatively curved Alexandrov space?

2.3. Tangent cones. Let Γ_x be the set of maximal unit speed geodesics starting at x . The *pre-tangent cone* $\hat{T}_x X$ at X is defined as the set $\Gamma_x \times [0, \infty)$ such that the points $(\gamma, 0)$ are identified. On $\hat{T}_x X$ we define the metric d_x as follows: Given geodesics $\gamma, \eta \in \Gamma_x$ there is an interval $I = [0, a]$ such that γ, η are both defined on I . Then define a metric d_x on $\hat{T}_x X$ by

$$d_x((\gamma, s), (\eta, t)) = \sup_{r \in [0, 1], \max\{rs, rt\} \leq a} \frac{d(\gamma(rs), \eta(rt))}{r}.$$

Lemma 2.17. *If (X, d) is Busemann concave then d_x is a well-defined metric on $\hat{T}_x X$. Furthermore, it holds $d_x((\gamma, \lambda s), (\eta, \lambda t)) = \lambda d_x((\gamma, s), (\eta, t))$.*

Proof. d_x is obviously non-negative and symmetric. The function $r \mapsto \frac{d(\gamma(sr), \eta(tr))}{r}$ is non-increasing by Busemann concavity so that the supremum in the definition of d_x is actually a limit w.r.t. $r \rightarrow 0$. This also implies that the triangle inequality holds. Also note that $d_x((\gamma, s), (\eta, t)) = 0$ implies that $d(\gamma(rs), \eta(rt)) = 0$ so that $\gamma(rs) = \eta(rt)$. As γ and η are unit speed geodesics starting at x we must have $s = t$ so that either $s = t = 0$ or $\gamma \equiv \eta$, i.e. d_x is definite. \square

It is possible to define an *exponential map* directly from the pre-tangent cone: Let $U_x \subset \hat{T}_x X$ be such that $(\gamma, t) \in U_x$ iff $\gamma(t)$ is defined. Then we define the exponential map $\exp_x : U_x \rightarrow X$ by

$$\exp_x(\gamma, t) = \gamma(t).$$

Note that by definition, \exp_x is onto. By Busemann concavity and the definition of d_x it is not difficult to show that \exp_x is 1-Lipschitz, i.e. $d_x(v, w) \geq d(\exp_x(v), \exp_x(w))$.

Definition 2.18 (Tangent cone). The *tangent cone* $(T_x X, d_x)$ at x is defined as the metric completion of $(\hat{T}_x X, d_x)$.

The tangent cone $(T_x X, d_x)$ is not necessarily the (pointed) Gromov-Hausdorff limit of the blow ups $(X, \frac{1}{\lambda} d, x)$ at x . Indeed, if (X, d) is compact and the tangent cone at some point is not locally compact then it cannot be the Gromov-Hausdorff limit of blow ups. An example is given by

$$K_c^p = \left\{ (x_i) \in \ell^p \mid \sum c^i x_i^p \leq 1 \right\}$$

where $c > 1$ and $p \in (1, \infty)$. The set K_c^p is a compact convex subset of ℓ^p , but the tangent cones at points (x_i) with $0 < \sum c^i x_i^p < 1$ are isometry to ℓ^p . However, if the blow-ups are precompact then their limit is uniquely given by the tangent cone.

Lemma 2.19. *If $\{(X, \frac{1}{\lambda} d, x)\}_{\lambda \in (0, 1]}$ is precompact then the limit as $\lambda \rightarrow 0$ exists in the pointed Gromov-Hausdorff topology and equals $(T_x X, d_x, 0_x)$.*

More generally, if the tangent cone $(T_x M, d_x)$ is locally compact we obtain the following.

Lemma 2.20. *If $(T_x X, d_x)$ is locally compact then for each $r > 0$ the sequence $\{(B_r(x), \frac{1}{\lambda}d, x)\}_{\lambda \in (0,1]}$ is precompact with respect to the pointed Gromov-Hausdorff topology. In particular, $(T_x X, d_x, 0_x)$ is the (unique) pointed Gromov-Hausdorff limit of $\{(X, \frac{1}{\lambda}d, x)\}_{\lambda \in (0,1]}$ as $\lambda \rightarrow 0$.*

Proof. By scale invariance of $T_x X$ any bounded subset is precompact. Let U_r be all $(\gamma, t) \in U_x$ with $t \leq r$. One can define a scaled exponential map $\exp_x^\lambda(\gamma, t) = \gamma(\lambda t)$. Then \exp_x^λ maps (U_r, d_x) onto $(B_r^\lambda(x), \frac{1}{\lambda}d)$ and is 1-Lipschitz where $B_r^\lambda(x)$ is the $(\frac{1}{\lambda}d)$ -ball of radius r with center x . By assumption U_r is bounded and thus precompact in $(T_x X, d_x)$. Therefore, for each $\epsilon > 0$ there is an $N(\epsilon) < \infty$ such that U_r can be covered by $N(\epsilon)$ d_x -balls of radius ϵ with center in U_r .

We claim that that B_r^λ can be covered by at most $N(\epsilon)$ $(\frac{1}{\lambda}d)$ -balls of radius ϵ . Indeed, let $\{v_1, \dots, v_{N(\epsilon)}\}$ be the centers of the d_x -balls of radius ϵ . This means for all $v \in U_r$ it holds $\inf_{i=1}^{N(\epsilon)} d_x(v, v_i) \leq \epsilon$. Set $x_i = \exp_x^\lambda v_i$. As \exp_x^λ is onto for each $x' \in B_r^\lambda(x)$ there is a $v \in (\exp_x^\lambda)^{-1}(x')$. Combining this with the 1-Lipschitz property we obtain

$$\begin{aligned} \inf_{i=1, \dots, N(\epsilon)} d(x', x_i) &= \inf_{i=1, \dots, N(\epsilon)} d(\exp_x^\lambda(v), \exp_x^\lambda(v_i)) \\ &\leq \inf_{i=1, \dots, N(\epsilon)} \lambda d_x(v, v_i) \leq \lambda \epsilon. \end{aligned}$$

Hence, $\{B_\epsilon^\lambda(x_i)\}_{i=1}^{N(\epsilon)}$ covers $B_r^\lambda(x)$. By definition $\text{diam } U_r, \text{diam } B_r^\lambda \leq 2r$ so that Gromov's precompactness theorem implies that $\{(B_r^\lambda(x), \frac{1}{\lambda}d)\}$ is precompact. Together with the previous lemma we see that the limit has to be $(\text{cl } U_r, d_x) = (B_r^{T_x M}(0_x), d_x)$. \square

Corollary 2.21. *If (X, d) is a complete Busemann concave (locally) doubling metric space then $(T_x X, d_x)$ is locally compact and the unique limit of the blowups $(X, \frac{1}{\lambda}d, x)_{\lambda \in (0, \epsilon]}$.*

By [LD11] it can be shown that for topologically and measure-theoretically almost all points $x \in X$ the tangent cone $(T_x M, d_x)$ is a finite dimensional Carnot group. However, the limit does not have to be Busemann concave so that Proposition 2.5 cannot be applied. Nevertheless, the results above show that a homogeneous tangent cone is necessarily both weakly Busemann concave and the homothetic contraction is affine. Both should imply that it has to be a Banach space. This, in particular, would imply that Busemann concave spaces are metric generalizations of Finsler manifolds.

2.4. Hausdorff measure, doubling and Poincaré. The Hausdorff measure is a natural measure associated to a metric space. For finite dimensional Alexandrov spaces it is known that there is an integer n such that the n -dimensional Hausdorff measure is non-trivial, i.e. non-zero and locally finite [BGP92]. Furthermore, if the space is non-negatively curved then this measure is doubling. For general Busemann concave spaces, we currently cannot show that the Hausdorff measure is non-trivial if the space is finite dimensional w.r.t. to any meaningful dimension definition. However, we will show that if the Hausdorff measure is non-trivial then it is doubling and satisfies a $(1, 1)$ -Poincaré inequality. Furthermore, it also satisfies the measure contraction property which is a (very) weak form of non-negative Ricci curvature. It is likely that further analysis shows that Busemann concavity implies

that the space satisfies $CD(0, n)$, i.e. Busemann concave spaces have non-negative n -dimensional Ricci curvature in the sense of Lott-Sturm-Villani.

Let $\delta > 0$ and S be a subset of X . Define

$$\mathcal{H}_\delta^n(S) = C_n \inf \left\{ \sum_{i \in \mathbb{N}} \left(\frac{1}{2} \text{diam } U_i \right)^n \mid S \subset \bigcup_{i \in \mathbb{N}} U_i, \text{diam } U_i < \delta \right\}$$

where C_n is a constant such that \mathcal{H}^n equals the Lebesgue measure on \mathbb{R}^n when $(X, d) = (\mathbb{R}^n, d_{\text{Euclid}})$. Note that $\mathcal{H}_\delta^n(S)$ is decreasing in δ so that we can define the n -dimensional Hausdorff measure of S as follows

$$\mathcal{H}^n(S) = \sup_{\delta > 0} \mathcal{H}_\delta^n(S) = \lim_{\delta \rightarrow 0} \mathcal{H}_\delta^n(S).$$

From the definition it follows that \mathcal{H}^n is an outer measure. Furthermore, one can show that each Borel set of X is \mathcal{H}^n -measurable and thus \mathcal{H}^n is a Borel measure.

Note that we have the following: if for some n it holds $\mathcal{H}^n(S) < \infty$ then $\mathcal{H}^{n'}(S) = 0$ for all $n' > n$. Also if $\mathcal{H}^n(S) > 0$ then $\mathcal{H}^{n'}(S) = \infty$ for all $0 < n' < n$. In particular, for each S there is at most one n with $0 < \mathcal{H}^n(S) < \infty$. Therefore, we can assign to each bounded set a number called *Hausdorff dimension*

$$\begin{aligned} \dim_H S &= \inf \{ n \in [0, \infty) \mid \mathcal{H}^n(S) = 0 \} \\ &= \sup \{ n \in [0, \infty) \mid \mathcal{H}^n(S) = \infty \} \end{aligned}$$

with conventions $\inf \emptyset = \infty$ and $\sup \emptyset = 0$. Now denote the *local Hausdorff dimension at x* by

$$\dim_H X(x) = \inf_{x \in U \text{ open}} \dim_H U.$$

Given points $x, y \in X$ we can choose a geodesic γ_{xy} connecting x and y such that for any $t \in [0, 1]$ the map

$$\Phi : (y, x, t) \mapsto \gamma_{xy}(t)$$

is a measurable function. Without loss of generality it is possible to choose Φ in a symmetric way, i.e. $\Phi(y, x, t) = \Phi(x, y, 1 - t)$.

Let Ω be some subset of X and define $\Omega_t = \Phi(\Omega, x, t)$. Denote its inverse by $g : \Omega_t \rightarrow \Omega$. Note that this map is onto and Busemann concavity implies it is t^{-1} -Lipschitz. Let $\{U_i\}_{i \in \mathbb{N}}$ be a δ -cover of Ω_t then $\{g(U_i)\}_{i \in \mathbb{N}}$ is a $t^{-1}\delta$ -cover of Ω . Furthermore, it holds

$$\sum (\text{diam } g(U_i))^n \leq \frac{1}{t^n} \sum (\text{diam } U_i)^n$$

and hence

$$\mathcal{H}_{t^{-1}\delta}^n(\Omega) \leq \frac{1}{t^n} \mathcal{H}_\delta^n(\Omega_t).$$

Taking the limit as $\delta \rightarrow 0$ on both sides we see that

$$(2.1) \quad \mathcal{H}^n(\Omega) \leq \frac{1}{t^n} \mathcal{H}^n(\Omega_t).$$

Note that this implies that if $\mathcal{H}^n(B_r(x)) < \infty$ then $\mathcal{H}^n(\Omega) < \infty$ for all bounded Ω . Indeed, there is a $t > 0$ depending only on Ω, x and r such that $\Omega_t = \Phi(\Omega, x, t) \subset B_r(x)$.

Lemma 2.22. *In a Busemann concave space (X, d) the Hausdorff dimension of bounded open subsets is equal to a fixed number $n \in [0, \infty]$. In particular, $\dim_H X(x) \equiv \text{const.}$*

Proof. Let Ω, Ω' be two bounded subsets with non-empty interior. Then there is an $x \in \Omega, x' \in \Omega'$ and $r, t > 0$ such that $\Omega_t \subset B_r(x') \subset \Omega'$ and $\Omega'_t \subset B_r(x) \subset \Omega$ where $\Omega_t = \Phi(\Omega, x', t)$ and $\Omega'_t = \Phi(\Omega', x, t)$. Hence, by 2.1 it holds

$$\mathcal{H}^n(\Omega) \leq \frac{1}{t^n} \mathcal{H}^n(\Omega')$$

and

$$\mathcal{H}^n(\Omega') \leq \frac{1}{t^n} \mathcal{H}^n(\Omega).$$

Now it is easy to see that $\dim_H \Omega = \dim_H \Omega'$ and that this number equals the local Hausdorff dimension at x . \square

In case the Hausdorff dimension is finite, it is still not clear if the corresponding measure is non-trivial, i.e. $0 < \mathcal{H}^n(B_r(x)) < \infty$ for some $x \in X$ and $r > 0$. However, if the n -dimensional Hausdorff measure is non-trivial, then the space enjoys nice properties.

Proposition 2.23. *Assume (X, d) is a complete Busemann concave metric space admitting a non-trivial Hausdorff measure. Then (X, d, \mathcal{H}^n) satisfies the measure contraction property $MCP(0, n)$, the Bishop-Gromov volume comparison $BG(0, n)$ and a (weak) $(1, 1)$ -Poincaré inequality. In particular, \mathcal{H}^n is a doubling measure with doubling constant 2^n .*

Remark. We refer to [Oht07b] for the exact definition of the measure contraction property and to [BB11, HKST15] for definitions of Poincaré and doubling conditions and their influence on the geometry and analysis of metric spaces.

Proof. Since Φ is measurable we can apply [Oht07b, Lemma 2.3].

The Bishop-Gromov volume comparison and the doubling property follows once we notice that $\Omega = B_R(x)$ implies that $\Omega_{\frac{r}{R}} \subset B_r(x)$. Equation 2.1 then implies

$$\frac{\mathcal{H}^n(B_R(x))}{\mathcal{H}^n(B_r(x))} \leq \frac{R^n}{r^n} = \frac{\mathcal{V}_{0,n}(R)}{\mathcal{V}_{0,n}(r)} \quad \text{for } 0 < r < R.$$

In particular, $r \mapsto \frac{\mathcal{H}^n(B_r(x))}{\mathcal{V}_{0,n}(r)}$ is non-increasing.

A standard argument implies that a weak $(1, 1)$ -Poincaré inequality holds (see e.g. [Hua10, Lemma 3.3]), i.e. it holds

$$\int_{B_r(x)} |u - u_{B_r(x)}| d\mathcal{H}^n \leq 2^{n+1} r \int_{B_{3r}(x)} g_u d\mathcal{H}^n$$

where g_u is a weak upper gradient of u .

We sketch the argument given in [Hua10, Lemma 3.3]: It suffices to assume $u \in \text{Lip}(X, d)$ and $g_u = \text{lip } u$ where $\text{lip } u$ is the local Lipschitz constant of u . Set $B = B_r(x)$ and $u_B = \frac{1}{\mathcal{H}^n(B)} \int_B u d\mathcal{H}^n$. Then for every geodesic $\gamma_{yz} : [0, 1] \rightarrow X$ connecting $y, z \in B_r(x)$ it holds

$$|u(y) - u(z)| \leq d(y, z) \int_0^1 g_u(\gamma_{yz}(t)) dt.$$

Thus

$$\begin{aligned} \int_B |u - u_B| d\mathcal{H}^n &\leq \frac{1}{\mathcal{H}^n(B)} \int_B \int_B |u(y) - u(z)| d\mathcal{H}^n(z) d\mathcal{H}^n(y) \\ &\leq \frac{2r}{\mathcal{H}^n(B)} \int_B \int_B \int_0^1 g_u(\gamma_{yz}(t)) dt d\mathcal{H}^n(z) d\mathcal{H}^n(y) \\ &\leq \frac{4r}{\mathcal{H}^n(B)} \int_B \int_B \int_{\frac{1}{2}}^1 g_u(\gamma_{yz}(t)) dt d\mathcal{H}^n(z) d\mathcal{H}^n(y) \end{aligned}$$

where the last inequality follows by choosing the geodesics γ_{yz} in a symmetric way. Note that the measure contraction property implies

$$\int_B f(\gamma_{xy}(t)) d\mathcal{H}^n(y) \leq \frac{1}{t^n} \int_{B_{tr}(x)} f(z) \mathcal{H}^n(z)$$

for all non-negative $f \in L^\infty_{\text{loc}}(X, \mathcal{H}^n)$ and $t \in (0, 1]$, see [Oht07b, Equations (2.2)]. Therefore, we obtain

$$\begin{aligned} \int_B |u - u_B| d\mathcal{H}^n &\leq \frac{4r}{\mathcal{H}^n(B)} \int_B \int_{\frac{1}{2}}^1 \int_{B_{2r}(y)} g_u(\gamma_{yz}(t)) d\mathcal{H}^n(z) dt d\mathcal{H}^n(y) \\ &\leq \frac{4r}{\mathcal{H}^n(B)} \int_B \int_{\frac{1}{2}}^1 \frac{1}{t^n} \int_{B_{2tr}(y)} g_u(w) d\mathcal{H}^n(w) dt d\mathcal{H}^n(y) \\ &\leq \frac{4r}{\mathcal{H}^n(B)} \int_B \int_{\frac{1}{2}}^1 \frac{1}{t^n} \int_{B_{3r}(x)} g_u(w) d\mathcal{H}^n(w) dt d\mathcal{H}^n(y) \\ &\leq 2^{n+1} r \int_{B_{3r}(x)} g_u d\mathcal{H}^n. \end{aligned}$$

□

Remark. In order to prove the Bishop inequality it remains to show that

$$\lim_{r \rightarrow 0} \frac{\mathcal{H}^n(B_r(x))}{r^n} = \omega_n$$

where $\omega_n = \mathcal{V}_{0,n}(1)$. This holds at points where the (blow-up) tangent space is isometric to an n -dimensional normed space as the Hausdorff measure of balls equals the volume of the n -dimensional Euclidean balls of same radius and the pointed Gromov-Hausdorff convergence is compatible with the measured Gromov-Hausdorff convergence if the reference measures are non-collapsing Hausdorff measures of the same dimension.

2.5. Bonnet-Myers theorem. Throughout this section we assume that geodesics are parametrized by arc length, i.e. they are unit speed geodesics. This will simplify some of the proofs below.

Recall a fact on triangles in S^2 : Let $a, b, c \in (0, \pi]$ with $a+b+c \leq 2\pi$. Then there is a triangle formed by unit speed geodesics $\tilde{\gamma}, \tilde{\eta}, \tilde{\xi}$ of length a, b, c with $\tilde{\gamma}_0 = \tilde{\eta}_0$, $\tilde{\gamma}_a = \tilde{\xi}_0$ and $\tilde{\eta}_b = \tilde{\xi}_c$. Furthermore, if $a = b \geq \frac{\pi}{2}$ and $a + b + c = 2\pi$ then

$$d_{S^2}(\tilde{\gamma}_{\frac{\pi}{2}}, \tilde{\eta}_{\frac{\pi}{2}}) = \pi$$

and $\tilde{\gamma}_0 = \tilde{\eta}_0$ is a midpoint of the pair $(\tilde{\gamma}_{\frac{\pi}{2}}, \tilde{\eta}_{\frac{\pi}{2}})$. The proof of the Bonnet-Myers theorem relies heavily on this rigidity.

We say that the complete geodesic metric space (X, d) has *Busemann curvature bounded below by 1* if for all unit speed geodesics γ, η, ξ in X of length a, b, c with

$\gamma_0 = \eta_0$, $\gamma_a = \xi_0$ and $\eta_b = \xi_c$ such that there is a corresponding comparison triangle in S^2 of length $a, b, c \geq 0$ then it holds

$$d(\gamma_{ta}, \eta_{tb}) \geq d_{S^2}(\tilde{\gamma}_{ta}, \tilde{\eta}_{tb})$$

for all $t \in [0, 1]$.

Using a slightly different notion of positive curvature Kann [Kan61] obtained a Bonnet-Myers theorem for two dimensional G -spaces which are positively curved in his sense. Note that his proof heavily relies on the notion of two dimensionality as well as local extendability of geodesics. The proof of the Bonnet-Myers theorem below is inspired by the one for $MCP(K, N)$ -spaces [Oht07b, Section 4]. The main idea is to replace the density estimates by length estimates. However, the technique is quite different and some steps are easier to prove in the current setting.

Before we start we need the following characterization of non-branching spaces that are not 1-dimensional: A non-branching geodesic metric space is said to be *not 1-dimensional* if for any unit speed geodesic $\gamma : [0, a] \rightarrow X$ and $\epsilon > 0$ there is a $y \in B_a(\gamma_0)$ with $d(\gamma_a, y) < \epsilon$ but $y \notin \gamma_{[0, a]}$.

Lemma 2.24. *Assume (X, d) is non-branching and not 1-dimensional. Then for all unit speed geodesics $\gamma : [0, a] \rightarrow X$ and $b \in (0, a)$ there is $z \in \partial B_b(\gamma_0)$ with $y \neq \gamma_b$.*

Proof. Choose $\epsilon = \frac{a-b}{2}$ and $y \in B_a(\gamma_0)$ as in the definition of not 1-dimensional. Let $\eta : [0, d(\gamma_0, y)] \rightarrow X$ be a unit speed geodesic connecting γ_0 and y . The choice of ϵ shows $b < d(\gamma_0, y)$. Since (X, d) is non-branching we see that $\gamma_b \neq \eta_b$ so that we may choose $z = \eta_b$. \square

For non-branching 1-dimensional spaces one gets the following rigidity. We leave the details to the interested reader, compare also with [Bus55, Theorem (9.6)].

Lemma 2.25. *If (X, d) is 1-dimensional and non-branching then it is isometric to a closed connected interval $I \subset X$ or a circle S_λ^1 of length $\lambda > 0$.*

Before proving the main theorem of this section we need the following technical lemmas.

Lemma 2.26. *Assume (X, d) has Busemann curvature bounded below by 1. If $\gamma, \eta : [0, \pi - a] \rightarrow X$ are two unit speed geodesics starting at $x \in X$ with $d(\gamma_{\frac{\pi}{2}}, \eta_{\frac{\pi}{2}}) < \pi$ and $a \in (0, \frac{\pi}{2}]$ then for any $s \in [a, \frac{\pi}{2}]$ it holds $d(\gamma_{\pi-s}, \eta_{\pi-s}) < 2s$.*

Proof. Not first that

$$\limsup_{s \rightarrow \frac{\pi}{2}^-} d(x, \gamma_{\pi-s}) + d(x, \eta_{\pi-s}) + d(\gamma_{\pi-s}, \eta_{\pi-s}) < 2\pi.$$

Thus

$$\limsup_{s \rightarrow \frac{\pi}{2}^-} (d(\gamma_{\pi-s}, \eta_{\pi-s}) - 2s) < 0.$$

In particular, for s sufficiently close to $\frac{\pi}{2}$ it holds $d(\gamma_{\pi-s}, \eta_{\pi-s}) < 2s$.

Assume the statement was not true. Then there is a largest $s_0 \in [a, \frac{\pi}{2})$ with $d(\gamma_{\pi-s_0}, \eta_{\pi-s_0}) = 2s_0$. The assumptions imply that there is a comparison triangle of the triangle formed by $(x, \gamma_{\pi-s_0}, \eta_{\pi-s_0})$ such that for some unit speed geodesics $\tilde{\gamma}, \tilde{\eta} : [0, \pi - s_0] \rightarrow S^2$ it holds $d(\gamma_{\pi-s_0}, \eta_{\pi-s_0}) = d(\tilde{\gamma}_{\pi-s_0}, \tilde{\eta}_{\pi-s_0})$ and

$$d(\gamma_t, \eta_t) \geq d_{S^2}(\tilde{\gamma}_t, \tilde{\eta}_t) \quad \text{for } t \in [0, \pi - s_0].$$

However, the comparison triangle satisfies

$$d_{S^2}(\tilde{\gamma}_{\frac{\pi}{2}}, \tilde{\eta}_{\frac{\pi}{2}}) = \pi$$

which would contradict the assumption $d(\gamma_{\frac{\pi}{2}}, \eta_{\frac{\pi}{2}}) < \pi$. \square

Lemma 2.27. *Assume (X, d) is not 1-dimensional and has Busemann curvature bounded below by 1. Then for $\epsilon > 0$ and all unit speed geodesics $\gamma, \eta : [0, \pi - s] \rightarrow X$ with $s \in (0, \frac{\pi}{2})$ there is a unit speed geodesic $\eta^\epsilon : [0, \pi - s_\epsilon] \rightarrow X$ such that $s_\epsilon \geq s$, $d(\eta_{\pi-s}, \eta_{\pi-s_\epsilon}^\epsilon) < \epsilon$ and*

$$d(\gamma_{\frac{\pi}{2}}, \eta_{\frac{\pi}{2}}^\epsilon) < \pi.$$

Proof. If $d(\gamma_{\frac{\pi}{2}}, \eta_{\frac{\pi}{2}}) < \pi$ then we can choose $\eta^\epsilon = \eta$. Assume $d(\gamma_{\frac{\pi}{2}}, \eta_{\frac{\pi}{2}}) = \pi$. Since (X, d) is not 1-dimensional there is a sequence of points $(y_n)_{n \in \mathbb{N}}$ in $B_{\pi-s}(x)$ such that $y_n \rightarrow \eta_{\pi-s}$ and $y_n \neq \eta(\pi - s_n)$ where $s_n = \pi - d(x, y_n)$. Let $\eta^n : [0, \pi - s_n] \rightarrow X$ be a unit speed geodesic connecting x and y_n . Note by non-branching $\eta_{\frac{\pi}{2}}^n \neq \eta_{\frac{\pi}{2}}$.

We claim that $d(\gamma_{\frac{\pi}{2}}, \eta_{\frac{\pi}{2}}^n) < \pi$. Indeed, equality would imply that x is a mid-point of both $(\gamma_{\frac{\pi}{2}}, \eta_{\frac{\pi}{2}})$ and $(\gamma_{\frac{\pi}{2}}, \eta_{\frac{\pi}{2}}^n)$ which is not possible by the non-branching assumption. \square

In the following we write $\text{diam } \emptyset = 0$. Thus $\text{diam } A = 0$ implies that A contains at most one element.

Corollary 2.28. *Assume (X, d) is not 1-dimensional and has Busemann curvature bounded below by 1. If γ, η are as in the lemma above then*

$$d(\gamma_{\pi-s}, \eta_{\pi-s}) \leq 2s$$

for $s \in [0, \frac{\pi}{2}]$. In particular, $\text{diam } \partial B_{\pi-s}(x) \leq 2s$ for $s \in [0, \frac{\pi}{2}]$.

Proof. Assume first $s > 0$. Using Lemma 2.27 we find a sequence of unit speed geodesics $\eta^n : [0, \pi - s_n] \rightarrow X$ with $s_n \geq s$, $\eta_{\pi-s_n}^n \rightarrow \eta_{\pi-s}$ and $d(\gamma_{\frac{\pi}{2}}, \eta_{\frac{\pi}{2}}^n) < \pi$. Thus we can apply Lemma 2.26 to $\gamma|_{[0, \pi-s_n]}$ and η^n and get

$$d(\gamma_{\pi-s_n}, \eta_{\pi-s_n}^n) < 2s_n.$$

Since $s_n \rightarrow s$ we see that

$$d(\gamma_{\pi-s}, \eta_{\pi-s}) = \lim_{n \rightarrow \infty} d(\gamma_{\pi-s_n}, \eta_{\pi-s_n}^n) \leq \lim_{n \rightarrow \infty} 2s_n = 2s.$$

The case $s = 0$ is obtain via approximation. \square

Combining the results we are able to prove the following theorem.

Theorem 2.29 (Bonnet-Myers Theorem). *Assume (X, d) is non-branching and has Busemann curvature bounded below by 1. If (X, d) is not 1-dimensional then the diameter of X is at most π .*

Remark. By scaling one sees that $\text{diam } X \leq \frac{\pi}{\sqrt{k}}$ if the space has Busemann curvature bounded below by $k > 0$.

Proof. Let $y \in X$ be such that $d(x, y) = \text{diam } X$ and γ be a unit speed geodesic connecting x and y . Assume by contradiction $\text{diam } X > \pi$. Corollary 2.28 shows $\partial B_\pi(x) = \{\gamma_\pi\}$. However, by Lemma 2.24 there is a $z \in \partial B_\pi(x)$ with $z \neq \gamma_\pi$ which is a contradiction. \square

By the same arguments it is possible to show that if (X, d) is not 1-dimensional then for all unit speed geodesics $\gamma : [0, a] \rightarrow X$ and $\eta : [0, b] \rightarrow X$ starting at x with $a, b \in [0, \pi)$, $a + b \geq \pi$ the triangle formed by (x, γ_a, η_b) has circumference at most 2π with strict inequality unless $d(\gamma_{ta}, \eta_{tb}) = \pi$ where $ta + tb = \pi$. In particular, given three points in X there is always a corresponding comparison triangle in S^2 .

Using this we can prove the following along the lines of [Oht07c, Section 5]: if $d(x, x^*) = \pi$ for some $x, x^* \in X$ then for any $z \in X$ it holds

$$d(x, z) + d(z, x^*) = \pi$$

and there is a unique unit speed geodesic γ connecting x and x^* such that $\gamma(d(x, z)) = z$. This implies that if X is compact then it is homeomorphic to a suspension of the space $(\partial B_{\frac{\pi}{2}}(x), d)$. We leave the details to the interested reader. An alternative proof can be done along the lines of [Ket15]. For this note that Euclidean cone $\text{Con}(X)$ is Busemann concave. If $\text{diam } X = \pi$ then $\text{Con}(X)$ contains a line and by Theorem 2.14 it splits so that X must be homeomorphic to a spherical suspension. As in the proof of the splitting theorem, it is not clear whether $\partial B_{\frac{\pi}{2}}(x)$ is totally geodesic.

3. UNIFORMLY SMOOTH SPACES

In this section we define a form global non-negative curvature via a “smoothness” assumption on the metric. Indeed, any Riemannian manifold whose distance is uniformly smooth (see below) must have non-negative sectional curvature on all planes spanned by tangent vectors which are tangent to a ray. However, locally the distance is C^∞ and automatically uniformly smooth.

The notion is inspired by the theory of Banach spaces: Uniform smoothness of the norm implies that Busemann functions are well-defined and linear. In particular they are given as duals of the corresponding vector which represents the ray. In this section we want to use uniform smoothness to show that Busemann functions are quasi-convex. A stronger condition, called p -uniformly convex, will give convexity. From this one can obtain by an argument of Cheeger–Gromoll [CG72] that such a space is an exhaustion of convex sets and can be retracted to a compact totally geodesic subspace, i.e. a variant of the soul theorem. Note, however, in the smooth setting this retract can have a non-empty boundary, so that the compact retract should rather be called a weak soul.

In the end of this section we try to give a local version of non-negative curvature inspired by Gromov’s characterization of non-negative curvature in terms of inward equidistant movements of convex hypersurfaces, see [Gro91].

3.1. Uniform smoothness and convexity of Busemann functions.

Definition 3.1 (Uniform smoothness). A geodesic metric space is said to be uniformly smooth if there is a non-decreasing function $\rho : (0, \infty) \rightarrow [0, \infty)$ such that $\frac{\rho(\epsilon)}{\epsilon} \rightarrow 0$ as $\epsilon \rightarrow 0$ and for all $x, y, z \in X$ with

$$d(y, z) \leq \epsilon \min\{d(x, y), d(x, z)\}$$

it holds

$$d(x, m) \geq (1 - \rho(\epsilon)) \min\{d(x, y), d(x, z)\}$$

whenever m is a midpoint of y and z .

We leave it to the interested reader to show that this is equivalent to the usual definition of uniform smoothness in case X is a Banach space. A stronger variant is the so-called p -uniform smoothness for $p \in (1, 2]$.

Definition 3.2 (p -uniform smoothness). A geodesic metric space is said to be p -uniformly smooth if there is a $C > 0$ such that for all $x, y, z \in X$ it holds

$$d(x, m)^p \geq \frac{1}{2}d(x, y)^p + \frac{1}{2}d(x, z)^p - \frac{C}{4}d(y, z)^p$$

whenever m is a midpoint of y and z .

Note that by Clarkson's inequality every L^p -space is p' -uniformly smooth for $p' = \min\{p, 2\}$. Furthermore, the dual of a q -uniformly convex Banach space is p -uniformly smooth with $\frac{1}{p} + \frac{1}{q} = 1$.

The following was proved by Ohta [Oht08, Theorem 4.2].

Lemma 3.3. *Any Berwald space of non-negative flag curvature is 2-uniformly smooth.*

However, as every Berwald space of non-negative flag curvature is affinely equivalent to a Riemannian manifold of non-negative curvature [Sza81, Sza06], one can obtain most topological and geometric properties directly from the affinely equivalent Riemannian manifold (see [Kel15]).

It is not difficult to show that p -uniform smoothness implies uniform smoothness. Indeed, one has

$$d(x, m)^p \geq (1 - \tilde{\rho})A$$

where $A = \frac{1}{2}d(x, y)^p + \frac{1}{2}d(x, z)^p$ and $A\frac{4}{C}\tilde{\rho} = d(y, z)^p$. Because $A \geq \min\{d(x, y), d(x, z)\}^p$, we have

$$d(x, m) \geq (1 - \rho) \min\{d(x, y), d(x, z)\}$$

where $\rho = \min\{1, \tilde{\rho}\}$. Thus we may choose

$$\rho(\epsilon) = \min\left\{\frac{C}{4}\epsilon^p, 1\right\}$$

to conclude.

An integral part of the soul theorem is the following function which we call *Cheeger–Gromoll function* (w.r.t. $x_0 \in X$)

$$b_{x_0}(x) = \sup b_\gamma(x)$$

where the supremum is taken over all rays starting at x_0 .

Proposition 3.4. *Assume (X, d) is uniformly smooth. Then any Busemann function b_γ associated to a ray γ is quasi-convex. In particular, all Cheeger–Gromoll functions are quasi-convex.*

Proof. Fix distinct points $x, y \in X$ and assume $x, y \notin \gamma([t, \infty))$ for some large t . Let $\epsilon_t = \frac{d(x, y)}{\min\{d(x, \gamma_t), d(y, \gamma_t)\}}$ and m be a midpoint of x and y . Then by uniform smoothness it holds

$$\begin{aligned} t - d(m, \gamma_t) &\leq t - (1 - \rho(\epsilon_t)) \min\{d(x, \gamma_t), d(y, \gamma_t)\}. \\ &= \max\{t - d(x, \gamma_t), t - d(y, \gamma_t)\} + \frac{\rho(\epsilon_t)}{\epsilon_t} d(x, y). \end{aligned}$$

Note $t \rightarrow \infty$ implies $\epsilon_t \rightarrow 0$ so that the rightmost term converges to zero. But then

$$\begin{aligned} b_\gamma(m) &= \lim_{t \rightarrow \infty} t - d(m, \gamma_t) \\ &\leq \lim_{t \rightarrow \infty} \max\{t - d(x, \gamma_t), t - d(y, \gamma_t)\} \\ &= \max\{b_\gamma(x), b_\gamma(y)\}. \end{aligned}$$

Since the Cheeger–Gromoll functions are suprema of quasi-convex functions they are quasi-convex as well. \square

Proposition 3.5. *Assume (X, d) is p -uniformly smooth. Then Busemann function b_γ associated to any ray γ is convex. In particular, all Cheeger–Gromoll functions are convex.*

Proof. Fix distinct point $x, y \in X$ and assume $x, y \notin \gamma([t, \infty))$. Then

$$t - \frac{d(m, \gamma_t)^p}{t^{p-1}} \leq \frac{1}{2} \left(t - \frac{d(x, \gamma_t)^p}{t^{p-1}} \right) + \frac{1}{2} \left(t - \frac{d(y, \gamma_t)^p}{t^{p-1}} \right) + \frac{C}{4} \frac{d(x, y)^p}{t^{p-1}}.$$

Note that the limit of the rightmost term converges to 0 as $t \rightarrow \infty$ and by Lemma 2.8

$$\lim_{t \rightarrow \infty} t - \frac{d(z, \gamma_t)^p}{t^{p-1}} = \lim_{t \rightarrow \infty} t - d(z, \gamma_t) = b_\gamma(z)$$

for any $z \in X$. Combining these gives

$$b_\gamma(m) \leq \frac{1}{2} b_\gamma(x) + \frac{1}{2} b_\gamma(y).$$

Since the Cheeger–Gromoll functions are suprema of convex functions they are convex as well. \square

Remark. By the same arguments the functions

$$\begin{aligned} \tilde{b}_{x_0}(x) &= \limsup_{t \rightarrow \infty} \sup_{y \in \partial B_t(x_0)} t - d(x, y) \\ \hat{b}_{x_0}(x) &= \limsup_{y_n \rightarrow \infty} d(x_0, y_n) - d(x, y_n) \end{aligned}$$

are both quasi-convex or resp. both convex.

Corollary 3.6. *Assume (X, d) is locally compact and p -uniformly smooth and geodesics in (X, d) can be extended locally. Then for any embedding line $\gamma : \mathbb{R} \rightarrow X$*

$$b_{\gamma^+} + b_{\gamma^-} = 0.$$

In particular, b_{γ^+} is affine. If, in addition, (X, d) is non-branching then X is homeomorphic to $b_{\gamma^+}^{-1}(0) \times \mathbb{R}$.

Proof. Let $\eta : [0, 1] \rightarrow X$ is a geodesic and set $f = b_{\gamma^+} + b_{\gamma^-}$. Convexity implies that $t \mapsto f(\eta(t))$ achieves its maximum at 0 or 1 if it is not constant. So assume $f(\eta(1)) < 0$. From the assumption we can extend η beyond $\eta(1)$ such that $\tilde{\eta} : [0, 1 + \epsilon] \rightarrow X$ is a local geodesics agreeing with η on $[0, 1]$. Let $a \in [0, 1)$ such that $\tilde{\eta}|_{[a, 1 + \epsilon]}$ is a geodesic. Then

$$\max_{t \in [a, 1 + \epsilon]} f(\tilde{\eta}(t)) = f(\eta(1)) = 0.$$

However, this implies that $f(\tilde{\eta})$ is constant on $[a, 1 + \epsilon]$. But then $f(\eta)$ also attains its maximum at $t = a$ implying that $f(\eta(t)) = 0$ for all $t \in [0, 1]$.

The equality shows that we may glue the asymptotic rays. Assume now that (X, d) is non-branching then there is at most one pair of asymptotic rays which can be glued together at a fixed point. In particular, for every $x \in X$ there are unique line η_x parallel to γ such that $x \in \eta_x$ and $\eta_x(b_{\gamma^+}(x)) = x$. Note also that $\eta_{\eta(t)} = \eta_{\eta(s)}$ for $s, t \in \mathbb{R}$.

Let $\Xi = \{\eta_x \mid x \in X\} \subset \text{Lip}(\mathbb{R}, X)$. By local compactness we can show that $\Psi : \Xi \rightarrow b_{\gamma^+}^{-1}(0)$ defined by $\Psi(\eta_x) = \eta_x(0)$ is a homeomorphism. In particular, the assignment $x \mapsto (\eta_x(0), b_{\gamma^+}(x))$ is a homeomorphism between X and $b_{\gamma^+}^{-1}(0) \times \mathbb{R}$. \square

Remark. Without local extendibility the result might be wrong. Indeed, if X is the product of a filled triangle and the real line then there exist two convex functions such that their sum is non-positive and somewhere negative, and they sum up to zero at the line formed by a vertex of the triangle. However, if we assume that (X, d) is Busemann concave then one can use the fact that

$$b_{\gamma^\pm}(\eta(0)) = b_{\eta^\mp}(\gamma(0))$$

to show affinity of the Busemann function. We leave the details to the interested reader.

The following is an analogue of the case of standard Busemann functions. The result also holds for \tilde{b}_{x_0} and \hat{b}_{x_0} .

Lemma 3.7. *Assume (X, d) is locally compact and unbounded. Then for any $x \in X$ there is a ray $\gamma_x : [0, \infty) \rightarrow X$ emanating from x such that*

$$b_{x_0}(\gamma_x(t)) = b_{x_0}(x) + t.$$

Proof. This is true for $b_{\gamma_{x_0}}$ where γ_{x_0} is a ray emanating from x_0 . From the definition there is a sequence $(\gamma_{x_0}^n)$ of rays emanating from x_0 such that $b_{x_0}(x) = \lim_{n \rightarrow \infty} b_{\gamma_{x_0}^n}(x)$. Let γ_x^n be the rays emanating from x with $b_{\gamma_{x_0}^n}(\gamma_x^n(t)) = b_{\gamma_{x_0}^n}(x) + t$. By local compactness we can assume $\gamma_{x_0}^n \rightarrow \gamma_{x_0}$ and $\gamma_x^n \rightarrow \gamma_x$. Thus

$$\begin{aligned} b_{x_0}(x) + t &= \lim_{n \rightarrow \infty} b_{\gamma_{x_0}^n}(x) + t \\ &= \lim_{n \rightarrow \infty} b_{\gamma_{x_0}^n}(\gamma_x^n(t)) \\ &= b_{\gamma_{x_0}}(\gamma_x(t)) \leq b_{x_0}(\gamma_x(t)). \end{aligned}$$

However, b_{x_0} is 1-Lipschitz implying $|b_{x_0}(\gamma_x(t)) - b_{x_0}(x)| \leq t$ and thus the inequality above is an equality. \square

Theorem 3.8. *Any uniformly smooth locally compact metric space (X, d) admits a quasi-convex exhaustion function $b : X \rightarrow \mathbb{R}$ with compact sublevels such that $S = b^{-1}(\min b)$ has empty interior.*

Proof. Let $b = b_{x_0}$ for some $x_0 \in X$. The fact that S_{x_0} has empty interior follows from the lemma above. Indeed, let $x \in S_{x_0}$. There is a ray γ_x emanating from x with $b(\gamma_x(t)) = b(x) + t$. Assume $\gamma_x(t) \in S_{x_0}$ then

$$\min b = b(\gamma_x(t)) = b(x) + t = \min b + t$$

which can only hold if $t = 0$. Therefore, $\text{int } S_x = \emptyset$. \square

We call S_{x_0} a *weak soul* as there is no way to dissect it further without an intrinsic notion of boundary (see also below).

Corollary 3.9. *If, in addition, b is strictly quasi-convex then S is a single point.*

Proof. Let $x, y \in S_{x_0}$ be two point and $\gamma : [0, 1] \rightarrow S_{x_0}$ be a geodesic connecting x and y . By strict quasi-convexity we have $b(\gamma_t) \leq \max\{b(x), b(y)\}$ with strict inequality if $x \neq y$. But that case cannot happen because $b(x), b(y) = \min b$. \square

Strict quasi-convexity of b implies that the sublevel sets are strictly convex. For Alexandrov spaces we can show the converse.

Proposition 3.10. *Assume (M, d) is an Alexandrov space of non-negative curvature. Then C_s is strictly convex (for some $s > \min b$) if and only if b is strictly quasi-convex on C_s .*

Remark. It is well possible that b is not strictly quasi-convex outside of C_s . An example is given as follows: glue the half cylinder $[0, \infty) \times S^{n-1}$ and the lower hemisphere S^{n-1}_2 along their boundaries which are isometric to S^{n-1} . Then b is up to a constant the distance from the south pole and is not strictly (quasi-)convex outside of the hemisphere.

Proof. Obviously strict quasi-convexity of b on C_s implies strict convexity of C_s . Assume instead C_s is strictly convex. We will use the rigidity of the distance from the boundary proven by [Yam12]. From Corollary 3.15 below we have $(b)|_{C_s} = s - b_{C_s}$ where $b_{C_s}(x) = d(x, \partial C_s)$. If b_{C_s} was not strictly quasi-convex then there is a non-constant geodesic γ in C_s such that b_{C_s} is constant along γ . The rigidity result in [Yam12, Proposition 2.1] shows that there is a non-constant geodesic $\eta : [0, 1] \rightarrow \partial C_s$ which is impossible by strict convexity of C_s . \square

The above actually gives a more general characterization: a closed convex set C in an Alexandrov space of non-negative curvature is strictly convex iff $b_C(\cdot) = d(\cdot, \partial C)$ is strictly quasi-convex on C .

3.2. An application of the technique in the smooth setting. In this section we apply the technique above in the smooth setting. We show that if a Finsler manifold with non-negative flag curvature has vanishing tangent curvature along a geodesic η then any Busemann function is convex along η . This can be used to simplify the proof of orthogonality of certain tangent vectors in [Lak14] and avoid a complicated Toponogov-like theorem proved in [KOT12]. In order to avoid a lengthy introduction, we refer the reader to [Oht08] for the notation used in this section. The focus will be on the proof of uniform smoothness of the distance [Oht08, Theorem 4.2, Corollary 4.4].

Note that in the Finsler setting a (forward) geodesic refers to an constant-speed auto-parallel curve $\gamma : [0, 1] \rightarrow M$ such that $d_F(\gamma_0, \gamma_1) = F(\dot{\gamma}_0)$ where d_F is the asymmetric metric induced by the Finsler structure F . Assuming smoothness of $\gamma : [0, 1] \rightarrow M$, this is equivalent to

$$d_F(\gamma_t, \gamma_s) = (s - t)d_F(\gamma_0, \gamma_1)$$

for $1 \geq s \geq t \geq 0$.

Lemma 3.11. *Let (M, F) be a connected forward geodesically complete C^∞ -Finsler manifold. Assume (M, F) has non-negative flag curvature and for all $x \in M$ the norms F_x are 2-uniformly smooth for some constant $S \geq 1$. If $\eta : [0, 1] \rightarrow M$ is a (forward) geodesic with $\mathcal{T} = 0$ on $T_\eta M$ then*

$$d^2(x, \eta_t) \geq (1 - t)d^2(x, \eta_0) + td^2(x, \eta_1) - (1 - t)tS^2d^2(\eta_0, \eta_1).$$

Proof. We only sketch the argument as the calculations are exactly those of [Oht08, Proof of Theorem 4.2]. The only time the assumption $\mathcal{T} \geq -\delta$ is applied is for estimating

$$g_{T(r)}(D_U^T U, T) = g_{T(r)}(D_U^U U, T) - \mathcal{T}_{T(r)}(v)$$

where $T(r) \in T_{\eta_0} M$ and $v = \frac{\dot{\eta}_0}{F(\dot{\eta}_0)}$. Since $D_U^U U(r) = 0$ and $\mathcal{T} = 0$ on $T_{\eta_0} M$ we see that $g_{T(r)}(D_U^T U, T) = 0$. In particular, it is possible to choose $\delta = 0$. Then following the calculation we obtain the result via [Oht08, Corollary 4.4]. \square

Along the lines of the proof of Proposition 3.5 we immediately obtain the following corollary.

Corollary 3.12. *Let (M, F) be as above. Then any Busemann function associated to a ray $\gamma : [0, \infty) \rightarrow M$ is convex along η , i.e. $t \mapsto b_\gamma(\eta(t))$ is convex on $[0, 1]$.*

The next result was proved in [Lak14, Lemma 4.8] for closed forward geodesics, i.e. a map $\eta : S^1 \rightarrow M$ such that η is locally geodesic. Note that we do not need the reversibility assumption of the closed geodesic [Lak14, Theorem 1.2]. The author wonders whether $\mathcal{T} = 0$ on $T_\eta M$ would imply that the reversed $\tilde{\eta} : t \mapsto \eta(1 - t)$ is a geodesic loop as well. This would be the case if $\mathcal{T} = 0$ in a neighborhood U of η , i.e. M is Berwaldian in U .

Corollary 3.13. *Let (M, F) be as above and assume $\eta : [0, 1] \rightarrow M$ is a (forward) geodesic loop, i.e. η is locally (forward) geodesic such that $\eta_0 = \eta_1$. Then for any ray $\gamma : [0, \infty) \rightarrow M$ with $\gamma_0 = \eta_0$ it holds*

$$g_{\dot{\gamma}_0}(\dot{\gamma}_0, \dot{\eta}_0) \leq 0 \quad \text{and} \quad g_{\dot{\gamma}_0}(\dot{\gamma}_0, \dot{\eta}_1) \geq 0.$$

In particular, if η is a (forward) closed geodesic then $g_{\dot{\gamma}_0}(\dot{\gamma}_0, \dot{\eta}_0) = 0$.

Proof. This is a direct consequence of the first variation formula and the convexity of the Busemann function. Indeed, by the first variation formula (see [BCS00, Exercise 5.2.4]) and uniqueness of geodesics between γ_0 and γ_t we have

$$\lim_{s \rightarrow 0^+} \frac{d(\eta_s, \gamma_t) - d(\eta_0, \gamma_t)}{d(\eta_0, \eta_s)} = -g_{\dot{\gamma}_0}(\dot{\gamma}_0, \frac{\dot{\eta}_0}{F(\dot{\eta}_0)})$$

and

$$\lim_{s \rightarrow 1^-} \frac{d(\eta_s, \gamma_t) - d(\eta_0, \gamma_t)}{d(\eta_1, \eta_s)} = g_{\dot{\gamma}_0}(\dot{\gamma}_0, \frac{-\dot{\eta}_1}{F(-\dot{\eta}_1)}).$$

Now convexity of the Busemann function b_γ associated to γ implies that

$$b_\gamma(\eta_s) \leq b_\gamma(\eta_0) = 0.$$

However, we have

$$d(\eta_0, \gamma_t) - d(\eta_s, \gamma_t) = t - d(\eta_s, \gamma_t) \leq b_\gamma(\eta_s)$$

which immediately shows that $g_{\dot{\gamma}_0}(\dot{\gamma}_0, \dot{\eta}_0) \leq 0$. A similar argument shows $g_{\dot{\gamma}_0}(\dot{\gamma}_0, -\dot{\eta}_1) \leq 0$.

If γ is a closed geodesic then also $\dot{\eta}_0 = \dot{\eta}_1$ so that

$$0 \geq g_{\dot{\gamma}_0}(\dot{\gamma}_0, \dot{\eta}_0) = g_{\dot{\gamma}_0}(\dot{\gamma}_0, \dot{\eta}_1) \geq 0.$$

\square

3.3. A strong deformation retract onto a weak soul. Having a weak soul shows that all geodesic loops starting in S_{x_0} must stay in S_{x_0} . Thus if all loops starting at some $x \in S_{x_0}$ are homotopic to a geodesic loop starting at x then the fundamental group of X and S_{x_0} are the same. In general, it is not clear how to use a similar argument for higher homotopy groups. If S_{x_0} was a strong deformation retract of X then the all fundamental group would agree.

Throughout this section we assume that (X, d) is a locally compact and uniformly smooth. Note that this implies that closed bounded sets are compact. We want to give a condition which implies that each sublevel of b_{x_0} is a strong deformation retract of X .

The first result is just a reformulation of [CG72, Proposition 1.3]. The only ingredient used in their proof is the fact that the sublevels of b_{x_0} are closed and totally geodesic.

Lemma 3.14. *The sublevel sets $C_s = b_{x_0}^{-1}([0, s])$ are compact and the boundaries have the following form: Let $0 < s < t$ then*

$$\partial C_s = \{x \in C_t \mid d(x, \partial C_t) = t - s\}.$$

Proof. Note by definition $b_{x_0}(\gamma(t)) = t$ for any ray starting at x_0 . In particular, b_{x_0} is unbounded when restricted to such a ray. Now suppose that sublevels of b_{x_0} are not compact. Then there is an s and a sequence $x_n \rightarrow \infty$ with $b_{x_0}(x_n) \leq s$. Let γ_n be a geodesic connecting x_0 and x_n . By local compactness there is a subsequence of (γ_n) converging to a ray γ starting at x_0 . As $C_s = b_{x_0}^{-1}([0, s])$ is closed convex, γ_n and γ lie entirely in C_s . This, however, implies that $b_{x_0}(\gamma(t)) \leq s$ which is a contradiction.

To see that second claim, note b_{x_0} is 1-Lipschitz so that $x \in \partial C_s$ and $y \in C_t$ with $d(x, y) < t - s$ implies $b_{x_0}(y) < t$. \square

The following shows that b_{x_0} can be described in a local manner. It can be used to show that a certain notion of positive curvature means that the weak soul S_{x_0} consists only of a single point, see Proposition 3.29 below.

Corollary 3.15. *For each $r > m$ where $m = \min b_{x_0}$ define the function $b_{C_r} : C_r \rightarrow [0, r - m]$ by*

$$b_{C_r}(x) = d(x, \partial C_r).$$

Then b_{C_r} is quasi-concave and it holds

$$b_{x_0}(x) = r - b_{C_r}(x)$$

for $x \in C_r$.

Proof. If $x \in \partial C_s$ for $s \leq r$ then $b_{x_0}(x) = s$. Thus the lemma implies

$$b_{x_0}(x) = r - (r - s) = r - b_{C_r}(x).$$

It remains to show that $b_{x_0}^{-1}(s) = \partial C_s$. Let $x \in b_{x_0}^{-1}(s)$. By Lemma 3.7 there is a ray γ_x emanating from x such that

$$b_{x_0}(\gamma_x(t)) = b_{x_0}(x) + t.$$

Since $\gamma_x(r - s) \in \partial C_r$ we obtain $d(x, \partial C_r) \leq r - s$. As b_{x_0} is 1-Lipschitz this must be an equality so that the previous lemma implies $x \in \partial C_s$. \square

Corollary 3.16. *Assume (X, d) is non-branching and let A_s be the set of points $x \in \partial C_s$ such that $d(x, S_{x_0}) = s - \min b_{x_0}$. Then $A_{r,s} = \cup_{r \leq r' \leq s} A_{r'}$ is homeomorphic to $A_s \times [0, 1]$ for all $r, s > \min b_{x_0}$. Furthermore, there is a continuous map $\Phi_r : A_r \rightarrow S_{x_0}$.*

Proof. We only indicate the proof as the construction is similar to the one above: The ray γ given by Lemma 3.7 can be extended backwards to a ray starting in S_{x_0} . More precisely, let γ' be a unit speed geodesic connecting y_x and x where y_x is a foot point of x in S_{x_0} . Because $d(x, S_{x_0}) = b_{x_0}(x) - \min b_{x_0}$, $\gamma \cup \gamma'$ is a ray emanating from y_x . By non-branching and compactness of S_{x_0} we see that there is exactly one $y \in S_{x_0}$ with $d(x, y) = d(x, S_{x_0})$. Furthermore, the assignment $x \mapsto y_x$ is continuous.

Furthermore, note that γ' intersects each A_r with $\min b_{x_0} < r \leq s$ in exactly one point. This shows that $A_{r,s}$ is homeomorphic to $A_s \times [0, 1]$. \square

In general it is not true that $A_s = \partial C_s$. Indeed, assume X is a space which is obtained by gluing a flat half-strip $\{(x, y) \in \mathbb{R}^2 \mid x \in [-1, 1], y \geq 0\}$ and half a disk $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1, y \leq 0\}$. Then for any $p \in X$ it holds $b_p^{-1}(\min b_p) = \{(0, -1)\}$ and $\cup_{s \geq \min b_p} A_s = \{(0, y) \in \mathbb{R}^2 \mid y \geq -1\}$ which corresponds to the ray starting at the ‘‘south pole’’ of the disk. However, $\partial C_s = \{(x, y) \in X \mid y = y_s\}$ for some y_s depending on s and p .

The following result shows that for a subclass with nice gradient flow behavior there is a non-expansive deformation retract. Note, however, it is expected that only Riemannian-like metric space have such property (see [OS12]). For a general overview of gradient flows on metric spaces we refer to [AGS08]. In the setting of Alexandrov spaces the non-expansiveness was shown in [PP94, Lyt06].

Proposition 3.17. *Assume (X, d) is p -uniformly smooth and gradient flows of convex functions exist and are non-expansive. Then there is a strong deformation retract $F : X \times [0, 1] \rightarrow X$ onto S_{x_0} such that F_t is a contraction onto a sublevel set of b_{x_0} . In particular, $F_1 : X \rightarrow S_{x_0}$ is a contraction.*

Remark. Note that only convexity of b_{x_0} is needed. More generally, a slight adjustment of proof works if b_{x_0} is p -convex. It is unclear whether quasi-convexity of b_{x_0} would be sufficient.

Proof. If b_{x_0} is convex so is $b_{x_0}^r = \max\{b_{x_0}, r\}$. Denote the gradient flows of $b_{x_0}^r$ by $\Phi_t^r : X \rightarrow X$. As the definition is local we see that for $x \in X$ with $b_{x_0}(x) > r \geq r'$ there is a $t_x^r \in (0, \infty)$ such that $\Phi_t^r(x) = \Phi_{t'}^{r'}(x)$ for $t \in [0, t_x^r]$. More precisely, the equality fails once $b_{x_0}(\Phi_t^r(x)) \leq r$. Similarly, one can show that Φ_t^r restricted to C_r is the identity. Note that any $r \geq \min b_{x_0}$ is reached in finite time. Furthermore, Φ_t^r is constant on C_r .

For any $x, y \in X$ and $t \geq \max\{t_x^r, t_y^r\}$ it holds

$$\begin{aligned} \Phi_t^r(x) &= \Phi_{t_x^r}^r(x) \\ \Phi_t^r(y) &= \Phi_{t_y^r}^r(y). \end{aligned}$$

Non-expansiveness implies

$$d(\Phi_t^r(x), \Phi_t^r(y)) \leq d(x, y).$$

Define now

$$F_r(x) = \lim_{t \rightarrow \infty} \Phi_t^r(x).$$

Since each x reaches C_r in finite time, the map is well-defined. Furthermore, we see that F_r is non-expansive and maps X onto C_r . Also note that F_r is the identity on C_r .

Let $\phi : [0, 1] \rightarrow [\min b_{x_0}, \infty)$ be a homeomorphism. We claim that the following map $F : X \times [0, 1] \rightarrow X$ satisfies the required properties:

$$F(x, s) = \begin{cases} F_{\phi(1-s)}(x) & s > 0 \\ x & s = 0. \end{cases}$$

It suffices to show that $(x_n, s_n) \rightarrow (x, 0)$ implies $F(x_n, s_n) \rightarrow x$. This follows because there is an $r > 0$ such that $x_n, x \in C_r$. Since $t_n = \phi(1 - s_n) \rightarrow \infty$ we see that for large n it holds $t_n \geq r$ and thus

$$F(x_n, s_n) = F_{t_n}(x_n) = x_n \rightarrow x.$$

□

For non-Riemannian-like metric spaces the above construction does not work. However, if we assume that locally projections onto convex sets are unique then it is still possible to construct a strong deformation retract. Such an assumption would imply a local weak upper curvature bound which is not satisfied on Alexandrov spaces. Indeed, Petrunin gave an example of an Alexandrov space which contains a convex set without such a property [Pet13]. As this is not published anywhere else here a short construction: Let X be the doubling of the (convex) region $F = \{(x, y) \in \mathbb{R}^2 \mid y \geq x^2\}$. Now let A be the (doubling) of F intersected with the ball with center $(0, -1)$ passing through $(1, 1)$. This set is convex in X . Every point on the boundary of F outside of A has two projection points onto A .

Definition 3.18 (strict convexity radius). The (strict) convexity radius $\rho(x)$ of a point x in a metric space (X, d) is the supremum of all $r \geq 0$ such that the closed ball $\bar{B}_s(x)$ is (strictly) convex for all $s \in (0, r]$.

Definition 3.19 (injectivity radius). The injectivity radius $i(x)$ of a point $x \in X$ is the supremum of all $r \geq 0$ such that for all $y \in B_r(x)$ there is only one geodesic connecting x and y .

Lemma 3.20. *Assume (X, d) has strict convexity radius locally bounded away from zero. Then the injectivity radius is locally bounded away from zero.*

Remark. The lemma shows that if (X, d) was in addition also a weak Busemann G -space then it is actually a Busemann G -space (see below).

Proof. Choose some large ball $B_R(x_0)$ such that $\rho(x) \geq \epsilon$ for all $x \in B_R(x_0)$ and $\epsilon \leq \frac{R}{4}$. We claim that $i(x) \geq 2\epsilon$. Indeed, let $y \in B_{2\epsilon}(x)$ for $x \in B_{\frac{R}{4}}(x_0)$ and set $2d = d(x, y) < 2\epsilon$. Then $B_d(x) \cap B_d(y) = \emptyset$, and $\bar{B}_d(x) \cap \bar{B}_d(y)$ is non-empty and consists entirely of midpoints of x and y . However, if there were two distinct points $m, m' \in \bar{B}_d(x) \cap \bar{B}_d(y)$ then their midpoint would be in $B_d(x) \cap B_d(y)$ which is a contradiction. This implies that midpoints in $B_R(x_0)$ are unique. Local compactness immediately gives uniqueness of the geodesics. □

Lemma 3.21. *Assume (X, d) is locally compact and has strict convexity radius locally bounded away from zero. Let C be a compact convex subset of X then there is an $\epsilon > 0$ such that any point y in the ϵ -neighborhood C_ϵ of C has a unique point $y_C \in C$ such that $d(y, y_C) = d(y, C)$.*

Proof. For every $x_0 \in X$ and $R > 0$ there is an $\epsilon > 0$ such that $\rho(x), i(x) \geq 2\epsilon$ for all $x \in B_R(x_0)$. Assume $C \subset B_{\frac{R}{2}}(x_0)$ and $\epsilon < \frac{R}{2}$. We claim that C_ϵ has the required property.

Let $y \in C_\epsilon$. Since C is compact, there is at least one such $y_C \in C$ with $d(y, y_C) = d(y, C)$. Assume there are two distinct $y_C, y'_C \in C$ with $d(y, y_C) = d(y, y'_C) = d(y, C) = \epsilon' < \epsilon$. From the definition we have $y_C, y'_C \in \partial B_{\epsilon'}(y)$. Assume m is the midpoint of y_C and y'_C . Since $\rho(y) \geq \epsilon'$, $\bar{B}_{\epsilon'}(y)$ is strictly convex. Thus $m \in \text{int } B_{\epsilon'}(y)$. However, this implies $d(m, y) < d(y, C)$ which is a contradiction as convexity of C shows $m \in C$. \square

Corollary 3.22. *Assume there is a convex set C' with $C \subset C' \subset C_\epsilon$. Then the projection induces a strong deformation retract of C' onto C .*

Proof. Denote the closest point projection by $p : C' \rightarrow C$. As the geodesic $\gamma : [0, 1] \rightarrow X$ connecting x and $p(x)$ is unique we may define a map $P : C' \times [0, 1] \rightarrow C'$ by

$$P(x, t) = \gamma_t.$$

Note that $P(x, 0) = x$, $P(x, 1) = p(x)$ and by convexity of C' we also have $P(x, t) \in C'$ for all $(x, t) \in C' \times [0, 1]$. Furthermore, local compactness together with local uniqueness of the geodesics shows that P is continuous and thus a strong deformation retract. \square

Theorem 3.23. *Assume (X, d) has strict convexity radius locally bounded away from zero and there a quasi-convex continuous function $b : X \rightarrow [0, \infty)$ with compact sublevels. Then for any $r > 0$ the sets $C_r = b^{-1}([0, r])$ are strong deformation retracts of (X, d) . In particular, X retracts onto the weak soul S .*

Proof. The construction is similar to [CG72, Section 2]. By the above lemma any point $x \in C_s$ admits a unique projection point $f_{s,r}^t(x) \in C_{(1-t)s+tr}$ if $s \geq r$ and $|s - r| \leq \epsilon(b, s)$ where ϵ is given as above by choosing R such that $C_s \subset B_{\frac{R}{2}}(x_0)$. Note that $s \mapsto \epsilon(b, s)$ is uniformly bounded away from zero on $[0, M]$ for all $M > 0$.

Now define $s_0 = r$ and $s_{n+1} = s_n + \frac{1}{2}\epsilon(b, s_n)$. We claim that $s_n \rightarrow \infty$ as $n \rightarrow \infty$. Indeed, if $s_n \rightarrow s < \infty$ then $\epsilon(b, s_n) \rightarrow 0$ which is a contradiction because $\epsilon(b, s_n) \geq \epsilon(b, M)$ whenever $s_n \leq M$.

Let $t \in [r, \infty)$. We define a family of functions $(f_n^t : C_{s_n} \rightarrow C_{s_n})_{n \in \mathbb{N}}$ as follows: Set $f_0^t = \text{id}$ and

$$f_{n+1}^t = \begin{cases} f_n^t \circ f_{s_n, s_{n-1}}^0 & t \leq s_{n-1} \\ f_{s_n, s_{n-1}}^\lambda & t \in [s_{n-1}, s_n] \text{ with } \lambda = \frac{t - s_{n-1}}{s_n - s_{n-1}} \\ \text{id} & t \geq s_n. \end{cases}$$

One can verify that f_{n+1}^t is continuous. Furthermore, let $m \leq n$ then

$$f_n^t = f_m^t \quad \text{on } C_{s_m}, t \geq s_m$$

and

$$f_n^t(x) = x \quad \text{if } (x, t) \in C_{s_m} \times [s_m, \infty).$$

Now let $\phi : [0, 1] \rightarrow [r, \infty)$ be an increasing homeomorphism. Then define

$$F(x, t) = \begin{cases} \lim_{n \rightarrow \infty} f_n^{\phi(t)}(x) & t < 1 \\ x & t = 1. \end{cases}$$

Note that F is continuous on $X \times [0, 1]$. So let $(x_n, t_n) \rightarrow (x, 1)$ then $x_n, x \in C_{s_m}$ for some large m . As $t_n \rightarrow 1$ we have $s_m \leq \phi(t_n)$ for $n \geq n_0$ and thus

$$F(x_n, t_n) = f_m^{\phi(t_n)}(x_n) = x_n \rightarrow x = f_m^{\phi(1)}(x) = F(x, 1).$$

This shows that F is a strong deformation retract because $f_n^0(C_{s_n}) = C_r$. \square

3.4. Towards a soul on uniformly smooth spaces. To complete the soul theorem along the lines of Cheeger–Gromoll one needs a proper notion of intrinsic boundary. We say that geodesics in (X, d) can be *locally extended* if for all geodesics $\gamma : [0, 1] \rightarrow X$ there is a local geodesic $\tilde{\gamma} : [-\epsilon, 1] \rightarrow X$ agreeing on $[0, 1]$ with γ . In the smooth setting, $x \in \partial M$ would imply that there is a geodesic starting at x such that $-\dot{\gamma}_x \neq T_x M$. In particular, γ cannot be extended beyond x .

Definition 3.24 (geodesic boundary). The *geodesic boundary* $\partial_g C$ of subset $C \subset X$ is the set of all $x \in C$ such that there is a geodesic in C that cannot be locally extended in C .

Remark. The notion differs from the boundary of Alexandrov spaces. Indeed, if the singular set is dense then the geodesic boundary is dense as well.

The following is a replacement for the notion of closed manifold.

Definition 3.25 (weak Busemann G -space). A geodesic metric space is a weak Busemann G -space if its geodesics can be locally extended in a unique way. In particular, $\partial_g X = \emptyset$.

Unique extendability implies non-branching. The Heisenberg group equipped with a left-invariant Carnot–Caratheodory metric is a weak Busemann G -space which is not a (strong) Busemann G -space. See [Bus55] for more on Busemann G -spaces. From Lemma 3.20 we immediately see that.

Lemma 3.26. *A weak Busemann G -space with strict convexity radius locally bounded away from zero is a Busemann G -space.*

By [CG72, Theorem 1.6] the geodesic boundary above agrees with the boundary of open convex subsets if (X, d) is a Riemannian manifold. Their proof also works for Finsler manifolds. In the current setting this is almost true.

Lemma 3.27. *If (X, d) is a weak Busemann G -space and $C = \text{cl}(\text{int } C)$ is convex then*

$$\partial C = \text{cl } \partial_g C.$$

Proof. The inclusion $\partial_g C \subset \partial C$ follows from local extendability and the fact that $C = \text{cl}(\text{int } C)$.

Let $x \in \partial C$ then there are $y_n \in \text{int } C$ and $z_n \in X \setminus C$ such that $y_n, z_n \rightarrow x$. Furthermore, there is a geodesic $\gamma_n : [0, 1] \rightarrow X$ connecting y_n and z_n . Observe that for any $\epsilon > 0$ and sufficiently large n it holds $\gamma_n(t) \in B_\epsilon(x)$.

By convexity of C there is a $t_0 > 0$ such that $\gamma_n([0, t_0]) \subset C$ and $\gamma_n((t_0, 1]) \subset (X \setminus C)$. But this means $\gamma_n(t_0) \in \partial_g C$ proving the claim. \square

Definition 3.28 (Gromov non-negative curvature). A weak Busemann G -space is said to be *non-negatively curved in the sense of Gromov* if for all closed convex sets C the functions

$$b_C : x \mapsto d(x, \partial_g C)$$

are quasi-concave on C , i.e. the superlevels of $b_C : C \rightarrow [0, \infty)$ are convex. If b_C is strictly quasi-concave at x_0 , then we say the space has *positive curvature in the sense of Gromov at x_0* . If it holds for all $x_0 \in X$ then we just say (X, d) has positive curvature in the sense of Gromov. Furthermore, we say it has *strong non-negative (positive) curvature* in the sense of Gromov if b_C is convex (strictly convex).

Remark. (1) In Gromov's terminology [Gro91, p.44] this means that the inward equidistant sets $(\partial C)_{-\epsilon}$ of the convex hypersurface ∂C remain convex.

(2) If $C = \text{cl}(\text{int } C)$ then $b_C(x) = d(x, \partial C)$ by the lemma above.

(3) The opposite curvature bound is usually called Pedersen convex, resp. Pedersen non-positive/negative curvature [Ped52, Bus55, (36.15)]. This property says that the r -neighborhood of convex sets remain convex. It is sometimes called "has convex capsules". Note that this characterization was rediscovered by Gromov [Gro91, p.44] as "the outward equidistant sets $(\partial C)_\epsilon$ of a convex convex hypersurface ∂C remain convex".

Note that Gromov non-negative curvature is rather weak. It is trivially satisfied on weak Busemann G -spaces whose only closed convex subset with non-trivial boundary are geodesic. Indeed, an example is given by Heisenberg group, see [MR05].

The author wonders if it is possible to define non-negative curvature in the sense of Gromov only in terms of (local/global) properties of the metric not relying on sets.

Assume in the following that (X, d) is proper, non-branching, uniformly smooth and non-negatively curved in the sense of Gromov. Furthermore, we assume $x_0 \in X$ is fixed and $C_r = b_{x_0}^{-1}((-\infty, r])$ are the sublevels of the Cheeger–Gromoll function (see above).

Proposition 3.29. *If (X, d) has positive curvature in the sense of Gromov then S_{x_0} is a point.*

Proof. Let $s > \min b_{x_0}$. Then Corollary 3.15 shows that b_{x_0} and $s - b_{C_s}$ agree on C_s . Positive curvature shows that b_{C_s} is strictly quasi-concave so by Corollary 3.9 the set S_{x_0} must be a point. \square

Now we want to show that one can successively reduce S_{x_0} . For such a reduction we need to assume that the geodesic boundary does not behave too badly. For this recall that a set A is nowhere dense in a closed subset B if its closure in B has empty interior w.r.t. B .

Definition 3.30 (non-trivial boundary). A metric space is said to have non-trivial boundary property if for all non-trivial closed convex set C the geodesic boundary $\partial_g C$ is nowhere dense in C .

By [CG72, Theorem 1.6] any Riemannian manifold satisfies the non-trivial boundary property.

Lemma 3.31. *Assume (X, d) has non-trivial boundary property and non-negative curvature in the sense of Gromov. Then for any closed convex set C the set $S = b_C^{-1}(\max b_C)$ is a closed convex subset without interior w.r.t. C . We call S the weak soul of C .*

If the Cheeger–Gromoll function b_{x_0} is seen as minus of the renormalized distance from the boundary at infinity then uniform smoothness implies Gromov non-negative curvature in the (very) large. Corollary 3.15 shows that the weak soul of S_{x_0} actually agrees with the weak soul of the sublevel set thus justifying the terminology.

Given a convex exhaustion function b , one can now start with X and obtain a weak soul $C_0 = S_{x_0}$. This set admits a weak soul C_1 as well. Repeatedly applying the lemma shows that there is a flag of closed convex set $C_0 \supset C_1 \supset \dots$. In order to show that this procedure eventually ends we need the following.

Definition 3.32 (topological dimension). A metric space (X, d) is said to have topological dimension n , denoted by $\dim_{\text{top}} X = n$, if every open cover $(U_\alpha)_{\alpha \in I}$ of X admits a cover $(V_\beta)_{\beta \in J}$ such that for all $\beta \in J$ there is an $\alpha \in I$ with $V_\beta \subset U_\alpha$ and each $x \in X$ is contained in at most n sets V_β .

Lemma 3.33. *Let C be convex and S its weak soul. Then*

$$1 + \dim_{\text{top}} S \leq \dim_{\text{top}} C.$$

If S contains a geodesic then $\dim_{\text{top}} S > 0$.

Proof. This follows along the lines of Corollary 3.16. Indeed, look at the sets $A_r \subset \partial C_r$ such that

$$d(x, S) = r \quad \text{for all } x \in A_r$$

where $C_r = b_C^{-1}([r, \infty))$.

Then as in Corollary 3.16 one can show that sets $A_r \times [0, 1]$ is homeomorphic to a subset of C . In particular, it holds

$$\begin{aligned} 1 + \dim_{\text{top}} A_r &= \dim_{\text{top}} A_r \times [0, 1] \\ &\leq \dim_{\text{top}} C. \end{aligned}$$

Also observe that there is a continuous map of A_r onto S so that $\dim_{\text{top}} A_r \geq \dim_{\text{top}} S$.

The last statement follows as an embedded line has topological dimension 1. \square

Corollary 3.34. *Any finite dimensional, proper, non-branching, uniformly smooth metric space (X, d) of non-negative curvature in the sense of Gromov and non-trivial boundary property admits a closed convex set S such that $\partial_g S = \emptyset$. We call S a soul of X .*

Note that the construction of the deformation retracts also works inside of convex sets when replacing the Cheeger–Gromoll exhaustion b_{x_0} by b_C . Thus we may summarize the results above as follows.

Theorem 3.35. *Assume (X, d) is uniformly smooth, strong non-negatively curved in the sense of Gromov and has strict convexity radius locally bounded away from zero. Then there is a strong deformation retract onto a closed convex set S with $\partial_g S = \emptyset$.*

Note that by Lemma 3.26 and [Ber77] the assumptions imply that (X, d) is finite dimensional.

REFERENCES

- [AGS08] L. Ambrosio, N. Gigli, and G. Savaré, *Gradient flows in metric spaces and in the space of probability measures*, second ed., Lectures in Mathematics ETH Zürich, Birkhäuser Verlag, Basel, 2008.
- [AKP] S. Alexander, V. Kapovich, and A. Petrunin, *Alexandrov geometry*, <http://anton-petrunin.github.io/book/>.
- [BB11] A. Björn and J. Björn, *Nonlinear Potential Theory on Metric Spaces*, European Mathematical Society Publishing House, Zuerich, Switzerland, 2011.
- [BCS00] D. Bao, S. Chern, and Z. Shen, *An Introduction to Riemann-Finsler Geometry*, Springer, 2000.
- [Ber77] V. N. Berestovskii, *The finite-dimensionality problem for Busemann G -spaces*, Siberian Mathematical Journal **18** (1977), no. 1, 159–161.
- [BGP92] Y. Burago, M. Gromov, and G. Perelman, *A.D. Alexandrov spaces with curvature bounded below*, Russian Mathematical Surveys **47** (1992), no. 2, 1–58.
- [BH99] M. R. Bridson and A. Häfliger, *Metric Spaces of Non-Positive Curvature*, Springer, 1999.
- [BP83] H. Busemann and B. B. Phadke, *Peakless and monotone functions on G -spaces*, Tsukuba Journal of Mathematics **7** (1983), no. 1, 105–135.
- [Bus55] H. Busemann, *The geometry of geodesics*, Academic Press, 1955.
- [CG72] J. Cheeger and D. Gromoll, *On the Structure of Complete Manifolds of Nonnegative Curvature*, Ann. of Math. (2) **96** (1972), no. 3, 413–443.
- [DL14] D. Descombes and U. Lang, *Convex geodesic bicondings and hyperbolicity*, Geometriae Dedicata **177** (2014), no. 1, 367–384.
- [FLS07] T. Foertsch, A. Lytchak, and V. Schroeder, *Nonpositive Curvature and the Ptolemy Inequality*, International Mathematics Research Notices **2007** (2007), 1–15.
- [Gig13] N. Gigli, *The splitting theorem in non-smooth context*, arxiv:1302.5555 (2013).
- [GKO13] N. Gigli, K. Kuwada, and S. Ohta, *Heat Flow on Alexandrov Spaces*, Communications on Pure and Applied Mathematics **66** (2013), no. 3, 307–331.
- [Gro91] M. Gromov, *Sign and geometric meaning of curvature*, Rendiconti del Seminario Matematico e Fisico di Milano **61** (1991), no. 1, 9–123.
- [HKST15] J. Heinonen, P. Koskela, N. Shanmugalingam, and J. T. Tyson, *Sobolev Spaces on Metric Measure Spaces: An Approach Based on Upper Gradients*, Cambridge University Press, 2015.
- [Hua10] B. Hua, *Harmonic functions of polynomial growth on singular spaces with nonnegative Ricci curvature*, Proceedings of the American Mathematical Society **139** (2010), no. 6, 2191–2205.
- [Kan61] E. Kann, *Bonnet’s theorem in two-dimensional G -spaces*, Communications on Pure and Applied Mathematics **14** (1961), no. 4, 765–784.
- [Kel14] M. Kell, *Uniformly convex metric spaces*, Analysis and Geometry in Metric Spaces **2** (2014), no. 1, 359–380.
- [Kel15] ———, *A note on non-negatively curved Berwald spaces*, arxiv:1502.03764 (2015).
- [Ket15] Ch. Ketterer, *Cones over metric measure spaces and the maximal diameter theorem*, Journal de Mathématiques Pures et Appliquées **103** (2015), no. 5, 1228–1275.
- [KK06] A. Kristály and L. Kozma, *Metric characterization of Berwald spaces of non-positive flag curvature*, Journal of Geometry and Physics **56** (2006), no. 8, 1257–1270.
- [KL97] B. Kleiner and B. Leeb, *Rigidity of quasi-isometries for symmetric spaces and Euclidean buildings*, Publications mathématiques de l’IHÉS **86** (1997), no. 1, 115–197.
- [KOT12] K. Kondo, S. Ohta, and M. Tanaka, *A Toponogov type triangle comparison theorem in Finsler geometry*, arxiv:1205.3913 (2012).
- [KS58] P. Kelly and E. Straus, *Curvature in Hilbert geometries*, Pacific Journal of Mathematics (1958).
- [Kuw13] K. Kuwae, *Jensen’s inequality on convex spaces*, Calculus of Variations and Partial Differential Equations **49** (2013), no. 3–4, 1359–1378.
- [Lak14] S. Lakzian, *On Closed Geodesics in Non-negatively Curved Finsler Structures with Large Volume Growth*, arxiv:1408.0214 (2014).
- [LD11] E. Le Donne, *Metric Spaces with Unique Tangents*, Annales Academiae Scientiarum Fennicae **36** (2011), no. 2, 683–694.

- [LV09] J. Lott and C. Villani, *Ricci curvature for metric-measure spaces via optimal transport*, Ann. of Math. (2) **169** (2009), no. 3, 903–991.
- [Lyt06] A. Lytchak, *Open map theorem in metric spaces*, St. Petersburg Mathematical Journal **17** (2006), no. 3, 477–491.
- [MR05] R. Monti and M. Rickly, *Geodetically Convex Sets in the Heisenberg Group*, Journal of Convex Analysis **12** (2005), no. 1, 187–196.
- [Oht07a] S. Ohta, *Convexities of metric spaces*, Geometriae Dedicata **125** (2007), no. 1, 225–250.
- [Oht07b] ———, *On the measure contraction property of metric measure spaces*, Commentarii Mathematici Helvetici **82** (2007), no. 4, 805–828.
- [Oht07c] ———, *Products, cones, and suspensions of spaces with the measure contraction property*, Journal of the London Mathematical Society **76** (2007), no. 1, 225–236.
- [Oht08] ———, *Uniform convexity and smoothness, and their applications in Finsler geometry*, Math. Ann. **343** (2008), no. 3, 669–699.
- [Oht09] ———, *Finsler interpolation inequalities*, Calc. Var. Partial Differential Equations **36** (2009), no. 2, 211–249.
- [Oht13] ———, *Weighted Ricci curvature estimates for Hilbert and Funk geometries*, Pacific Journal of Mathematics **265** (2013), no. 1, 185–197.
- [OS12] S. Ohta and K.-Th. Sturm, *Non-Contraction of Heat Flow on Minkowski Spaces*, Arch. Ration. Mech. Anal. **204** (2012), no. 3, 917–944.
- [Ots97] Y. Otsu, *Differential geometric aspects of Alexandrov spaces*, Comparison Geometry, Math. Sci. Res. Inst. Publ, 1997, pp. 135–148.
- [Ped52] F. P. Pedersen, *On spaces with negative curvature*, Mater. Tidsskr. B (1952).
- [Pet11] A. Petrunin, *Alexandrov meets Lott–Villani–Sturm*, Münster J. of Math. **4** (2011), 53–64.
- [Pet13] ———, *Alexandrov space with zero reach*, <http://mathoverflow.net/questions/123922/does-convex-set-in-alexandrov-space-has-positive-reach> (2013).
- [PP94] G. Perelman and A. Petrunin, *Quasigeodesics and gradient curves in Alexandrov spaces*, preprint (1994).
- [Stu06] K.-Th. Sturm, *On the geometry of metric measure spaces*, Acta Math. **196** (2006), no. 1, 65–131.
- [Sza81] Z. I. Szabó, *Positive definite Berwald spaces*, Tensor N. S. **35** (1981), 25–39.
- [Sza06] ———, *Berwald metrics constructed by Chevalley’s polynomials*, arxiv:0601522 (2006).
- [Yam12] T. Yamaguchi, *Collapsing 4-manifolds under a lower curvature bound*, arxiv:1205.0323 (2012).

MATHEMATISCHES INSTITUT, UNIVERSITÄT TÜBINGEN, TÜBINGEN, GERMANY
E-mail address: martin.kell@math.uni-tuebingen.de