TRANSPORT MAPS, NON-BRANCHING SETS OF GEODESICS AND MEASURE RIGIDITY

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ABSTRACT. In this paper we investigate the relationship between a general existence of transport maps of optimal couplings with absolutely continuous first marginal and the property of the background measure called essentially non-branching introduced by Rajala–Sturm (*Calc. Var. PDE* 2014). In particular, it is shown that the qualitative non-degeneracy condition introduced by Cavalletti–Huesmann (*Ann. Inst. H. Poincaré Anal. Non Linéaire* 2015) implies that any essentially non-branching metric measure space has a unique transport maps whenever the initial measure is absolutely continuous. This generalizes a recently obtained result by Cavalletti–Mondino (*Commun. Contemp. Math.* 2017) on essentially non-branching spaces with the measure contraction condition $\mathsf{MCP}(K, N)$.

In the end we prove a measure rigidity result showing that any two essentially non-branching, qualitatively non-degenerate measures on a fixed metric spaces must be mutually absolutely continuous. This result was obtained under stronger conditions by Cavalletti–Mondino (*Adv. Math.* 2016). It applies, in particular, to metric measure spaces with generalized finite dimensional Ricci curvature bounded from below.

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1. Introduction

In the theory of optimal transport one of the first questions one asks is whether an optimal coupling π_{opt} that minimizes the functional

$$\pi \mapsto \int c(x,y) d\pi(x,y)$$

among all π with fixed marginals $(p_1)_*\pi = \mu$ and $(p_2)_*\pi = \nu$ can be written as a coupling induced by a transport map, i.e. whether there is a measurable map $T: M \to M$ such that $\pi = (\mathrm{id} \times T)_*\mu$. On sufficiently nice spaces this result can be deduced from Rademacher's Theorem, the (weak) differential structure and the exponential map whenever μ is absolutely continuous, see Brenier [Bre91] in the Euclidean setting, McCann [McC01] on Riemannian manifolds, Ambrosio–Rigot [AR04] and Figalli–Rigot [FR10] on a sub-class of sub-Riemannian manifolds and also Bertrand [Ber08] on Alexandrov spaces.

A first proof for non-smooth non-branching spaces with generalized Ricci curvature bounded from below was obtained by Gigli [Gig12]. He showed that the non-existence of a transport map would imply that two disjoint parts of a Wasserstein geodesic whose initial densities with respect to the background measure m are bounded would overlap at intermediate times. This, however, cannot happen for non-branching spaces (compare Lemma 2.9 below).

Consequently, Gigli's idea was adapted to essentially non-branching spaces with generalized Ricci curvature bounded from below (see [RS14, GRS16, CM17]). Here essentially non-branching is a weak version of the non-overlapping property described above, i.e. it prohibits that initially disjoint parts of a Wasserstein geodesic overlap at intermediate times whenever the initial and final measures are absolutely continuous.

Both Gigli–Rajala–Sturm [GRS16] and Cavalletti–Mondino [CM17] had to prove that there are absolutely continuous interpolations along which a corresponding interpolation inequality holds.

Rather than using density bounds of intermediate measures, Cavalletti–Huesmann [CH15] showed that if not too much mass is lost in a uniform way when transported towards a fixed point (compare Definition 5.1), then, together with the non-overlapping property implied by non-branching property, the non-existence of transport maps would yield a contradiction.

Whereas the interpolation inequality implied density bounds, Cavalletti–Huesmann's approach only relied on an "easier to measure" quantity. In this paper we want to combine this approach with the one of Cavalletti–Mondino [CM17]. The difficulty is that an arbitrary interpolation might only "see" part of the interpolation points. We avoid this by proving the following:

- at finitely many fixed times $\{t_i\}_{i=1}^n$ there are absolutely continuous interpolations $\mu_{t_n} \ll \mathsf{m}$ (see first part of Theorem 4.10 and Corollary 4.11)
- at a fixed time t the interpolation μ_t "sees" m-almost every possible interpolation point Γ_t (see second part of Theorem 4.10)

Whereas the first part gives us the non-overlapping if the space is essentially non-branching, the second part makes sure that the set-wise interpolation Γ_t does not contain a set of positive m-measure that is not seen by the interpolation μ_t.

The remaining parts follow along the line of Cavalletti–Huesmann [CH15], more precisely, if the background measure is *qualitatively non-degenerate*, i.e. there is a

function $f : (0,1) \to (0,\infty)$ with $\limsup_{t\to 0} f(t) > \frac{1}{2}$ such that for all Borel set A and all $x \in X$ it holds $\mathsf{m}(A_{t,x}) \geq f(t)\mathsf{m}(A)$ where $A_{t,x} = \{\gamma_t \,|\, \gamma_0 \in A, \gamma_1 = x\}$, then
- every optimal coupling between $\mu_0 \ll \mathsf{m}$ and $\mu_1 = \sum \lambda_i \delta_{x_i}$ is induced by a transport map (Lemma 5.4 and Corollary 5.6)
- any c_p-cyclically monotone set Γ satisfies a qualitative non-degeneracy condition, i.e. $\mathsf{m}(\Gamma_t) \geq f(t)\mathsf{m}(\Gamma_0)$ (Lemma 5.7)
- every optimal coupling between $\mu_0 \ll \mathsf{m}$ and μ_1 is induced by a transport map (Theorem 5.8).

As it turns out the general existence of transport maps implies not only uniqueness of the optimal coupling, but also unique geodesics between almost all points coupled via the optimal transport, see Lemma 3.4. This in return gives uniqueness of the interpolation measures. Furthermore, only assuming that any optimal coupling between an absolutely continuous measure and another measure is induced by a transport map already implies that the space must be essentially non-branching (Proposition 3.6). We call spaces with such an *a priori existence of transport maps* spaces having *good transport behavior* $(\mathsf{GTB})_p$, see Definition 3.1.

We may summarize one of the results of this note as follows.

Theorem (see Proposition 3.6 and Theorem 5.8). *Let (M, d, m) be a metric measure space and assume m is qualitatively non-degenerate then for fixed $p \in (1, \infty)$ the following properties are equivalent:*

(i) *(M, d, m) is p-essentially non-branching, i.e. any p-optimal dynamical coupling $\pi \in \mathcal{P}(\mathsf{Geo}_{[0,1]}(M, d))$ with $(e_0)_*\sigma, (e_1)_*\sigma \ll \mathsf{m}$ is concentrated on a set of non-branching geodesics.*

(ii) *for every $\mu_0, \mu_1 \in \mathcal{P}_p(M)$ with $\mu_0 \ll \mathsf{m}$ there is a unique p-optimal dynamical coupling $\sigma \in \mathcal{P}_p(\mathsf{Geo}_{[0,1]}(M, d))$ between μ_0 and μ_1 and for this coupling σ the p-optimal coupling $(e_0, e_1)_*\sigma$ is induced by a transport map and every interpolation $\mu_t = (e_t)_*\sigma$, $t \in [0, 1)$, is absolutely continuous, i.e. $\mu_t \ll \mathsf{m}$.*

It is well-known that on smooth spaces, there is an abundance of measures satisfying all but the last property of the second statement in the theorem above, in particular, they have the good transport behavior $(\mathsf{GTB})_p$.

We call the last property of the second statement in the theorem above *strong interpolation property* $(\mathsf{sIP})_p$. Note that if two reference measures m_1 and m_2 have the strong interpolation property and $\mu_0 \ll \mathsf{m}_1$ and $\mu_1 \ll \mathsf{m}_2$ then the interpolation $(e_t)_*\sigma$ must be absolutely continuous with respect to both m_1 and m_2 implying m_1 and m_2 cannot be mutually singular. This property can be used to obtain the following measure rigidity theorem.

Theorem (Measure Rigidity). *Let $p \in (1, \infty)$. Then any two measures on a complete separable metric space (M, d) which are p-essentially non-branching and qualitatively non-degenerate must be mutually absolutely continuous.*

Under a more involved inversion property and a stronger qualitative non-degeneracy with $\limsup_{t\to 0} f(t) = 1$ such a statement was obtained by Cavalletti–Mondino [CM16].

By [RS14, AGS14] the statement applies in particular to spaces with finite dimensional Ricci curvature bounded from below.

Corollary. *If both (M, d, m_1) and (M, d, m_2) are $\mathsf{RCD}^*(K, N)$-spaces with $N \in [1, \infty)$ then m_1 and m_2 must be mutually absolutely continuous.*

A slightly different kind of measure rigidity of the background measure m for metric measure spaces satisfying the $\mathsf{RCD}(K, N)$-condition with $N \in [1, \infty)$ was obtain independently by Gigli–Pasqualetto [GP16] and by Mondino and the author [KM17] using the weak converse of Rademacher's theorem proven in [DPR16].

The section dealing with the measure rigidity theorem (Section 6) only relies on few properties which are restated in the beginning of that section and can be read independently of the rest of this note. We also present a proof of the rigidity theorem that relies on a bounded density property and applies to strong $\mathsf{CD}_p(K, \infty)$-spaces with the strong interpolation property, see Theorem 6.5.

2. Preliminaries

Throughout this paper we always assume that (M, d, m) is a *geodesic metric measure space*, i.e. (M, d) is a complete separable geodesic metric space and m is measure on M which is finite on bounded sets. It is *proper* if every closed bounded set $B \subset M$ is compact.

Selection Dichotomy and Disintegration Theorem. We present two technical results which help to classify couplings that are induced by a transport map. The first can be obtained by combining the Measurable Selection Theorem and Lusin's Theorem. This form of the selection dichotomy was used by Cavalletti–Huesmann [CH15] to select p-optimal couplings that overlap at the initial measure.

In the following for a set $\Gamma \subset M \times M$ we let $\Gamma(x) = \{y \in M \mid (x, y) \in \Gamma\}$. We say a map $T : M \to M$ is a selection of Γ if $(x, T(x)) \in \Gamma$ for all $x \in p_1(\Gamma)$.

Theorem 2.1 (Selection Dichotomy of Sets)**.** *Assume μ is a probability measure on M and $\Gamma \subset M \times M$ a Borel set with $\mu(p_1(\Gamma)) = 1$. Then exactly one of the following holds:*

(i) *For μ-almost all $x \in M$ the set $\Gamma(x)$ contains exactly one element. Furthermore, if $\pi \in \mathcal{P}(M \times M)$ with $\operatorname{supp} \pi \subset \Gamma$ and $(p_1)_* \pi = \mu$ then $\pi = (\operatorname{id} \times T)_* \mu$ for a μ-measurable selection $T : M \to M$ of Γ. In particular, T is unique up to μ-measure zero and π is unique among all measures $\tilde{\pi} \in \mathcal{P}(M \times M)$ which are concentrated in Γ and have first marginal μ.*

(ii) *There are a compact set $K \subset \operatorname{supp} \mu$ with $\mu(K) > 0$ and two μ-measurable selections $T_1, T_2 : M \to M$ of Γ which are continuous when restricted to K and $T_1(K) \cap T_2(K) = \varnothing$. Furthermore, if the function*

$$\varphi_\Gamma(x) = \begin{cases} \sup_{(x,y) \in \Gamma} d(x, y) - \inf_{(x,y) \in \Gamma} d(x, y) & x \in p_1(\Gamma) \\ 0 & otherwise. \end{cases}$$

is positive on a set of positive μ-measure then K can be chosen such that for some $\delta > 0$

$$\sup_{(x,y_1) \in K \times T_1(K)} d(x, y_1) + \delta \leq \inf_{(x,y_2) \in K \times T_2(K)} d(x, y_2).$$

Proof. It is easy to see that the conditions are mutually exclusive. Indeed, the measures $(\operatorname{id} \times T_1)_* \mu$ and $(\operatorname{id} \times T_2)_* \mu$ are distinct and concentrated in Γ.

By the Measurable Selection Theorem there is a μ-measurable map T such that $(x, T(x)) \in \Gamma$ for all $x \in p_1(\Gamma)$. Using Lusin's Theorem one can show that there is a Borel set $\Omega \subset M$ of full μ-measure such that

$$\operatorname{graph}_\Omega T := \{(x, T(x) \mid x \in \Omega\}$$

is a Borel subset of $M \times M$. Thus $\Gamma' = \Gamma \backslash \operatorname{graph}_\Omega T$ is a Borel set and there is a μ-measurable selection $S : M \to M$ with $(x, S(x)) \in \Gamma'$ for all $x \in p_1(\Gamma')$. Since $p_1(\Gamma')$ is μ-measurable, we can redefine S outside of $p_1(\Gamma')$ and assume $T(x) = S(x)$ for all $x \in M \backslash p_1(\Gamma')$.

If $\mu(p_1(\Gamma')) = 0$ then $\mu(p_1(\Gamma') \cap \Omega) = 0$ and thus $S(x) = T(x)$ for μ-almost all $x \in M$. In particular, the first case holds.

Otherwise, by Lusin's Theorem there is a compact set $K_1 \subset \operatorname{supp} \mu \cap p_1(\Gamma') \cap \Omega$ with $\mu(K_1) > 0$ such that T and S are continuous on K_1 and $S(x) \neq T(x)$ for $x \in K_1$. Thus for sufficiently small $r > 0$ and a fixed $x_0 \in K_1$, it holds

$$T(x) \neq S(x') \quad \text{for all } x, x' \in \bar{B}_r(x_0) \cap K_1.$$

In case $\mu(\{\varphi_\Gamma > 0\}) = 0$ we can choose $K = \bar{B}_r(x_0) \cap K_1$, $T_1 = T$ and $T_2 = S$ and conclude.

If $\mu(\{\varphi_\Gamma > 0\}) > 0$ then there are $\epsilon > 0$ and a compact set $K_2 \subset \operatorname{supp} \mu$ with $\mu(K_2) > 0$ and $\varphi_{\operatorname{supp} \pi}(x) > \epsilon$ for all $x \in K_2$. Note also that the sets

$$\Gamma^+ = \{(x, y) \in \Gamma \mid x \in K_2, d(x, y) \geq \sup_{(x, y') \in \Gamma} d(x, y') - \frac{\epsilon}{2}\}$$

$$\Gamma^- = \{(x, y) \in \Gamma \mid x \in K_2, d(x, y) \leq \inf_{(x, y') \in \Gamma} d(x, y') + \frac{\epsilon}{2}\}$$

are non-empty Borel subsets of Γ with $p_1(\Gamma^+) = p_1(\Gamma^-) = K_2$. Thus there are two μ-measurable selections T^+ and T^- with $(x, T^\pm(x)) \in \Gamma^\pm$ for all $x \in K_2$. As above we may assume that $T^\pm$ agree with T outside of K_2.

Choose another compact $K_3 \subset K_2$ such that the maps $T^\pm$ are continuous on K_3. In particular, for some $x_0 \in K_3$ and sufficiently small $r > 0$ it holds

$$d(x, T^-(x')) + \delta \leq d(x, T^+(x'')) \qquad \text{for all } x, x', x'' \in \bar{B}_r(x_0) \cap K_3.$$

To conclude observe that the compact set $K = \bar{B}_r(x_0) \cap K_3$ and the maps $T_1 = T^-$ and $T_2 = T^+$ satisfy the last part of the second statement. $\qquad \square$

Note that in general the second possibility of the Selection Dichotomy above does not say anything about the relationship of the measures $(\operatorname{id} \times T_i)_* \mu$ and a fixed measure $\pi \in \mathcal{P}(M \times M)$ with $(p_1)_* \pi = \mu$ and $\operatorname{supp} \pi \subset \Gamma$. More precisely, in general, $(T_i)_* \mu$ might be singular with respect to $(p_2)_* \pi$, or more generally, it is possible that $(T_i)_* \mu \perp \mathsf{m}$ even if $(p_2)_* \pi \ll \mathsf{m}$.

The following lemma is a more general version of the Selection Dichotomy and can be extracted from Gigli's work [Gig12, Proof of Theorem 3.3]. It shows that any measure π, regarded as a generalized transport map $\int \delta_x \otimes \mu_x d\mu(x)$, is either already induced by a transport map, i.e. $\mu_x = \delta_{T(x)}$, or can be decomposed into (at least) two partial transport with target transport on a compact set K of positive μ-measure. We give a simpler proof relying on the Selection Dichotomy for Sets.

First, recall the the statement of the Disintegration Theorem. Let (X, d) and (Y, d) be two complete separable metric spaces. Denote the Borel σ-algebra of X and Y by $\mathcal{B}(X)$ and resp. $\mathcal{B}(Y)$.

Definition 2.2 (Disintegration over S). Let σ a probability measure on X, $S : X \to Y$ a Borel map and $\varpi = S_* \sigma$. An assignment $\sigma : \mathcal{B}(X) \times Y \to [0, 1]$, denoted $(B, y) \mapsto \sigma_y(B)$, is called *a disintegration of σ over S* if

(1) $\sigma_y(\cdot)$ is a probability measure on X for all $y \in Y$.
(2) $y \mapsto \sigma_y(B)$ is ϖ-measurable for all Borel sets $B \in \mathcal{B}(X)$.

(3) $\sigma_y(S^{-1}(y)) = 1$ for all $y \in Y$

and it holds

$$\sigma(C \cap S^{-1}(B)) = \int_C \sigma_y(B)d\varpi(y).$$

Regarding the assignment $y \mapsto \sigma_y = \sigma_y(\cdot)$ as a map from Y to $\mathcal{P}(X)$ we abbreviate this as

$$\sigma = \int \sigma_y d\varpi(y).$$

Remark. (1) A disintegration of σ over S as above is usually called a disintegration of σ which is strongly consistent with S.

(2) If $\pi \in \mathcal{P}(M \times M)$ for some complete separable metric space M with $\mu = (p_1)_*\pi$, then any disintegration $\pi = \int \pi_x d\mu(x)$ must satisfy $\operatorname{supp} \pi_x \subset \{x\} \times M$ and hence $\pi_x = \delta_x \otimes \mu_x$ for a measure $\mu_x \in \mathcal{P}(M)$.

The following theorem can be deduced from the general Disintegration Theorem [Fre06, Section 452].

Lemma 2.3 (Disintegration Theorem). *For every $\sigma \in \mathcal{P}(X)$ and every Borel map $S : X \to Y$ there exists a disintegration $\sigma_\cdot(\cdot)$ of σ over S which is almost everywhere uniquely defined, i.e. for any disintegration $\tilde{\sigma}_\cdot(\cdot)$ of σ over S it holds $\sigma_y(\cdot) = \tilde{\sigma}_y(\cdot)$ for $S_*\sigma$-almost all $y \in Y$.*

The theorem allows us to say that up to a μ-null set $\sigma = \int \sigma_x d\mu$ is *the* disintegration of σ over S.

Theorem 2.4 (Selection Dichotomy for Measures). *Let π be a probability measure on $M \times M$ and $\mu = (p_1)_*\pi$. Then exactly one of the following holds:*

(i) *There is a μ-measurable map $T : M \to M$ such that $\pi(\operatorname{graph} T) = 1$.*

(ii) *There are a compact set $K \subset M$ and two closed bounded sets $A_1, A_2 \subset M$ with $A_1 \cap A_2 = \varnothing$ such that*

$$\pi(K \times A_1), \pi(K \times A_2) > 0.$$

Furthermore, there are two measures $\pi_1, \pi_2 \in \mathcal{P}(M \times M)$ with $\frac{1}{\mu(K)}\mu\big|_K = (p_1)_*\pi_1 = (p_1)_*\pi_2$ and $\pi_1, \pi_2 \ll \pi$ such that π_1 and π_2 have disjoint support.

Remark. (1) The construction shows that for some $\epsilon > 0$ it holds

$$(A_1)_\epsilon \cap (A_2)_\epsilon = \varnothing$$

where $A_\epsilon = \bigcup_{x \in A} B_\epsilon(x)$ for a set $A \subset M$. Furthermore, it is possible to choose K, A_1 and A_2 such that for some $\delta > 0$

$$\sup_{(x,y_1) \in K \times A_1} d(x, y_1) + \delta \leq \inf_{(x,y_2) \in K \times A_2} d(x, y_2).$$

(2) If $\pi = \int \delta_x \otimes \mu_x d\mu(x)$ is the disintegration over p_1 and π is not induced by a map then for μ-almost all $x \in K$ the measure μ_x is not a delta measure. Indeed, for μ-almost all $x \in K$ it holds

$$\mu_x = \mu_x\big|_{A_1} + \mu_x\big|_{A_2} + \mu_x\big|_{M \backslash (A_1 \cup A_2)}$$

and the choice of K shows that $\mu_x\big|_{A_1}$ and $\mu_x\big|_{A_2}$ are non-trivial for μ-almost all $x \in K$.

Proof. We apply Theorem 2.1 to $\Gamma = \operatorname{supp} \pi$. If the first option of the dichotomy holds then $\pi = (\operatorname{id} \times T)_* \mu$ and thus $\pi(\operatorname{graph} T) = 1$.

Otherwise let K, T_1 and T_2 as in the second possibility of Theorem 2.1. We may restrict K further and assume $\operatorname{supp}(\mu\big|_K) = K$.

We claim that for all $\epsilon > 0$ and $i = 1, 2$ it holds

$$\pi(K \times (T_i(K))_\epsilon) > 0.$$

Indeed, note that $\pi\big|_K = \pi(\cdot \cap (K \times M)) \neq \mathbf{0}$, so that $(x, T_1(x)), (x, T_2(x)) \in \operatorname{supp}(\pi\big|_K) = \operatorname{supp} \pi \cap (K \times M)$. Because $B_\epsilon^{M \times M}(x, y) \subset B_\epsilon(x) \times B_\epsilon(y)$ for $i = 1, 2$ it holds

$$\begin{aligned}
0 &< \pi\big|_K (B_\epsilon^{M \times M}(x, T_i(x))) \\
&\leq \pi((B_\epsilon(x) \cap K) \times B_\epsilon(T(x))) \\
&\leq \pi(K \times B_\epsilon(T(x))) \leq \pi(K \times (T_i(K))_\epsilon).
\end{aligned}$$

Since T_1 and T_2 are continuous on K and K is compact there is an $\epsilon > 0$ such that

$$(T_1(K))_{2\epsilon} \cap (T_2(K))_{2\epsilon} = \varnothing.$$

Choosing $A_1 = \operatorname{cl}(T_1(K))_\epsilon$ and $A_1 = \operatorname{cl}(T_2(K))_\epsilon$ gives first part of the claim.

To obtain the second part, note that there are a $\delta > 0$ and compact K' of positive μ-measure such that

$$\mu_x(A_1), \mu_x(A_2) \in (\delta, 1 - \delta) \qquad \text{for all } x \in K'.$$

Restricting K' again, assume $K' = \operatorname{supp}(\mu\big|_{K'})$ and define two non-trivial measures $\pi_1, \pi_2 \in \mathcal{P}(M \times M)$ as follows

$$\begin{aligned}
\pi_1 &= \frac{1}{\mu(K')} \int_{K'} \frac{1}{\mu_x(A_1)} \delta_x \otimes \mu_x\big|_{A_1} d\mu(x) \\
\pi_2 &= \frac{1}{\mu(K')} \int_{K'} \frac{1}{\mu_x(A_2)} \delta_x \otimes \mu_x\big|_{A_2} d\mu(x).
\end{aligned}$$

It is easy to see that $\pi_1, \pi_2 \ll \pi$ and $\frac{1}{\mu(K')} \mu\big|_{K'} = (p_1)_* \pi_1 = (p_1)_* \pi_2$ which proves the claim. $\qquad \square$

For completeness we present the following more general form of the Selection Dichotomy.

Corollary 2.5 (General Selection Dichotomy). *Assume (X, d) and (Y, d) are complete separable metric spaces. Let σ be a measure on X and $S : X \to Y$ a Borel map. Then exactly one of the following holds:*

(i) *There is a measurable map $T : Y \to X$ such that $S(T(y)) = y$ and $T_* \varpi = \sigma$ where $\varpi = S_* \sigma$. In particular, the disintegration of σ via S is given by*

$$\sigma = \int \delta_{T(y)} d\varpi(y).$$

(ii) *There are a compact set $K \subset X$ with $\sigma(K) > 0$ and two closed bounded sets $A_1, A_2 \subset X$ with $A_1 \cap A_2 = \varnothing$ such that*

$$\sigma_x\big|_{A_i} \neq 0, \ x \in K, i = 1, 2.$$

In particular, for ϖ-almost all $x \in K$ the measures σ_x are not delta measures.

Proof. Just note that the proofs above did not rely on the product structure of $M \times M$ and that p_1 is a projection. Thus replace $M \times M$ by X and p_1 by S we can follow the proofs above line by line. $\square$

Wasserstein spaces on geodesic spaces. Let (X, d) be a complete, separable metric space. A map $\gamma : [0, 1] \to X$ satisfying

$$d(\gamma_t, \gamma_s) = |t - s| d(\gamma_0, \gamma_1) \qquad \text{for } t, s \in [0, 1]$$

is called a *geodesic connecting γ_0 and γ_1*. Note that our terminology implies that any geodesic is the curve of minimal length between its endpoints. Denote by $\mathsf{Geo}_{[0,1]}(X, d)$ the set of geodesics. On $\mathsf{Geo}_{[0,1]}(X, d)$ there are natural evaluation maps $e_t : \mathsf{Geo}_{[0,1]}(M, d) \to M$, $t \in [0, 1]$, defined by $e_t : \gamma \mapsto \gamma_t$. Denote the length of a geodesic γ by $\ell(\gamma) := d(\gamma_0, \gamma_1)$ and define a restriction map $\mathsf{restr}_{s,t} : \mathsf{Geo}_{[0,1]}(M, d) \to \mathsf{Geo}_{[0,1]}(M, d)$ for all $0 \le s, t \le 1$ by

$$(\mathsf{restr}_{s,t}\gamma)(r) = \gamma_{s+(t-s)r}.$$

We say the metric space (X, d) is a *geodesic metric space* if between each $x, y \in X$ there is a geodesic connecting x and y, i.e. $(e_0, e_1)(\mathsf{Geo}_{[0,1]}(X, d)) = X \times X$.

In the following we introduce the main concepts used from the theory of optimal transport. For a comprehensive introduction we refer the reader to Villani's book [Vil08]. As we rely on the strict convexity of $r \mapsto |r|^p$ we assume throughout this note that $p \in (1, \infty)$.

Recall that (M, d) is a complete separable geodesic metric space. Let $\mathcal{P}(M)$ be the set of probability measures on M and for a fixed $x_0 \in M$ let

$$\mathcal{P}_p(M) = \left\{ \mu \in \mathcal{P}(M) \,|\, \int d(x, x_0)^p d\mu(x) \right\}$$

the space of probability measures with *finite p-th moment*. On $\mathcal{P}_p(M)$ we define the *p-Wasserstein metric W_p* as follows

$$W_p(\mu_0, \mu_1) = \left(\inf_{\pi \in \Pi(\mu_0, \mu_1)} \int d(x, y)^p d\pi(x, y) \right)^{\frac{1}{p}}$$

where $\Pi(\mu_0, \mu_1)$ is the set of $\pi \in \mathcal{P}(M \times M)$ with $(p_1)_*\pi = \mu_0$ and $(p_2)_*\pi = \mu_1$. This defines a complete metric on $\mathcal{P}_p(M)$ with a topology which is strictly stronger than the subspace topology induced by $\mathcal{P}_p(M) \subset \mathcal{P}(M)$ unless (M, d) is bounded. We call the convergence induced by the subspace topology *weak convergence*.

One can show that for each $\mu_0, \mu_1 \in \mathcal{P}_p(M)$ there is a $\pi_{\mathrm{opt}} \in \Pi(\mu_0, \mu_1)$ such that

$$W_p(\mu_0, \mu_1) = \left(\int d(x, y)^p d\pi_{\mathrm{opt}}(x, y) \right)^{\frac{1}{p}}.$$

In this case we say π_{opt} is a *p-optimal coupling*. Let $\mathsf{Opt}_p(\mu_0, \mu_1)$ denote the set of all p-optimal couplings between μ_0 and μ_1. A general measure $\pi \in \mathcal{P}(M \times M)$ is said to be p-optimal if it is a p-optimal coupling between $(e_0)_*\pi$ and $(e_1)_*\pi$.

Since (M, d) is geodesic it is possible to show that $(\mathcal{P}_p(M), W_p)$ is geodesic as well. Just note that $(x, y) \mapsto (e_0, e_1)^{-1}(x, y) \subset \mathsf{Geo}_{[0,1]}(M, d)$ is a measurable closed-valued map and any measurable selection $\mathcal{T}$ will lift a coupling π to a dynamical coupling $\sigma = \mathcal{T}_*\pi \in \mathcal{P}(\mathsf{Geo}_{[0,1]}(M, d))$ between two measures μ_0 and μ_1. If π is p-optimal then we say $\sigma \in \mathcal{P}(\mathsf{Geo}_{[0,1]}(M, d))$ is a p-optimal dynamical coupling. Now one may readily verify that $t \mapsto (e_t)_*\sigma$ is a geodesic connecting $(e_0)_*\sigma$ and

$(e_1)_*\sigma$. Denote the set of p-optimal dynamical couplings between μ_0 and μ_1 by $\mathsf{OptGeo}_p(\mu_0, \mu_1)$.

Also note that each geodesic $t \mapsto \mu_t$ in $\mathcal{P}_p(M)$ is induced by a measure $\sigma \in \mathcal{P}(\mathsf{Geo}_{[0,1]}(M,d))$ such that $(e_t)_*\sigma = \mu_t$. In this case it is easy to see that $(e_t, e_s)_*\sigma$ is a p-optimal coupling between μ_t and μ_s.

Recall that disintegrating a dynamical coupling σ over $(e_0, e_1) : \mathsf{Geo}_{[0,1]}(M,d) \to M \times M$ shows that

$$\sigma = \int \sigma_{x,y} d\pi(x,y)$$

where $\pi = (e_0, e_1)_*\sigma$ and $(x,y) \mapsto \sigma_{x,y}$ is a measurable assignment of dynamical couplings between δ_x and δ_y. Similarly, we can disintegrate σ over e_0 to obtain $\sigma = \int \sigma_{x_0} d\mu_0(x_0)$ such that for μ_0-almost all $x_0 \in M$ the probability measure σ_{x_0} is a dynamical coupling between δ_{x_0} and a probability measure μ_{x_0} with $\pi = \int \delta_{x_0} \otimes \mu_{x_0} d\mu(x_0)$. Furthermore, if σ is p-optimal then σ_{x_0} is p-optimal for μ_0-almost all $x_0 \in M$.

The following is the well-known restriction property of optimal couplings, see [Vil08]. Compare the following notation also the concept push-forward via a plan, see e.g. [AGS14, Definition 2.1].

Lemma 2.6. *Assume $\mu_0, \mu_1 \in \mathcal{P}_p(M)$, $p \in (1, \infty)$, and σ is a p-optimal dynamical coupling between μ_0 and μ_1. Then for $f : M \times M \to [0, \infty)$ with $\lambda = \int f d\pi \in (0, \infty)$ the measure*

$$\sigma_f = \lambda^{-1} \int \sigma_{x,y} f(x,y) d\pi(x,y)$$

is a p-optimal coupling between μ_0^f and μ_1^f where

$$\mu_0^f = (p_1)_*\sigma_f$$
$$\mu_1^1 = (p_2)_*\sigma_f.$$

Furthermore, if, in addition $f \le 1, \int f d\pi \in (0, 1)$ and $\tilde{\sigma}_f$ is another p-optimal dynamical coupling between μ_0^f and μ_1^f then

$$\tilde{\sigma} = \lambda \tilde{\sigma}_f + (1 - \lambda)\sigma_{1-f}$$

is also a p-optimal coupling between μ_0 and μ_1.

Remark. (1) If f is a function depending only on the first coordinate then $\mu_0^f = \lambda^{-1} f \mu_0$.

(2) If $\Gamma \subset M \times M$ is Borel set with $\sigma(\Gamma) > 0$ then we write $\sigma_\Gamma = \sigma_{\chi_\Gamma} = \frac{1}{\pi(\Gamma)}\sigma\big|_{\hat{\Gamma}}$ where $\hat{\Gamma} = (e_0, e_1)^{-1}(\Gamma)$.

Proof. The first part follows from the restriction property of optimal transport and the second from linearity of the cost functional

$$\sigma \mapsto \int d(\gamma_0, \gamma_1)^p d\sigma(\gamma)$$

and the fact that $\tilde{\sigma}$ is still a dynamical coupling between μ_0 and μ_1. $\square$

It is easy to see that whenever there are two distinct p-optimal couplings π_1 and π_2 between μ_0 and μ_1 then there are at least two distinct p-optimal dynamical couplings σ_1 and σ_2 between μ_0 and μ_1. By convexity this would actually give a continuum of p-optimal (dynamical) couplings. If, however, the dynamical coupling is unique we get the following for restrictions of the endpoints.

Corollary 2.7. *Let μ_0 and μ_1 be two measures in $\mathcal{P}_p(M)$, $p \in (1, \infty)$. If there is a unique p-optimal dynamical coupling σ between μ_0 and μ_1 then for any $f \in L^\infty(\pi)$ the p-optimal dynamical coupling σ_f is unique between μ_0^f and μ_1^f.*

Proof. Note that $\sigma_f = \sigma_{cf}$ for all $c > 0$. Thus if $f \in L^\infty(\pi)$ then it is possible to replace f by $\frac{1}{\|f\|_\infty} f$ and assume without loss of generality $f \leq 1$ and $\int f d\mathsf{m} \in (0, 1)$. Thus the dynamical coupling $\tilde{\sigma} := \lambda \tilde{\sigma}_f + (1 - \lambda)\sigma_{1-f}$ is p-optimal between μ_0 and μ_1 whenever $\tilde{\sigma}_f$ is p-optimal between μ_0^f and μ_1^f. Furthermore, if $\tilde{\sigma}_f$ is distinct from σ_f then $\tilde{\sigma}$ is also distinct from σ proving the claim. $\qquad\square$

Non-branching geodesics. Given a set $\Gamma \subset M \times M$ we frequently use the following abbreviations

$$\hat{\Gamma} = (e_0, e_1)^{-1}(\Gamma)$$

and for $t, s \in [0, 1]$

$$\Gamma_t = e_t(\hat{\Gamma})$$

$$\Gamma_{t,s} = (e_t, e_s)(\hat{\Gamma}).$$

Thus $\hat{\Gamma}$ is the set of geodesics with endpoints $(x, y) \in \Gamma$ and Γ_t is the set of t-midpoints, where z is a t-midpoint of x and y if $\gamma_t = z$ for a geodesic connecting x and y.

For a set $A \subset M$ and $x \in M$ we also use the abbreviation

$$A_{t,x} = \{\gamma_t \,|\, \text{for some } \gamma \in \mathsf{Geo}_{[0,1]}(M, d) \text{ with } \gamma_0 \in A \text{ and } \gamma_1 = x\}.$$

Let L be a subset of geodesics, then we denote by L^{-1} the set of reversed geodesics, i.e.

$$\mathsf{L}^{-1} = \{t \mapsto \gamma_{1-t} \,|\, \gamma \in \mathsf{L}\}.$$

Similarly, let $\Gamma^{-1} = \{(y, x) \,|\, (x, y) \in \Gamma\}$. It is easy to see that $(\Gamma^{-1})^\wedge = (\hat{\Gamma})^{-1}$.

Definition 2.8 (non-branching set)**.** A set of geodesics $\mathsf{L} \subset \mathsf{Geo}_{[0,1]}(M, d)$ is *non-branching to the right* if for all $\gamma, \eta \in \mathsf{L}$ with $\mathsf{restr}_{0,t}\gamma = \mathsf{restr}_{0,t}\eta$ for some $t \in (0, 1]$ it holds $\gamma \equiv \eta$. Similarly, L is *non-branching to the left* if L^{-1} is non-branching to the right. Furthermore, L is *non-branching* if it is both non-branching to the left and to the right.

Remark. Non-branching to the left is the same as Rajala–Sturm's non-branching condition [RS14, Section 2.2]. This lack of symmetry is irrelevant in their study as essentially non-branching is a symmetric condition when one changes initial and final points (see below).

The following is well-known and follows from strict convexity of $r \mapsto r^p$ and the triangle inequality [CH15, Kel17].

Lemma 2.9. *Let $p \in (1, \infty)$ and assume for $\gamma, \eta \in \mathsf{Geo}_{[0,1]}(M, d)$ it holds*

$$d^p(\gamma_0, \gamma_1) + d^p(\eta_0, \eta_1) \leq d^p(\gamma_0, \eta_1) + d^p(\eta_0, \gamma_1).$$

Then $\gamma_t = \eta_t$ for some $t \in (0, 1)$ implies $\ell(\gamma) = \ell(\eta)$ and if (M, d) is, in addition, non-branching then $\gamma \equiv \eta$.

Finally recall some properties of Wasserstein geodesics on essential non-branching spaces. We collect results which can be deduced from [RS14, CM16]. The reader may consult the appendix for a proof of Theorem 2.11 and Corollary 2.12.

Definition 2.10 (*p*-essentially non-branching)**.** A metric measure space (M, d, m) is *p-essentially non-branching* if for all $\mu_0, \mu_1 \in \mathcal{P}_p(M)$, , $p \in (1, \infty)$, with $\mu_0, \mu_1 \ll \mathsf{m}$, any optimal dynamical coupling $\sigma \in \mathsf{OptGeo}_p(\mu_0, \mu_1)$ is concentrated on a set of non-branching geodesics, i.e. there is a measurable non-branching set $\mathsf{L} \subset \mathsf{Geo}_{[0,1]}(M, d)$ such that $\sigma(\mathsf{L}) = 1$. For brevity we say a measure m is *p*-essentially non-branching if (M, d, m) is a metric measure space which is *p*-essentially non-branching.

Remark. The property essentially non-branching introduced in [RS14] is equivalent to the property 2-essentially non-branching.

Theorem 2.11. *Assume* (M, d, m) *is a p-essentially geodesic measure space for some* $p \in (1, \infty)$. *Then for every p-optimal dynamical coupling* $\sigma \in \mathcal{P}(\mathsf{Geo}_{[0,1]}(M, d))$ *with* $(e_0)_* \sigma, (e_1)_* \sigma \ll \mathsf{m}$ *the following holds:*

   (i) *For each* $t \in (0, 1)$ *there is a Borel map* $\mathsf{T}_t : M \to \mathsf{Geo}_{[0,1]}(M, d)$ *such that the disintegration of* σ *over* e_t

$$\sigma = \int \delta_{\mathsf{T}_t(x)} d\mu_t(x)$$

   where $\mu_t = (e_t)_* \sigma$.
   (ii) *For each* $t \in (0, 1)$ *there is a measurable set of geodesics* $\mathsf{L} \subset \mathsf{Geo}_{[0,1]}(M, d)$ *with* $\sigma(\mathsf{L}) = 1$ *and whenever* $\gamma_t = \eta_t$ *for* $\gamma, \eta \in \mathsf{L}$ *then* $\gamma \equiv \eta$.
   (iii) *For all disjoint Borel sets* $\Gamma^{(1)}, \Gamma^{(2)} \subset M \times M$ *of positive* $(e_0, e_1)_* \sigma$-*measure the t-midpoints of the restricted geodesics* $s \mapsto (e_s)_* \sigma_{\Gamma^{(i)}}$, $i = 1, 2$, *are mutually singular, i.e.*

$$(e_t)_* \sigma_{\Gamma^{(1)}} \perp (e_t)_* \sigma_{\Gamma^{(2)}}.$$

Corollary 2.12. *Assume* (M, d, m) *is p-essentially non-branching for some* $p \in (1, \infty)$. *Then for each geodesic* $t \mapsto \mu_t$ *in* $\mathcal{P}_p(M)$ *connecting* $\mu_0, \mu_1 \ll \mathsf{m}$ *and* $t \in (0, 1)$ *the geodesics* $s \mapsto \mu_{st}$ *and* $s \mapsto \mu_{t+s(1-t)}$ *are the unique geodesics connecting* μ_0 *and* μ_t *and respectively* μ_t *and* μ_1. *Furthermore, the (unique) p-optimal couplings of* (μ_t, μ_0) *and* (μ_t, μ_1) *are induced by transport maps* $T_{t,0}, T_{t,1} : M \to M$.

3. Spaces with good transport behavior

In this section we study spaces where we assume a priori that every optimal coupling is induced by a transport map whenever its first marginal is absolutely continuous. It turns out that such spaces are already essentially non-branching. Throughout this section assume $p \in (1, \infty)$.

Definition 3.1 (Good transport behavior)**.** A metric measure space (M, d, m) has *good transport behavior* $(\mathsf{GTB})_p$ if for all $\mu, \nu \in \mathcal{P}_p(M)$ with $\mu \ll \mathsf{m}$ any optimal transport plan between μ and ν is induced by a map.

Remark. The condition was used in a recent work by F. Galaz-García, A. Mondino, G. Sosa and the author [GGKMS17] in order to study the orbit structure of groups acting isometrically on metric measure spaces with $(\mathsf{GTB})_p$. In particular, it can be used to exclude isometries with too large fixed point set, see also [Sos16, Lemma 4.1].

Proposition 3.2. *The following spaces have* $(\mathsf{GTB})_p$:

(i) *Essentially non-branching* $\mathsf{MCP}(K, N)$*-spaces for* $p = 2$, $K \in \mathbb{R}$, *and* $N \in [1, \infty)$. *In particular, this includes essentially non-branching* $\mathsf{CD}^*(K', N')$*-spaces, essentially non-branching* $\mathsf{CD}(K, N)$*-spaces, and* $\mathsf{RCD}^*(\mathsf{K}', \mathsf{N}')$*-spaces, see* [GRS16, CM17].

(ii) *Non-branching, qualitatively non-degenerate spaces for all* $p \in (1, \infty)$*, see* [CH15] *and Definition* 5.1 *below.*

(iii) *Any (local) doubling measure* m *on* $(\mathbb{R}^n, \|\cdot\|_{\mathrm{Euclid}})$ *or more generally on a Riemannian manifold, see* [GM96].

The last example shows that there is an abundance of spaces with good transport behavior. However, we will show that the existence of transport maps prevents too much branching and excludes therefore normed spaces whose norm is not strictly convex. Note that the main theorem of this note extends the list above to p-essentially non-branching, qualitatively non-degenerate spaces, see Theorem 5.8.

The first two lemmas were proved in a slightly different form in [GGKMS17]. Recall that for $\Gamma \subset M \times M$ and $s, t \in [0, 1]$ we define $\Gamma_{s,t} := (e_s, e_t)\big((e_0, e_1)^{-1}\Gamma\big)$.

Lemma 3.3. *Let* $\Gamma \subset M \times M$ *be a* c_p*-cyclically monotone set. Then for any* $s, t \in [0, 1]$ *the set* $\Gamma_{s,t}$ *is* c_p*-cyclically monotone.*

Proof. Choose $(x_s^i, x_t^i) \in \Gamma_{s,t}$, $i = 1, \ldots, n$, and note that there are geodesics $\gamma^{(i)} \in \hat{\Gamma}$, $i = 1, \ldots, n$, with $(\gamma_s^{(i)}, \gamma_t^{(i)}) = (x_s^{(i)}, x_t^{(i)})$. By assumption

$$\bigcup_{i=1}^{n} \{(\gamma_0^{(i)}, \gamma_1^{(i)})\} \subset \Gamma$$

is c_p-cyclically monotone and hence

$$\sigma = \frac{1}{n} \sum \delta_{\gamma^{(i)}}$$

is a p-optimal dynamical coupling. Observe that

$$\bigcup_{n=1}^{n} \{(\gamma_s^{(i)}, \gamma_t^{(i)})\} = \mathrm{supp}(e_s, e_t)_* \sigma$$

is c_p-cyclically monotone because $(e_s, e_t)_* \sigma$ is p-optimal. Since $\bigcup_{n=1}^{n} \{(\gamma_s^{(i)}, \gamma_t^{(i)})\} \subset \Gamma_{s,t}$ this shows that $\Gamma_{s,t}$ is c_p-cyclically monotone. $\square$

Recall that for a subset $\Gamma \subset M \times M$ we define for $x \in M$

$$\Gamma(x) = \{y \in M \mid (x, y) \in \Gamma\}.$$

Lemma 3.4 ([GGKMS17, Lemma 4.5]). *A metric measure space* (M, d, m) *has* $(\mathsf{GTB})_p$ *if and only if for every* c_p*-cyclically monotone* Γ*, the set* $\Gamma(x)$ *contains at most one point for* m*-almost all* $x \in M$.

In particular, if (M, d, m) *has* $(\mathsf{GTB})_p$ *then for any closed* c_p*-cyclically monotone set* Γ *and* m*-almost all* $x \in M$ *there exists a unique geodesic connecting* x *and* $\Gamma(x)$, *whenever the set* $\Gamma(x)$ *is non-empty.*

Remark. The lemma applies in particular to the c_p-superdifferential $\partial^{c_p} \varphi$ of c_p-concave functions φ.

Proof. The first part follows from the Selection Dichotomy for Sets (Theorem 2.1) and the fact that the support $\Gamma = \mathrm{supp}\, \pi$ of a p-optimal coupling π is c_p-cyclically monotone. Indeed, the second possibility of the Selection Dichotomy applied to Γ

and $\mu = (p_1)_*\pi$ would imply that there is a compact set $K \subset p_1(\Gamma)$ and two maps $T_1, T_2 : K \to M$ with $T_1(x) \neq T_2(x)$ for $x \in K$ and $\{T_1(x), T_2(x)\} \subset \Gamma(x)$ for $x \in p_1(\Gamma)$ and that for $\mu_0 = \frac{1}{\mu(K)}\mu\big|_K$ the following coupling

$$\frac{1}{2}\left((\mathrm{id}\times T_1)\mu_0 + (\mathrm{id}\times T_1)\mu_0\right)$$

is p-optimal and not induced by a transport map. Therefore, either of the conditions implies that μ_0 cannot be absolutely continuous with respect m.

To prove the last statement suppose (M, d, m) has $(\mathsf{GTB})_p$ and observe that by the previous lemma whenever γ is a geodesic connecting x and $y \in \Gamma(x)$ then $\gamma_t \in \Gamma_{0,t}(x)$.

Let $D_t = \{x \in M \,|\, \Gamma_{0,t}(x) \neq \varnothing\}$ and note that $D_t \subset D_{t'}$ whenever $0 \leq t' \leq t \leq 1$. Let $(t_n)_{n\in\mathbb{N}}$ be dense in $(0,1]$ with $t_1 = 1$ and choose a measurable set $\Omega_n \subset D_1$ of full m-measure in D_1 such that $\Gamma_{0,t_n}(x)$ is single-valued for all $x \in \Omega_n$. Then $\Omega = \cap_{n\in\mathbb{N}}\Omega_n$ also has full m-measure in D_1. Let γ and η be two geodesics connecting $x \in \Omega$ and $y \in \Gamma_{0,1}(x)$. If γ and η were distinct then there is an open interval $I \subset (0,1)$ such that $\gamma_s \neq \eta_s$ for all $s \in I$. In particular, there is an $n > 0$ such that $t_n \in I$. Hence $\gamma_{t_n} \neq \eta_{t_n}$ and $\Gamma_{0,t_n}(x)$ is not single-valued. However, this is a contradiction as $x \in \Omega \subset \Omega_n$ implies that $\Gamma_{0,t_n}(x)$ is single-valued. $\qquad\square$

Lemma 3.5. *Let φ be a c_p-concave function and $(x_0, x_1), (y_0, y_1) \in \partial^{c_p}\varphi$ be such that for some $t_0 \in (0,1)$ it holds $x_{t_0} = y_{t_0}$, where x_t and y_t are t-midpoints of (x_0, x_1) and (y_0, y_1) respectively. Then $(x_0, y_1), (y_0, x_1) \in \partial^{c_p}\varphi$.*

Proof. Choose geodesics $s \mapsto x_s$ and $s \mapsto y_s$ between x_0 and x_1 and resp. y_0 and y_1 and define

$$\mu_s = \frac{1}{2}\left(\delta_{x_s} + \delta_{y_s}\right).$$

Note that (φ, φ^{c_p}) is a dual solution for the measures μ_0 and μ_1.

We write $\varphi_t = t^{p-1}\varphi$ and note that the function φ_t is c_p-concave and $(\varphi_t, \varphi_t^{c_p})$ a dual solution for the measure μ_0 and μ_t ([Kel17, 2.9 and Remark after 2.1]). Denote the c_p-duals of φ and φ_t by ψ and ψ_t respectively.

Since $\partial^{c_p}\varphi$ is c_p-cyclically monotone, Lemma 2.9 shows that

$$d(x_0, x_1) = d(y_0, y_1) = d(x_0, y_1) = d(y_0, x_1).$$

Furthermore, $(x_0, x_t), (y_0, y_t) \in \partial^{c_p}\varphi_t$ by the choice of geodesics $s \mapsto x_s$ and $s \mapsto y_s$. Hence

$$\varphi_t(x_0) + \psi_t(x_t) = d^p(x_0, x_t) = d^p(y_0, y_t) = \varphi_t(y_0) + \psi_t(y_t).$$

For $t = 1$ we obtain

$$\varphi(x_0) + \psi(x_1) = \varphi(y_0) + \psi(y_1),$$

and because $x_{t_0} = y_{t_0}$ for some $t_0 \in (0,1)$, we also have

$$\varphi_{t_0}(x_0) + \psi_{t_0}(x_t) = \varphi_{t_0}(y_0) + \psi_{t_0}(x_t)$$

implying

$$\varphi(x_0) = t_0^{1-p}\varphi_{t_0}(x_0) = t_0^{1-p}\varphi_{t_0}(y_0) = \varphi(y_0).$$

Therefore,

$$\varphi(x_0) + \psi(y_1) = d^p(x_0, x_1) = d^p(x_0, y_1)$$

which shows $(x_0, y_1) \in \partial^{c_p}\varphi$. Similarly, it holds $(y_0, x_1) \in \partial^{c_p}\varphi$. $\qquad\square$

The following is a direct application of the previous lemma.

Proposition 3.6. *Assume (M, d, m) has $(\mathsf{GTB})_p$ and let $\mu_0, \mu_1 \in \mathcal{P}_p(M)$ with $\mu_0 \ll \mathsf{m}$. Then there is a unique p-optimal dynamical plan $\sigma \in \mathrm{OptGeo}_p(\mu_0, \mu_1)$ and a measurable set L with $\sigma(\mathsf{L}) = 1$ which is non-branching to the right. In particular, a metric measure space with $(\mathsf{GTB})_p$ is p-essentially non-branching.*

Proof. Let μ_0, μ_1, and π be as above and T be a p-optimal transport map between μ_0 and μ_1. Assume $\mathcal{T} : M \times M \to \mathsf{Geo}(M, d)$ is a measurable selection such that $\mathcal{T}(x, y)_0 = x$ and $\mathcal{T}(x, y)_0 = y$ and define a measurable map $\mathcal{S} : M \to \mathsf{Geo}(M, d)$ by

$$\mathcal{S}(x) = \mathcal{T}(x, T(x)).$$

Note that if φ is a dual solution then $\operatorname{supp} \pi \subset \partial^{c_p} \varphi$.

By Lemma 3.4 there is a Borel set A of full μ_0-measure such that

$$(A \times M) \cap \partial^{c_p} \varphi = (A \times M) \cap \operatorname{graph} T$$

and for all $x \in A$ the geodesic $\mathcal{S}(x)$ is the unique geodesic connecting x and $T(x)$. This implies immediately that the dynamical coupling $\sigma = \mathcal{S}_* \mu_0$ is the unique p-optimal dynamical coupling between μ_0 and μ_1.

It suffices to show that $\mathsf{L} = \mathcal{S}(A)$ is non-branching to the right. For this let $\gamma, \eta \in \mathsf{L}$ be two geodesics with $\mathsf{restr}_{0,t} \gamma = \mathsf{restr}_{0,t} \eta$ for some $t \in (0, 1)$. Lemma 3.5 implies that $(\gamma_0, \eta_1) \in \partial^{c_p} \varphi$. However, since $\gamma_0 = \eta_0 \in A$ this means $\gamma_1 = \eta_1 = T(\gamma_0)$ and $S(\gamma_0) = \gamma = \eta$. As γ and η are arbitrary we see that L is non-branching to the right. $\square$

Remark. The conclusion in the first part of Proposition 3.6 above is stronger than the ordinary p-essentially non-branching property as it takes into account arbitrary final measures rather than just absolutely continuous ones.

Corollary 3.7. *Assume (M, d, m) has good transport behavior $(\mathsf{GTB})_p$ and is strongly non-degenerate $(\mathsf{sND})_p$ (see Definition 4.4 below). Then for any $\mu_0, \mu_1 \in \mathcal{P}_p(M)$ with $\mu_0 \ll \mathsf{m}$ there is a unique p-optimal dynamical coupling σ between μ_0 and μ_1 and this coupling is concentrated on a set of non-branching geodesics. Furthermore, $(e_t)_* \sigma \ll \mathsf{m}$ for all $t \in (0, 1)$. In particular, m has the strong interpolation property $(\mathsf{sIP})_p$.*

Example of essentially non-branching spaces with bad geometric behavior. In this section we construct a measure on the tripod that is essentially non-branching and for any two absolutely continuous measures there is a unique transport map. However, the obvious branching in the tripod shows that there is no measure that makes the tripod into a space with good transport behavior.

Definition 3.8. A metric measure space (M, d, m) has the *weak good transport behavior* $(\mathsf{GTB})_{w,p}$ for $p \in (1, \infty)$, if for all $\mu_0, \mu_1 \in \mathcal{P}_p(M)$ with $\mu_0, \mu_1 \ll \mathsf{m}$ any optimal transport plan between μ_0 and μ_1 is induced by a map.

Let (T, d) be the tripod, i.e. T is obtained by gluing together three intervals $I_i = [0_i, 1_i]$, $i = 1, 2, 3$ at $\mathbf{0} = 0_1 = 0_2 = 0_3$ and d is the corresponding length metric. Denote by T_i the natural inclusions $[0, 1] \to I_i \subset \mathsf{T}$.

Example. There is continuum of measures m of full support on T such that $(\mathsf{T}, d, \mathsf{m})$ is p-essentially non-branching and has the weak good transport behavior $(\mathsf{GTB})_{w,p}$ for all $p \in (1, \infty)$.

Sketch of the construction. Let ν_0, ν_1, ν_2 be three non-atomic probability measures on $[0,1]$ with full support and Ω_i three disjoint sets such that $\nu_i(\Omega_j) = \delta_{ij}$. Define a measure m on T by

$$\mathsf{m}\big|_{I_i} = (T_i)_* \nu_i$$

and a set $\Omega = \cup T_i(\Omega_i)$. Note that $\mathsf{m}(M \backslash \Omega) = 0$.

If $x, y \in \Omega$ satisfy $d(\mathbf{0}, x) = d(\mathbf{0}, y)$ then $x, y \in \Omega_i$ for exactly one $i = 1, 2, 3$ and $x = y$. Thus since branching can only happen at $\mathbf{0}$, any two geodesics with endpoints in a c_p-cyclically monotone set $\Gamma \subset \Omega \times \Omega$ which intersect at a point $t \in (0, 1)$ must be equal. In particular, any such Γ is already non-branching. Note that whenever $\mu_0, \mu_1 \ll \mathsf{m}$ and π is a p-optimal coupling then

$$\pi(\Omega \times \Omega) = 1$$

so that π is concentrated on the non-branching set $\operatorname{supp} \pi \cap (\Omega \times \Omega)$.

To obtain transport maps it is sufficient to assume $\mu_0 \ll \mathsf{m}\big|_{I_i}$. In that case let $S_i : \mathsf{T} \to [-1, 1]$ be the map that collapses I_j and I_k for $i \neq k, j$ where we assume I_i corresponds to $[-1, 0]$. Note also that S_i restricted to Ω is invertible hence $(S_i)_* \mathsf{m}$ is a non-atomic measure. This can be used to show that the (unique) p-optimal transport map between $(S_i)_* \mu_0$ and $(S_i)_* \mu_1$ can be pulled back to a p-optimal transport map.

This construction works more generally for all cost functions $h(d(\cdot, \cdot))$ with h strictly convex and increasing. $\qquad\square$

4. Existence of absolutely continuous interpolations for non-degenerate measures

In this section we prove the existence of absolutely continuous interpolation measures if the initial measure is absolutely continuous and the background measure satisfies certain non-degeneracy conditions. In order to avoid proving very similar results for final measures supported on finite sets and then on general sets, we generalize the construction to optimal couplings concentrated on so called non-degenerate sets.

Non-degenerate measures and sets. The following condition was introduced in [CM16] and is based on stronger variant called *qualitative non-degenericty* (see below) introduced earlier in [CH15].

Definition 4.1 (non-degenerate measure). A metric measure space (M, d, m) is called *non-degenerate* if for all Borel sets A with $\mathsf{m}(A) > 0$ it holds $\mathsf{m}(A_{t,x}) > 0$ for $t \in (0, 1)$.

Our main goal is to prove existence of absolutely continuous interpolations. We formalize the general a priori existence by the following condition.

Definition 4.2 (interpolation property). A metric measure space (M, d, m) is said to have the *interpolation property* $(\mathsf{IP})_p$ for some $p \in (1, \infty)$ if for all $\mu_0, \mu_1 \in \mathcal{P}_p(M)$ with $\mu_0 \ll \mathsf{m}$, all p-optimal couplings $\pi \in \mathsf{Opt}_p(\mu_0, \mu_1)$ and all $t \in (0, 1)$ there is a p-optimal dynamical coupling σ between μ_0 and μ_1 with $(e_0, e_1)_* \sigma = \pi$ and $(e_t)_* \sigma \ll \mathsf{m}$.

It has the *strong interpolation property* $(\mathsf{sIP})_p$ if for all p-optimal dynamical couplings σ between μ_0 and μ_1 and all $t \in (0, 1)$ it holds $(e_t)_* \sigma \ll \mathsf{m}$ whenever $\mu_0 \ll \mathsf{m}$.

In order to show that the interpolation property $(\mathsf{IP})_p$ holds we will study the supports of optimal couplings and need a non-degeneracy condition of sets $\Gamma \subset M \times M$.

The following notation will be used: Given a set $A \subset M$ and $s \in [0,1]$ define the set $\Gamma^{A,s}$ by
$$((e_0, e_1)e_s^{-1}(A)) \cap \Gamma,$$
i.e. we throw out all endpoints which cannot be reached via geodesics having an s-midpoint in A. As above $\Gamma_t^{A,s}$ equals $e_t(\Gamma^{A,s})$ whenever $t \in [0,1]$. Observe that if A is analytic then $\Gamma_t^{A,s}$ is analytic for all $s, t \in [0,1]$. Also in case $s = t = 0$ this simplifies to $\Gamma_0^{A,0} = \Gamma_0 \cap A$.

Definition 4.3 (non-degenerate set). A Borel set $\Gamma \subset M \times M$ is *non-degenerate* (with respect to m) if for all Borel sets A with $\mathsf{m}(\Gamma_0 \cap A) > 0$ it holds $\mathsf{m}(\Gamma_t^{A,0}) > 0$ whenever $t \in (0,1)$.

It is easy to see that $B \times \{x\}$ is non-degenerate for all $x \in M$ and all Borel sets $B \subset M$ whenever m is non-degenerate.

Definition 4.4 (strongly non-degenerate measure). A metric measure space (M, d, m) is *strongly non-degenerate* $(\mathsf{sND})_p$ for some $p \in (1, \infty)$ if every c_p-cyclically monotone Borel set Γ is non-degenerate.

Remark. By abuse of notation we say m is (strongly) non-degenerate or has the (strong) interpolation property if (M, d, m) is (resp. has) the corresponding property.

It is easy to see that any strongly non-degenerate measure m is also non-degenerate. Furthermore, a measure with strong interpolation property $(\mathsf{sIP})_p$ is necessarily strongly non-degenerate $(\mathsf{sND})_p$. The converse is true if the p-optimal dynamical coupling σ is unique. In general the converse is false as can be seen by the metric measure space $(\mathbb{R}^n, \| \cdot - \cdot \|_\infty, \lambda^n)$ which has many non-absolutely continuous interpolations from the Lebesgue measures restricted to the unit ball to the delta measure at the origin, see also Remark after the proof of Lemma 5.7.

However, the interpolation property is sufficient to show that the space is strongly non-degenerate. Via the existence of absolutely continuous interpolations in the next section one can show that both properties are actually equivalent.

Lemma 4.5. *Assume (M, d, m) is a metric measure space having the interpolation property $(\mathsf{IP})_p$. Then (M, d, m) is strongly non-degenerate $(\mathsf{sND})_p$.*

Proof. Let A be a Borel set and Γ be a c_p-cyclically monotone Borel set with $\mathsf{m}(p_1(\Gamma) \cap A) > 0$. Without loss of generality $A \subset p_1(\Gamma) = \Gamma_0$. Let $\mu_0 = \frac{1}{\mathsf{m}(A)}\mathsf{m}\big|_A$ and choose a measurable selection T of $\Gamma \cap (A \times M)$. Then $\pi = (\mathrm{id} \times T)_*\mu_0$ is a p-optimal coupling. Let σ be given by the interpolation property. Then $(e_t)_*\sigma(\Gamma_t^{A,0}) = 1$ and $(e_t)_*\sigma \ll \mathsf{m}$ implying $\mathsf{m}(\Gamma_t^{A,0}) > 0$. Because A and Γ are arbitrary we conclude that (M, d, m) is strongly non-degenerate $(\mathsf{sND})_p$. $\square$

The GKS-Construction. In this section we construct an absolutely continuous interpolation μ_t given a p-optimal coupling π which is concentrated (in a consistent way) on a non-degenerate set Γ and its first marginal $(p_1)_*\pi$ is absolutely continuous. Furthermore, we find a Borel set $\tilde{\Gamma} \subset \Gamma$ of full π-measure, such that μ_t "sees" all sets in $\tilde{\Gamma}_t$ of positive m-measure. The last property turns out to be crucial in

order to apply the idea of Cavalletti–Huesmann [CH15] in the setting of essentially non-branching spaces, see proof of Lemma 5.4 and Theorem 5.8.

The proof of Theorem 4.10 below is based on the following generalized form of the Lebesgue decomposition which can be found in [Rud08, Section 9.4]. One part of the result was proven by Glicksberg and the other by König and Sievers owing the name *GKS-Decomposition*, see [Rud08, 9.4.1].

Lemma 4.6 (GKS-Decomposition [Rud08, 9.4.4]). *Let (M, d) be a locally compact complete separable metric space and $\mathcal{B} \subset \mathcal{P}(M)$ be a weakly compact and linearly convex subset of probability measures. Then every non-negative finite measure $\tilde{\mathsf{m}}$ has a unique decomposition*

$$\tilde{\mathsf{m}} = \tilde{\mathsf{m}}_a + \tilde{\mathsf{m}}_s$$

such that $\tilde{\mathsf{m}}_a \ll \mu$ for some $\mu \in \mathcal{B}$ and there is a Borel set F which is a countable union of closed subsets such that $\tilde{\mathsf{m}}_s$ is concentrated on F and, in addition, F is $\mathcal{B}$-null, i.e. it holds $\tilde{\mathsf{m}}_s(M \backslash F) = 0$ and $\nu(F) = 0$ for all $\nu \in \mathcal{B}$.

Remark. (1) Linear convexity of $\mathcal{B}$ means that whenever $\mu, \nu \in \mathcal{B}$ then also $(1 - \lambda)\mu + \lambda\nu \in \mathcal{B}$ for all $\lambda \in [0, 1]$.

(2) The lemma is usually stated for compact Hausdorff spaces. However, one can embed M into the one-point compactification $M^* = \{*\} \cup M$ such that $\mathcal{B}$ is still compact in $\mathcal{P}(M^*)$. Note that M^* is a compact Hausdorff space. Since each of the involved measures gives zero measure to the set $\{*\}$, we see that the lemma also holds for general locally compact Hausdorff spaces. In particular, it holds for proper metric spaces.

(3) Recall that m is a locally bounded measure if (M, d, m) is a proper metric measure space. In that case there is a continuous function $\varphi : [0, \infty) \to (0, 1]$ such that $\tilde{\mathsf{m}} = \varphi(d(x_0, \cdot))\mathsf{m}$ is a finite measure. Then the unique decomposition of m with respect to $\mathcal{B}$ is given by $\mathsf{m} = (\varphi(d(x_0, \cdot))^{-1}\tilde{\mathsf{m}}_a + (\varphi(d(x_0, \cdot))^{-1}\tilde{\mathsf{m}}_s$.

Before stating the main theorem of this section we need the following technical lemmas.

Lemma 4.7. *If $A \subset M$ is an analytic set and π is a coupling concentrated on $\Gamma^{A,t}$ then there is a dynamical coupling σ concentrated on $e_t^{-1}(A) \cap \hat{\Gamma}$. In particular, $(e_t)_*\sigma(A) = 1$.*

Proof. Since A is analytic, the set

$$\Lambda = \{(\gamma_0, \gamma_1, \gamma) \in \Gamma \times \mathsf{Geo}_{[0,1]}(M, d) \,|\, \gamma \in e_t^{-1}(A) \cap \hat{\Gamma}\}$$
$$= (e_0, e_1, \mathrm{id}) \left(e_t^{-1}(A) \cap (e_0, e_1)^{-1}(\Gamma)\right)$$

is also analytic. Thus by von Neumann's Measurable Selection Theorem there is a measurable selection $S : \Gamma \to \mathsf{Geo}_{[0,1]}(M, d)$ such that $(x, y, S(x, y)) \in \Lambda$ for all $(x, y) \in \Gamma$. In particular, $S(x, y)_t \in A$. To conclude just observe that $\sigma = S_*\pi$ is concentrated on $e_t^{-1}(A) \cap \hat{\Gamma}$. $\square$

Lemma 4.8. *Assume (M, d) is a proper geodesic space. Then for all measures $\mu_0, \mu_1 \in \mathcal{P}_p(M)$ and every p-optimal coupling π the following set of t-midpoints*

$$\mathcal{B} = \{(e_t)_*\sigma \,|\, \sigma \in \mathsf{OptGeo}(\mu_0, \mu_1), (e_0, e_1)_*\sigma = \pi\}$$

is linearly convex, compact in $\mathcal{P}_p(M)$ and weakly compact in $\mathcal{P}(M)$.

Remark. Similar arguments also show that for finitely many $\{s_1, \ldots, s_n\} \subset [0,1]$ and a measure $\boldsymbol{\pi} \in \mathcal{P}(M^n)$ the set

$$\mathcal{C} = \{(e_t)_* \sigma \mid \sigma \in \mathsf{OptGeo}(\mu_0, \mu_1), (e_{s_1}, \ldots e_{s_n})_* \sigma = \boldsymbol{\pi}\}$$

is linearly convex, compact in $\mathcal{P}_p(M)$ and weakly compact in $\mathcal{P}(M)$.

Note that the result shows that the GKS-Decomposition can be applied to the set $\mathcal{B}$. The proof of Lemma 4.8 is given at the end of this section.

In order to make the main theorem more readable we introduce the following condition. It won't be used anywhere else but here.

Definition 4.9. A coupling $\pi \in \mathcal{P}(M \times M)$ is *strongly consistent* if for all $\tilde{\pi} \ll \pi$ and every measurable set Γ' with $\tilde{\pi}(\Gamma') = 1$ there is a non-degenerate, c_p-cyclically monotone Borel set $\Gamma \subset \Gamma'$ with $\tilde{\pi}(\Gamma) = 1$.

Note that whenever π is strongly consistent then any coupling π' with $\pi' \ll \pi$ is strongly consistent as well.

Theorem 4.10 (GKS-Construction)**.** *Let (M, d, m) be a proper metric measure space. Assume π is a strongly consistent, p-optimal coupling between $\mu_0 \ll \mathsf{m}$ and μ_1. Then for every $t \in (0, 1)$ there is a p-optimal dynamical coupling σ such that $(e_0, e_1)_* \sigma = \pi$ and*

$$\mu_t = (e_t)_* \sigma \ll \mathsf{m}.$$

Furthermore, μ_t is maximal in the following sense: Let Γ be a Borel set of full π-measure and

$$\mathsf{m}\big|_{\Gamma_t} = g\mu_t + \mathsf{m}\big|_F$$

be the Lebesgue decomposition of $\mathsf{m}\big|_{\Gamma_t}$ with respect to μ_t where $F \subset \Gamma_t$ is a Borel with $\mu_t(F) = 0$. Then π is concentrated on a Borel set $\tilde{\Gamma} \subset \Gamma \backslash \Gamma^{F,t}$ and it holds

$$\mathsf{m}\big|_{\tilde{\Gamma}_t} \ll \mu_t.$$

Remark. Absolute continuity and maximality imply $\mathsf{m}\big|_{\tilde{\Gamma}_t} \ll \mu_t \ll \mathsf{m}\big|_{\tilde{\Gamma}_t}$ as π is concentrated on $\tilde{\Gamma}$.

Corollary 4.11. *Suppose $\mu_t \ll \mathsf{m}$ and $\tilde{\Gamma}$ are constructed from π as above. If there is a strongly consistent coupling $\pi_{t,1}$ of μ_t and μ_1 then for each $s \in (t, 1)$ there is a p-optimal dynamical coupling $\hat{\sigma}$ such that $(e_0, e_1)_* \hat{\sigma} = \pi$, $(e_t)_* \hat{\sigma} = \mu_t$ and $(e_s)_* \hat{\sigma} \ll \mathsf{m}$.*

Proof of Theorem 4.10. Let $\mathcal{B}$ be defined as in Lemma 4.8 above. We split the proof into two steps.

STEP 1: There is a $\mu_t \in \mathcal{B}$ which is maximal in the sense of the theorem and $\rho_t \neq 0$ where $\mu_t = \rho_t \mathsf{m} + \mu_t^s$ is the Lebesgue decomposition of μ_t with respect to m.

Let $\Gamma \subset \operatorname{supp} \pi$ be a non-degenerate, c_p-cyclically monotone Borel set with $\pi(\Gamma) = 1$. Since $\mathcal{B}$ is weakly compact and linearly convex we can apply the GKS-Decomposition to $\mathsf{m}\big|_{\Gamma_t}$ and obtain a measure $\mu_t \in \mathcal{B}$ such that

$$\mathsf{m}\big|_{\Gamma_t} = g\mu_t + \mathsf{m}_s$$

and there is a Borel set $F \subset \Gamma_t$ such that $\tilde{\mu}_t(F) = 0$ for all $\tilde{\mu}_t \in \mathcal{B}$ and $\mathsf{m}_s(M \backslash F) = 0$. Thus $\mathsf{m}_s = \mathsf{m}\big|_F$ and $g(x) > 0$ for μ_t-almost all $x \in M$.

We claim that π is concentrated on $\Gamma \backslash \Gamma^{F,t}$. Assume, by contradiction, that

$$\lambda = \pi(\Gamma^{F,t}) > 0.$$

Let $\sigma \in \mathsf{OptGeo}_p(\mu_0, \mu_1)$ with $(e_t)_* \sigma = \mu_t$ and $(e_0, e_1)_* \pi$. Then for $f = \chi_{\Gamma^{F,t}}$ the coupling σ_f is a p-optimal dynamical coupling between $(e_0)_* \sigma_f$ and $(e_1)_* \sigma_f$, and $\pi_F = (e_0, e_1)_* \sigma_f$ is concentrated on $\Gamma^{F,t}$. Also note that $\pi = \lambda \pi_F + (1 - \lambda) \check{\pi}$ where $\check{\pi} = (e_0, e_1)_* \sigma_{1-f}$.

By Lemma 4.7 there is a p-optimal dynamical coupling $\tilde{\sigma}_f$ induced by π_F which concentrated on $\hat{\Gamma}^{F,t}$ such that $(e_t)_* \tilde{\sigma}_f(F) = 1$. However, by Lemma 2.6 the dynamical coupling

$$\tilde{\sigma} = \lambda \tilde{\sigma}_f + (1 - \lambda) \sigma_{1-f}$$

is also p-optimal with

$$(e_0, e_1)_* \tilde{\sigma} = \lambda \pi_F + (1 - \lambda) \check{\pi} = \pi.$$

Thus $\tilde{\mu}_t = (e_t)_* \tilde{\sigma} \in \mathcal{B}$ with

$$\tilde{\mu}_t(F) \geq \lambda \mu_t^f(F) = \lambda > 0$$

contradicting the properties of GKS-Decomposition. Hence $\pi(\Gamma \backslash \Gamma^{F,t}) = 1$.

Let

$$\mu_t = \rho_t \mathsf{m} + \mu_t^s$$

be the Lebesgue decomposition of μ_t with respect to m with $\mu_t^s \perp \mathsf{m}$. Note that $\rho_t(x) > 0$ for m-almost all $x \in A_t = (\Gamma \backslash \Gamma^{F,t})_t$. By assumption π is concentrated on a non-degenerate c_p-cyclically monotone Borel set $\tilde{\Gamma} \subset \Gamma \backslash \Gamma^{F,t}$. Since $\mu_0(\tilde{\Gamma}_0) = 1$ and $\mu_0 \ll \mathsf{m}$ it holds $\mathsf{m}(\tilde{\Gamma}_0) > 0$ and thus $\mathsf{m}(\tilde{\Gamma}_t) > 0$ by non-degeneracy of $\tilde{\Gamma}$ implying $\rho_t \neq 0$, i.e. the absolutely continuous part of μ_t is non-trivial. Finally observe that $\rho_t(x) > 0$ for m-almost all $x \in A_t$ shows that $\mathsf{m}(A_t \backslash \tilde{\Gamma}_t) = 0$ hence $\mathsf{m}|_{\tilde{\Gamma}_t} \ll \mu_t$ yields maximality of μ_t.

STEP 2: Given $\mu_t = \rho_t \mathsf{m} + \mu_t^s$ and $\tilde{\Gamma}$ as in STEP 1, there is a $\mu_t^* \in \mathcal{B}$ with $\mu_t^* = \rho_t^* \mathsf{m}$ and $\rho_t \leq \rho_t^*$, and μ_t^* is maximal in the sense of the theorem.

We define a partial ordering on subsets of $\mathcal{B}$ and show that maximal elements exist and are absolutely continuous: For $\rho \in L^1_{\geq 0}(\mathsf{m})$ with $\int \rho d\mathsf{m} \in [0, 1]$ set

$$\mathcal{B}_\rho = \{\mu \in \mathcal{B} \mid \mu = \rho \mathsf{m} + \mu^s\}$$

where $\mu = \rho \mathsf{m} + \mu^s$ is the Lebesgue decomposition of μ with respect to m. Note that we identify two $L^1(\mathsf{m})$-functions which agree m-almost everywhere.

Let

$$\mathbb{K} = \{\rho \in L^1_{\geq 0}(\mathsf{m}) \mid \int \rho d\mathsf{m} \in [0, 1], \mathcal{B}_\rho \neq \varnothing\}$$

and write

$$\rho' \succ \rho :\Longleftrightarrow \rho' \geq \rho, \rho' \neq \rho.$$

This is a partial ordering of $\mathbb{K}$. Also note that

$$\bigcup_{\rho \in \mathbb{K}} \mathcal{B}_\rho$$

is a partition of $\mathcal{B}$. Hence the partial order $\succ$ induces one on this partition.

Assume $\mu \in \mathcal{B}_\rho$ is not absolutely continuous with respect to m and let $\sigma \in \mathsf{OptGeo}_p(\mu_0, \mu_1)$ be a p-optimal dynamical coupling with $(e_0, e_1)_*\sigma = \pi$ and $(e_t)_*\sigma = \mu$. Then the decomposition $\mu = \rho\mathsf{m} + \mu^s$ induces a decomposition of σ as follows

$$\sigma = (1 - \lambda)\sigma_a + \lambda\sigma_s$$

where σ_a and σ_s are uniquely defined measures in $\mathcal{P}(\mathsf{Geo}_{[0,1]}(M, d))$ with

$$(e_t)_*(1 - \lambda)\sigma_a = \rho\mathsf{m}$$

and

$$(e_t)_*\lambda\sigma_s = \mu^s \neq 0.$$

Let

$$\tilde{\mu}_i = (e_i)_*\sigma_s \quad i = 0, 1$$

and

$$\pi_s = (e_0, e_1)_*\sigma_s.$$

Since $\pi_s \ll \pi$, we see that π_s is a strongly consistent p-optimal coupling between $\tilde{\mu}_0 \ll \mathsf{m}$ and $\tilde{\mu}_1$. Thus STEP 1 above is applicable to $(\tilde{\mu}_0, \tilde{\mu}_1)$ and there is a t-midpoint $\tilde{\mu}_t$

$$\tilde{\mu}_t = \tilde{\rho}\mathsf{m} + \tilde{\mu}^s$$

with $\tilde{\rho} \neq 0$ such that

$$\mu_t^{'} = (\rho + \lambda\tilde{\rho})\mathsf{m} + \lambda\tilde{\mu}^s$$

is still in $\mathcal{B}$. Hence $\rho + \lambda\tilde{\rho} \in \mathbb{K}$ and $\rho + \lambda\tilde{\rho} \succ \rho$. In particular, any $\rho \in \mathbb{K}$ with $\int \rho d\mathsf{m} \neq 1$ is not maximal with respect to the partial order $\succ$. Also note that any element $\rho \in \mathbb{K}$ satisfying $\int \rho d\mathsf{m} = 1$ is automatically maximal. To finish the proof it suffices to show that there are maximal elements above any $\rho \in \mathbb{K}$.

For this we want to apply Zorn's Lemma: Let $\{\rho_i\}_{i \in I}$ be a totally ordered chain where I is a totally ordered index set. Then choose $\mu_i \in \mathcal{B}_{\rho_i}$ and observe by compactness of $\mathcal{B}$ there is a subnet $I' \subset I$ such that $\lim_{i \in I'} \mu_i = \mu \in \mathcal{B}$. Then the net $(\rho_i)_{i \in I'}$ is an increasing family of non-negative $L^1(\mathsf{m})$-function so that by monotone convergence there is a $\rho \in L^1(\mathsf{m})$ such that

$$\rho = \lim_{i \in I'} \rho_i = \lim_{i \in I} \rho_i$$

and $\int \rho d\mathsf{m} \in [0, 1]$. Since

$$\rho_i \mathsf{m} \leq \mu_j$$

whenever $i \leq j$, it holds $\rho_i \mathsf{m} \leq \mu$ and thus $\rho\mathsf{m} \leq \mu$ where $\mu \leq \nu$ means $\mu(A) \leq \nu(A)$ for all Borel sets A. In particular, the Lebesgue decomposition of μ is given by

$$\mu = \hat{\rho}\mathsf{m} + \mu^s$$

for some $\hat{\rho} \in \mathbb{K}$ with $\hat{\rho} \succeq \rho \succeq \rho_i$, i.e. the chain $\{\rho_i\}_{i \in I}$ has a maximal element in $\mathbb{K}$. Therefore, Zorn's Lemma applied to $(\mathbb{K}, \succeq)$ gives the existence of at least one maximal element $\rho_t^* \in \mathbb{K}$ with $\rho_t^* \geq \rho_t$. Choosing $\mu_t^* = \rho_t^*\mathsf{m}$ gives a measure satisfying the statements of the theorem.

Finally, the properties of μ_t and $\tilde{\Gamma}$ imply $\mathsf{m}\big|_{\tilde{\Gamma}_t} \ll \rho_t\mathsf{m} \ll \mu_t^* \ll \mathsf{m}\big|_{\tilde{\Gamma}_t}$. Thus for any other Borel set Γ with $\pi(\Gamma) = 1$ there is a Borel set $\Gamma^{'} \subset \Gamma \cap \tilde{\Gamma}$ with $\pi(\Gamma^{'}) = 1$. But then $\mathsf{m}\big|_{\tilde{\Gamma}_t} \ll \mu_t^* \ll \mathsf{m}\big|_{\Gamma_t^{'}}$ implying $\mathsf{m}(\tilde{\Gamma}_t \backslash \Gamma_t^{'}) = 0$. This yields immediately $\mathsf{m}\big|_{\tilde{\Gamma}_t} \ll \mu_t^*$ and thus maximality of μ_t^*. $\square$

In order to apply Theorem 4.10 we need to prove that we find p-optimal couplings satisfying the assumptions of the theorem.

Lemma 4.12. *Assume (M, d, m) is non-degenerate. Then any c_p-cyclically monotone set Γ such that $\Gamma_1 = p_2(\Gamma)$ is finite is non-degenerate. In particular, if π is a p-optimal coupling with $(p_1)_*\pi \ll \mathsf{m}$ and $(p_2)_*\pi = \sum_{i=1}^n a_i \delta_{x_i}$ then any measurable set Γ of full π-measure contains a non-degenerate, c_p-cyclically monotone Borel set $\tilde{\Gamma} \subset \operatorname{supp} \pi \cap \Gamma$ of full π-measure.*

Proof. Observe that

$$\Gamma = \bigcup_{i=1}^n B^i \times \{x_i\}$$

for measurable subsets $B^i \subset M$. Since $\Gamma_1 = \mathsf{m}(\cup_{i=1}^n B^i)$ the condition $\mathsf{m}(\Gamma_0^A) > 0$ implies that there is at least one $i \in \{1, \ldots, n\}$ with $\mathsf{m}(B^i \cap A) > 0$. But then

$$\mathsf{m}(\Gamma_t^A) \geq \mathsf{m}(B_{t,x_i}^i) > 0$$

showing that Γ is non-degenerate.

For the last statement note that if $\pi(\Gamma') = 1$ then $\pi(\Gamma' \cap \operatorname{supp} \pi) = 1$ so that there is a Borel set $\tilde{\Gamma} \subset \Gamma' \cap \operatorname{supp} \pi$ with $\pi(\tilde{\Gamma}) = 1$. Since

$$p_2(\tilde{\Gamma}) \subset p_2(\operatorname{supp} \pi) = \{x_i\}_{i=1}^n$$

is finite, the result follows. $\qquad\square$

The same argument also holds more generally if the background measure is strongly non-degenerate.

Lemma 4.13. *If (M, d, m) is strongly non-degenerate $(\mathsf{sND})_p$ then for any p-optimal coupling π with $(p_1)_*\pi \ll \mathsf{m}$ the following holds: Whenever $\pi(\Gamma') = 1$ for a π-measurable set Γ' then there is a c_p-cyclically monotone, non-degenerate Borel subset $\Gamma \subset \Gamma'$ of full π-measure.*

Proof. Just note that π is concentrated on $\Gamma' \cap \operatorname{supp} \pi$ which is measurable, non-degenerate and c_p-cyclically monotone. Hence there is a Borel subset $\Gamma \subset \Gamma' \cap \operatorname{supp} \pi$ with $\pi(\Gamma) = 1$. By strong non-degeneracy Γ is non-degenerate. $\qquad\square$

The two lemmas allow us to apply Theorem 4.10 and Corollary 4.11.

Corollary 4.14. *Let (M, d, m) be a proper metric measure space and $\mu_0, \mu_1 \in \mathcal{P}_p(M)$ with $\mu_0 \ll \mathsf{m}$. Assume either (M, d, m) is non-degenerate and $\mu_1 = \sum_{i=1}^n \lambda_i \delta_{x_i}$ or that (M, d, m) is strongly non-degenerate $(\mathsf{sND})_p$. Then for any $0 < t < s < 1$ there is a p-optimal dynamical coupling σ between μ_0 and μ_1 with $(e_t)_*\sigma, (e_s)_*\sigma \ll \mathsf{m}$ and $(e_t)_*\sigma$ is maximal in the sense of Theorem 4.10. In particular, (M, d, m) is strongly non-degenerate $(\mathsf{sND})_p$ if and only if it has the interpolation property $(\mathsf{IP})_p$.*

Proof. For strongly non-degenerate measures the result follows immediately from the previous lemma.

For the case of m being non-degenerate and $\mu_1 = \sum_{i=1}^n \lambda_i \delta_{x_i}$ just observe that any p-optimal coupling $\pi_{t,1}$ between $\mu_t \ll \mathsf{m}$ and μ_1 is strongly consistent. Indeed, if $\tilde{\pi} \ll \pi_{t,1}$ then the set $p_2(\operatorname{supp} \tilde{\pi} \cap \Gamma')$ is finite hence contains a c_p-cyclically monotone Borel set Γ with $\tilde{\pi}(\Gamma) = 1$. $\qquad\square$

Proof of Lemma 4.8. Since (M, d) is geodesic $\mathcal{B}$ is non-empty and closed in $\mathcal{P}_p(M)$. Furthermore, properness of M together with $\mathcal{B}$ being bounded implies that $\mathcal{B}$ is weakly precompact. If μ_t^0 and μ_t^1 are measures in $\mathcal{B}$ then there are two p-optimal dynamical couplings σ^0 and σ^1 such that $(e_t)_*\sigma^i = \mu_t^i$ and $(e_0, e_1)_*\sigma^i = \pi$, $i = 0, 1$.

It holds

$$\mu_t^\lambda = (e_t)_* \sigma^\lambda$$

for

$$\mu_t^\lambda = (1 - \lambda)\mu_t^0 + \lambda \mu_t^1$$
$$\sigma^\lambda = (1 - \lambda)\sigma^0 + \lambda \sigma^1$$

so that

$$W_p^p(\mu_t^\lambda, \mu_1) \leq \int d(\gamma_t, \gamma_1)^p d\sigma^\lambda(\gamma)$$
$$= (1 - \lambda)\int t^p d(\gamma_0, \gamma_1)^p d\sigma^0(\gamma) + \lambda \int t^p d(\gamma_0, \gamma_1)^p d\sigma^1(\gamma)$$
$$= t^p W_p^p(\mu_0, \mu_1)$$

and similarly $W_p^p(\mu_0, \mu_t^\lambda) \leq (1 - t)^p W_p^p(\mu_0, \mu_1)$. Thus $W_p(\mu_0, \mu_t^\lambda) + W_p(\mu_t^\lambda, \mu_1) \leq W_p(\mu_0, \mu_1)$ which implies that μ_t^λ is a t-midpoint. Since $(e_0, e_1)_* \sigma^\lambda = \pi$ we have $\mu_t^\lambda \in \mathcal{B}$ implying linear convexity of $\mathcal{B}$.

Now let $(\mu_t^n)_{n \in \mathbb{N}}$ be a sequence in $\mathcal{B}$. By weak compactness we can assume after picking a subsequence and relabeling that $(\mu_t^n)_{n \in \mathbb{N}}$ converges weakly to some $\mu \in \mathcal{P}(M)$. Since

$$W_p(\mu_0, \mu) \leq \liminf_{n \to \infty} W_p(\mu_0, \mu_t^n) = t W_p(\mu_0, \mu_1)$$
$$W_p(\mu, \mu_1) \leq \liminf_{n \to \infty} W_p(\mu_t^n, \mu_1) = (1 - t) W_p(\mu_0, \mu_1)$$

we get

$$W_p(\mu_0, \mu_1) \leq W_p(\mu_0, \mu) + W_p(\mu, \mu_1) \leq W_p(\mu_0, \mu_1)$$

so that μ is a t-midpoint as well. Hence $W_p(\mu_0, \mu_t^n) \to W_p(\mu_0, \mu)$ and the sequence $(\mu_t^n)_{n \in \mathbb{N}}$ also converges in the p-th moment. This shows that $\mu_t^n \to \mu$ in $\mathcal{P}_p(M)$ (see [Vil08, Definition 6.8]). Thus any sequence in $\mathcal{B}$ has a subsequence converging in $\mathcal{P}_p(M)$. In particular, $\mathcal{B}$ is compact in $\mathcal{P}_p(M)$. $\qquad \square$

5. EXISTENCE OF TRANSPORT MAPS

In this section we want to prove the existence of transport maps using a combined approach of [CH15] and [CM17].

Qualitatively non-degenerate measures. Non-degeneracy and the GKS-Construction in the previous section imply that there are absolutely continuous interpolations between $\mu_0 \ll \mathsf{m}$ and $\mu_1 = \sum \lambda_i \delta_{x_i}$. However, the non-degeneracy condition is too weak to use approximation arguments for general μ_1. For this we need the following uniform variant which was introduced by Cavalletti–Huesmann [CH15] and represents a weak form of the measure contraction condition $\mathsf{MCP}(K, N)$, see e.g. [Stu06a, Oht07, CM17] and references therein.

Definition 5.1. The measure m is said to be *qualitatively non-degenerate* if for all $R > 0$ and $x_0 \in M$ there is a function $f_{R,x_0} : (0, 1) \to (0, \infty)$ with

$$\limsup_{t \to 0} f_{R,x_0}(t) > \frac{1}{2}$$

such that for every measurable $A \subset B_R(x_0)$ and all $x \in B_R(x_0)$ and $t \in (0,1)$ it holds

$$\mathsf{m}(A_{t,x}) \geq f_{R,x_0}(t)\mathsf{m}(A).$$

Corollary 5.2. *Any qualitatively non-degenerate measure is non-degenerate.*

The following proposition shows that qualitatively non-degenerate spaces are proper and make it possible to use GKS-Construction of the previous section.

Proposition 5.3. *A qualitatively non-degenerate measure m is locally doubling, i.e. for each $R > 0$ and $x_0 \in M$ there is a constant $C_{R,x_0} > 0$ such that*

$$\mathsf{m}(B_{2r}(x)) \leq C_{R,x_0} \cdot \mathsf{m}(B_r(x))$$

whenever $B_{2r}(x) \subset B_R(x_0)$. In particular, (M,d) is a proper metric space.

Proof. Just note that $B_r(x) \subset (B_{2r}(x))_{\frac{1}{2},x}$ for all $x \in M$ and $r > 0$. Thus qualitative non-degeneracy implies for $B_{2r}(x) \subset B_R(x_0)$

$$\mathsf{m}(B_{2r}(x)) \leq \frac{1}{f_{R,x_0}(\frac{1}{2})}\mathsf{m}(B_r).$$

Finally, properness follows from m being locally doubling (see e.g. [Hei01]).   □

Lemma 5.4. *Assume (M,d,m) is p-essentially non-branching for some $p \in (1,\infty)$ and m is qualitatively non-degenerate. If for a Borel set A, the set $A \times \{x,y\}$ is $c_{p'}$-cyclically monotone for $x \neq y \in M$ and $p' \in (1,\infty)$ then $\mathsf{m}(A) = 0$.*

Remark. One may replace the qualitative non-degeneracy by the following pointwise variant

$$\liminf_{t \to 0} \mathsf{m}(A_{t,x}) > \frac{1}{2}\mathsf{m}(A).$$

This condition is, however, too weak to do approximations of general c_p-cyclically monotone sets as in Lemma 5.7.

Proof. By inner regularity we can assume A is compact and $\{x,y\} \cup A \subset B_R(x_0)$ for some $R > 0$. By compactness of A we find t close to 0 and s close to 1 such that $f_{R,x_0}(t) \geq \frac{1}{2} + \epsilon$,

$$\mathsf{m}(A_\delta) \leq (1 + \epsilon)\mathsf{m}(A),$$

$$A_{t,x} \cup A_{t,y} \subset A_\delta$$

and

$$(A_{s,x})_\epsilon \cap (A_{s,y})_\epsilon = \varnothing$$

for some $\epsilon, \delta > 0$.

Decompose A into two Borel sets

$$A^{eq} = \{z \in A \,|\, d(z,x) = d(z,y)\}$$

and

$$A^{ne} = A \backslash A^{eq} = \{z \in A \,|\, d(z,x) \neq d(z,y)\}.$$

It suffices to show that the claim is true for the cases $A = A^{eq}$ and $A = A^{ne}$.

First assume $A = A^{ne}$ and observe that by $c_{p'}$-cyclic monotonicity and the fact that $d(z,x) \neq d(z,y)$ for all $z \in A$ it holds

$$A_{t,y} \cap A_{t,x} = \varnothing \quad \text{for all } t \in (0,1).$$

Hence

$$(1 + \epsilon)\mathsf{m}(A) \geq \mathsf{m}(A_\delta)$$
$$\geq \mathsf{m}(A_{t,x} \cup A_{t,y})$$
$$= \mathsf{m}(A_{t,x}) + \mathsf{m}(A_{t,y})$$
$$\geq 2f_{R,x_0}(t)\mathsf{m}(A) = (1 + 2\epsilon)\mathsf{m}(A)$$

which implies that $\mathsf{m}(A) = 0$.

For the case $A = A^{eq}$ assume by contradiction $\mathsf{m}(A) > 0$. Set

$$\mu_0 = \frac{1}{\mathsf{m}(A)}\mathsf{m}\big|_A$$

and observe that $A = A^{eq}$ implies that $A \times \{x, y\}$ is $c_{p''}$-cyclically monotone for all $p'' \in [1, \infty)$. In particular, $A \times \{x, y\}$ is c_p-cyclically monotone.

Apply Corollary 4.14 to (μ_0, δ_x) and (μ_0, δ_y) to get two dynamical couplings σ^x and σ^y whose interpolations at times s and t are absolutely continuous. The choice of s shows that $\mu_s^x = (e_s)_*\sigma^x$ and $\mu_s^y = (e_s)_*\sigma^y$ have disjoint support. Furthermore, the measures $\mu_t^x = (e_t)_*\sigma^x$ and $\mu_t^y = (e_t)_*\sigma^y$ are maximal with respect to $\tilde{A} \times \{x\}$ and resp. $\tilde{A} \times \{y\}$ for some $\tilde{A} \subset A$ with $\mathsf{m}(A \backslash \tilde{A}) = 0$. Since the set $\tilde{A} \times \{x, y\}$ is still c_p-cyclically monotone, the dynamical coupling $\frac{1}{2}(\sigma^x + \sigma^y)$ is p-optimal between μ_0 and $\frac{1}{2}(\delta_x + \delta_y)$. Because $\mu_s = \frac{1}{2}(\mu_s^x + \mu_s^y)$ is a decomposition into mutually singular measures, Theorem 2.11 shows

$$\mu_t^x \perp \mu_t^y.$$

By maximality of μ_t^x and μ_t^y it holds $\mathsf{m}\big|_{\tilde{A}_{t,x}} \ll \mu_t^x$ and $\mathsf{m}\big|_{\tilde{A}_{t,y}} \ll \mu_t^y$ so that $\mathsf{m}(\tilde{A}_{t,x} \cap \tilde{A}_{t,y}) = 0$. In particular, since m is qualitatively non-degenerate

$$\mathsf{m}(\tilde{A}_{t,x} \cup \tilde{A}_{t,y}) = \mathsf{m}(\tilde{A}_{t,x}) + \mathsf{m}(\tilde{A}_{t,y})$$
$$\geq 2f(t)\mathsf{m}(A) \geq (1 + 2\epsilon)\,\mathsf{m}(A).$$

Combining those facts we obtain

$$(1 + \epsilon)\mathsf{m}(A) \geq \mathsf{m}(A_\delta)$$
$$\geq \mathsf{m}(\tilde{A}_{t,x} \cup \tilde{A}_{t,y})$$
$$\geq (1 + 2\epsilon)\mathsf{m}(A)$$

which is a contradiction. This shows that $\mathsf{m}(A) = 0$. $\qquad\square$

Corollary 5.5. *Assume (M, d, m) is p-essentially non-branching and m qualitatively non-degenerate. If for some $p' \in (1, \infty)$ the set Γ is a $c_{p'}$-cyclically monotone set in $B_R(x_0) \times B_R(x_0)$ and Γ_1 is finite then*

$$\mathsf{m}(\Gamma_t) \geq f_{R,x_0}(t)\mathsf{m}(\Gamma_0).$$

Proof. Let $\{x_i\}_{i=1}^n = \Gamma_1$ with $x_i \neq x_j$ whenever $i \neq j$ and set $\Gamma^i = \Gamma \cap (M \times \{x_i\})$. Then the previous theorem shows

$$\mathsf{m}(\Gamma_0^i \cap \Gamma_0^j) = 0$$

for $i \neq j$. Thus there are disjoint sets $A^i \subset \Gamma_0^i$, $i = 1, \ldots, n$, such that

$$\mathsf{m}(\Gamma_0 \backslash \cup A^i) = 0.$$

Similarly one may replace A^i by a possibly smaller set which has full m-measure in A^i satisfying the condition above and, in addition, it holds

$$\mathsf{m}(A^i_{t,x_i} \cap A^j_{t,x_j}) = 0$$

for $i \neq j$. Setting

$$\Gamma' = \bigcup_{i=1}^n A^i \times \{x_i\}$$

we conclude

$$\mathsf{m}(\Gamma_t) \geq \mathsf{m}(\Gamma'_t) = \sum_{i=1}^n \mathsf{m}(A^i_{t,x_i})$$

$$\geq \sum_{i=1}^n f_{R,x_0}(t)\mathsf{m}(A^i) = f_R(t)\mathsf{m}(\Gamma_0).$$

$\square$

A similar argument also shows the following. As the result is not used below we leave the proof to the interested reader.

Corollary 5.6. *Assume (M,d) is p-essentially non-branching and m is qualitatively non-degenerate. Then for any $p' \in (1,\infty)$ and any p'-optimal coupling π with $(p_1)_*\pi \ll \mathsf{m}$ and $(p_2)_*\pi = \sum a_i \delta_{x_i}$ is induced by a transport map.*

Via an approximation argument of Cavalletti–Huesmann [CH15, Proposition 4.3] qualitative non-degeneracy implies strong non-degeneracy.

Lemma 5.7. *Assume (M,d,m) is p-essentially non-branching and m is qualitatively non-degenerate. Then for any $c_{p'}$-cyclically monotone Borel set Γ in $B_R(x_0) \times B_R(x_0)$ it holds*

$$\mathsf{m}(\Gamma_t) \geq f_{R,x_0}(t)\mathsf{m}(\Gamma_0).$$

In particular, (M,d,m) is strongly non-degenerate $(\mathsf{sND})_{p'}$ for all $p' \in (1,\infty)$.

Proof. For compact Γ the argument is as in [CH15, Proposition 4.3]. For completeness, we present the argument: Let $(\Gamma^{(n)})_{n\in\mathbb{N}}$ be a sequence of c_p-cyclically monotone sets such that $\Gamma_0^{(n)} = \Gamma_0$ and $\Gamma_1^{(n)} \subset \Gamma_1$ is finite. More precisely, choose a countable dense sequence $y_n \in \Gamma_1$ and define

$$E_i^{(n)} = \{x \in \Gamma_0 \,|\, d(x,y_i)^p - \varphi^{c_p}(y_i) \leq d(x,y_j)^p - \varphi^{c_p}(y_j), j = 1,\ldots,n\}$$

and

$$\Gamma^{(n)} = \bigcup_{i=1}^n E_i^{(n)} \times \{y_i\}.$$

From the definition of $E_i^{(n)}$ it follows that $\Gamma^{(n)}$ is $c_{p'}$-cyclically monotone. Furthermore, compactness of Γ shows that for all $\epsilon > 0$ there is an N_ϵ such that for all $n \geq N_\epsilon$ it holds

$$\Gamma_t^{(n)} \subset (\Gamma_t)_\epsilon = \bigcup_{x\in\Gamma_t} B_\epsilon(x)$$

This yields immediately the result for compact Γ as follows

$$\begin{aligned}
\mathsf{m}(\Gamma_t) &= \lim_{\epsilon \to 0} \mathsf{m}((\Gamma_t)_\epsilon) \\
&\geq \limsup_{n \to \infty} \mathsf{m}(\Gamma_t^{(n)}) \\
&\geq f_{R,x_0}(t)\mathsf{m}(\Gamma_0).
\end{aligned}$$

For arbitrary c_p-monotone Borel sets Γ in $B_R(x_0) \times B_R(x_0)$ we can use the Measurable Selection Theorem and Lusin's Theorem to reduce the result to compact set. If $\mathsf{m}(\Gamma_0) = 0$ then there is nothing to prove. So assume $\mathsf{m}(\Gamma_0) > 0$. Choose a measurable selection T of Γ. As m is locally bounded and $\Gamma_0 \subset B_R(x_0)$ we obtain by Lusin's Theorem a family of compact sets $K^1 \subset K^2 \subset \ldots \subset \Gamma_0$ with $\mathsf{m}(K^i) \to \mathsf{m}(\Gamma_0)$ such that T restricted to K^i is continuous. Define $\Gamma^i = \operatorname{graph}_{K^i} T \subset \Gamma$ and note that Γ^i is compact with $\Gamma_0^i = K^i$ so that

$$\begin{aligned}
\mathsf{m}(\Gamma_t) &\geq \limsup_{i \to \infty} \mathsf{m}(\Gamma_t^i) \\
&\geq \limsup_{i \to \infty} f_{R,x_0}(t)\mathsf{m}(K^i) = f_{R,x_0}(t)\mathsf{m}(\Gamma_0).
\end{aligned}$$

It remains to show that m is strongly non-degenerate $(\mathsf{sND})_p$. First observe

$$\bigcup_{R>0} \Gamma^R = \Gamma$$

where $\Gamma^R = \Gamma \cap (B_R(x_0) \times B_R(x_0))$. As Γ^R is bounded we have

$$\mathsf{m}(\Gamma_t) \geq \mathsf{m}(\Gamma_t^R) \geq f_{R,x_0}(T)\mathsf{m}(\Gamma_0^R).$$

Assume now $\mathsf{m}(\Gamma_0) > 0$ then $\mathsf{m}(\Gamma_0^R) \in (0, \infty)$ for all large $R > 0$ so that

$$\mathsf{m}(\Gamma_t) \geq \mathsf{m}(\Gamma_t^R) > 0.$$

Non-degeneracy of Γ follows by observing that for all Borel sets A the set $\Gamma^{A,0} = \Gamma \cap (A \times M)$ is still a c_p-cyclically monotone Borel set. Thus $\mathsf{m}(\Gamma_0^{A,0}) > 0$ implies $\mathsf{m}(\Gamma_t^{A,0}) > 0$. $\qquad\square$

Remark. The proof of the results above relies only on the qualitative non-degeneracy of m and that

$$\mathsf{m}(\{z \in M \mid d(z,x) = d(z,y)\}) = 0$$

for all $x \neq y$. In particular, it holds for $(\mathbb{R}^n, \|\cdot - \cdot\|_\infty, \lambda^n)$ which is highly branching.

By combining the previous lemma and the idea of the proof of Lemma 5.4 we obtain the main theorem of this section.

Theorem 5.8. *Assume (M, d, m) is p-essentially non-branching and m is qualitatively non-degenerate. Then any p-optimal coupling $\pi \in \mathcal{P}_p(M \times M)$ with $(p_1)_*\pi \ll \mathsf{m}$ is induced by a transport map. In particular, any such space has good transport behavior $(\mathsf{GTB})_p$.*

Combined with Proposition 3.6 and the existence of absolutely continuous interpolations (Corollary 4.14) we get the following two corollaries.

Corollary 5.9. *If (M, d, m) is p-essentially non-branching and m qualitatively non-degenerate then between any two measure $\mu_0, \mu_1 \in \mathcal{P}_p(M)$ with $\mu_0 \ll \mathsf{m}$ there is a unique p-optimal dynamical coupling σ and this coupling satisfies $(e_t)_*\sigma \ll \mathsf{m}$ for all $t \in [0,1)$. In particular, it has the strong interpolation property $(\mathsf{sIP})_p$.*

Corollary 5.10. *Assume* m *is qualitatively non-degenerate. Then* m *is* p-*essentially non-branching if and only if it has good transport behavior* $(\mathsf{GTB})_p$.

Proof of Theorem 5.8. Note by Proposition 5.3, (M, d) is proper so that we can apply the GKS-Construction of the previous section.

Let $\Gamma = \operatorname{supp}\pi$ and note that $\mathsf{m}(p_1(\Gamma)) = \mathsf{m}(\operatorname{supp}((p_1)_*\pi)) > 0$. It suffices to show that $\Gamma(x)$ is single-valued for m-almost all $x \in M$. This holds, if for all $R > 0$, $\Gamma^R(x)$ is single-valued for m-almost all $x \in M$ where $\Gamma^R = \Gamma \cap (\bar{B}_R(x_0) \times \bar{B}_r(x_0))$. Note that for large $R > 0$ it holds $\mathsf{m}(p_1(\Gamma^R)) \in (0, \infty)$.

Assume, by contradiction, that for some $R > 0$ there is a Borel set A with $\mathsf{m}(A) > 0$ and the set $\Gamma^R(x)$ is non-empty and not single-valued for all $x \in A$. Then by the Selection Dichotomy of Sets (Theorem 2.1), there is a compact set $K \subset A$ of positive m-measure, and two continuous maps $T_1, T_2 : M \to M$ with $T_1(K) \cap T_2(K) = \varnothing$ and

$$(x, T_1(x)), (x, T_2(x)) \in \Gamma^R \subset \operatorname{supp}\pi \cap (\bar{B}_R(x_0) \times \bar{B}_R(x_0)).$$

Restricting K further, we can also assume $\operatorname{supp}(\mathsf{m}\big|_K) = K \subset \bar{B}_R(x_0)$.

Define now $\mu_0 = \frac{1}{\mathsf{m}(K)}\mathsf{m}\big|_K$, $\pi_i = (\operatorname{id} \times T_i)_*\mu_0$ and $\mu_1^i = (p_2)_*\pi_i$ for $i = 1, 2$. Let $\Gamma^{(i)} = \operatorname{supp}\pi_i$, $i = 1, 2$, and note that $\Gamma_0^{(i)} = K$ and both $\Gamma^{(1)}$ and $\Gamma^{(2)}$ are compact and c_p-cyclically monotone.

Choose $\delta > 0$, t close to 0 and s close to 1 such that $f_{R,x_0}(t) \geq \frac{1}{2} + \epsilon$,

$$\mathsf{m}(K_\delta) \leq (1 + \epsilon)\mathsf{m}(K),$$

$$\Gamma_t^{(1)} \cup \Gamma_t^{(2)} \subset K_\delta$$

and

$$(\Gamma_s^{(1)})_\epsilon \cap (\Gamma_s^{(2)})_\epsilon = \varnothing.$$

Corollary 4.14 applied to (μ_0, μ_1^1) and (μ_0, μ_1^2) gives two p-optimal dynamical couplings $\sigma^{(1)}$ and $\sigma^{(2)}$ such that $(e_s)_*\sigma^{(1)}$ and $(e_s)_*\sigma^{(2)}$ are absolutely continuous with respect to m and have disjoint support. The choice of $\Gamma^{(1)}$ and $\Gamma^{(2)}$ implies that $\frac{1}{2}(\pi_1 + \pi_2)$ is supported on $\Gamma^{(1)} \cup \Gamma^{(2)} \subset \operatorname{supp}\pi$. Hence $(\operatorname{restr}_{0,s})_*\frac{1}{2}(\sigma^{(1)} + \sigma^{(2)})$ is a p-optimal dynamical coupling between μ_0 and $\frac{1}{2}(\mu_s^1 + \mu_s^2)$ so that Theorem 2.11 shows

$$\mu_t^{(1)} = (e_t)_*\sigma^{(1)} \perp (e_t)_*\sigma^{(2)} = \mu_t^{(2)}.$$

Maximality at time t shows that for $i = 1, 2$ there are measurable subsets $\tilde{\Gamma}^{(i)} \subset \Gamma^{(i)}$ with $\mathsf{m}(K \backslash \tilde{\Gamma}_0^{(i)}) = 0$ and $\mathsf{m}\big|_{\tilde{\Gamma}^{(i)}} \ll \mu_t^{(i)} \ll \mathsf{m}\big|_{\tilde{\Gamma}^{(i)}}$. Since $\mu_t^{(1)} \perp \mu_t^{(2)}$ we must have $\mathsf{m}(\tilde{\Gamma}_t^{(1} \cap \tilde{\Gamma}_t^{(2)}) = 0$. In combination with Lemma 5.7 this yields

$$\mathsf{m}(\tilde{\Gamma}_t^{(1)} \cup \tilde{\Gamma}_t^{(2)}) = \mathsf{m}(\tilde{\Gamma}_t^{(1)}) + \mathsf{m}(\tilde{\Gamma}_t^{(2)}) \geq 2f_R(t)\mathsf{m}(K).$$

This, however, leads to the following contradiction

$$(1 + \epsilon)\mathsf{m}(K) \geq \mathsf{m}(K_\delta)$$
$$\geq \mathsf{m}(\tilde{\Gamma}_t^{(1)} \cup \tilde{\Gamma}_t^{(2)})$$
$$= \mathsf{m}(\tilde{\Gamma}_t^{(1)}) + \mathsf{m}(\tilde{\Gamma}_t^{(2)})$$
$$\geq 2f_{R,x_0}(t)\mathsf{m}(K) = (1 + 2\epsilon)\mathsf{m}(K).$$

Thus we have proved that the $\Gamma^R(x)$ is at most single-valued for m-almost all $x \in M$ proving that π is induced by a transport map. $\square$

The proof relies heavily on the p-essentially non-branching property of dynamical couplings between absolutely continuous measures. In contrast to the case of a discrete target measures we cannot show that that general p'-optimal couplings with absolutely continuous first marginals are induced by transport maps. Nevertheless, p-essentially non-branching and the idea of Lemma 5.4 still exclude a too general behavior of the support of p'-optimal couplings.

Theorem 5.11. *Assume (M, d, m) is p-essentially non-branching for some $p \in (1, \infty)$, m is qualitatively non-degenerate and $p' \in (1, \infty)$. Then for any p'-optimal $\pi \in \mathcal{P}(M \times M)$ with $\mu_0 = (p_1)_*\pi \ll \mathsf{m}$ and for μ_0-almost every $x \in M$ it holds*

$$d(x, y_1) = d(x, y_2) \quad \text{whenever } (x, y_1), (x, y_2) \in \operatorname{supp} \pi.$$

Corollary 5.12. *The $c_{p'}$-superdifferential $\partial^{c_{p'}}\varphi$ of a $c_{p'}$-concave function φ satisfies for m-almost every $x \in M$*

$$d(x, y_1) = d(x, y_2) \quad \text{for all } y_1, y_2 \in \partial^{c_{p'}}\varphi(x).$$

Remark. The property p-essentially non-branching is used only to show that $c_{p'}$-cyclically monotone sets are non-degenerate. As mentioned above, this holds if we replace p-essentially non-branching by the assumption

$$\mathsf{m}(\{z \in M \,|\, d(z, x) = d(z, y)\}) = 0$$

for all $y \neq z$,

Proof. If the claim was false then π is not induced by a transport map and as above we get a compact set K of positive m-measure and measurable selections T_1 and T_2 as above which, in addition, satisfy

$$\sup_{(x,y_1)\in K\times T_1(K)} d(x, y_1) < \inf_{(x,y_2)\in K\times T_2(K)} d(x, y_2).$$

Let $\mu_0 = \frac{1}{\mathsf{m}(K)}\mathsf{m}\big|_K$ and for $i = 1, 2$ define $\pi_i = (\mathrm{id} \times T_i)_*\tilde{\mu}_0$ and $\Gamma^{(i)} = \operatorname{supp} \pi_i$. Again $\Gamma^{(i)}$ is c_p-cyclically monotone, but satisfies, in addition, the following

$$\Gamma_t^{(1)} \cap \Gamma_t^{(1)} = \varnothing \quad \text{for all } t \in (0, 1)$$

Choosing ϵ, δ and t as in the previous proof, we arrive at the following contradiction

$$(1 + \epsilon)\mathsf{m}(K) > \mathsf{m}(K_\delta)$$
$$\geq \mathsf{m}(\Gamma_t^{(1)} \cup \Gamma_t^{(2)})$$
$$= \mathsf{m}(\Gamma_t^{(1)}) + \mathsf{m}(\Gamma_t^{(2)})$$
$$\geq 2 f_{R,x_0}(t)\mathsf{m}(K) = (1 + 2\epsilon)\mathsf{m}(K).$$

$\square$

Density bounds of qualitatively non-degenerate measures. In [CM17] Cavalletti–Mondino showed that the measure contraction property $\mathsf{MCP}(K, N)$ as defined in [Stu06b, Definition 5.1] (see also [CM17, Definition 2.5]) implies the existence of absolutely continuous interpolations with controlled L^∞-bounds on their density. This was then used to prove the general existence of transport maps.

A version of the measure contraction property, which we denote here by $\mathsf{MCP}_{Ohta}(K, N)$, was introduced by Ohta [Oht07, Definition 2.1]. This conditions requires that for any set A of bounded and positive m-measure and any $x_0 \in M$ there is an absolutely continuous interpolation μ_t of $\mu_0 = \frac{1}{\mathsf{m}(A)}\mathsf{m}\big|_A$ and $\mu_1 = \delta_{x_0}$ whose density is

bounded in a controlled way depending on $\mathsf{m}(A)$ and the distance of $y \in A$ and x_0. For $K = 0$ and N the property says that

$$f_t \leq t^N \frac{1}{\mathsf{m}(A)} \equiv t^N f_0$$

where f_t and f_0 are the densities of μ_t and resp. μ_0. We may weaken this property as follows.

Definition 5.13. The measure m has *bounded density property* if for all $R > 0$ and $x_0 \in M$ there is a function $g_{R,x_0} : (0,1) \to (0,\infty]$ with

$$\limsup_{t \to 0} g_{R,x_0}(t) < 2$$

such that for some $p \in (1,\infty)$ and for every $\mu_0 = f_0 \mathsf{m} \in \mathcal{P}_p(M)$ with $\|f_0\|_\infty < \infty$, $\operatorname{supp} \mu_0 \subset B_R(x_0)$ and $x \in B_R(x_0)$ there is a geodesic $t \mapsto \mu_t = f_t \mathsf{m}$ between μ_0 and δ_x in $\mathcal{P}_p(M)$ such that

$$\|f_t\|_\infty \leq g_{R,x_0}(t)\|f_0\|_\infty.$$

It is easy to see that the definition does not depend on $p \in (1,\infty)$. In addition, one may readily verify that the $\mathsf{MCP}_{Ohta}(K,N)$-condition implies the bounded density property.

In [Raj12b, Theorem 1.4] Rajala showed that $\mathsf{CD}(K,N)$-spaces satisfy the $\mathsf{MCP}_{Ohta}(K,N)$-condition. Note Rajala's proof also works for the $\mathsf{MCP}(K,N)$-condition which was shown by Cavalletti–Mondino in [CM17, Theorem 3.1].

In the general setting observe that the bounded density property is stronger than qualitative non-degeneracy.

Lemma 5.14. *Every measure* m *with bounded density property is qualitatively non-degenerate.*

Proof. Let $\mu_0 = \frac{1}{\mathsf{m}(A_0)} \mathsf{m}\big|_{A_0}$ and note that

$$\operatorname{supp} \mu_t \subset A_{t,x}$$

and

$$\|f_0\|_\infty = \frac{1}{\mathsf{m}(A_0)}$$

we obtain

$$1 = \int_{A_{t,x}} f_t \mathsf{m} \leq g_{R,x_0}(t) \frac{1}{\mathsf{m}(A_0)} \mathsf{m}(A_{t,x}).$$

Choosing $f_{R,x_0} = g_{R,x_0}^{-1}$ we obtain the result. $\square$

Assuming m is p-essentially non-branching, the following result implies that the bounded density property is equivalent to qualitative non-degeneracy and, in addition, a pointwise density bound is obtained.

Proposition 5.15. *Assume* (M, d, m) *is* p-*essentially non-branching,* m *is qualitatively non-degenerate and* $\mu_0, \mu_1 \in \mathcal{P}_p(M)$ *with* $\mu_0 = f_0 \mathsf{m}$ *and* $\operatorname{supp} \mu_0, \operatorname{supp} \mu_1 \subset B_R(x_0)$. *Then for the unique* p-*optimal dynamical coupling* $\sigma \in \mathsf{OptGeo}_p(\mu_0, \mu_1)$ *it holds*

$$f_t(\gamma_t) \leq \frac{1}{f_{R,x_0}(t)} f_0(\gamma_0) \quad \text{for } \sigma\text{-almost all } \gamma \in \mathsf{Geo}(M, d)$$

where $(e_t)_*\sigma = f_t\mathsf{m}$. In particular, it holds

$$\|f_t\|_\infty \leq \frac{1}{f_{R,x_0}(t)}\|f_0\|_\infty$$

so that m has the bounded density property.

Corollary 5.16. *In a p-essentially non-branching metric measure space (M, d, m) the following are equivalent:*

- *The measure m is qualitatively non-degenerate.*
- *The measure m has the bounded density property.*

Proof of Proposition 5.15. We first assume $f_0 \equiv \frac{1}{\mathsf{m}(A_0)}$. If the claim was wrong then there is a compact set $\mathsf{L} \subset \mathsf{Geo}(M, d)$ with $\sigma(\mathsf{L}) > 0$ such that

$$\frac{1}{f_{R,x_0}(t)}\frac{1}{\mathsf{m}(A_0)} \leq (1-\epsilon)f_t(\gamma_t) \qquad \text{for all } \gamma \in \mathsf{L}.$$

In particular, by restricting σ to L we see that for $\tilde{A}_0 = e_0(\mathsf{L}) \subset A_0$ it holds

$$\frac{1}{f_{R,x_0}(t)}\frac{1}{\mathsf{m}(\tilde{A}_0)} \leq (1-\epsilon)\tilde{f}_t(\gamma_t) \quad \text{for } \sigma\text{-almost all } \gamma \in \mathsf{L}$$

where $\tilde{f}_t\mathsf{m} = (e_t)_*\sigma_\mathsf{L}$. The qualitative non-degeneracy yields

$$\mathsf{m}(e_t(\mathsf{L})) \geq f_{R,x_0}(t)\mathsf{m}(\tilde{A}_0).$$

Note that we always have

$$\text{ess}\inf_{\mathsf{m}|e_t(\mathsf{L})} \tilde{f}_t \leq \frac{1}{\mathsf{m}(e_t(\mathsf{L}))}.$$

This, however, leads to the following contradiction

$$\text{ess}\inf_{\mathsf{m}|e_t(\mathsf{L})} \tilde{f}_t \leq \frac{1}{f_{R,x_0}(t)}\frac{1}{\mathsf{m}(\tilde{A}_0)} \leq (1-\epsilon)\tilde{f}_t(\gamma_t) \quad \text{for } \sigma\text{-almost all } \gamma \in \mathsf{L}.$$

For general μ_0 we may assume that the set $\{f_0 > 0\}$ has finite measure. Then

$$t \mapsto \tilde{\mu}_t = \frac{1}{\mathsf{m}(\{f_0 > 0\})}\int_{\{f_0>0\}} \frac{1}{f_0(x)}\delta_{T_t(x)}d\mu_0(x)$$

is a geodesic in $\mathcal{P}_p(M)$ such that $\tilde{\mu}_0$ has constant density, i.e. $\tilde{\mu}_0 = \frac{1}{\mathsf{m}(\{f_0>0\})}\mathsf{m}|_{\{f_0>0\}}$. Note that $\tilde{\sigma} = \frac{1}{f_0\circ e_0\cdot\mathsf{m}(\{f_0>0\})}\sigma$ is the unique p-optimal dynamical coupling between $\tilde{\mu}_0$ and $\tilde{\mu}_1$, and $\tilde{f}_t$ satisfies

$$\tilde{f}_t(\gamma_t) = \frac{f_t(\gamma_t)}{f_0(\gamma_0)}\tilde{f}_0(\gamma_0) \quad \text{for } \tilde{\sigma}\text{-almost every } \gamma \in \mathsf{Geo}_{[0,1]}(M, d)$$

Since $\tilde{f}_0(\gamma_0) = \frac{1}{\mathsf{m}(\{f_0>0\})}$ for $\tilde{\sigma}$-almost all $\gamma \in \mathsf{Geo}_{[0,1]}(M, d)$ we obtain

$$\frac{f_t(\gamma_t)}{f_0(\gamma_0)}\frac{1}{\mathsf{m}(\{f_0 > 0\})} = \tilde{f}_t(\gamma_t) \leq \frac{1}{f_{R,x_0}(t)}\tilde{f}_0(\gamma_0) = \frac{1}{f_{R,x_0}(t)}\frac{1}{\mathsf{m}(\{f_0 > 0\})}$$

for $\tilde{\sigma}$-almost every $\gamma \in \mathsf{Geo}_{[0,1]}(M, d)$. This proves the claim as σ and $\tilde{\sigma}$ are mutually absolutely continuous. $\qquad\square$

Recall that the $\mathsf{MCP}(0, N)$-condition holds if for all $\mu_0 = \rho_0 \mathsf{m} \in \mathcal{P}_2(M)$ and all $x \in M$ there is geodesic $t \mapsto \mu_t = \rho_t \mathsf{m} + \mu_t^s$ between μ_0 and δ_x such that

$$\int \rho_t^{1-\frac{1}{N}} \geq (1 - t) \int \rho_0^{1-\frac{1}{N}} \, d\mathsf{m}.$$

Cavalletti–Mondino showed that $\mathsf{MCP}(0, N)$-spaces have the bounded density property with $g_{R,x_0}(t) = (1 - t)^{-N}$, see [CM17, Theorem 3.1]. Thus we obtain the following equivalent characterization of essentially non-branching $\mathsf{MCP}(0, N)$-spaces.

Corollary 5.17. *A p-essentially non-branching metric measure space satisfies the measure contraction property* $\mathsf{MCP}(0, N)$ *if and only if it is qualitatively non-degenerate with* $f_{R,x_0}(t) = (1 - t)^N$, *i.e.* $\mathsf{m}(A_{t,x}) \geq (1 - t)^N \mathsf{m}(A)$ *for all* $x \in M$ *and all Borel set* $A \subset M$ *of finite* m*-measure.*

There are similar versions for the general measure contraction property $\mathsf{MCP}(K, N)$, $K \in \mathbb{R}$ and $N \in [1, \infty)$. This actually shows that one can regard the measure contraction property as a directional version of Bishop–Gromov volume comparison condition which for $K = 0$ says that $\mathsf{m}(B_r(x)) \geq (1 - t)^N \mathsf{m}(B_{(1-t)r}(x))$.

Remark (Removing essentially non-branching I). Using a construction of good geodesics as in Rajala [Raj12b] and Cavalletti–Mondino [CM17] combined with the GKS-Construction (Theorem 4.10) it might be possible to show that a measure m that is not necessarily essentially non-branching is qualitatively non-degenerate if and if it has the bounded density property. We leave the details to a future work.

Along the lines of [Raj12a] we also obtain local versions of the Poincaré inequality with constant

$$C_{R,x_0} = \sup_{t \in (0,1)} \min\{\frac{1}{f_{R,x_0}(t)}, \frac{1}{f_{R,x_0}(1-t)}\},$$

i.e. for all Lipschitz functions $f : M \to \mathbb{R}$ and $B_r(x) \subset B_R(x_0)$ it holds

$$\int_{B_r(x)} |f - \bar{f}_{B_r(x)}| d\mathsf{m} \leq 4r C_{R,x_0} \int_{B_{2r}(x)} \mathrm{Lip} \, f d\mathsf{m}$$

where

$$\bar{f}_A = \frac{1}{\mathsf{m}(A)} \int_A f d\mathsf{m}$$

and

$$\mathrm{Lip} \, f(x) = \limsup_{y \to x} \frac{|f(y) - f(x)|}{d(x, y)}.$$

Note that if the m satisfies the bounded density property then the metric measure space (M, d, m) satisfies the connectedness assumptions of Eriksson-Bique, see [EB16, Section 4]. In particular, it will satisfy a local $(1, p)$-Poincaré inequality in $B_R(x_0)$ for some $p > 1$ depending only on the distortion function $g_{x_0,R}$. If the distortion function is independent of x_0 and R this would even yield a global $(1, p)$-Poincaré inequality.

Remark (Removing essentially non-branching II). Similar to Lemma 5.7 it is possible to show that under the condition

$$\mathsf{m}(\{z \in M \,|\, d(z, x) = d(z, y)\}) = 0$$

for all $x \neq y$ the bounded density property holds between every $\mu_0, \mu_1 \in \mathcal{P}_p(M)$ whenever $\mathrm{supp}\, \mu_0, \mathrm{supp}\, \mu_1 \subset B_R(x_0)$ and the function g_{R,x_0} is upper semi-continuous

in $(0,1)$. In particular, if the density bounds are sufficiently nice then a local doubling condition and local Poincaré inequality holds.

We quickly sketch the argument: Note first that Lemma 5.4 holds for those spaces so that one obtains for $\mu_1 = \sum_{i=1}^n \lambda_i \delta_{x_i}$ a geodesic in $\mathcal{P}_p(M)$ between μ_0 and μ_1 which has uniform density bounds only depending on the density of μ_0. Now let $\mu_1^n \rightharpoonup \mu_1$. At a fixed time $t \in (0,1)$ there is a $\mu_t^n = \rho_t^n \mathsf{m}$ with $\int \rho_t^n \mathsf{m} = 1$ and $\|\rho_t^n\|_\infty \leq g_{R,x_0}(t)\|\rho_0\|_\infty$ implying that $(\rho_t^n)_{n\in\mathbb{N}}$ is precompact in $L^1(\mathsf{m})$. Hence up to extracting a subsequence $\rho_t^n \to \rho_t$ in $L^1(\mathsf{m})$, $\|\rho_t\|_\infty \leq C_t\|\rho_0\|_\infty$ and $\mu_t = \rho_t \mathsf{m}$ being a t-midpoint of μ_0 and μ_1. The same argument then gives a geodesic $t \mapsto \mu_t$ which is absolutely continuous with uniform density bounds at all points $t \in \mathbb{Q} \cap (0,1)$. By upper semi-continuity of g_{R,x_0} and the same argument, this time applied to $\mu_{t_n} \rightharpoonup \mu_t$ with $t_n \in \mathbb{Q} \cap (0,1)$ and $t_n \to t \in (0,1)$, shows that μ_t is absolutely continuous with uniform density bound, compare also with [CM17, Proof of Theorem 4.1].

Generalizations to $N = \infty$. As it turns out the idea of the proof of existence of transport maps can be easily generalized to a more general situation. Compare the results of this section with [Gig12, Theorem 3.3(ii)] where non-branching spaces were treated. Recall that the $\mathsf{CD}_p(K,\infty)$-condition (see [LV09, Stu06b, Kel17]) requires that for $\mu_0, \mu_1 \in \mathcal{P}_p(M)$ with $\mu_i \ll \mathsf{m}$ there is a geodesic $t \mapsto \mu_t \ll \mathsf{m}$ such that

$$\int f_t \log f_t d\mathsf{m} \leq (1-t)\int f_0 \log f_0 d\mathsf{m} + t\int f_1 \log f_1 d\mathsf{m} - Kt(1-t)W_p(\mu_0,\mu_1)^2$$

where f_t is the density of μ_t.

If we choose $\mu_0 = \frac{1}{\mathsf{m}(A_0)}\mathsf{m}\big|_{A_0}$ and apply Jensen's inequality on the left-hand side, then it holds

$$\log \mathsf{m}(\Gamma_t) \geq (1-t)\log \mathsf{m}(A_0) - t\int f_1 \log f_1 d\mathsf{m} + Kt(1-t)W_2(\mu_0,\mu_1)^2$$

where $\Gamma_t = \operatorname{supp}\mu_t$. Thus

$$\lim_{t\to 0}\mathsf{m}(\Gamma_t) = \mathsf{m}(A_0)$$

whenever $\int f_1 \log f_1 d\mathsf{m} < \infty$ and A_0 is compact.

Remark. We can replace the $\mathsf{CD}_p(K,\infty)$-condition by the $\mathsf{CD}_p^*(K,N)$-condition with $N < 0$ as defined by Ohta in [Oht16]. Indeed, following the proof of [Oht16, Theorem 4.8] gives a stronger variant of the Brunn–Minkowski inequality (replace A_t by Γ_t) which for $K = 0$ and $r = -\frac{1}{N} > 0$ says

$$\mathsf{m}(\Gamma_t)^{-r} \leq (1-t)\mathsf{m}(A_0)^{-r} - t\int f_1^{1+r} d\mathsf{m}.$$

implying again $\lim_{t\to 0}\mathsf{m}(\Gamma_t) = \mathsf{m}(A_0)$.

Lemma 5.18. *Let A be a bounded Borel set and $\mu_1 \in \mathcal{P}_p(M)$ with $\mu_1 \ll \mathsf{m}$. Assume (M,d,m) is p-essentially non-branching and satisfies the $\mathsf{CD}_p(K,\infty)$-condition. If the geodesic connecting $\mu_0 = \frac{1}{\mathsf{m}(A)}\mathsf{m}\big|_A$ and μ_1 is unique then the (unique) p-optimal coupling π of μ_0 and μ_1 is induced by a transport map.*

Remark. Strictly speaking the assumptions imply that the strong $\mathsf{CD}_p(K,\infty)$-condition holds between μ_0 and μ_1 thus the argument of the proof of [RS14, Corollary 1.4] can be used. For completeness we present the arguments based on the ideas above.

Proof. Assume by contradiction that the claim is false for $\mu_0 = \frac{1}{\mathsf{m}(A)}\mathsf{m}\big|_A$ and $\mu_1 \ll \mathsf{m}$. Then by the Selection Dichotomy (Theorem 2.4) there are a compact set $K \subset A$ and two disjoint bounded closed set A_1 and A_2 such that $\pi(K \times A_1) > 0$ and

$$\mu_0 = (p_1)_*\pi_i = \frac{1}{\mathsf{m}(K)}\mathsf{m}\big|_K.$$

$$\mu_1^i = (p_2)_*\pi_i \ll \mu_1 \ll m$$

where $\pi_i = \frac{1}{\pi(K \times A_i)}\pi\big|_{K \times A_i}$ and $i = 1, 2$. Denote the density of μ_1^i by f_1^i and note that for large $n \in \mathbb{N}$

$$\mu_1^1(\{f_1^1 \le n\}), \mu_1^2(\{f_1^2 \le n\}) > 0.$$

Thus we may restrict K further (and obtain new A_i, π_i, and μ_1^i) and assume that for $i = 1, 2$, $f_1^i(y)$ is bounded by n for μ_1^i-almost all $y \in M$. Since each A_1^i, $i = 1, 2$, is bounded we obtain

$$\int \tilde{f}_1^i \log \tilde{f}_1^i d\mathsf{m} \in \mathbb{R}.$$

Corollary 2.7 implies that the p-optimal dynamical coupling between μ_0 and μ_1^i is still unique so that the interpolation inequality implies

$$\log \mathsf{m}(\Gamma_t^{(i)}) \ge (1 - t) \log \mathsf{m}(K) - t \int f_1^i \log f_1^i d\mathsf{m} + KW_p(\mu_0, \mu_1^i)^2$$

where $\Gamma^{(i)} = \operatorname{supp} \pi_i$ is the support of the unique p-optimal dynamical coupling of μ_0 and μ_0^i. In particular, it holds

$$\lim_{t \to 0} \mathsf{m}(\Gamma_t^{(i)}) = \mathsf{m}(K) \quad \text{for } i = 1, 2.$$

Also note that $\Gamma^{(1)} \cup \Gamma^{(2)}$ is c_p-cyclically monotone, so that (M, d, m) being p-essential non-branching shows for a sequence $t_n \to 0$ we may replace $\Gamma^{(1)}$ and $\Gamma^{(2)}$ by smaller sets $\tilde{\Gamma}^{(1)} \subset \Gamma^{(1)}$ and $\tilde{\Gamma}^{(2)} \subset \Gamma^{(2)}$ such that $\pi_1(\tilde{\Gamma}^{(1)}) = \pi_2(\tilde{\Gamma}^{(2)}) = 1$ and $\mathsf{m}(\Gamma_{t_n}^{(1)} \cap \Gamma_{t_n}^{(2)}) = 0$ for all large $n \in \mathbb{N}$. Note that still $\lim_{t \to 0} \mathsf{m}(\tilde{\Gamma}_t^{(i)}) = \mathsf{m}(K)$. But then

$$\mathsf{m}(K) \ge \lim_{\delta \to 0} \mathsf{m}(K_\delta)$$
$$\ge \lim_{t \to 0} \mathsf{m}(\tilde{\Gamma}_t^{(1)} \cup \tilde{\Gamma}_t^{(2)})$$
$$= \lim_{t \to 0} \left[\mathsf{m}(\tilde{\Gamma}_t^{(1)}) + \mathsf{m}(\tilde{\Gamma}_t^{(2)}) \right]$$
$$= 2\mathsf{m}(K)$$

which is a contradiction. $\square$

Theorem 5.19. *Assume (M, d, m) is p-essentially non-branching and satisfies the $\mathsf{CD}_p(K, \infty)$-condition. If $\mu_0, \mu_1 \in \mathcal{P}_p(M)$ are such that there is a p-optimal dynamical coupling $\sigma \in \mathsf{OptGeo}_p(\mu_0, \mu_1)$ with $\mu_0, (e_{t_0})_*\sigma, \mu_1 \ll \mathsf{m}$ for some $t_0 \in M$ then the p-optimal coupling $(e_0, e_1)_*\sigma$ is induced by a transport map T.*

Proof. We reduce the general case to Lemma 5.18 above. Let $t \mapsto \mu_t = (e_t)_*\sigma$ be a geodesic between $\mu_0 \ll \mathsf{m}$ and $\mu_1 \ll \mathsf{m}$ with $\mu_{t_0} \ll \mathsf{m}$. By Corollary 2.12 we see that $s \mapsto \mu_{st_0}$ is the unique geodesic connecting μ_0 and μ_{t_0} and there is a unique p-optimal transport map $T_{t_0, 1}$ between μ_{t_0} and μ_1. Hence, it suffices to show that $(e_0, e_{t_0})_*\sigma$ is induced by a transport map. Since $s \mapsto \mu_{st_0}$ is unique

between its absolutely continuous endpoints it suffices to show the claim for μ_0 and μ_1 connected by a unique geodesic $t \mapsto \mu_t$.

Let σ be the unique p-optimal dynamical coupling induced by $t \mapsto \mu_t$ and $\pi = (e_0, e_1)_* \sigma$. Denote the densities of μ_0 and μ_1 with respect to m by f_0 and f_1 respectively. Since

$$\pi(\bigcup_{n \in \mathbb{N}} C_n) = 1$$

where $C_n = (\{f_0 \geq \frac{1}{n}\} \cap B_n(x_0)) \times M$ for a fixed $x_0 \in M$, it suffices to show that the claim holds for π restricted to C_n. Note that by Corollary 2.7 the geodesic connecting the marginals of $\frac{1}{\pi(C_n)} \pi\big|_{C_n}$ is still unique.

Thus we can assume $\mu_0 = f_0 \mathsf{m}$ has bounded support and there is a Borel set $A \subset M$ of full μ_0-measure such that f_0 is bounded below by an $\epsilon > 0$. Let σ be the unique p-optimal dynamical coupling between μ_0 and μ_1 and define a function $f : M \times M \to [0, \infty)$ by

$$f(x, y) = \chi_A(x) \frac{1}{f_0(x)}.$$

Corollary 2.7 shows that σ_f is still unique between $\mu_0^f = \frac{1}{\mathsf{m}(A)} \mathsf{m}\big|_A$ and μ_1^f. It is easy to see that $(e_0, e_1)_* \sigma_f$ is induced by a transport map if and only if $(e_0, e_1)_* \sigma$ is induced by a transport map. We conclude by noticing that $A = \operatorname{supp} \mu_0^f$ and μ_1^f satisfy the conditions of the previous lemma and hence $(e_0, e_1)_* \sigma_f$ is induced by a p-optimal transport map. $\square$

Since by Rajala–Sturm [RS14] strong $\mathsf{CD}_p(K, \infty)$-spaces are p-essentially non-branching, we recover their result on the existence of transport maps [RS14, Corollary 1.4].

Corollary 5.20. *If $(M, d\mathsf{m})$ satisfies the strong $\mathsf{CD}_p(K, \infty)$-condition between every $\mu_0, \mu_1 \in \mathcal{P}_p(M)$ with $\mu_0, \mu_1 \ll \mathsf{m}$ there is a p-optimal dynamical coupling σ and $(e_0, e_1)_* \sigma$ is induced by a transport map.*

If, instead, we know that between an absolutely continuous initial measure and an arbitrary measure every interpolation is absolutely continuous then we can show general existence of transport maps.

Corollary 5.21. *Assume $(M, d\mathsf{m})$ has the strong interpolation property $(\mathsf{sIP})_p$. Then (M, d, m) is p-essentially non-branching and satisfies the (weak) $\mathsf{CD}_p(K, \infty)$-condition if and only if it satisfies the strong $\mathsf{CD}_p(K, \infty)$-condition. Furthermore, if either of the cases hold then (M, d, m) has good transport behavior $(\mathsf{GTB})_p$ as well.*

In the more general setting there could be more than one geodesic connecting two absolutely continuous measures. However, it is possible to show that the absolutely continuous part of a geodesic connecting measures with finite entropy is just a restriction of a geodesic given by the interpolation inequality. Hence this geodesic is unique.

Corollary 5.22. *Assume (M, d, m) is p-essentially non-branching and satisfies the $\mathsf{CD}_p(K, \infty)$-condition. Then for all $\mu_0, \mu_1 \in \mathcal{P}_p(M)$ with $\mu_0, \mu_1 \ll \mathsf{m}$ and $\int f_0 \log f_0 d\mathsf{m}, \int f_1 \log f_1 d\mathsf{m} < \infty$ there is a unique p-optimal dynamical coupling $\sigma \in \mathsf{OptGeo}_p(\mu_0, \mu_1)$ along which the $\mathsf{CD}_p(K, \infty)$-interpolation inequality holds and for this dynamical coupling the p-optimal coupling $(e_0, e_1)_* \sigma$ is induced by a transport map.*

Furthermore, for any other p-optimal dynamical coupling $\tilde{\sigma} \in \mathsf{OptGeo}_p(\mu_0, \mu_1)$ such that for some $t \in (0, 1)$ the t-midpoint $\mu_t = \rho_t \mathsf{m} + \mathsf{m}$ with $\rho_t \not\equiv 0$, there are a function $f : M \to [0, \infty)$ which is positive m-almost everywhere on $\{\rho_t > 0\}$ and a Borel set $A_t \subset M$ such that

$$\sigma_f = \tilde{\sigma}_{\tilde{f}}$$

where $\tilde{f}(\gamma) = \chi_{A_t}(\gamma_t)$, i.e. the absolutely continuous part of $\tilde{\sigma}$ is obtained via a restriction of σ.

Proof. Let σ be a p-optimal dynamical coupling along which the $\mathsf{CD}_p(K, \infty)$-interpolation inequality holds and $\tilde{\sigma}$ be another p-optimal dynamical coupling such that at time $t \in (0, 1)$ the interpolation $\tilde{\mu}_t = (e_t)_* \tilde{\sigma}$ has density with respect to m, i.e.

$$\tilde{\mu}_t := \tilde{\mu}_t^a + \tilde{\mu}_t^s$$

with $\tilde{\mu}_t^s \perp \mathsf{m}$, $\tilde{\mu}_t^a \ll \mathsf{m}$ and $\tilde{\mu}_t^a(M) > 0$. Let A_t be a Borel set such that $\tilde{\mu}_t^a(M \backslash A_t) = 0$ and $\tilde{\mu}_t^s(A_t) = 0$.

Set $\tilde{f}(\gamma) = \chi_{A_t}(\gamma_t)$, then by Theorem 5.19 the p-optimal coupling $(e_0, e_{t_0})_* \tilde{\sigma}_{\tilde{f}}$ between $\tilde{\mu}_0^{\tilde{f}}$ and $\tilde{\mu}_0^{\tilde{f}}$ which is induced by a transport map $\tilde{T}$. Similarly, $(e_0, e_t)_* \sigma$ is induced by a transport map T.

We claim that $T = \tilde{T}$ on $A_0 = \tilde{T}^{-1}(A_t)$. Indeed, this would imply the result because $\tilde{\mu}_0^{\tilde{f}} \leq \mu_0|_{A_0}$.

The claim follows by observing that between $\hat{\mu}_0 = \frac{1}{2}(\mu_0 + \tilde{\mu}_0^{\tilde{f}})$ and $\hat{\mu}_0 = \frac{1}{2}(\mu_t + \tilde{\mu}_t^a)$ there is a unique p-optimal coupling which is induced by a transport map. $\square$

Remark. The corollary allows us to do localization so that we can show the following: between any two measure $\mu_0 = f_0 \mathsf{m}$ and $\mu_1 = f_1 \mathsf{m}$ with finite entropy there is an interpolation $\mu_t = f_t d\mathsf{m}$ such that

$$\log f_t(\gamma_t) \leq (1 - t) \log f_0(\gamma_0) + t \log f_1(\gamma_1) - Kd(\gamma_0, \gamma_1)^2.$$

This can be used to obtain density bounds of f_t and thus a weak Poincaré inequality.

6. Measure rigidity

In this section we prove the measure rigidity theorem, i.e. we will show that two qualitatively non-degenerate measures on a p-essentially non-branching space are mutually absolutely continuous.

For convenience of the reader we recall the main properties of p-essentially non-branching, qualitatively non-degenerate measures m which are implied by Theorem 5.8 and Corollary 5.9.

Theorem. *Let (M, d, m) be p-essentially non-branching and qualitatively non-degenerate. Then (M, d, m) satisfies the good transport behavior $(\mathsf{GTB})_p$ and has the strong interpolation property $(\mathsf{sIP})_p$. In particular, for every $\mu_0, \mu_1 \in \mathcal{P}_p(M)$ with $\mu_0 \ll \mathsf{m}$ there is a unique p-optimal dynamical coupling $\sigma \in \mathsf{OptGeo}_p(\mu_0, \mu_1)$ and this dynamical coupling satisfies $(e_t)_* \sigma \ll \mathsf{m}$ for $t \in [0, 1)$.*

Define

$$R_t(x) := e_t\left((e_0, e_t)^{-1}(\{x\} \times M)\right)$$

to be the set of t-midpoints of geodesics starting at x. This set is analytic and hence measurable. Furthermore, for $0 < t \leq s \leq 1$ it holds

$$R_t(x) \subset R_s(x).$$

Thus the following set

$$R_{(0,t)}(x) := \bigcup_{0<s<t} R_s(x) = \bigcup_{s<t,s\in\mathbb{Q}} R_s(x)$$

is also analytic and measurable. We also write $R_{(0,t]}(x) = R_t(x)$. Finally define the set of strict t-midpoints by

$$R_t^*(x) := R_t(x) \backslash R_{(0,t)}(x).$$

Recall that the measure m is non-degenerate if for all measurable sets $A \subset M$, $x \in M$ and $t \in (0,1)$ it holds $\mathsf{m}(A_{t,x}) > 0$ whenever $\mathsf{m}(A) > 0$ where $A_{t,x} = e_t((e_0, e_1)^{-1}(A \times \{x\}))$.

Lemma 6.1. *Assume m is non-degenerate. Then $\mathsf{m}(R_t^*(x)) = 0$ for all $t \in (0,1]$.*

Proof. First note that

$$f(t) = \mathsf{m}(R_{(0,t]}(x) \cap B_R(x))$$

is a non-decreasing function and finite for fixed $R > 0$ so that for some set $\Omega \subset [0,1]$, whose complement is at most countable, it holds

$$\mathsf{m}(R_t^*(x) \cap B_R(x)) = \lim_{\epsilon \to 0} f(t+\epsilon) - f(t) = 0 \quad \text{for all } t \in \Omega.$$

Assume by contradiction $\Omega \neq (0,1]$. Then there is a $t \notin \Omega$ with $\mathsf{m}(R_t^*(x) \cap B_R(x)) > 0$. It always holds

$$(R_t^*(x) \cap B_R(x))_{s,x} \subset R_{st}^*(x) \cap B_R(x) \quad \text{for all } s \in (0,1).$$

Also note that every $t \in (0,1]\backslash\Omega$ there is an $s \in (0,1)$ with $st \in \Omega$. In combination with the non-degeneracy of m this leads to the following contradiction

$$0 = \mathsf{m}(R_{st}^*(x) \cap B_R(x)) \geq \mathsf{m}(R_t^*(x) \cap B_R(x))_{s,x} > 0.$$

$\square$

Proposition 6.2. *Assume m is a non-degenerate measure. For every measurable $A \subset M$ of finite measure and every $\epsilon > 0$ there is a compact subset $K \subset A$ with $\mathsf{m}(A\backslash K) < \epsilon$ and $t \in (0,1)$ such that*

$$K \subset R_t(x).$$

Furthermore, there is an m-measurable map $T : K \to M$ such that

$$d(x, T(y)) = \frac{d(x,y)}{t} = d(x,y) + d(y, T(y)).$$

In particular, there is a dynamical coupling σ which is p-optimal for all $p \in [1,\infty)$ such that $(e_0)_*\sigma = \delta_x$, $(e_t)_*\sigma = \frac{1}{\mathsf{m}(K)}\mathsf{m}\big|_K$ and $(e_1)_*\sigma = T_*((e_t)_*\sigma)$.

Proof. Just note that because $\mathsf{m}(R_1^*(x)) = 0$ and $t \mapsto R_t(x)$ is monotone we have

$$\mathsf{m}(A) = \mathsf{m}(A \cap R_1(x)) = \mathsf{m}(A \cap \cup_{t<1} R_t(x)) = \lim_{t\to 1} \mathsf{m}(A \cap R_t(x)).$$

Now for all $\epsilon > 0$ there is a $t \in (0,1)$ close to 1 and a compact set $K \subset A \cap R_t(x)$ such that $\mathsf{m}(A\backslash K) < \epsilon$. Since $K \subset R_t(x)$ there is a measurable map $T : K \to M$ such that

$$(e_0, e_t, e_1)^{-1}(x, y, T(y)) \neq \varnothing \quad \text{for all } y \in K$$

proving the first part of the claim. For the second just note that $\mu_t = \frac{1}{\mathsf{m}(K)}\mathsf{m}\big|_K$ is a t-midpoint of δ_x and $T_*\mu_t$.

$\square$

Theorem 6.3. *Assume* m_i, $i = 1, 2$, *are both p-essentially non-branching and qualitatively non-degenerate measures on* (M, d). *Then* m_1 *and* m_2 *are mutually absolutely continuous.*

Proof. Assume first m_1 and m_2 are mutually singular. Then we immediately arrive at a contradiction: For measures $\mu_0 \ll \mathsf{m}_1$ and $\mu_1 \ll \mathsf{m}_2$ the interpolations must be absolutely continuous with respect to both m_1 and m_2 which is not possible.

If m_1 and m_2 are not mutually singular then there is a non-trivial, non-negative function $f \in L^1_{\mathrm{loc}}(\mathsf{m}_2)$ such that

$$\mathsf{m}_1 = f\mathsf{m}_2 + \mathsf{m}_1^s$$

where m_2 and m_1^s are mutually singular. As both measures are finite on bounded sets, there is a measurable set Ω such that $\mathsf{m}_2(\Omega) = 0$ and $\mathsf{m}_1^s(A \cap \Omega) = \mathsf{m}_1^s(A)$ for all measurable sets A.

Assume by contradiction $\mathsf{m}_1^s \not\equiv 0$. Let $A \subset \Omega$ be a compact set with $\mathsf{m}_1^s(A) > 0$. We claim that $\mathsf{m}_2(A_{t,x}) = 0$ for all x and $t \in (0, 1)$. Indeed, if this was not the case then for some $x \in M$ and $t \in (0, 1)$ there is a compact $K \subset A$ and σ as in the previous proposition such that $\mathsf{m}_2(K_{s,x}) > 0$ and $\mathsf{m}_1^s(K_{s,x}) = 0$ for $s \in (0, 1)$. In that case it holds $(e_t)_*\sigma = \frac{1}{\mathsf{m}_1^s(K)}\mathsf{m}_1^s\big|_K$, $(e_{st})_*\sigma \perp \mathsf{m}_1^s$ and $(e_{st})_*\sigma \ll \mathsf{m}_2$.

However, by the strong interpolation property $(\mathsf{sIP})_p$ between the measures $(e_{st})_*\sigma$ and $(e_1)_*\sigma$ this would imply that $(e_t)_*\sigma \ll \mathsf{m}_2$ which is a contradiction as m_2 and m_1^s are mutually singular.

This shows that

$$\mathsf{m}_1^s(A_{t,x}) = \mathsf{m}_1(A_{t,x}) \geq f(t)\mathsf{m}_1(A) = f(t)\mathsf{m}_1^s(A).$$

Because $A \subset \Omega$ is arbitrary and Ω is a set of full m_1^s-measure, we see that m_1^s is qualitatively non-degenerate and p-essentially non-branching.

We arrive at a contradiction by observing that m_1^s and m_2 are mutually singular and both are qualitatively non-degenerate and p-essentially non-branching. This shows that m_1^s must be trivial and hence $\mathsf{m}_1 \ll \mathsf{m}_2$. Similarly, exchanging the roles of m_1 and m_2 shows $\mathsf{m}_2 \ll \mathsf{m}_1$ which proves the theorem. $\qquad\square$

Since any $\mathsf{RCD}^*(K, N)$-space with $N \in [1, \infty)$ is 2-essentially non-branching and qualitatively non-branching, we see that any two measures m_1 and m_2 on a metric space (M, d) must be mutually absolutely continuous if the metric measure spaces (M, d, m_i), $i = 1, 2$, are $\mathsf{RCD}^*(K, N)$-spaces.

Using the strong interpolation property in combination of bounds on the density of interpolation measures obtained from the $\mathsf{CD}_p(K, \infty)$-condition we will give an alternative proof of the measure rigidity theorem. The following technical lemma can be extracted from the work of Cavalletti–Huesmann [CH14]. For convenience of the reader we include its short proof.

Lemma 6.4 (Self-Intersection Lemma)**.** *Assume* $t \mapsto \mu_t = f_t\mathsf{m}$ *is a geodesic in* $\mathcal{P}_p(M)$ *such that* $\mu_0 = \frac{1}{\mathsf{m}(K)}\mathsf{m}\big|_K$ *for some compact set* K, $\mathrm{supp}\,\mu_1$ *having bounded support and it holds* $C := \sup_{t \in [0,\delta]} \|f_t\|_\infty < \infty$ *for* $\delta \in (0, 1)$ *then there is a* $t_0 \in (0, \delta)$ *such that for all* $t \in [0, t_0)$ *it holds* $\mu_t(K) > 0$. *In particular,* μ_t *and* μ_0 *cannot be mutually singular.*

Proof. Assume this is not the case. Then there is a sequence $t_n \to 0$ such that $\mu_{t_n} \perp \mu_0$. In particular, there are Borel sets $A_0 \subset K$ and $A_n \subset \mathrm{supp}\,\mu_{t_n}$ with $A_n \cap A_0 = \varnothing$ and $\mu_{t_n}(A_n) = \mu_0(A_0) = 1$. Note that this shows $\mathsf{m}(A_0) = \mathsf{m}(K)$.

Since the support of μ_0 and μ_t are bounded for all $\epsilon > 0$ there is a $t_\epsilon \in (0, \delta)$ such that all $t_n \leq t_\epsilon$

$$A_n \subset \operatorname{supp}\mu_{t_n} \subset K_\epsilon.$$

Also note that

$$\mathsf{m}(A_n) \geq \frac{1}{C}\mu_{t_n}(A_n) = \frac{1}{C}$$

But then

$$\begin{aligned}
\mathsf{m}(K) &= \lim_{\epsilon \to 0}\mathsf{m}(K_\epsilon) \\
&\geq \limsup_{n \to \infty}\mathsf{m}(A_0 \cup A_n) \\
&\geq \mathsf{m}(K) + \limsup_{n \to \infty}\mathsf{m}(A_n) \geq \mathsf{m}(K) + \frac{1}{C}
\end{aligned}$$

which is a contradiction. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Theorem 6.5. *Assume (M, d, m_1) and (M, d, m_2) have the strong interpolation property $(\mathsf{sIP})_p$ and satisfy the $\mathsf{CD}_p(K, \infty)$-condition. Then m_1 and m_2 are mutually absolutely continuous.*

Remark. The proof below also works in the setting of p-essentially non-branching, qualitatively non-degenerate measures. Indeed, Corollary 5.9 implies the strong interpolation property $(\mathsf{sIP})_p$ and by Proposition 5.15 the density f_t of the t-interpolation μ_t is uniformly bounded by the density of f_0 if t is close to 1, compare this also to [Raj12b, Theorem 4.2] and [CM17, Theorem 4.1].

However, the strong interpolation property $(\mathsf{sIP})_p$, which follows from qualitative non-degeneracy, is essential in order to combine the singular part of one of the measures with the bounded density property. It is unclear whether without this property there could be two $\mathsf{CD}_p(K, \infty)$-measures which are not mutually absolutely continuous.

Proof. As above two such measures m_1 and m_2 cannot be mutually singular and there is a non-trivial, non-negative function $f \in L^1_{\mathrm{loc}}(\mathsf{m}_2)$ such that

$$\mathsf{m}_1 = f\mathsf{m}_2 + \mathsf{m}_1^s$$

where m_2 and m_1^s are mutually singular. As both measures are finite on bounded sets, there is a measurable set Ω such that $\mathsf{m}_2(\Omega) = 0$ and $\mathsf{m}_1^s(A \cap \Omega) = \mathsf{m}_1^s(A)$ for all measurable sets A.

Assume by contradiction $\mathsf{m}_1^s \not\equiv 0$ and let $A \subset \Omega$ and $B \subset \{f > 0\}$ be compact sets with $\mathsf{m}_1^s(A) > 0$, $\mathsf{m}_2(A) = 0$, $\mathsf{m}_1^s(B) = 0$ and $\mathsf{m}_1(B) = \int_B f d\mathsf{m}_2 > 0$. Let

$$\mu_0 = \frac{1}{\mathsf{m}_1^s(A)}\mathsf{m}_1^s\big|_A \ll \mathsf{m}_1$$

and

$$\mu_1 = \frac{1}{\mathsf{m}_1(B)}\mathsf{m}_1\big|_B \ll \mathsf{m}_2.$$

Then there is a unique geodesic $t \mapsto \mu_t$ connecting μ_0 and μ_1. The strong interpolation property of m_1 and m_2 shows that for $t \in (0, 1)$ the interpolation measures μ_t must be absolutely continuous with respect to both m_1 and m_2. In particular, $\mu_t(A) = 0$ for all $t \in (0, 1]$.

Now the $\mathsf{CD}_p(K, \infty)$-condition implies

$$\|f_t\|_\infty \leq C(K, \operatorname{diam} A, \operatorname{diam} B) \cdot \max\{\|f_0\|_\infty, \|f_1\|_\infty\}.$$
$$= C(K, \operatorname{diam} A, \operatorname{diam} B) \cdot \max\{\frac{1}{\mathsf{m}_1^s(A)}, \frac{1}{\mathsf{m}_1(B)}\}$$

where $\mu_t = f_t \mathsf{m}_1$.

We arrive at a contradiction by observing that $\mu_t(A) > 0$ by the Self-Intersection Lemma above. Thus m_1^s must be trivial showing $\mathsf{m}_1 \ll \mathsf{m}_2$. Exchanging the roles of m_1 and m_2 also shows $\mathsf{m}_2 \ll \mathsf{m}_1$ proving the theorem. $\qquad\square$

APPENDIX A. PROOF OF THEOREM 2.11 AND COROLLARY 2.12

Before we prove the theorem we need the following technical lemmas.

Lemma A.1. *Let σ be a p-optimal dynamical coupling of μ_0 and μ_1 such that $(e_t)_*\sigma = \delta_{x_t}$ for some $t \in (0,1)$ and $x_t \in M$ then $\mu_0 \otimes \mu_1$ is a p-optimal coupling and $d(\cdot, \cdot)$ is constant on $\operatorname{supp}\mu_0 \times \operatorname{supp}\mu_1$. In particular, if $\mu_0 \otimes \mu_1$ is not a delta measure then there is a p-optimal dynamical coupling $\tilde{\sigma}$ such that $\tilde{\sigma}(\mathsf{L}) < 1$ for all measurable non-branching $\mathsf{L} \subset \operatorname{Geo}_{[0,1]}(M, d)$.*

Proof. First note that the trivial coupling $\mu_0 \otimes \mu_1$ is a p-optimal coupling of μ_0 and μ_1 for all $p \in [1, \infty)$. Indeed, the assumptions imply that for each $\gamma, \eta \in \operatorname{supp}\sigma$ it holds $\gamma_t = \eta_t = x_t$ and hence $\ell(\gamma) = \ell(\eta)$. But then $\ell(\operatorname{restr}_{0,t}\gamma) = W_p(\mu_0, \delta_{x_t})$ and $\ell(\operatorname{restr}_{t,1}\gamma) = W_p(\delta_{x_t}, \mu_1)$ and thus for all $x_0 \in \operatorname{supp}\mu_0$ and $x_1 \in \operatorname{supp}\mu_1$

$$d(x_0, x_1) = \frac{d(x_0, x_t)}{t} = \frac{W_p(\mu_0, \delta_{x_t})}{t} = W_p(\mu_0, \mu_1).$$
$$= \frac{d(x_t, x_1)}{1-t} = \frac{W_p(\delta_{x_t}, x_1)}{1-t} = W_p(\mu_0, \mu_1)$$

implying that $\operatorname{supp}\mu_0 \times \operatorname{supp}\mu_1$ is c_p-cyclically monotone.

Let $T_{0,t} : M \to \operatorname{Geo}_{[0,1]}(M, d)$ be a measurable map such that $T_{0,t}(x_0)$ is a geodesic between x_0 and x_t. Then $\sigma_{0,t} := (T_{0,t})_*\mu_0$ is a p-optimal dynamical coupling between μ_0 and δ_{x_t}. Similarly, let $T_{t,1} : M \to \operatorname{Geo}_{[0,1]}(M, d)$ be a Borel map such that $T_{t,1}(x_1)$ is a geodesic between x_t and x_1. Note that for each $x_0 \in \operatorname{supp}\mu_0$ and $x_1 \in \operatorname{supp}\mu_1$

$$\gamma_s^{x_0, x_1} = \begin{cases} T_{0,t}(x_0)\left(\frac{s}{t}\right) & s \in [0, t] \\ T_{t,1}(x_1)\left(\frac{s-t}{1-t}\right) & s \in [t, 1] \end{cases}$$

is a geodesic between x_0 and x_1. Since $T_{0,t}$ and $T_{t,1}$ are Borel maps, so is $T_{0,1}^{x_t} : (x_0, x_1) \mapsto \gamma^{x_0, x_1}$. In particular, $\tilde{\sigma} = (T_{0,1}^{x_t})_*\mu_0 \otimes \mu_1$ is a p-optimal dynamical coupling of μ_0 and μ_1.

If $\mu_0 \otimes \mu_1$ is not a delta measure then either μ_0 or μ_1 (or both) is not a delta measure. Assume μ_1 is not a delta measure. Then for each set L with $\tilde{\sigma}(\Gamma) = 1$ and for μ_0-almost all $x_0 \in e_0(\Gamma)$ there are at least two distinct geodesics $\gamma, \eta \in \Gamma$ with

$$\operatorname{restr}_{0,t}\gamma = \operatorname{restr}_{0,t}\eta = T_{0,t}(x_0).$$

In particular, $\mu_0 \otimes \mu_1$ is not concentrated on a non-branching set. $\qquad\square$

Remark. Assume $x \mapsto (\mu_0^x, \mu_1^x)$ is a measurable map such that δ_x is the t-midpoint of μ_0^x and μ_1^x. Then $x \mapsto (T_{0,1}^x)_*(\mu_0^x \otimes \mu_1^x)$ is also measurable.

Lemma A.2. *Let μ_0 and μ_1 be probability measures such that any p-optimal dynamical coupling between μ_0 and μ_1 is concentrated on a set of non-branching geodesics. Then for any t-midpoint μ_t of μ_0 and μ_1, any p-optimal dynamical coupling between μ and μ_t is concentrated on a set of non-branching geodesics.*

Proof. It is easy to see that any p-optimal dynamical coupling $\sigma \in \mathsf{OptGeo}_p(\mu, \mu_t)$ is obtained by restricting a p-optimal dynamical coupling $\tilde{\sigma} \in \mathsf{OptGeo}_p(\mu, \nu)$, i.e.

$$(\mathsf{restr}_{0,t})_* \tilde{\sigma} = \sigma$$

and hence

$$(\mathsf{restr}_{0,t})_* \mathsf{OptGeo}_p(\mu_0, \mu_1) = \mathsf{OptGeo}_p(\mu_0, \mu_t).$$

Furthermore, if L is non-branching and measurable then $\mathsf{restr}_{0,t}(\mathsf{L})$ is also non-branching and measurable. In particular, choosing L such that $\tilde{\sigma}(L) = 1$ implies $\sigma(\mathsf{restr}_{0,t}(\mathsf{L})) = 1$. $\qquad\square$

Proposition A.3. *Let μ_0 and μ_1 be probability measures such that any p-optimal dynamical coupling between μ_0 and μ_1 is concentrated on a set of non-branching geodesics. Then for any p-optimal dynamical coupling $\sigma \in \mathsf{OptGeo}_p(\mu_0, \mu_1)$ and any $t \in (0,1)$ and $s \in [0,1]$ the p-optimal coupling $(e_t, e_s)_* \sigma$ is induced by a transport map.*

Proof. Let $\mu_s = (e_s)_* \sigma$, $\pi_{t,0} = (e_t, e_0)_* \sigma$ and $\pi_{t,1} = (e_t, e_1)_* \sigma$. It suffices to show that $\pi_{t,0}$ is induced by a transport map T, i.e. $(\mathrm{id} \times T)_* \mu_t = \pi_{t,0}$. By disintegrating $\pi_{t,0}$ and $\pi_{t,1}$ over (e_t, e_0) and resp. (e_t, e_1) we get

$$\pi_{0,t} = \int \mu_x \otimes \delta_x d\mu_t(x)$$

$$\pi_{t,1} = \int \delta_x \otimes \nu_x d\mu_t(x).$$

Let $\sigma = \int \sigma_x d\mu_t(x)$ denote the disintegration of σ over e_t and define a new dynamical coupling

$$\tilde{\sigma} = \int (T^x_{0,1})_*(\mu_x \otimes \nu_x) d\mu_t(x)$$

where $T^x_{0,1}$ is defined as in the proof of the previous lemma. Note that $(x_0, x_1, x_t) \mapsto T^{x_t}_0(x_0, x_1)$ is measurable on $(e_0, e_1, e_t)(\mathsf{Geo}_{[0,1]}(M, d))$ and hence $x \mapsto (T^x_{0,1})_*(\mu_x \otimes \nu_x)$ measurable on $\operatorname{supp} \mu_t$.

We claim $\tilde{\sigma}$ is p-optimal. Indeed, by the previous lemma it holds

$$\int d(\gamma_0, \gamma_1)^p d\tilde{\sigma}(\gamma) = \int \int \int d(y,z)^p d\mu_x(y) d\nu_x(z) d\mu_t(x)$$

$$= \int \frac{1}{t^p} d(y,x)^p d\mu_x(y) d\mu_t(x)$$

$$= \frac{1}{t^p} \int d(y,x)^p d\pi_{0,t}(y,x) = W_p(\mu, \nu)^p.$$

If $\pi_{0,t}$ is not induced by a transport map then there is a Borel set $B \subset M$ of positive μ_t-measure such that for all $x \in B$ the measure $\mu_x \otimes \nu_x$ is not a delta measure. This, however, implies that for all $x \in B$ the measure $\tilde{\sigma}_x = (T^x_{0,1})_*(\mu_x \otimes \nu_x)$ is not concentrated on a set of non-branching geodesics. The assumption shows that

there is a non-branching measurable set $\mathsf{L} \subset \mathsf{Geo}_{[0,1]}(M,d)$ with $\tilde{\sigma}(\mathsf{L}) = 1$. But this is a contradiction since $\sigma_x(\mathsf{L}) < 1$ for μ_t-almost all $x \in B$ implies

$$\tilde{\sigma}(\mathsf{L}) = \int \tilde{\sigma}_x(\mathsf{L}) d\mu_t(x) < 1.$$

$\square$

Corollary A.4. *Let μ_0 and μ_1 be probability measures such that any p-optimal dynamical coupling between μ_0 and μ_1 is concentrated on a set of non-branching geodesics. Then for any p-optimal dynamical coupling $\sigma \in \mathsf{OptGeo}_p(\mu_0, \mu_1)$ and any $t \in (0,1)$ there is a Borel map $\mathsf{T}_t : M \to \mathsf{Geo}_{[0,1]}(M,d)$ such that*

$$\sigma = \int \delta_{\mathsf{T}_t(x)} d\mu_t(x)$$

where $\mu_t = (e_t)_* \sigma$. In particular, whenever $\tilde{\sigma} \in \mathsf{OptGeo}_p(\mu_0, \mu_1)$ with $\mu_t = (e_t)_* \tilde{\sigma}$ then $\sigma \equiv \tilde{\sigma}$.

Proof. Let $\sigma = \int \sigma_x d\mu_t(x)$ be the disintegration of σ over e_t. The proof above shows that σ is unique among all dynamical couplings $\tilde{\sigma} \in \mathsf{OptGeo}_p(\mu_0, \mu_1)$ with $\mu_t = (e_t)_* \tilde{\sigma}$. For fixed $s \in [0,1]$ there is a transport map $T_{t,s}$ such that

$$(e_t, e_s)_* \sigma = \int (e_t, e_s)_* \sigma_x d\mu_t(x) = \int \delta_x \otimes \delta_{T_{t,s}(x)} d\mu_t(x).$$

In particular, there is a Borel set $\Omega_s \subset M$ with $\mu_t(\Omega_s) = 1$ and

$$(e_t, e_s)_* \sigma_x = \delta_x \otimes \delta_{T_{t,s}(x)}$$

for all $x \in \Omega_s$. Let $(s_n)_{n \in \mathbb{N}}$ be dense in $(0,1)$ and note $\mu_t(\Omega) = 1$ where $\Omega = \cap_{n \in \mathbb{N}} \Omega_{s_n}$. Define $\gamma_s^x = T_{t,s}(x)$ and observe that $\gamma^x \in \mathsf{Geo}_{[0,1]}(M,d)$ and

$$(e_t, e_{s_n})_* \sigma_x = (e_t, e_{s_n}) \delta_{\gamma^x}.$$

This shows that $\sigma_x = \delta_{\gamma^x}$ on Ω and thus

$$\sigma = \int \delta_{\mathsf{T}_t(x)} d\mu_t(x)$$

where $\mathsf{T}_t : M \to \mathsf{Geo}_{[0,1]}(M,d)$ is any measurable map with $\mathsf{T}_t(x) = \gamma^x$ on Ω. $\square$

Proof of Theorem 2.11. By the p-essentially non-branching property we see that the second statement follows directly from the previous corollary. Furthermore, for Ω as in the previous proof we can choose

$$\mathsf{L} = \operatorname{supp} \sigma \cap e_t^{-1}(\Omega) \subset \mathsf{Geo}_{[0,1]}(M,d).$$

Then σ is concentrated on L and whenever $\gamma, \eta \in \mathsf{L}$ with $\gamma_t = \eta_t$ then $\gamma_t, \eta_t \in \Omega$ so that $\mathsf{T}_t(\gamma_t) \equiv \gamma \equiv \eta$. $\square$

Proof of Corollary 2.12. Assume $t \mapsto \tilde{\mu}_s^{0,t}$ and $t \mapsto \tilde{\mu}_s^{t,1}$ are geodesics connecting μ_0 and μ_t and resp. μ_t and μ_1. Then

$$t \mapsto \hat{\mu}_s = \begin{cases} \tilde{\mu}_{\frac{s}{t}}^{0,t} & s \in [0,t] \\ \tilde{\mu}_{\frac{s-t}{1-t}}^{t,1} & s \in [t,1] \end{cases}$$

is also a geodesic connecting μ_0 and μ_1. Denote the induced p-optimal dynamical coupling by $\hat{\sigma}$ and note that $\mathring{\sigma} = \frac{1}{2}(\sigma + \hat{\sigma})$ is also a p-optimal dynamical coupling between $(e_0)_*\sigma$ and $(e_1)_*\sigma$. Thus it holds

$$\mathring{\sigma} = \int \delta_{\mathring{\mathsf{T}}_t(x)} d\mu_t(x) = \frac{1}{2} \int \delta_{\mathsf{T}_t(x)} + \delta_{\tilde{\mathsf{T}}_t(x)} d\mu_t(x)$$

where T_t, $\tilde{\mathsf{T}}_t$ and $\mathring{\mathsf{T}}_t$ are the maps given by Theorem 2.11. This, however, shows that the three maps agree μ_t-almost everywhere implying $\hat{\sigma} = \sigma$ and thus $\mu_s = \hat{\mu}_s$ for $s \in [0,1]$. In particular, $t \mapsto \mu_{ts}$ and $t \mapsto \mu_{s+(1-s)t}$ are the unique geodesics between μ_0 and μ_t and resp. μ_t and μ_1. $\square$

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