Notes of the lecture “Limits of Spaces”

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Contents

Important note: The following notes will be/were written during the course “Limits of Spaces” during the summer term 2019 at the University of Tübingen. It will be continuously extended and corrected and may contain errors in its current state, not only typos but also inaccuracies in the formulation as well as the proofs. I’m always happy for anyone informing me about errors.

1 Technical details and a notion based on Arzela–Ascoli

Convention: the notions $A \subset B$ and $A \subseteq B$ are equivalent. The latter one will be avoided below.

1.1 Topological and metrical definitions

In the following let M be any set. A set of subsets $\tau \subset 2^M$ is called a *topology* if $\varnothing, M \in 2^M$, τ is closed under finite intersections and closed under arbitrary unions, i.e. if $U_i \in \tau$ for some index set I then $U_i \cap U_j \in \tau$ for $i, j \in I$ and $\cup_{i\in I} U_i \in \tau$. An element of τ will be called *open* and its complement *closed*. A set $A \subset M$ is called a *neighborhood* of $x \in M$ if there is an open set $U \subset A$ with $x \in U$. The topology τ is called *Hausdorff* if for all distinct $x, y \in M$ there are open neighborhoods U_x and U_y of x and resp. y such that $x \notin U_y$ and $y \notin U_x$.

For an arbitrary set A we define the closure $\operatorname{cl} A$ of A and its interior $\operatorname{int} A$ as follows

$$\operatorname{cl} A = \bigcap_{A \subset C \text{ closed}} C$$
$$\operatorname{int} A = \bigcup_{A \supset U \text{ open}} U,$$

i.e. $\operatorname{cl} A$ is the smallest closed set containing A and $\operatorname{int} A$ is the largest open set contained in A. It is easy to show that $\operatorname{cl} A$ is closed and int A is open.

The tuple (M, τ) will be called a *topological space*. With the help of a topology we can define the notion of *convergence of sequences*[1]: Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in M. We say $(x_n)_{n\in\mathbb{N}}$ *converges to* x, written $x_n \to x$, if for all open set U there is an $N > 0$ such that $x_n \in U$ for all $n \geq N$.

A function $f : (M, \tau) \to (M, \tau')$ between two topological spaces is called *continuous* if for all open set $U' \in \tau'$ if $f^{-1}(U') \in \tau$.

Let $A \subset B$ be two subsets of M. We say A *is dense in* B if for all neighborhoods U of $x \in B$ it holds $U \cap A \neq \varnothing$. Call M *separable* if there is a countable subset $A = \{x_n\}_{n\in\mathbb{N}}$ which is dense in M.

A family $\{U_i\}_{i\in I}$ is called *open cover* of a set $A \subset M$ if

$$A \subset \bigcup_{i\in I} U_i.$$

[1] Without countability properties the closure of a subset A may be larger than the set of all accumulation points of sequences in A. Extending the notion of convergences to nets this is indeed true.

A subset K is compact if every open cover $\{U_i\}_{i\in I}$ of K admits a finite subcover, i.e. there is a finite subset $I' \subset I$ with

$$K \subset \bigcup_{i\in I'} U_i.$$

From the definition it follows that any compact subset is closed. An arbitrary subset $A \subset M$ is called *precompact* if its closure is compact. A topological space (M, τ) is called locally compact if every $x \in M$ admits a compact neighborhood.

If A is a subset of M then the topology τ induces a topology τ_A on A as follows

$$\tau_A = \{U \cap A \,|\, U \in \tau\}.$$

This allows us to use topological notions relative to A, e.g. $\Omega \subset A$ is called open/closed/compact in A if it holds for the topological space (A, τ_A).

A function $d : M \times M \to [0, \infty)$ (resp. $d : M \times M \to [0, \infty]$) is called *metric (extended metric)* if for all $x, y, z \in M$ it holds

- (DEFINITENESS) $d(x, y) = 0 \iff x = y$
- (SYMMETRY) $d(x, y) = d(y, x)$
- (TRIANGLE INEQUALITY) $d(x, z) \le d(x, y) + d(y, z)$.

Remark. It is possible to drop the symmetry assumption. However, the notions of (open/closed) balls and Cauchy sequence as well as topology might depend on the order, i.e. whether $d(x, x_0) < \epsilon$ or $d(x_0, x) < \epsilon$.

The tuple (M, d) will be called *metric space*. Every metric spaces induces a topology τ_d on M as follows

$$\tau_d = \{A \in 2^M \,|\, \forall x \in M \exists r > 0 : B_r(x) \subset A\}$$

where $B_r(x)$, the *open ball at x of radius r*, is defined as follows

$$B_r(x) = \{y \in M \,|\, d(x, y) < r\}.$$

Similarly the *closed*[2] *ball $\bar{B}_r(x)$ at x of radius r* is defined as

$$\bar{B}_r(x) = \{y \in M \,|\, d(x, y) \le r\}.$$

It is easy to see that every open ball of positive radius is an open set (w.r.t. τ_d) and every closed ball is a closed set. Note, however, that the closure of an open ball might not be the closed ball of the same radius. We observe that any topology induced by a metric space is necessarily Hausdorff.

Using the metric it is possible to show that a sequence $(x_n)_{n\in\mathbb{N}}$ converges to x (w.r.t. τ_d) if and only if

$$\lim_{n\to\mathbb{N}} d(x_n, x) = 0.$$

By definiteness this shows that the limit of a converging sequence is unique. In particular, any metric induces a Hausdorff topology. Furthermore, the usual $\epsilon - \delta$-notion of convergence is equivalent to either of the two convergences as well.

[2]

Lemma 1.1. *A subset $A \subset M$ is closed if and only if for all $x_n \to x$ with $x_n \in A$ it holds $x \in A$.*

Proof. Note that it suffices to show that $A = \operatorname{cl} A$. Indeed, if $x_n \to x$ with $x_n \in A$ then automatically $x \in \operatorname{cl} A$.

Let $x \in \operatorname{cl} A \backslash A$. Assume by contradiction that there is an $n_0 \in \mathbb{N}$ with $B_{\frac{1}{n_0}}(x) \cap A = \varnothing$. Then $A \subset \operatorname{cl} A \backslash B_{\frac{1}{n_0}}(x)$ and $\operatorname{cl} A \backslash B_{\frac{1}{n_0}}(x)$ is closed. This, however, is a definition of $\operatorname{cl} A$ as being the smallest closed subset containing A.

Hence we have shown that for all $n \in \mathbb{N}$ the sets $B_{\frac{1}{n}}(x) \cap A$ are non-empty. Now choose a sequence $x_n \in B_{\frac{1}{n}}(x) \cap A$ and observe that $x_n \to x$. In particular, any $x \in \operatorname{cl} A \backslash A$ is a limit point of a converging sequence in A. $\square$

A sequence $(x_n)_{n \in \mathbb{N}}$ is called *Cauchy sequence* if for all $\epsilon > 0$ there is an $N_\epsilon > 0$ such that for all $n, m \geq N_\epsilon$

$$d(x_n, x_m) < \epsilon.$$

A metric space (M, d) is called complete if every *Cauchy sequence* is convergent.

In a metric space compactness can be characterized as follows:

Proposition 1.2. *Let (M, d) be complete and $C \subset M$ be a closed subset. Then the following are equivalent:*

- *(*COVERING COMPACT*) C is compact, i.e. every open covering of C admits a finite subcover.*
- *(*SEQUENTIALLY COMPACT*) Every sequence in C admits a converging subsequence with limit in C.*
- *(*TOTALLY BOUNDED *- externally) For every $\epsilon > 0$ there are finitely many $x_1, \ldots, x_n \in M$ with*
$$C \subset \bigcup_{i=1}^{n} B_\epsilon(x_i)$$
- *(*TOTALLY BOUNDED *- internally) For every $\epsilon > 0$ there are finitely many $x_1, \ldots, x_n \in C$ with*
$$C \subset \bigcup_{i=1}^{n} B_\epsilon(x_i).$$

Furthermore, (M, d) is locally compact if and only if for every $x \in M$ there is an $r > 0$ such that $\bar{B}_r(x)$ is compact.

The definition of total boundedness immediately implies that every totally bounded set as well as its closure is separable. Also note that the proposition may be applied to precompact sets.

Corollary 1.3. *Let (M, d) be complete and $A \subset M$ be any subset. Then the following are equivalent:*

- *(*COVERING PRECOMPACT*) A is precompact, i.e. every open covering of* $\operatorname{cl} A$ *admits a finite subcover.*
- *(*SEQUENTIALLY COMPACT*) Every sequence in A admits a converging subsequence (with limit in* $\operatorname{cl} A$*).*
- *(*TOTALLY BOUNDED - *externally) For every* $\epsilon > 0$ *there are finitely many* $x_1, \ldots, x_n \in M$ *with*
$$A \subset \bigcup_{i=1}^{n} B_\epsilon(x_i)$$
- *(*TOTALLY BOUNDED - *internally) For every* $\epsilon > 0$ *there are finitely many* $x_1, \ldots, x_n \in C$ *with*
$$A \subset \bigcup_{i=1}^{n} B_\epsilon(x_i).$$

We also observe the following corollary which might turn out to be useful later on. Using the metric it is possible to strengthen the notion of continuity as follows:

Definition 1.4 (uniformly continuous)**.** A function $f : (X, d_X) \to (Y, d_Y)$ is called *uniformly continuous* if for all $\epsilon > 0$ there is a $\delta = \delta(\epsilon) > 0$ such that

$$d(x, y) < \delta \implies d(f(x), f(y)) < \epsilon.$$

The function is called *locally uniformly continuous* every point $p \in M$ admits a neighborhood $U \ni p$ such that $f\big|_U$ is uniformly continuous.

The following lemma is left as an exercise.

Lemma 1.5. *If X is compact then any continuous function is uniformly continuous.*

In order to describe uniform continuity it often helps to quantify the $\epsilon - \delta$-notion as follows:

Lemma 1.6. *A function f is uniformly continuous if and only if there is an non-decreasing function $\omega : (0, \infty) \to [0, \infty)$ with $\omega(t) \to 0$ as $t \to 0$ such that for all $x, y \in M$ it holds*
$$d(f(x), f(y)) < \omega(d(x, y)).$$
In this case we say f is ω-uniformly continuous or f has modulus of uniform continuity equal to ω.

Proof. If for some $\epsilon_0, t > 0$ it holds$\delta(\epsilon) > t$ for all $\epsilon \le \epsilon_0$ then it is possible to show that f is constant on all balls $B_t(x)$, i.e. f is locally constant. I Similarly, if f is ω-uniformly continuous with $\omega(t) = 0$ for some $t > 0$ then it is locally constant. In either of such a case the equivalence holds trivially. Thus w.l.o.g. we may assume f is not locally constant.

Let f be *uniformly continuous* and define

$$\omega(t) = \inf\{\epsilon \,|\, t < \delta(\epsilon)\}.$$

It is easy to see that ω is non-decreasing and we have $\omega(t) > 0$ for all $t > 0$ as f is not locally constant.

From the definition of we have for all n there is an $\epsilon > \omega(t)$ with

$$\epsilon \le \omega(d_X(x,y)) - \frac{1}{n}$$

with $d(x,y) < \delta(\epsilon)$. Then uniform continuity of f shows

$$d_Y(f(x), f(y)) < \epsilon < \omega(d_X(x,y)).$$

It remains to show that $\omega(t) \to 0$ as $t \to 0$. Assume this is not the case. Then for some $\epsilon_0 > 0$ it holds $\omega(\frac{1}{n}) \ge \epsilon_0$ for all $\frac{1}{n}$. However, this implies $\delta(\epsilon_0) \le \frac{1}{n}$ which is not possible from the definition of uniform continuity.

Assume now f is not locally constant and ω-uniformly continuous and let $\epsilon > 0$. Since ω is positive and non-decreasing and $\omega(t) \to 0$ as $t \to 0$ there is a $t > 0$ such that $\omega(t) \in (0, \epsilon)$. Choosing $\delta = t$ we obtain

$$d_Y(f(x), f(y)) \le \omega(d_X(x,y)) \le \omega(t) \le \epsilon$$

which proves the claim and thus the lemma. □

Example 1.7. If $\omega = Ct^\alpha$ for some $\alpha \in (0,1]$ then f is called *Hölder continuous* (with modulus (C,α)). If $\alpha = 1$ then it is called Lipschitz continuous. The smallest possible C in the definition of Lipschitz continuity will be called the Lipschitz constant.

Definition 1.8 (Uniform/biHölder/biLipschitz equivalence)**.** Two metric spaces (X, d_X) and (Y, d_Y) will be called *uniformly equivalent* if there is a bijective map $\varphi : X \to Y$ such that φ and its inverse φ^{-1} are uniformly continuous. If both are Hölder or resp. Lipschitz continuous then we call the two metric spaces biHölder (resp. biLipschitz) equivalent. Either of the notion may holds locally if there are bijective maps such that φ and φ^{-1} are locally uniformly continuous.

We say (X, d_X) and (Y, d_Y) are isometric if φ and φ^{-1} are Lipschitz continuous with constant 1. In that case φ is bijective and

$$d_X(x,y) = d_Y(\varphi(x), \varphi(y))$$

for all $x, y \in X$.

Proposition 1.9 (Extension Lemma)**.** *Assume A is dense in a metric space (X, d_Y) and (Y, d_Y) is complete. If $f : A \to Y$ is an ω-continuous map then there is a unique ω-continuous map $\tilde{f} : X \to Y$ with $\tilde{f}\big|_A = f$.*

Proof. Let $x \in X$. Since $\operatorname{cl} A = X$ by Lemma 1.1 there is a sequence $x_n \to x$ with $x_n \in A$. Set $y_n = f(x_n)$. Since $(x_n)_{n\in\mathbb{N}}$ is Cauchy and $\omega(t) \to 0$ as $t \to 0$ we have

$$\lim_{n,m\to\infty} d(y_n, y_m) \le \lim_{n,m\to\infty} \omega(d(x_n, x_m)) = 0$$

we see that $(y_n)_{n\in\mathbb{N}}$ is also Cauchy. By completeness there is a y with $y_n \to y$. If y is unique then setting $\tilde{f}(x) = y$ gives a unique value at y. We leave it to the reader to show that $\tilde{f}$ is ω-uniformly continuous.

To see that y is unique let $(\tilde{x}_n)_{n\in\mathbb{N}}$ be another sequence with $\tilde{x}_n \to 0$. Then there is a $\tilde{y} \in Y$ with $f(\tilde{x}_n) = \tilde{y}_n \to \tilde{y}$. Since $d(x_n, \tilde{x}_n) \to 0$ we see that

$$d(y_n, \tilde{y}_n) \le \omega(d(x_n, \tilde{x}_n)) \to 0$$

implying that $y = \tilde{y}$. $\square$

Definition 1.10 (proper metric space)**.** A metric space (M, d) is called *proper* or *boundedly compact* if every bounded closed subset is compact.

By Corollary 1.3 we see that a variant of the classical Heine–Borel Theorem applies to any proper metric space, i.e. any bounded sequence has a convergent subsequence. Furthermore, a proper metric space is separable and complete. Indeed, completeness follows as such spaces are locally compact. To see separability observe that $M = \cup_{n\in\mathbb{N}} \bar{B}_N(x_0)$ and that $\bar{B}_N(x_0)$ is compact and thus separable.

A proof of the following using ultralimits can be found in the next chapter.

Theorem 1.11 (Arzela–Ascoli)**.** *Assume $(f_n : (X, d_X) \to (Y, d_Y))_{n\in\mathbb{N}}$ is a sequence of ω-uniformly continuous functions between a separable metric space (X, d_X) and a proper metric space (Y, d_Y) such that for some x_0 the sequence $(f_n(x_0))_{n\in\mathbb{N}}$ is bounded. Then there is a subsequence (f_{n_k}) such that for all $x \in X$ the sequences $(f_{n_k}(x))_{n_k\in\mathbb{N}}$ are convergent to points $f(x)$ such that $f : (X, d_X) \to (Y, d_Y)$ is ω-uniformly continuous. If (X, d) is compact then this pointwise convergence is uniform, i.e. $\sup_{x\in M} |f_{n_k}(x) - f(x)| \to 0$ as $k \to \infty$.*

1.2 Quantifying compactness

Definition 1.12 (ϵ-net)**.** A subset $A \subset M$ is an *ϵ-net of the metric space* (M, d) if for all $y \in M$ there is an $x \in A$ with $d(x, y) < \epsilon$. A ϵ-net A is *minimal* if for all $x \in A$ the net $A\backslash\{x\}$ is no an ϵ-net. Define $N^{(M,d)}(\epsilon) \in \mathbb{N} \cup \{\infty\}$ to be the minimal number of points needed to form an ϵ-net.

We often use the notation $A \subset B$ is an ϵ-net if A is an ϵ-net of $(B, d\big|_{B\times B})$.

Definition 1.13 (ϵ-separated sets)**.** A subset $B \subset M$ is an *ϵ-separated set of the metric space* (M, d) if for $x, y \in B$ it holds $d(x, y) \ge \epsilon$. An ϵ-separated set B is maximal for $x \in M\backslash B$ the set $B \cup \{x\}$ is not an ϵ-separated set. Define $M^{(M,d)}(\epsilon) \in \mathbb{N} \cup \{\infty\}$ to be the maximal number of points that can form an ϵ-separared set.

It is easy to see that $N^{(M,d)}$ and $M^{(M,d)}$ are non-increasing

Lemma 1.14. *Every maximal ϵ-separated set B is an ϵ-net. In particular, $N^{(M,d)}(\epsilon) \le M^{(M,d)}(\epsilon)$.*

Proof. If this was wrong then there is an $x \in M$ with $d(x,y) \ge \epsilon$ for all $y \in B$. But then $B \cup \{x\}$ forms an ϵ-separated set which is not possible by maximality of B. □

Lemma 1.15. *Any 2ϵ-separated set has at most $N^{(M,d)}(\epsilon)$ number of points. In particular, $M^{(M,d)}(2\epsilon) \le N^{(M,d)}(\epsilon)$.*

Proof. Let B is an 2ϵ-separated set and A an ϵ-net with $\#A = N^{(M,d)}(\epsilon)$, i.e. A is a minimal ϵ-net. Let $a \in A$ and $b, b' \in B$ be such that $d(a,b'), d(a,b) \le \epsilon$. Then $d(b,b') < 2\epsilon$ implying $b = b'$. In particular, for all $b \in B$ there is a unique $a(b) \in B$. Thus $\#B \le \#A$ proving the claim. □

The following observation follows from the fact that total boundedness is equivalent to being precompact.

Corollary 1.16. *The completion of a metric space (M,d) is compact if and only if $N^{(M,d)}$ (or $M^{(M,d)}$) are finite-valued.*

1.3 Product and Quotient spaces, and limits of spaces via Arzela–Ascoli

Given finitely many metric spaces (M_n, d_n), $n = 1, \ldots, N$, we may define product metric on the product $\times_{n=1}^N M_n$ as follows: Let $F : \mathbb{R}^N \to [0,\infty)$ be a norm on N. Then for $(x_n), (y_n) \in \times_{n=1}^N M_n$

$$d_F((x_n),(y_n)) = F(d(x_n,y_n))$$

defines a metric on $\times_{n=1}^N M_n$ such that the natural "coordinate" projections $\pi_k : (x_n) \mapsto x_k$ are Lipschitz continuous with constant $F(e_k)$. Since any two norms on $\mathbb{R}^N$ are biLipschitz equivalent[3] we see that any two metric products are biLipschitz equivalent. In particular, the notion of uniform continuity of on two such products only differs up to a constant. As an abbreviation we use $M_1 \times_p M_2 = (M \times M, d_{\ell_p})$ where $\|(x,y)\|_{\ell_p} = \sqrt[p]{x^p + y^p}$ and resp. $\|(x,y)\|_\infty = \max\{|x|,|y|\}$.

Lemma 1.17. *If (M,d) is a metric space then $d : M \times M \to [0,\infty)$ is 1-Lipschitz on $M \times_1 M$.*

Proof. Assume w.l.o.g. $d(x_1,y_1) \ge d(x_2,y_2)$. Then applying the triangle inequality twice we get

$$\begin{aligned} |d(x_1,y_1) - d(x_2,y_2)| &= d(x_1,y_1) - d(x_2,y_2) \\ &\le \{d(x_1,x_2) + d(x_2,y_2) + d(y_2,y_1)\} - d(x_2,y_2) \\ &\le d(x_1,x_2) + d(y_1,y_2) = d((x_1,y_1),(x_2,y_2)). \end{aligned}$$

This proves the claim. □

[3] Usually it is just called "any two norms are equivalent".

A *pseudo metric* is a function $\delta : M \times M \to [0,\infty)$ on a set M that is symmetric and satisfies the triangle inequality. As for a metric any pseudo metric δ induces a topology on M. However, this topology is Hausdorff if and only if δ is a metric.

The following lemma is left as an exercise.

Lemma 1.18. *Let (M,τ) be a topological space and $\{C_q\}_{q\in N} \subset 2^M\backslash\tau$ be a closed partition for an index set N, i.e C_q is closed $C_q \cap C_r = \varnothing$ for $q \neq r$ and*

$$\bigcup_{q\in N} C_i = M.$$

Then the following set

$$\tau_N = \{\Omega \subset N \mid \bigcup_{q\in\Omega} C_q \in \tau\}$$

is a topology on N and the natural projection $\pi : M \to N$ is continuous where π is defined by $\pi(x) = q$ whenever $x \in C_q$.

Corollary 1.19. *If $\delta : M \times M \to [0,\infty)$ is a continuous pseudo metric on (M,τ) then is a metric space (N,d) and a continuous projection $\pi : M \to N$ such that*

$$\delta(x,y) = d(\pi(x),\pi(y)).$$

Furthermore, if (M,τ) is compact then (N,d) is a compact metric space.

Proof. By symmetry $x \sim_\delta y$ defines an equivalence relation. Since δ is continuous we see that the sets of equivalence classes $[x]$ is closed. Hence the index set N of the equivalence classes can be made into a topological space (N,τ_N).

Assume $x \sim_\delta x'$ and $y \sim_\delta y'$. Since $\delta(x,x') = \delta(y,y') = 0$ we see

$$\begin{aligned}\delta(x,y) &\le \delta(x,x') + \delta(x',y') + \delta(y',y)\\ &= \delta(x',y')\\ &\le \delta(x',x) + \delta(x,y) + \delta(y,y') = \delta(x,y)\end{aligned}$$

implying that $d(x,y) = d(x',y')$. Thus there is a uniquely define function $d : N \times N \to [0,\infty)$ with $\delta(x,y) = d(\pi(x),\pi(y))$. This function is symmetric and satisfies the triangle inequality. To see that it is definite observe that whenever $d(\pi(x),\pi(y)) = 0$ then $\delta(x,y) = 0$ so that $\pi(x) = \pi(y)$. Here we used the fact that π is onto. □

In the following we will denote by (N_δ, d_δ) the metric space obtained from a pseudo metric on (M,τ).

Given a modulus of continuity ω and a metric space (M,d) define the following space

$$\mathfrak{X}_\omega = \{[(N_\delta,d_\delta)] \mid \delta \text{is an } \omega\text{-uniformly continuous pseudo metric on } M \times M\}$$

where $[(N,d)]$ denotes the equivalence class of all metric spaces isometric to (N,d).

On $\mathfrak{X}_\omega$ we define the following notion of convergences: We say $[N_n,d_n] \to [N_\infty,\delta_\infty]$ if for some $[N_{\delta_n},d_{\delta_n}] = [N_n,d_n]$ for $n \in \mathbb{N}\cup\{\infty\}$ and $\delta_n \to \delta$ pointwise. Observe that the pointwise limit of a sequence of pseudo metrics is itself a pseudo metric.

We first need the following lemma whose proof relies on the Gromov–Hausdorff convergence which will be introduced later on.

Lemma 1.20. *Assume (M,d) is compact,$\delta_n \to \delta$, δ_n is ω-uniformly continuous and $[N,d]=[N_{\delta_n},d_{\delta_n}]$. Then $[N,d]=[N_\delta,d_\delta]$.*

Proof. Let $\{x_k\}_{k=1}^{L_\epsilon}$ be an ϵ-dense set in M. Then $\{[x_k]_{\delta_n}\}_{k=1}^{L_\epsilon}$ and $\{[x_k]_\delta\}_{n=1}^{L_\epsilon}$ are γ-dense in their corresponding spaces where $\gamma=\omega(\epsilon)$. Choose n large enough so that $\sup|\delta_n-\delta|<\gamma$. Thus whenever $\delta_n(x_k,x_l)=0$ then $\delta(x_k,x_l)<\gamma$. So for each $k\in\{1,\ldots,L_\epsilon\}$ we may choose $\varphi_n(k)$ among $1,\ldots,N_\epsilon$ such that $\varphi_n(k)=\varphi_n(k')$ and $[x_k]_{\delta_n}=[x_{\varphi_n(k)}]_{\delta_n}$ whenever $[x_k]_{\delta_n}=[x_{k'}]_{\delta_n}$.

For each $[x]_{\delta_n}\neq[x_k]_{\delta_n}$ choose $k([x]_{\delta_n})$ such $k([x_k]_{\delta_n})=[x_k]_{\delta_n}$ and $\delta_n(x,x_{k([x])})\le\gamma$. Now define a map $\Phi_n:[x]_{\delta_n}\mapsto[x_{\varphi_n(k([x]_{\delta_n}))}]_\delta$ and observe that $\Phi_n(\{[x_k]_{\delta_n}\}_{k=1}^{L_\epsilon})$ (and hence $\Phi_n(N_{\delta_n})$) is an 2γ-net of N_δ. Indeed, if $[x]\in N_\delta$ then there is an x_l with $\delta(x,x_l)<\gamma$. From the triangle inequality we obtain

$$\delta(x,x_{\varphi_n(l)})\le\delta(x,x_l)+\delta(x_l,x_{\varphi_n(l)})<2\gamma.$$

Also observe for $k=k([x]_{\delta_n})$ and $l=k([y]_{\delta_n})$ we have

$$\begin{aligned}|d(\Phi_n([x]_{\delta_n}),\Phi_n([y]_{\delta_n}))-d_\delta([x]_\delta,[y]_\delta)| &= |\delta_n(x_{\varphi_n(k)},x_{\varphi_n(l)})-\delta(x,y)\\ &\le\gamma+|\delta(x_{\varphi_n(k)},x_{\varphi_n(l)})-\delta(x,y)|\\ &\le\gamma+\delta(x_{\varphi_n(k)},x)+\delta(x_{\varphi_n(l)},y)\le5\gamma.\end{aligned}$$

Hence $\Phi_n:N_{\delta_n}\to N_\delta$ is a 2γ-approximation.

A similar argument gives a 2γ-approximation $\Psi_n:N_\delta\to N_{\delta_n}$. Since γ will be arbitrary small as $n\to\infty$ we see that $d_{GH}((N_{\delta_n},d_{\delta_n}),(N_{\delta,}d_\delta))\to0$. But by assumption $d_{GH}((N_{\delta_n},d_{\delta_n}),(N_{\delta_1},d_{\delta_1}))=0$ so that $d_{GH}((N_\delta,d_\delta),(N_{\delta_1},d_{\delta_1}))=0$ which by completeness of the spaces proves that (N_δ,d_δ) and $(N_{\delta_1},d_{\delta_1})$ are isometric. □

Proposition 1.21. *Assume (M,d) is compact. Then the notion of convergence described above is induced by a metric $\mathfrak{d}_{(\omega)}$ making $(\mathcal{X}_\omega,\mathfrak{d}_\omega)$ into a compact metric space.*

Proof. Define $\mathfrak{d}_\omega$ as follows:

$$\mathfrak{d}_\omega([N,d],[N',d'])=\inf\{\sup_{x,y\in X}|\delta(x,y)-\delta'(x,y)|\,|\,[N,d]=[N_\delta,d_\delta],[N',d']=[N_{\delta'},d_{\delta'}]\}.$$

It is easy to see that $\mathfrak{d}_\omega$ is symmetric and satisfies the triangle inequality. By Arzela–Ascoli the infimum is actually attained by pseudo metrics δ and δ'. Indeed, if the tuples $(\delta_n,\delta_n^{'})$ form a minimizing sequences then by Arzela–Ascoli a subsequences converges to a tuple of pseudo metrics (δ,δ') and by the previous lemma $[N,d]=[N_\delta,d_\delta]$ as well as $[N',d']=[N_{\delta'},d_{\delta'}]$. The same argument also yields that any sequence in $\mathfrak{X}_\omega$ has a convergent subsequence.

From this we conclude definiteness of $\mathfrak{d}_\omega$ as follows: If $\mathfrak{d}_\omega([N,d],[N',d'])=0$ then for some $\delta=\delta'$ it holds $[N,d]=[N_\delta,d_\delta]=[N',d']$, i.e. (N,d) and (N',d') are isometric. □

The following lemma helps to verify whether a given pseudo metric is ω-uniformly continuous.

Lemma 1.22. *Assume δ is a pseudo metric and for all $x \in M$ the functions $\delta_x = \delta(x,\cdot) : M \to [0,\infty)$ are ω-uniformly continuous. Then δ is (2ω)-uniformly continuous on $M \times_\infty M$.*

Proof. Choose points $x_1, x_2, y_1, y_2 \in M$. Adding $\delta(x_1, y_2) - \delta(x_1, y_2)$ we obtain

$$\begin{aligned}|\delta(x_1,y_1) - d(x_2,y_2)| &= |\delta(x_1,y_1) - \delta(x_1,y_2) + \delta(x_1,y_2) - d(x_2,y_2)| \\ &\le \omega(d(y_1,y_2)) + \omega(d(x_1,x_2)) \\ &\le 2\omega(\max\{d(x_1,x_2), d(y_1,y_2)\}).\end{aligned}$$

□

1.4 Basics on length and geodesic spaces

Let (M,d) be a length space and $\gamma : [a,b] \to M$ be a (continuous) curve. A *reparametrization* $\gamma^\varphi$of a curve $\gamma : [a,b] \to M$ is a a surjective monotone function $\varphi : [a,b] \to [c,d]$ such that $\gamma^\varphi = \gamma \circ \varphi$ is a continuous curve in M. If not specified otherwise we usually assume $\varphi(a) = c$ and $\varphi(b) = d$.

Given two curves $\gamma^i : [0,1] \to M$, $i = 1,2$, with $\gamma^1_1 = \gamma^2_0$ we define the *glueing* $\eta = \gamma^1 \cup \gamma^2 : [0,1] \to M$ of the two curves as follows

$$\eta_t = \begin{cases}\gamma^1_{2t} & t \in [0,\frac{1}{2}] \\ \gamma^2_{2t-1} & t \in [\frac{1}{2},1].\end{cases}$$

We say that γ is *rectifiable* if $\ell(\gamma) < \infty$ where

$$\ell(\gamma) = \sup_{(t_0,\ldots,t_n)\in\mathcal{I}_{[a,b]}} \sum_{i=1}^n d(\gamma_{t_{i-1}}, \gamma_{t_i})$$

where

$$\mathcal{I}_{[a,b]} = \{(t_0,\ldots,t_n)\,|\, n\in\mathbb{N}, t_i \le t_{i+1}, t_0 = a, t_n = b\}.$$

The following two facts are easy to verify

$$\begin{aligned}\ell(\gamma^\varphi) &= \ell(\gamma) \\ \ell(\gamma^1 \cup \gamma^2) &= \ell(\gamma^1) + \ell(\gamma^2).\end{aligned}$$

Lemma 1.23. *If $\gamma : [0,1] \to M$ is a rectifiable curve then for decreasing sequence of closed connected sets $I_n \subset [0,1]$ with $\operatorname{diam} I_n \to 0$ it holds $\ell(\gamma\big|_{I_n}) \to 0$.*

Proof. We leave it to the reader to show that for a rectifiable curve $\ell(\gamma) = \mathcal{H}^1(\gamma([0,1]))$ where $\mathcal{H}^1$ is the one-dimensional Hausdorff measures (see below for definition of the Hausdorff measures). By the property of being a measure it holds

$$0 = \mathcal{H}^1(\{t_0\}) = \inf_{n\in\mathbb{N}} \mathcal{H}^1(I_n) = \lim_{n\to\infty} \mathcal{H}^1(I_n) = \lim_{n\to\infty} \ell(\gamma\big|_{I_n}).$$

□

Proposition 1.24. *Assume $\gamma : [0,1] \to M$ is rectifiable. Then there is a constant speed reparametrization of γ, i.e. there is a continuous surjective and monotone function $\varphi : [0,1] \to [0,1]$ such that*

$$\ell(\gamma^\varphi\big|_{[0,t]}) = t\ell(\gamma).$$

Proof. Note that the function $L : t \mapsto \ell(\gamma\big|_{[0,t]})$ is non-decreasing. By the previous lemma we have

$$\limsup_{s\searrow t} L(s) - L(t) = \limsup_{s\searrow t} \ell(\gamma\big|_{[t,s]}) = 0$$

and

$$\liminf_{r\nearrow t} L(t) - L(r) = \liminf_{r\nearrow t} \ell(\gamma\big|_{[r,t]}) = 0$$

so that L is also continuous.

Note that L is constant on $[t,s]$ if and only if γ is constant on $[t,s]$. Thus we can find a non-decreasing, possibly discontinuous function $\varphi : [0,1] \to [0,1]$ such that $L \circ \varphi(s) = s\ell(\gamma)$. To finish the proof it remains to show that γ^φ is continuous. Observe that whenever φ is not continuous to the left at a point s then there is a $t_0 < t$ such that L is constant on $[t_0, \varphi(s)]$. In this case $\gamma(t_0) = \gamma(\varphi(s))$. Choosing the minimal t_0 and setting $\varphi_{\min}(s) = t_0$. Do this for all points of left-discontinuity and set the function outside of those points equal to φ to find a non-decreasing function $\varphi_{\min}$ that is left-continuous at all points and satisfies $L \circ \varphi_{\min}(s) = s\ell(\gamma)$ for $s \in [0,1]$. This shows, in particular, that $\gamma^{\varphi_{\min}}(s) = \gamma^\varphi(s)$ for all $s \in [0,1]$ and that $\gamma^{\varphi_{\min}}$ is left-continuous. Doing this for points of right discontinuity we also obtain a function $\varphi_{\max}$ such that $\gamma^{\varphi_{\max}} = \gamma^\varphi$ is right-continuous at all points t. Thus we have shown that γ^φ is both rigth and left continuous and satisfies

$$s \mapsto \ell(\gamma^\varphi\big|_{[0,s]}) = L(\varphi(s)) = s\ell(\gamma).$$

□

We now define the induced length metric d_L of a metric spaces as follows:

$$d_L(x,y) = \inf\{\ell(\gamma) \,|\, \gamma : [0,1] \to M \text{ is a curve with } \gamma_0 = x \text{ and } \gamma_1 = y\}.$$

A metric space (M,d) such that $d_L = d$ is called a *length space.*

Lemma 1.25. *For any curve γ it holds $\ell_d(\gamma) = \ell_{d_L}(\gamma)$.*

Proof. We always have $d_L \geq d$ so that $\ell_{d_L} \geq \ell_d$. In particular, we may assume γ is (d-)rectifiable. Thus for $t,s \in [0,1]$ it holds

$$d_L(\gamma_t, \gamma_s) \leq \ell_d(\gamma_{[t,s]}).$$

But then for all $(t_0, \ldots, t_n) \in \mathcal{I}_{[0,1]}$ it holds

$$\sum_{i=1}^n d_L(\gamma_{t_{i-1}}, \gamma_{t_i}) \leq \ell_d(\gamma_{[0,1]}).$$

Taking the supremum on the left hand side we obtain $\ell_{d_L}(\gamma) \leq \ell_d(\gamma)$. □

Corollary 1.26. *It holds* $(d_L)_L = d_L$.

Proposition 1.27. *The space* (M, d_L) *is an extended metric space. Furthermore,* (M, d_L) *is a length space if and only if every two points can be connected by a rectifiable curve.*

Proof. Since $d_L \geq d$ we see that d_L must be definite. Symmetry is also obvious. Choose $x, y, z \in M$ and observe that if there are no rectifiable curve γ or η between x and y or resp. y and z then the triangle inequality holds trivially. In the other case $\gamma \cup \eta$ is a rectifiable curve between x and z. Taking the infimum over all such curves we see that the triangle inequality holds for d_L.

The last claim follows from the definition of d_L together with the previous corollary. □

Note that the topologies of (M, d) and (M, d_L) can be quite different.

Definition 1.28 (geodesics)**.** A curve $\gamma : [0, 1] \to M$ is called a *geodesic* between x and y if

$$d(\gamma_t, \gamma_s) = |t - s| d(\gamma_0, \gamma_1) \qquad \text{for all } s, t \in [0, 1].$$

A metric space is called a *geodesic* space if every two points can be connected by a geodesic.

We also say a curve $\gamma : [a, b] \to M$ is a $[a, b]$*-parametrized geodesic* if $t \mapsto \gamma_{(b-a)t+a}$ is a geodesic. Similar definitions exist for curves of open or half open intervals. We say $\gamma : [a, b] \to M$ has *unit speed* if $d(\gamma_t, \gamma_s) = |s - t|$ for $t, s \in [a, b]$. A unit speed geodesic $\gamma : [0, \infty) \to M$ will be called a *(geodesic) ray* and a unit speed $\gamma : \mathbb{R} \to M$ will be called a *(geodesic) line.*

A curve $\eta : I \to M$ over an open interval is a *local geodesic* if for all $t \in I$ there is a neighborhood $[a_t, b_t] \subset I$ of t such that $\eta\big|_{[a_t, b_t]}$ is an $[a_t, b_t]$-parametrized geodesic.

Proposition 1.29. *If* γ *is a geodesic between* x *and* y *then* $\ell(\gamma) = d(x, y) = d_L(x, y)$. *In particular, a geodesic space is a length space.*

We leave the following three statements as an exercise to the reader.

Lemma 1.30. *A complete metric space* (M, d) *is a length space if and only if it admits approximate midpoints, i.e. for all* $x, y \in M$ *and* $\epsilon > 0$ *there is an* $m = m(x, y, \epsilon) \in M$ *such that*

$$\begin{aligned} d(x, m) + d(m, y) &\leq d(x, y) + \epsilon. \\ |d(x, m) - d(m, y)| &\leq \epsilon \end{aligned}$$

Lemma 1.31 (Menger convexity)**.** *A complete metric space is a geodesic space if for all distinct* $x, y \in M$ *there is an* $m \in M \backslash \{x, y\}$ *such that*

$$d(x, m) + d(m, y) = d(x, y).$$

Proposition 1.32 (Hopf–Rinow)**.** *A metric space is a proper length space if and only if it is a locally compact geodesic space.*

1.5 Manifolds and their length structure

TO COME

Fact 1.33 (Blackbox Theorem - geodesic extendability). *Assume (M, F) is a Finsler manifold. Then any point $x \in M$ admits a neighrborhood U such that for all $K \subset\subset U$ there is an $\epsilon > 0$ such that any (minimizing) geodesic $\gamma : [0,1] \to K$ there is a (minimizing) geodesic $\eta : [0, 1+\epsilon] \to U$ such that $\gamma \equiv \eta\big|_{[0,1]}$.*

Corollary 1.34. *If (M, d_F) complete then it is geodesically complete, i.e. any geodesic $\gamma : [0,1] \to M$ can be extended to a local geodesic $\gamma : \mathbb{R} \to M$.*

2 Ultralimits

2.1 Ultrafilters

A non-empty subset $\mathcal{F} \subset 2^X$ of a set X is called a *filter* if

- $\varnothing \notin \mathcal{F}$
- if $A \in \mathcal{F}$ and $A \subset B$ then $B \in \mathcal{F}$
- if $A, B \in \mathcal{F}$ then $A \cap B \in \mathcal{F}$.

We say $\mathcal{F}$ is a *principle filter* if $\{x\} \in \mathcal{F}$.

A filter that is maximal w.r.t. to inclusion is called an *ultrafilter*.

Proposition 2.1. *Let $\mathcal{F}$ be the filter. The following properties are equivalent to being an ultrafilter*

1. *If $\mathcal{F} \subset \mathcal{F}'$ for some filter $\mathcal{F}'$ then $\mathcal{F} = \mathcal{F}'$.*
2. *If $A \cup B \in \mathcal{F}$ then either $A \in \mathcal{F}$ or $B \in \mathcal{F}$.*
3. *For all $A \subset X$ either $A \in \mathcal{F}$ or $X \backslash A \in \mathcal{F}$.*
4. *There is a non-trivial finitely additive measure $\alpha : 2^X \to \{0, 1\}$ with $\alpha(A) = 1$ iff $A \in \mathcal{F}$.*

Before we prove this proposition the reader may verify that the following lemma is true.

Lemma 2.2. *Let $\mathcal{F}$ be a filter and $A \subset M$ such that for all $B \in \mathcal{F}$ it holds $A \cap B \neq \varnothing$. Then the following defines a filter containing A:*

$$\mathcal{F}_A = \{C \cap D \,|\, A \subset C \text{ and } D \in \mathcal{F}\}.$$

In particular, $\mathcal{F} \subset \mathcal{F}_A$ and $A \in \mathcal{F}_A$.

Proof of the proposition. Assume first $\mathcal{F}$ satisfies the first property, i.e. $\mathcal{F}$ is an ultrafilter and assume $A \cup B \in \mathcal{F}$ for two sets $A, B \subset X$. We claim that either $A \cap D \neq \varnothing$ for all $D \in \mathcal{F}$ or $B \cap D \neq \varnothing$ for all $D \in \mathcal{F}$. Indeed, by renaming we may assume $A \cap D = \varnothing$ for some $D \in \mathcal{F}$. Since $A \cup B \in \mathcal{F}$ and $\varnothing \notin \mathcal{F}$ we must have $B \cap D \neq \varnothing$ by the third property of being a filter. Assume now by contradiction $B \cap D' = \varnothing$ for some $D' \in \mathcal{F}$ then $D' \cap D \in \mathcal{F}$. However, this leads to the contradiciton $\varnothing = D \cap D' \cap (A \cup B) \in \mathcal{F}$.

Since $B \cap D \neq \varnothing$ for all $D \in \mathcal{F}$ we see that $\mathcal{F}_B$ is a filter which is larger than $\mathcal{F}$ and contains B. But this means $\mathcal{F}_B = \mathcal{F}$ implying the claim,.

If the second property holds then the third one follows since $X \in \mathcal{F}$ holds since filters are always non-empty. Thus assume the third property holds for $\mathcal{F}$. Assume $\mathcal{F} \subset \mathcal{F}'$ for some filter $\mathcal{F}'$. Assume by contradiction there is an $A \in \mathcal{F}' \backslash \mathcal{F}$. Then the third property implies $X \backslash A \in \mathcal{F} \subset \mathcal{F}'$. But this leads to the contradiction $\varnothing = A \cap (X \backslash A) \in \mathcal{F}'$.

It remains to show that the third and fourth properties are equivalent. First note that any non-trivial finitely additive measure $\alpha : 2^X \to \{0, 1\}$ induces a subset $\mathcal{F} \subset 2^X$ satisfying the third property. The properties of being a measure and the fact that $\alpha(\varnothing) = 0$ and $1 \leq \alpha(A) \leq \alpha(B) \leq 1$ for some $A \in 2^X$ shows that $\mathcal{F}$ is non-empty, $\varnothing$ and the second property of a filter holds. Being fintiely additive then implies the last property of a filter.

Now let $\mathcal{F}$ be an ultrafilter and define $\alpha(A) = 1$ if $A \in \mathcal{F}$ and otherwise 0. Let A, B be disjoint non-empty sets. If neither $A, B \notin \mathcal{F}$ then $(X \backslash A), (X \backslash B) \in \mathcal{F}$ as well as $(X \backslash A) \cap (X \backslash B) \in \mathcal{F}$ so that $0 = \alpha(A \dot{\cup} B) = \alpha(A) = \alpha(B)$ which is obviously additive. If $A \in \mathcal{F}$ then $A \dot{\cup} B \in \mathcal{F}$ and $B \notin \mathcal{F}$ as $B \subset X \backslash A \notin \mathcal{F}$. Thus $1 = \alpha(A \dot{\cup} B) = \alpha(A)$ and $\alpha(B) = 0$ which implies additivity. A similar argument holds for $B \in \mathcal{F}$. $\square$

Proposition 2.3 (Ultrafilter Lemma)**.** *For any filter $\mathcal{F}$ there is a ultrafilter $\mathcal{F}' \supset \mathcal{F}$.*

Proof. Let $\mathcal{P}$ be the set of ultrafilters ordered by inclusion. The statement follows from Zorn's Lemma. Let I be a totally ordered set and $\{\mathcal{F}_i\}_{i \in I}$ a chain, i.e. $\mathcal{F}_i \subset \mathcal{F}_j$ whenever $i \leq j$. We claim that

$$\mathcal{F} = \bigcup_{i \in I} \mathcal{F}_i$$

defines filter which we can use to apply Zorn's Lemma which would yield the claim.

Obviously $\mathcal{F}$ is non-empty and $\varnothing \notin \mathcal{F}$. If $A, B \in \mathcal{F}$ then $A \in \mathcal{F}_i$ and $B \in \mathcal{F}_j$. W.l.o.g. assume $i \leq j$. If $C \supset A$ then $C \in \mathcal{F}_i \subset \mathcal{F}$. Also since $i \leq j$ we have $A \in \mathcal{F}_j$ so that $A \cap B \in \mathcal{F}_j \subset \mathcal{F}$ which proves that $\mathcal{F}$ is a filter. $\square$

Corollary 2.4. *On any infinite set X there are non-principle ultrafilters.*

Proof. We define the following set

$$\mathcal{F}_{co} = \{A \subset X \mid X \backslash A \text{ is finite, i.e. } A \text{ is cofinite}\}$$

and claim that this defines a filter. The previous proposition then implies the claim of the corollary.

First observe that $\varnothing \notin \mathcal{F}_{co}$ as X is infinite. Also if A and B are cofinite then $X \backslash (A \cap B) \subset (X \backslash A) \cup (X \backslash B)$ is also finite, i.e. $A \cap B \in \mathcal{F}_{co}$. Similarly, if $C \supset A$ then $X \backslash C \subset X \backslash A$ is finite so that $C \in \mathcal{F}_{co}$. $\square$

2.2 Ultralimits

In this section we regard ultrafilters on $\mathbb{N}$ as both subsets of $2^{\mathbb{N}}$ as well as finitely additive measures, i.e. $A \in \omega$ is equivalent to $\omega(A) = 1$.

A number $a \in \mathbb{N}$ will be called an ultralimit of a sequence $(a_n)_{n\in\mathbb{N}}$ w.r.t. a non-principle ultrafilter if for all $\epsilon > 0$ it holds

$$\{n \in \mathbb{N} \,|\, |a_n - a_\omega| < \epsilon\} \in \omega.$$

If a sequence admits an ultralimit a we use the notation$\lim_\omega a_n := a$.

Theorem 2.5. *For any non-principle ultrafilter ω on $\mathbb{N}$ and any bounded sequence $(a_n)_{n\in\mathbb{N}}$ admits a unique ultralimit a_ω. Furthermore, there is a subsequence $(n_k)_{k\in\mathbb{N}}$ such that $a_{n_k} \to a_\omega$.*

Proof. $M = \sup_{n\in\mathbb{N}} |a_n|$. By induction we define sets I_n and A_n satisfying $\omega(I_n) = 1$, $I_{n+1} \subset I_n$ and $A_{n+1} \subset A_n$ as well as $\operatorname{diam} A_n = 2 \cdot 2^{-n} \cdot M$ and $\omega(I_n) = 1$: $I_0 = \mathbb{N}$ and $A_0 = [-M, M]$. Note that $\omega(I_0) = 1$.

Assume I_n and A_n are defined. Let $m_n = \frac{\sup A_n + \inf A_n}{2}$ and define two sets

$$A^+_{n+1} = [\inf A_n, m_n]$$
$$A^-_{n+1} = (m_n, \sup A_n]$$

and two index sets

$$I^\pm_n = \{n \in I_n \,|\, a_n \in A^\pm_{n+1}\}.$$

Note that those sets define disjoint partitions of A_n and resp. I_n. Thus

$$1 = \omega(I_n) = \omega(I^+_n) + \omega(I^-_n).$$

Since ω has values in $\{0, 1\}$ we set $I_{n+1} = I^+_n$ and $A_{n+1} = A^+_{n+1}$ if $\omega(I^+_n) = 1$ and $I_{n+1} = I^-_{n+1}$ and $A_{n+1} = A^-_{n+1}$ if $\omega(I^-_n) = 1$.

Note that any sequence $b_n \in A_n$ will be Cauchy and converging to the unique point $a_\omega \in \cap_{n\in\mathbb{N}} \operatorname{cl} A_n$.

Now pick $\epsilon > 0$ and choose n large such that $4 \cdot 2^{-n} \cdot M < \epsilon$ then

$$I_n \subset \{n \in \mathbb{N} \,|\, |a_n - a_\omega| < 4 \cdot 2^{-n} \cdot M\} \subset \{n \in \mathbb{N} \,|\, |a_n - a_\omega| < \epsilon\}$$

implying $\{n \in \mathbb{N} \,|\, |a_n - a_\omega| < \epsilon\} \in \omega$. To see that a_ω must be unique, assume there is a $b \neq a$ then $\{n \in \mathbb{N} \,|\, |a_n - a| < \frac{|b-a|}{2}\}, \{n \in \mathbb{N} \,|\, |a_n - b| < \frac{|b-a|}{2}\} \in \omega$. However, this would imply

$$\{n \in \mathbb{N} \,|\, |a_n - a| < \frac{|b - a|}{2}\} \cap \{n \in \mathbb{N} \,|\, |a_n - b| < \frac{|b - a|}{2}\} = \varnothing \notin \omega.$$

To conclude note I_n is infinite since ω is non-principle. Thus let $n_k = \inf I_k \cap [k, \infty)$ will define a sequence $a_{n_k} \in I_k$ converging to a_ω. □

Corollary 2.6. *If $(a_n)_{n\in\mathbb{N}}$ then either (a_n) admits an ultralimit or one of the following holds*

- *for all $M > 0$*
$$\{n \in \mathbb{N} \,|\, a_n > M\} \in \omega$$
for all $M > 0$
$$\{n \in \mathbb{N} \,|\, a_n < -M\} \in \omega.$$
In the first case we write $\lim_\omega a_n = \infty$ and in the second case $\lim_\omega a_n = -\infty$.

Remark. Using a transformation $b_n = \arctan a_n$ we see that $\lim_\omega b_n = \pm\frac{\pi}{2}$ if and only $\lim_\omega a_n = \pm\infty$. Hence the values $\pm\infty$ may still be called ultralimits if $\lim_\omega a_n = \pm\infty$.

Proposition 2.7. *If ω is a non-principle ultrafilter and (M, d) is a proper metric space then for any bounded sequence $(x_n)_{n\in\mathbb{N}}$ there is a unique $x_\omega \in X$ such that $\lim_\omega d(x, x_n) = 0$. We call x the ultralimit of (x_n) and write $\lim_\omega x_n = x$.*

Proof. Assume $x_n \in \bar{B}_R(x_0)$ for some $R > 0$. Pick a countable dense subset $\{y_k\}_{k\in\mathbb{N}}$. We construct numbers N_l, a disjoint partition $\{A_{i,l}\}_{i=1}^{N_l}$of $\bar{B}_R(x)$ with $\operatorname{diam} A_{i,l} \le 2^{-l}R$ and index sets $I_{l+1} \subset I_l$ with $I_l \in \omega$ as follows: $N_0 = 1$, $A_{1,0} = \bar{B}_R(x_0)$ and $I_0 = \mathbb{N}$. Assume the sets are constructed for $l-1$. Define N_l the infimum among all N such that
$$\bar{B}_R(x) \subset \bigcup_{i=1}^{N} \bar{B}_{2^{-l}R}(y_i).$$
and define
$$A_{i,l} = \bar{B}_{2^{-l}R}(y_i) \backslash \bigcup_{j=1}^{i-1} \bar{B}_{2^{-l}R}(y_j).$$
There is a unique index i_l such that
$$\{n \in I_{l-1} \,|\, x_n \in A_{i_l,l}\} \in \omega.$$
We set $I_l = \{n \in I_{l-1} \,|\, x_n \in A_{i_l,l}\}$.

Now pick $n_l = \inf I_l \cap [l, \infty)$ and observe that $d(x_{n_b}, x_{n_a}) \le 2^{-l}R$ whenever $a, b \ge l$. Thus $(x_{n_l})_{l\in\mathbb{N}}$ is Cauchy and converging to some x.

We claim $\lim_\omega d(x_n, x) = 0$. Let $\epsilon > 0$ and choose l large such that $4 \cdot 2^{-l}R < \epsilon$. If $n \in I_l$ then
$$\begin{aligned}
d(x_n, x) &\le d(x_n, x_{n_l}) + d(x_{n_l}, x) \\
&\le d(x_n, x_{n_l}) + \sum_{k=l}^{N} d(x_{n_k}, x_{n_{k+1}}) + d(x_{n_N}, x) \\
&\le 2^{-l}R + \sum_{k=l}^{\infty} 2^{-l}R \\
&\le 4 \cdot 2^{-l}R < \epsilon.
\end{aligned}$$
Therefore, $I_l \subset \{n \in \mathbb{N} \,|\, |d(x_n, x) - 0| < \epsilon\} \in \omega$ which yields the claim. ∎

Proof of Arzela–Ascoli via ultralimit. Let $f_n : (M,d) \to (N,d)$ be ω-uniformly continuous maps such that (N,d) is proper and $(f_n(x_0))_{n\in\mathbb{N}}$ is bounded for some fixed x_0, i.e. $d(f_n(x_0), y_0) \le R$ for some $y_0 \in N$ and $R > 0$.

For $x \in M$ observe that

$$d(f_n(x), y_0) \le d(f_n(x), f_n(x_0)) + d(f_n(x_0), y_0) \le \omega(d(x, x_0)) + R.$$

Thus for any fixed $x \in M$ the sequence $(f_n(x))_{n\in\mathbb{N}}$ is bounded. Define

$$f(x) := \lim_{\omega} f_n(x).$$

Then

$$\begin{aligned} d(f(x), f(y)) &\le d(f(x), f_n(x)) + d(f_n(x), f_n(y)) + d(f_n(y), f(y)) \\ &\le d(f(x), f_n(x)) + d(f_n(y), f(y)) + \omega(d(x,y). \end{aligned}$$

Taking the ultralimit of the right hand side shows f is ω-uniformly continuous. □

Proof of Hopf–Rinow via ultralimits. A proper length space (M,d) has approxmate midpoints. Let m_n be an approximate midpoint such that

$$d(x, m_n), d(m_n, y) \le \frac{1}{2} d(x,y) + \frac{1}{n}.$$

This implies $(m_n)_{n\in\mathbb{N}}$ is bounded so that by properness it has an ultralimit $m = \lim_\omega m_n$ where ω is some non-principle ultrafilter on $\mathbb{N}$. Furthermore, m satisfies

$$\max\{2d(x,m), 2d(m,y), d(x,m) + d(m,y)\} \le d(x,y) \le d(x,m) + d(y,m)$$

which shows $d(x,m) = d(m,y) = \frac{1}{2} d(x,y)$, i.e. m is a midpoint showing (M,d) is a proper geodesic space and in particular a locally compact geodesic space.

Assume (M,d) is a locally compact geodesic space. Define

$$r(x) = \sup\{r \ge 0 \,|\, \bar{B}_{r'}(x) \text{ is compact for all } r' \in [0,r]\}.$$

By local compactness $r(x) > 0$ for all $x \in M$. If $r(x) = \infty$ then (M,d) is proper.

Assume $\bar{B}_r(x)$ is compact. Then $\partial \bar{B}_r(x)$ is compact so that it can be covered by finitely many balls $\bar{B}_{r_i}(y_i)$ with $r_i \le \frac{r(y_i)}{3}$, $i = 1, \ldots, N$. In particular, $\min r_i > 0$.

Let $\epsilon \in (0, \min r_i)$ and observe that for all $y \in \bar{B}_{r+\epsilon}(x) \backslash \bar{B}_r(x)$ there is a point $\tilde{y} \in \partial \bar{B}_r(x)$ on the geodesic connecting x and y such that $d(y, \tilde{y}) = d(y,x) - r \le \epsilon$. Since $\cup_i \bar{B}_{r_i}(y_i)$ is a cover there is an i with $\tilde{y} \in \bar{B}_{r_i}(y_i)$

$$d(y_i, y) \le d(y_i, \tilde{y}) + d(y, \tilde{y}) \le r_i + \epsilon \le \frac{2r(y_i)}{3}.$$

Which implies

$$\bar{B}_{r+\epsilon}(x) \subset \bar{B}_r(x) \cup \bigcup_{i=1}^{N} \bar{B}_{\frac{2r(y_i)}{3}}(y_i).$$

Since the right hand side is a union of finitely many compact set we are done. This shows that $\bar{B}_{r(x)}(x)$ must be non-compact if $r(x) < \infty$.

Assume now by contradiction $R = r(x) < \infty$ for some $x \in M$. Let $(y_n)_{n\in\mathbb{N}}$ be a sequence in $\bar{B}_r(x)$ and $(\gamma^n)_{n\in\mathbb{N}}$ be a sequence of geodesics connecting x and y_n. Note that for all $t \in [0,1)$ it holds

$$\gamma_t^n \in \bar{B}_{tR}(x).$$

Since $\bar{B}_{tR}(x)$ is compact for $t < \infty$ the ultralimits of $(\gamma_t^n)_{n\in\mathbb{N}}$ for $t \in [0,1)$ are well-defined, i.e. $\gamma_t := \lim_\omega \gamma_t^n$ is well-defined. We may also verify that $\gamma : [0,1) \to M$ defines a geodesic in $B_R(x)$ starting from x. By assumption γ can be extended to a geodesic γ. Set $y = \gamma_1$. We claim $y = \lim_\omega y_n$. Indeed, it holds

$$\begin{aligned} d(y_n, y) &\le d(y_n, \gamma_t^n) + d(\gamma_t^n, \gamma_t) + d(\gamma_t, y) \\ &\le d(\gamma_t^n, \gamma_t) + 2(1-t)R. \end{aligned}$$

Thus for all $\epsilon > 0$ and $t \in [0,1)$ with $2(1-t)R < \epsilon$ we get

$$\{n \in \mathbb{N} \,|\, d(y_n, y) < \epsilon\} \supset \{n \in \mathbb{N} \,|\, d(\gamma_t^n, \gamma_t) < \epsilon - 2(1-t)R\} \in \omega.$$

By the properties of being a filter we see that $\{n \in \mathbb{N} \,|\, d(y_n, y) < \infty\} \in \omega$ which implies y is an ultralimit of y_n. From this we obtain a subsequence $y_{n_k} \to y$ which implies compactness of $\bar{B}_R(x)$. □

2.3 Ultralimits of metric spaces

Let (X_n, d_n) be a sequence of metric spaces. Set

$$\mathcal{X}_\infty = \{(x_n)_{n\in\mathbb{N}} \,|\, x_n \in X_n\}$$

and define a function $d_\omega : \mathcal{X}_\infty \times \mathcal{X}_\infty \to [0,\infty]$ by

$$d_\omega((x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}) = \lim_\omega d_n(x_n, y_n).$$

It is not difficult to see that d_ω is an extended pseudo-metric, i.e. it is symmetric and satisfies the triangle inequality. Now defined equivalence classes on $\mathcal{X}_\infty$ as follows

$$[x_n]_{n\in\mathbb{N}} := \{(y_n)_{n\in\mathbb{N}} \,|\, d_\omega((x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}) = 0\}.$$

and set

$$X_\infty = \{[x_n]_{n\in\mathbb{N}} \,|\, x_n \in X_n\}$$

and

$$d_\omega([x_n]_{n\in\mathbb{N}}, [y_n]_{n\in\mathbb{N}}) = d_\omega((x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}).$$

Then d_ω defines an extended metric on X_∞. We say (X_∞, d_ω) is the ultralimit of the sequence $(X_n, d_n)_{n\in\mathbb{N}}$. One can show that $\lim_\omega \operatorname{diam}(X_n, d_n) < \infty$ if and only if (X_∞, d_ω) is a metric space.

In general we need to exclude the case $d_\omega([x_n]_{n\in\mathbb{N}},[y_n]_{n\in\mathbb{N}}) = \infty$. For this pick a sequence of "origins" $p_n \in X_n$ and define

$$X_\infty^{(p_n)} = \{[x_n]_{n\in\mathbb{N}} \,|\, d_\omega([x_n]_{n\in\mathbb{N}},[y_n]_{n\in\mathbb{N}}) < \infty\}.$$

Then $(X_\infty^{(p_n)}, d_\omega)$ is a metric space and will be called the pointed ultralimit of the sequence of pointed metric spaces (X_n, d_n, p_n).

Lemma 2.8. *Any ultralimit space is complete.*

Proof. Let $([x_n^k]_{n\in\mathbb{N}})_{k\in\mathbb{N}}$ be a Cauchy sequence. We may replace the sequence by a subsequence such that $d_\omega([x_n^k],[x_n^{k-1}]) < \frac{1}{2^k}$. Thus

$$S_k = \{n \in \mathbb{N} \,|\, d_n(x_n^l, x_n^{l-1}) < \frac{1}{2^l}, l \le k\} \in \omega.$$

Note if $n \in S_\infty = \cap S_k$ then $d_n(x_n^l, x_n^{l-1}) = 0$. Thus if $S_\infty \in \omega$ then $d_\omega([x_n^k]_{n\in\mathbb{N}},[x_n^l]_{n\in\mathbb{N}}) = 0$ so that there is nothing to prove.

In the other case we $\tilde{S}_k = S_k \backslash S_\infty \in \omega$ so that $\cap_k \tilde{S}_k = \varnothing$. Set $k(-1) = 0$ and $\tilde{S}_0 = \mathbb{N}$. Let $k(n)$ be the largest k such that $n \in \tilde{S}_{k(n)}$. Since $\tilde{S}_k \backslash \tilde{S}_{k+1} \notin \omega$ one may readily verify that $\lim_\omega k(n) = \infty$.

We claim $[x_n^{k(n)}]_{n\in\mathbb{N}}$ is the limit point of the Cauchy sequence: Pick k_0 and observe that for $k_0 \le k \le k(n)$ we have $n \in S_{k(n)}$ so that

$$d_n(x_n^k, x_n^{k(n)}) < \sum_{l=k_0+1}^{k(n)} d(x_n^l, x_n^{l-1}) < 2^{-k_0}.$$

Thus for $k \ge k_0$ it holds

$$\{n \in \mathbb{N} \,|\, d_n(x_n^k, x_n^{k(n)}) < 2^{-k_0}\} \supset \{n \in \mathbb{N} \,|\, k(n) \ge k \ge k_0\} \cap S_k \in \omega$$

which implies $\lim_\omega d_n(x_n^k, x_n^{k(n)}) < 2^{-k_0}$. This proves the claim. □

Lemma 2.9. *If (X_n, d_n) are length spaces then any points of finite distance in the ultralimit space can be connected by a geodesic. In particular, a pointed ultralimit of length spaces is geodesic.*

Proof. It suffices to look at the sequence of $\frac{1}{n}$-approximate midpoints m_n of x_n and y_n

$$\begin{aligned}\lim_\omega d_n(x_n, y_n) &= \lim_\omega d_n(x_n, m_n) + d_n(m_n, y_n) + \frac{2}{n}\\ &= 2\lim_\omega d(x_n, m_n) = 2\lim_\omega d(m_n, y_n)\end{aligned}$$

implying that $[m_n]$ is a midpoint of $[x_n]$ and $[y_n]$ whenever $d_\omega([x_n],[y_n]) < \infty$. By completeness we obtain the claim. □

Proposition 2.10. *Let (X,d) be a metric space. If $(X_\infty, d_\infty, (p)_{n\in\mathbb{N}}) = \lim_\omega(X,d,p)_{n\in\mathbb{N}}$ is proper then (X_∞, d_ω) is isometric to the completion$(\bar{X}, d)$ of (X,d).*

Remark. The proof shows that whenever the completion of a metric space is compact then it equals the ultralimit of the constant sequence of this space.

Proof. The map $x \mapsto [x]_{n\in\mathbb{N}}$ defines a natural isometric embedding i of (X,d) into (X_∞, d_ω). Thus if $(x_k)_{k\in\mathbb{N}}$ is a bounded sequence in (X,d) then there is a subsequence $([x_{k_l}]_{n\in\mathbb{N}})_{l\in\mathbb{N}}$ which is Cauchy in (X_∞, d_ω). But then already $(x_{k_l})_{l\in\mathbb{N}}$ is Cauchy in X which means it converges to a point $x \in \bar{X}$. This proves $\bar{X}$ is proper. This shows that the embedding extends naturally[1] to i.

Now choose $[x_n]_{n\in\mathbb{N}} \in X_\infty$ and let x_ω be the ultralimit of $(x_n)_{n\in\mathbb{N}}$ in $\bar{X}$. Then

$$d_\omega([x_\omega]_{n\in\mathbb{N}}, [x_n]_{n\in\mathbb{N}}) = \lim_\omega d(x_\omega, x_n) = 0$$

proving that $x \mapsto [x]_{n\in\mathbb{N}}$ is also onto, i.e. the embedding defines an isometry of $\bar{X}$ onto X_∞. □

Lemma 2.11. *There is an uncountable set $I \subset \mathbb{N}^\mathbb{N}$ such that*

$$\{n \in \mathbb{N} \,|\, a_n \neq b_n\} \in \omega$$

for all $(a_n), (b_n) \in I$.

Proof. For each $r > 0$ let $a_n^r = \lfloor n^r \rfloor$. Then $\lim_{n\to\infty} a_n^r - a_n^s > 0$ for all $r > s$. Thus $I = \{(a_n^r)_{n\in\mathbb{N}}\}_{r>0}$ defines the required set. □

Proposition 2.12. *The ultralimit (X_∞, d_ω) of the constant sequence (X,d) is not separable if and only if (X,d) is contains a bounded set that is not totally bounded. In particular, (X_∞, d_ω) is separable if and only if it is proper if and only if the completion of (X,d) is proper.*

Proof. If A is a bounded set that is not totally bounded then there is a countable infinite subset $\{x_n\}_{n\in\mathbb{N}}$ with $\inf_{n\neq m} d(x_n, x_m) \geq \epsilon$ for some $\epsilon > 0$.

Let be as in the previous lemma and define for $(a_k) \in I$ a sequence $y_n^{(a_k)} = x_{a_n}$. Then

$$\{n \in \mathbb{N} \,|\, d(y_n^{(a_k)}, y_n^{(b_k)}) \geq \epsilon\} \supset \{n \in \mathbb{N} \,|\, a_n \neq b_n\} \in \omega$$

we see that $\lim_\omega d(y_n^{(a_k)}, y_n^{(b_k)}) \geq \epsilon$. Since I is uncountable, we see no countable set can approximate the uncountable set $\{[y_n^{(a_k)}]_{n\in\mathbb{N}}\}_{(a_k)\in I}$. In particular, X_∞ cannot be separable.

On the other hand if each bounded set in X is totally bounded then the completion of X must be proper. But the previous proposition shows that $(X_\infty, d_\omega) = (\bar{X}, d)$ is proper and thus separable. □

The following corollary is left as an exercise to the reader.

Corollary 2.13. *If (X_n, d_n) are complete length spaces that are not geodesic then the ultralimit cannot be uniquely geodesic. Even worse (X_∞, d_ω) contains two points $(x_n), (y_n)$ admitting uncountably many midpoints.*

[1] The embedding extend without requiring $\bar{X}$ to be proper.

3 Spaces with curvature bounds

3.1 Non-positive and non-negative curvature bounds

Definition 3.1. A geodesic space (M, d) is said to be locally *non-negatively curved* if any point $x \in M$ admits a neighrbohood U_x such that for all geodesics $\gamma, \eta : [0,1] \to U$ with $\gamma_0 = \eta_0$ it holds

$$d(\gamma_t, \eta_t) \geq t d(\gamma_1, \eta_1).$$

Similarly, it is said locally non-positively curved if any any point $x \in M$ admits a neighrbohood U_x such that for all geodesics $\gamma, \eta : [0,1] \to U$ with $\gamma_0 = \eta_0$ it holds

$$d(\gamma_t, \eta_t) \leq t d(\gamma_1, \eta_1).$$

Lemma 3.2. *A locally non-negatively curved geodesic space is non-branching.*

Proof. Let γ and η be two points with $\gamma\big|_{[0,t]} \equiv \eta\big|_{[0,t]}$ for some $t \in [0,1]$. Assume t is maximal in $[0,1]$. If $t = 1$ then $\gamma \equiv \eta$. Assume $t \in (0,1)$. Choose $U = U_{\gamma_t}$ and observe there is an ϵ such that γ and η restricted to $[t-\epsilon, t+\epsilon]$ has image in U. Let $\tilde{\gamma}$ and $\tilde{\eta}$ the $[0,1]$-reparametrized geodesics $\gamma\big|_{[t-\epsilon,t+\epsilon]}$ and $\eta\big|_{[t-\epsilon,t+\epsilon]}$. However, we arrive at the following contradiction

$$0 = d(\tilde{\gamma}_{\frac{1}{2}}, \tilde{\eta}_{\frac{1}{2}}) \geq \frac{1}{2} d(\tilde{\gamma}_1, \tilde{\eta}_1) > 0.$$

□

Lemma 3.3. *A locally non-positively curved geodesic spaces has positive injectivity radius and sufficiently small balls are convex.*

3.2 Curvature on manifolds

Fact 3.4 (Blackbox Theorem - Rauch I)**.** *Assume (M, g) is a geodesically complete Riemannian manifold with sectional curvature bounded above by k (bounded below by k). Then for all $x \in M$ there is a neighborhood U such that for all geodesics $\gamma^s : [0, a] \to U$ with $\gamma_0^s = \gamma^0$ for all $s \in (-\epsilon, \epsilon)$ that vary smoothly in s, i.e. $s \mapsto \gamma_t^s$ is smooth for all $s \in (-\epsilon, \epsilon)$ the function*

$$t \mapsto \frac{b_t}{b_t^k}$$

is non-decreasing (non-increasing) where

$$b_t = \frac{d}{ds}\bigg|_{s=0} d(\gamma_t^s, \gamma_t^0)$$

and b_t^k is obtained by a geodesic variation starting at a fixed point in the 2-dimensional model space of constant curvature k such that

$$\frac{b_t}{b_t^k} \to 1 \text{ as } t \to 0.$$

Observe if $k = 0$ then $b_t^k = ct$ for some t depending only on b_t. Thus for $t < t'$

$$\frac{b_t}{ct} \geq \frac{b_{t'}}{ct'}$$

and in particular

$$b_{t'} \leq \frac{t'}{t} b_t$$

Theorem 3.5. *If (M, g) has non-negative sectional curvature then it is locally non-negatively curved.*

Proof. Choose a ball $U = B_r(x)$ such that geodesic starting at x can be extended to a (minimizing) geodesic in $B_{2r}(x)$. Let $\gamma, \eta : [0,1] \to U$ be two geodesics starting at x. Let ζ^t be geodesics connecting γ_t and η_t.

Fix $t \in (0,1)$ and $\epsilon \in (0, t-1)$ and define a family of geodesics $\xi^s : [0,1] \to M$ such that $\xi_0^s = x$ and $\xi_t^s = \zeta_s^t$. Observe that this variation is smooth.

Let $s_0 \leq s_1 \leq \ldots \leq s_N = 1$ then for all $s \in (0,1)$ and $t' \in (0,1)$ it holds

$$d(\gamma_{t'}, \eta_{t'}) \leq \sum_{i=1}^{N} d(\xi_t^{s_i}, \xi_t^{s_{i-1}}).$$

Since the geodesic variation is smooth, for all $\epsilon > 0$ there is a $\delta > 0$ such that $|s_i - s_{i-1}| < \delta$ it holds[1]

$$d(\xi_{t'}^{s_i}, \xi_{t'}^{s_{i-1}}) \leq |s_i - s_{i-1}| b_{t'}^{s_i} + \epsilon\delta$$

where $b_{t'}^s = \frac{d}{ds'}\big|_{s'=s} d(\xi_{t'}^{s'}, \xi_{t'}^s)$. Observe that $b_t^s = d(\gamma_t, \eta_t)$.

Then by the remark after Rauch I and for $t' \in (t, 1]$ we get

$$d(\xi_{t'}^{s_i}, \xi_{t'}^{s_{i-1}}) \leq |s_i - s_{i-1}| b_{t'}^{s_i} + \epsilon \leq \frac{t'}{t} |s_i - s_{i-1}| b_t^{s_i} + \epsilon\delta.$$

[1]Note if $(f_s)_{s\in I}$ are uniformly differentiable at 0 then $f_s(t) = f_s(t) + tf_s^{'}(s) + R_s(t)$ with $\sup_{s\in I} \lim_{t\to 0} R_s(t) = 0$.

Assume now $|s_i - s_{i-1}| = \frac{1}{N}$ we get

$$\begin{aligned} d(\gamma_{t'}, \eta_{t'}) &\le \sum_{i=1}^{N} d(\xi_t^{s_i}, \xi_t^{s_{i-1}}) \\ &\le \sum_{i=1}^{N} (\frac{1}{N} b_{t'}^{s_i} + \frac{1}{N}\epsilon) \\ &\le \frac{t'}{t}\left(\sum_{i=1}^{N} \frac{1}{N} b_t^{s_i}\right) + \epsilon. \\ &= \frac{t'}{t} d(\gamma_t, \eta_t) + \epsilon \end{aligned}$$

Letting $\epsilon \to 0$ gives the claim for $t < t' \le 1$. □

Choosing in the proof $t' \in (0, t)$ then Rauch I gives also the following:

Theorem 3.6. *If (M, g) has non-positive sectional curvature then it is locally non-positively curved.*

Before we continue we need the following fact about Riemannian manifolds.

Lemma 3.7. *If (M, g) is a Riemannian manifold such that $x \mapsto g_x$ is continuous then $\lim_\omega(M, \frac{1}{t_k} d_g, m) = (\mathbb{R}^n, \| \cdot - \cdot \|_{g_m})$ for all $t_k \to 0$.*

Proof. Let $\varphi : U \to V \subset \mathbb{R}^n$ be a chart at m. W.l.o.g. assume $\varphi(m) = 0$. We will identify $x \mapsto g_x$ with a locally defined map $x \mapsto g_x^U$ defined on $\mathbb{R}^n$. We also assume d is defined on $V \subset \mathbb{R}^n$. Note that the set of scalar product on $\mathbb{R}^n$ can be metrized by the following

$$d_{scal}(g, \tilde{g}) = \inf\{\log L \,|\, \forall v \in \mathbb{R}^n : L^{-1} g(v, v) \le \tilde{g}(v, v) \le L g(v, v)\}.$$

Since $x \mapsto g_x$ is continuous we see that for all $\epsilon > 0$ there is an δ such that such that for all $x \in B_\delta(m)$

$$(1 + \epsilon)^{-1} g_x \le g_m \le (1 + \epsilon) g_x.$$

In particular, for all curves $\gamma : [0, 1] \to B_\delta(m)$ it holds

$$(1 + \epsilon)^{-1} \ell_g(\gamma) \le \ell_{g_m}(\gamma) \le (1 + \epsilon)\ell_g(\gamma).$$

Choosing δ even small this shows

$$(1 + \epsilon)^{-1} d(x, y) \le \|x - y\|_{g_m} \le (1 + \epsilon) d(x, y)$$

for all $x, y \in B_\delta(x)$.

We need to show

$$(B_{t_k R}(m), \frac{1}{t_k} d, m) \to (B_R^{\mathbb{R}^n}(m), \| \cdot - \cdot \|_{g_m}, 0).$$

To see this observe that define on $V_k = \frac{1}{t_k} B_{t_k R}(m) \subset \mathbb{R}^n$ a metric $d_k(v,w) = \frac{1}{t_k} d(t_k v, t_k w)$ and note this gives a natural isometry $\psi_k : (V_k, d_k) \to (B_{t_k R}(m), d_k)$ and it holds

$$(1+\epsilon_k)^{-1}\|v-w\|_{g_m} \le d_k(t_k v, t_k w) \le (1+\epsilon_k)\|v-w\|_{g_m}$$

for some $\epsilon \to 0$.

Let $B_r^{g_m}(0)$ denote the ball of radius r around 0 w.r.t. $\|\cdot - \cdot\|_{g_m}$ and observe that

$$B^{g_m}_{(1+\epsilon_k)^{-1}R}(0) \subset V_k \subset B^{g_m}_{(1+\epsilon_k)R}(0).$$

We also observe that for $k \to \infty$ we may assume $\epsilon \to 0$. This implies for $v_k, w_k \in V_k \subset B^{g_m}_{(1+\epsilon)R}(0)$.

$$\lim_\omega d_k((v_k),(w_k)) - \|v_k - w_k\| = 0.$$

Thus $d_\omega((v_k),(w_k)) = \|v-w\|$ where v and w are the ultralimits of (v_k) and (w_k). Here we used properness of $(\mathbb{R}^n, \|\cdot - \cdot\|_{g_m})$. Note we may choose $v_k^{'}, w_k^{'} \in B^{g_m}_{(1+\epsilon_k)^{-1}R}(0)$ such that $d_k(v_k, v_k^{'}), d_k(w_k, w_k^{'}) \le (1-(1+\epsilon_k)^{-1})R$. Then $(v_k) = (v_k^{'})$ and $(w_k) = (w_k^{'})$ showing that (V_k, d_k) converges to $(B_R^{g_m}, \|\cdot - \cdot\|_{g_m})$. □

Definition 3.8. We say a geodesic space (M,d) has curvature locally bounded below by 0, abbreviated $(\mathsf{CBB})_{loc}(0)$, if all $x_0 \in M$ admits a neighborhood U such that for all $x,y,z \in U$

$$d^2(x,m) \ge \frac{1}{2}d^2(x,y) + \frac{1}{2}d^2(x,z) - \frac{1}{4}d^2(y,z)$$

for any midpoint m of y and y. Similarly, it has local curvature curvature locally bounded below by 0, abbreviated $(\mathsf{CBA})_{loc}(0)$,

$$d^2(x,m) \le \frac{1}{2}d^2(x,y) + \frac{1}{2}d^2(x,z) - \frac{1}{4}d^2(y,z).$$

Remark. One may easily verify that $(\mathsf{CBB})_{loc}$ implies

$$d^2(x,\gamma_t) \ge (1-t)d^2(x,\gamma_0) + td^2(x,\gamma_1) - (1-t)td^2(\gamma_0,\gamma_1)$$

for any $x \in U$ and any geodesic $\gamma : [0,1] \to U$.

Lemma 3.9. *The condition $(\mathsf{CBB})_{loc}$ implies local non-negative curvature. Similarly, $(\mathsf{CBA})_{loc}$ implies local non-negative curvature.*

Proof. Let $\gamma, \eta : [0,1] \to U$ be two geodesics where U is given by the definition of $(\mathsf{CBB})_{loc}$. Then

$$\begin{aligned} d^2(\gamma_1,\eta_t) &\ge (1-t)d^2(\gamma_1,\eta_0) + td^2(\gamma_1,\eta_1) - (1-t)td^2(\eta_0,\eta_1) \\ &= td^2(\gamma_1,\eta_1) + (1-t)\left(d^2(\gamma_0,\gamma_1) - td^2(\eta_0,\eta_1)\right) \end{aligned}$$

so that

$$\begin{aligned} d^2(\eta_t,\gamma_t) &\ge (1-t)d^2(\eta_t,\gamma_0) + td^2(\eta_t,\gamma_1) - (1-t)td^2(\gamma_0,\gamma_1) \\ &= td^2(\eta_t,\gamma_1) + (1-t)\left(td^2(\eta_0,\eta_1) - td^2(\gamma_0,\gamma_1)\right) \\ &\ge t^2 d^2(\gamma_1,\eta_1). \end{aligned}$$

Replacing $\ge$ by $\le$ gives the second claim. □

Proposition 3.10. *A Riemannian manifold of non-negative sectional curvature satisfies* $(\mathsf{CBB})_{loc}$. *Similarly, a Riemannian manifold of non-positive sectional curvature satisfies* $(\mathsf{CBA})_{loc}$.

Proof. The induced geodesic space has non-negative curvature. Choose $x, y, z \in B_\epsilon(x_0) \subset B_{6\epsilon}(x_0) \subset U$ where U is given by the definition of non-negative curvature. Choose a geodesic ζ connecting y and z and set $m = \zeta_{\frac{1}{2}}$. Define $y_t = \zeta_{\frac{1}{2}-\frac{1}{2}t}$ and $z_t = \zeta_{\frac{1}{2}(2-t)}$. Let γ be a geodesic connecting m and x and and set $x_t = \gamma_t$. Also set

$$
\begin{aligned}
a_t &= \frac{1}{t} d(x_t, y_t) \\
b_t &= \frac{1}{t} d(x_t, y_t) \\
c_t &= \frac{1}{t} d(y_t, z_t) \\
d_t &= \frac{1}{t} d(x_t, m).
\end{aligned}
$$

Then $a_t \geq a_1$, $b_t \geq b_1$ and $c_t = c_1$ and $d_t = d_1$ for all $t \in (0, 1]$.

For $t_n = \frac{1}{n}$, let e_∞ be the ultralimits of $(e_{t_n})_{n \in \mathbb{N}}$ for $e \in \{a, b, c, d\}$. Note all those numbers are finite. Since $(M, nd)_{n \in \mathbb{N}}$ converges to a Euclidean space we have

$$
\begin{aligned}
0 &\geq \frac{1}{2} a_\infty^2 + \frac{1}{2} b_\infty^2 - d_\infty^2 - \frac{1}{4} c_\infty^2 \\
&= \lim_\omega \frac{1}{2} a_{t_n}^2 + \frac{1}{2} b_{t_n}^2 - d_{t_n}^2 - \frac{1}{4} c_{t_n}^2 \\
&\geq \frac{1}{2} a_1^2 + \frac{1}{2} b_1^2 - d_1^2 - \frac{1}{4} c_1^2
\end{aligned}
$$

which proves the first claim. The second claim follows again by replacing $\geq$ by $\leq$. $\square$

Corollary 3.11. *If* (M, d) *has non-negative (non-positive) curvature and for each point* m *in the interior of a geodesic either admits a neighborhood satisfying* (CBB) *(resp.* (CBA)*) or any blow-up* $\lim_\omega(M, nd, m)$ *satisfies* (CBB) *(resp.* (CBA)*) then* (M, d) *satisfies* (CBB) *(resp.* (CBA)*).*

Remark. Let (M, F) is a Finsler manifold having (local) non-negative curvature. Then a minor adaptation shows that if F is 2-uniformly smooth with constant $C \geq 1$, i.e.

$$
F\left(\frac{v+w}{2}\right)^2 \geq \frac{1}{2} F(v)^2 + \frac{1}{2} F(w)^2 - \frac{C}{4} F(v-w)^2
$$

then a similar inequality holds for the distance d_F. The corresponding non-positive version is called 2-uniformly convex with constant $D \leq 1$.

Note, however, by a result of Ivanov–Lytchak if (M, F) is smooth then (M, d_F) is Berwald, i.e. there is a Riemannian structure g such that $d_g(\gamma_t, \gamma_s) = |t - s| d_g(\gamma_0, \gamma_1)$ for all geodesics γ of (M, d_F).

3.3 Non-positive curvature

Proposition 3.12. *Assume (M, d) has local non-positive curvature. Then for all $x \in M$ there is an $\epsilon_x > 0$ such that $(B_{\epsilon_x}(x), d)$ is a geodesic space with non-positive curvature.*

Proof. Let U be the neighborhood given by the definition of non-positive curvature. Choose ϵ such that $B_\epsilon(x) \subset B_{3\epsilon}(x) \subset U$. Let γ and η be two geodesics in $B_\epsilon(x)$ and find a geodesic ζ connecting γ_0 and η_1. Then η is a geodesic in $B_{3\epsilon}(x)$. Thus

$$\begin{aligned} d(\gamma_t, \eta_t) &\le d(\gamma_t, \zeta_t) + d(\zeta_t, \eta_t) \\ &\le td(\gamma_1, \zeta_1) + (1-t)d(\zeta_0, \eta_0) \\ &= (1-t)d(\gamma_0, \eta_0) + td(\gamma_1, \eta_1). \end{aligned}$$

Choosing η be the constant geodesic we see that $t \mapsto \gamma_t$ stays in $B_\epsilon(x)$ whenever$\gamma_0, \gamma_1 \in B_\epsilon(x)$. This proves the claim. □

Remark. Note that $x \mapsto \epsilon_x$ is continuous in x.

Proposition 3.13. *Assume (M, d) is proper, has local non-positive curvature and its injectivity radius $i_0 = i_0(M) := \inf_{x \in M} i_M(x)$ is bounded from below. Then $B_{\frac{i_0}{2}}(x)$ is a geodesically convex set. One may replace properness by the assumption of continuously varying geodesics in $B_{\frac{i_0}{2}}(x)$.*

Proof. By properness we see that geodesics in $B = B_{\frac{i_0}{2}}(x)$ are unique and vary continuously.

Let γ and η be two geodesics in B. Let $\epsilon > 0$ be a lower bound of $t \mapsto \epsilon_{\gamma_t}$. Assume first that $\sup d(\gamma_t, \eta_t) < \epsilon$. Then there is a $\delta > 0$ such that for all $t \in [\delta, 1-\delta]$ the functions

$$t' \mapsto d(\gamma_{t'}, \eta_{t'})$$

restricted to $[t-\delta, t+\delta]$ stay in $B_\epsilon(\gamma_t)$. Hence they are convex in $[t-\delta, t+\delta]$. But then the function is already convex in $[0, 1]$.

To finish, let ζ and ξ be geodesics connecting γ_0 and η_0 and resp. γ_1 and η_1. Define geodesics γ^s connecting ζ_s and ξ_s. Then γ^s is a unique continuous variation between $\gamma = \gamma^0$ and $\eta = \gamma^1$. Let $0 = s_0 \le s_1 \le \ldots \le s_N = 1$ with $s_n - s_{n-1} < \epsilon$. Then $t \mapsto d(\gamma_t^{s_n}, \gamma_t^{s_{n-1}})$ is convex so that

$$\begin{aligned} d(\gamma_t, \eta_t) &\le \sum_{n=1}^N d(\gamma_t^{s_n}, \gamma_t^{s_{n-1}}) \\ &\le (1-t) \sum_{n=1}^N d(\gamma_0^{s_n}, \gamma_0^{s_{n-1}}) + t \sum_{n=1}^N d(\gamma_1^{s_n}, \gamma_1^{s_{n-1}}) \\ &= (1-t)d(\gamma_0, \eta_0) + td(\gamma_1, \eta_1). \end{aligned}$$

□

Corollary 3.14. *If (M,d) satisfies $(\mathsf{CBA})_{loc}$ and $i_0 = i_0(M) > 0$ then $(B_{\frac{i_0}{2}}(x), d)$ satisfies (CBA).*

Proposition 3.15. *Assume $(M_n, d_n, p_n)_{n\in\mathbb{N}}$ are proper and satisfy $(\mathsf{CBA})_{loc}$ and $i_0 = \lim_\omega i_0(M_n) > 0$. Then the pointed ultralimit $(M_\infty, d_\omega, (p_n))$ satisfies $(\mathsf{CBA})_{loc}$ and $i_0(M_\infty) > i_0$.*

Proof. Let (q_n) be arbitrary. Choose $(x_n), (y_n), (z_n) \in$Let (m_n) be a sequence of midpoints $y_n, z_n \in B_{\frac{i_0}{2}}(q_n)$. Then $(m_n) \in B_{\frac{i_0}{2}}(q_n)$ and it is a midpoint of (y_n) and (z_n).

Assume by contradiction there is another midpoint $(\tilde{m}_n)$ of (y_n) and (z_n). If k_n denotes midpoints of m_n and $\tilde{m}_n$. From the (CBA)-inequality and the fact that $d_\omega((m_n),(\tilde{m}_n)) > 0$ we get

$$\begin{aligned} d_\omega((y_n),(z_n)) \le d_\omega((y_n),(k_n)) + d((z_n),(k_n)) &< d_\omega((y_n),(m_n)) + d_\omega((m_n),(z_n)) \\ &= d_\omega((y_n),(z_n)) \end{aligned}$$

which is a contradiction.

The argument shows that any geodesic/midpoint in M_∞ is given as a limit of geodesic/midpoint. In particular, the the (CBA)-inequality will hold in $i_0(M_\infty)$. ☐

Corollary 3.16. *If, in addition, (M_n, d_n) is geodesically complete then so is the ultralimit.*

Remark. Non-branching is not preserved by taking ultralimits: blowing down the hyperbolic plane one obtains a metric tree with uncountably many edges issuing from a fixed vertex.

Proposition 3.17. *Assume (M,d) is non-positively curved and $s \mapsto d(\gamma_s, \eta_s)$ is an affine function for geodesics $\gamma, \eta : I \to M$ and some closed connected set $I \subset \mathbb{R}$. Let $\zeta^s : [0,1] \to M$ be geodesics connecting γ_s and η_s. Then there is a normed space $(\mathbb{R}^2, \|\cdot\|)$ with strictly convex norm and an isometric embedding $\varphi : \zeta^s_t \mapsto \varphi(\zeta^s_t)$ into $\mathbb{R}^2$, i.e. the convex hull of γ and η will be isometric to a convex subset of $\mathbb{R}^2$.*

Proof. We first claim that for $s, \tilde{s} \in I$ the curve $t \mapsto \zeta_t^{(1-t)s+t\tilde{s}}$ is a geodesic connecting γ_s and $\eta_{\tilde{s}}$: Let ξ be the geodesic connecting γ_s and $\eta_{\tilde{s}}$. Then

$$\begin{aligned} d(\gamma_{(1-t)s+t\tilde{s}}, \eta_{(1-t)s+t\tilde{s}}) &\le d(\gamma_{(1-t)s+t\tilde{s}}, \xi_t) + d(\xi_t, \eta_{(1-t)s+t\tilde{s}}) \\ &\le t d(\gamma_{\tilde{s}}, \eta_{\tilde{s}}) + (1-t) d(\gamma_s, \eta_s) \\ &= d(\gamma_{(1-t)s+t\tilde{s}}, \eta_{(1-t)s+t\tilde{s}}) \end{aligned}$$

implying that ξ_t is a t-midpoint of $\gamma_{(1-t)s+t\tilde{s}}$ and $\eta_{(1-t)s+t\tilde{s}}$. By uniqueness of geodesic we must have $\xi_t = \zeta_t^{(1-t)s+t\tilde{s}}$.

By a similar argument we see that $r \mapsto \zeta^{(1-r)s+r\tilde{s}}_{(1-r)t+r\tilde{t}}$ is a geodesic between ζ^s_t and $\zeta^{\tilde{s}}_{\tilde{t}}$ for all $s, \tilde{s} \in I$ and $t, \tilde{t} \in [0,1]$.

This implies that the convex hull of γ and η is homeomorphic to a convex subset of $\mathbb{R}^2$.

TO BE CONTINUED! ☐

3.4 Non-negative curvature

Let $\gamma : [0,\infty) \to M$ be a geodesic ray. Note for $t, s \geq 0$ it holds

$$\begin{aligned}(t+s) - d(x,\gamma_{t+s}) &\geq (t+s) - (d(x,\gamma_t) + d(\gamma_t,\gamma_{t+s})) \\ &= t - d(x,\gamma_t).\end{aligned}$$

Hence $t \mapsto t - d(x,\gamma_t)$ is monotone non-decreasing so that the following function

$$b_\gamma(x) = \lim_{t\to\infty} t - d(x,\gamma_t)$$

is well-defined. We call b_γ the *Busemann function* associated to the ray γ.

Given a point $x \in M$ we say $\gamma^{(x)}$ is a *co-ray* associated to γ if for some $t_n \to \infty$ the sequence of unit speed geodesics $\gamma^{(n)}$ connecting x and γ_{t_n} converges locally to $\gamma^{(x)}$. A geodesic line $\gamma^{(x)}$ is called a co-line associated to a geodesic line γ if for all $s \in \mathbb{R}$ the rays $t \mapsto \gamma^{(x)}_{\pm t+s}$ are co-rays associated to $t \mapsto \gamma_{\pm t}$.

Lemma 3.18. *If $\gamma^{(x)}$ is a co-ray associated to γ then $b_\gamma(\gamma^{(x)}_t) = b_\gamma(x) + t$ for $t \geq 0$.*

Proof. Let $\gamma^{(n)}$ be given by the definition of $\gamma^{(x)}$ being a co-ray.

Since b_γ is a pointwise limit of 1-Lipschitz functions $x \mapsto (t - d(x,\gamma_t))$ it must be 1-Lipschitz itself, i.e.

$$b_\gamma(\gamma^{(x)}_{\pm t}) - b_\gamma(x) \leq t$$

Note also that for $t \geq 0$

$$\begin{aligned}b_\gamma(\gamma^{(x)}_t) &= \lim_{n\to\infty} t_n - d(\gamma^{(x)}_t, \gamma_{t_n}) \\ &\geq \lim_{n\to\infty} t_n - d(\gamma^{(n)}_t, \gamma_{t_n}) - d(\gamma^{(x)}_t, \gamma^{(n)}_t) \\ &\geq \lim_{n\to\infty} t_n - d(x, \gamma_{t_n}) - d(\gamma^{(x)}_t, \gamma^{(n)}_t) + t \\ &= b_\gamma(x) + t.\end{aligned}$$

This proves $b_\gamma(\gamma^{(x)}_t) = b_\gamma(x) + t$. □

Corollary 3.19. *Assume (M,d) is non-branching and $\gamma^{(x)}$ a co-ray associated to γ. Then for all $s > 0$ the ray $t \mapsto \gamma^{(x)}_{t+s}$ is the unique co-ray associated to γ which issues from $y = \gamma^{(x)}_s$.*

Proof. Assume $\gamma^{(y)}$ is a co-ray associated to γ which issues from y. Then

$$\begin{aligned}b_\gamma(\gamma^{(y)}_t) - b_\gamma(x) &\leq d(\gamma^{(y)}_t, x) \\ &\leq d(\gamma^{(y)}_t, y) + d(y,x) \\ &= t + s \\ &= b_\gamma(\gamma^{(y)}_t) - b_\gamma(y) + b_\gamma(y) - b_\gamma(x)\end{aligned}$$

proving the following $\tilde{\gamma} : [0, \infty) \to M$ is a geodesic ray

$$\tilde{\gamma}_t = \begin{cases} \gamma_t^{(x)} & t < s \\ \gamma_s^{(y)} & t \geq s. \end{cases}$$

which agrees with the ray $\gamma^{(x)}$. But non-branching implies $\gamma^{(x)} \equiv \tilde{\gamma}$ proving the claim of the corollary. $\square$

Proposition 3.20. *Assume (M, d) is non-negatively curved and proper. If γ is a geodesic line then for all $x \in M$ there is a unique co-line $\gamma^{(x)}$ associated to γ and issuing from x. Furthermore, it holds*

$$t \mapsto d(\gamma_{t+s}^{(x)}, \gamma_{t+r}^{(y)})$$

is constant for all $s, s' \in \mathbb{R}$. In particular, (M, d) admits a 1-parameter group $t \mapsto \varphi_t$ of isometries such that

$$d(x, \varphi_t(x)) = t.$$

Proof. Fix $x \in M$ and let $\gamma^{(n)} : [0, d(x, \gamma_{t_n})] \to M$, $n \in \mathbb{Z}$ be unit speed geodesics connecting x and γ_{-t_n}. Via a diagonal argument using Arzela–Ascoli it is possible to prove that there are rays $\gamma^{(x,\pm)}$ issuing from x in the direction γ_{t_n}, $n \to \pm\infty$. Via ultralimits this construction is particularly simple: Fix a non-principle ultrafilter ω on $\mathbb{N}$ and define

$$\gamma_t^{(x)} = \begin{cases} \lim_\omega \gamma_t^{(n)} & t \geq 0 \\ \lim_\omega \gamma_{-t}^{(-n)} & t \leq 0. \end{cases}$$

It is easy to verify that $\gamma^{(x)}$ restricted to $[0, \infty)$ and to $(-\infty, 0]$ are co-ray associated to γ issuing from x.

In order to prove that $\gamma^{(x)}$ is a geodesic line it suffices to prove $d(\gamma_s^{(x)}, \gamma_{-s}^{(x)}) = 2s$ whenever $s > 0$. Let $a_n = d(x, \gamma_n)$ and $b_n = d(x, \gamma_n)$. Using the triangle inequality we have

$$\{a_n, b_n\} \leq n + d(x, \gamma_0) \leq \{a_n, b_n\} + 2d(x, \gamma_0).$$

Thus each of the following sequence will converge to 1: $\frac{a_n}{b_n}$, $\frac{a_n}{t_n}$ and $\frac{b_n}{t_n}$.

Choosing n sufficiently large we may assume $\frac{s}{a_n} < 1$. Now define

$$\begin{aligned} x_n &= \gamma_{a_n \frac{s}{a_n}}^{(n)} = \gamma_s^{(n)} \\ y_n &= \gamma_{b_n \frac{s}{a_n}}^{(-n)}. \end{aligned}$$

Since $r \mapsto \gamma_{a_n r}^{(n)}$ and $r \mapsto \gamma_{b_n r}^{(-n)}$ are $[0, 1]$-parametrized geodesics so that the definition of non-negative curvature with $r = \frac{s}{a_n}$

$$d(x_n, y_n) \geq r d(\gamma_n, \gamma_{-n}) = \frac{2t_n}{a_n} \cdot s.$$

Note that

$$d(\gamma_s^{(-n)}, y_n) = s|1 - \frac{b_n}{a_n}| \to 0$$

so that

$$\gamma_s^{(x)} = \lim_\omega y_n.$$

But then

$$\begin{aligned} d(\gamma_s^{(x)}, \gamma_{-s}^{(x)}) &= \lim_\omega d(x_n, y_n). \\ &\geq 2 = d(\gamma_s^{(x)}, x) + d(x, \gamma_{-s}^{(x)}) \\ &\geq d(\gamma_s^{(x)}, \gamma_{-s}^{(x)}) \end{aligned}$$

proving that $\gamma^{(x)}$ is a geodesic line.

We claim that the above construction neither depends on t_n nor on the ultrafiliter ω: Indeed, if $\eta^{(x)}$ is obtain by a similar procedure as above but with either different t_n or different ultrafilter then for $s > 0$ the rays $t \mapsto \eta^{(x)}_{\pm(t+s)}$ are co-rays associated to $t \mapsto \gamma_{\pm t}$. By Corollary 3.19 we must have $\gamma^{(x)}\big|_{[s,\infty)} \equiv \eta^{(x)}\big|_{[s,\infty)}$ and resp. $\gamma^{(x)}\big|_{(-\infty,-s]} \equiv \eta^{(x)}\big|_{(-\infty,-s]}$ proving that the geodesic lines are unique co-lines associated to γ.

Now let $\gamma^{(n)}$ and $\eta^{(n)}$ be geodesic connecting x and γ_{t_n} and resp. y and γ_{t_n}. Let $a_n^\pm = d(x, \gamma_{t_n})$ and $b_n^\pm = d(y, \gamma_{t_n})$.

$$\begin{aligned} d(\gamma_{\pm s}^{(x)}, \gamma_{\pm s}^{(y)}) &= \lim_\omega d(\gamma_{\pm s}^{(n)}, \gamma_{s\frac{a_n}{b_n}}^{(n)}) \\ &\geq \lim_\omega (1 - \frac{s}{a_n}) d(x,y) = d(x,y). \end{aligned}$$

Because the line $t \mapsto \gamma^{(x)}_{\pm s+t}$ is the unique co-line issuing from $\gamma_s^{(x)}$ we also get

$$d(x,y) = d(\gamma_{\mp s}^{(\gamma_{\pm s}^{(x)})}, \gamma_{\mp s}^{(\gamma_{\pm s}^{(y)})}) \geq d(\gamma_{\pm s}^{(x)}, \gamma_{\pm s}^{(y)})$$

we see that the last claim of the proposition holds. □

Corollary 3.21. *For all $x, y \in M$ the lines $\gamma^{(x)}$ and $\gamma^{(y)}$ are co-line w.r.t. each other. In particular, the co-line relation is an equivalence relation.*

Proof. Use the 1-parameter group of isometries we have for $x, y \in M$ the following

$$\begin{aligned} b_\gamma^\pm(x) &= \lim_{t\to\infty} t - d(x, \gamma_{\pm t}) \\ &= \lim_{t\to\infty} t - d(\gamma_{\mp t}^{(x)}, \gamma_0) = b^\mp_{\gamma^{(x)}}(\gamma_0). \end{aligned}$$

Similarly, for all $t \in \mathbb{R}$ it holds

$$b^+_{\gamma^{(x)}}(\gamma_s) = b^-_\gamma(x) + s = s$$

if $b_\gamma^-(x) = 0$.

Let ζ be the co-line associated to $\gamma^{(x)}$ with $\zeta_0 = \gamma_0$. Then for $t \geq 0$ and $b^-(x) = 0$ it holds

$$\begin{aligned} d(\gamma_{-t}, x) + d(x, \zeta_t) = 2t &= b^+_{\gamma^{(x)}}(\gamma_{-t}) + b^+_{\gamma^{(x)}}(\zeta_t) \\ &\leq d(\gamma_{-t}, \zeta_t) \leq d(\gamma_{-t}, x) + d(x, \zeta_t) \end{aligned}$$

so that by non-branching $\zeta \equiv \gamma$. Therefore, the notion of co-line is symmetric.

Let $\zeta^{(y)}$ be the co-line associated to $\gamma^{(x)}$. Since $\zeta^{(\gamma_0)} = \gamma$ we have

$$\begin{aligned} b^\pm_\gamma(y) &= \lim_{t\to\infty} (t \pm s) - d(y, \gamma_{\pm(t\pm s)}) \\ &= \lim_{t\to\infty} t - d(\zeta^{(y)}_{-t}, \gamma_s) \pm s \\ &= b^\mp_{\zeta^{(y)}}(\gamma_s) \pm s. \end{aligned}$$

Thus $b^\pm_{\zeta^{(y)}}(\gamma_s) = b^\pm_{\zeta^{(y)}}(\gamma_0) \pm s$. This shows that γ is a co-line to $\zeta^{(y)}$. Thus by symmetry $\zeta^{(y)}$ must be also a co-line to γ. But the unique co-line to γ passing y is given by $\gamma^{(y)}$ implying $\zeta^{(y)} = \gamma^{(y)}$. □

Corollary 3.22. *If, in addition, (M, d) has Euclidean tangent spaces then it splits isometrically, i.e. there is an isometry*

$$\Phi : (M, d) \to (M' \times_2 \mathbb{R})$$

where $M' \subset M$ is a closed geodesically convex subset of M.

Proof. Let b_γ be the Busemann function associated to $t \mapsto \gamma_t$. For $x, y \in M$ choose a midpoint m and observe

$$\begin{aligned} b^\pm(m) &= \lim_{t\to\infty} t - d(m, \gamma_{\pm t}) \\ &= \lim_{t\to\infty} t - \frac{d(m, \gamma_{\pm t})^2}{t} \\ &\leq \lim_{t\to\infty} t - \frac{\frac{1}{2}d(x, \gamma_{\pm t})^2 + \frac{1}{2}d(y, \gamma_{\pm t})^2 - \frac{1}{4}d(x, y)^2}{t} \\ &= \frac{1}{2}b^+(x) + \frac{1}{2}b^+(y) \end{aligned}$$

proving that $b^\pm$ is geodesically convex. Also note

$$\begin{aligned} b^+(x) + b^-(x) &= \lim_{t\to\infty} 2t - d(x, \gamma_t) - d(x, \gamma_{-t}) \\ &\leq \lim_{t\to\infty} 2t - d(\gamma_{-t}, \gamma_t) = 0. \end{aligned}$$

Furthermore,

$$f : t \mapsto d(x, \gamma_t)^2$$

satisfies $f" \leq 2$ and thus

$$f(t) \leq t^2 + at + b$$

for some $a, b \in \mathbb{R}$. But then

$$\begin{aligned} b^+(x) + b^-(x) &= \lim_{t\to\infty} 2t - \sqrt{f(t)} - \sqrt{f(-t)} \\ &\geq \lim_{t\to\infty} 2t - 2t\sqrt{1 + \frac{a}{t} + \frac{b}{t^2}} = 0 \end{aligned}$$

proving $b^+ \equiv -b^-$. But then $b^\pm$ are affine functions, i.e. $b(\eta_t) = (1-t)b(\eta_0) + tb(\eta_1)$ for all geodesics $\eta : [0,1] \to M$. In particular, $M' := b^{-1}(0)$ is a closed geodesically convex subset of M. Note that the 1-parameter group of isometries shows that M' is isometric to $b^{-1}(r)$ for all $r \in \mathbb{R}$.

We claim that for all $x \in M$ with $b_{\gamma^{(y)}}(x) = 0$ it holds

$$d(\gamma_r^{(x)}, \gamma_s^{(y)})^2 = d(x,y)^2 + |r-s|^2.$$

Observe that

$$d(\gamma_r^{(x)}, \gamma_s^{(y)}) = d(x, \gamma_{s-r}^{(y)})$$

so that it suffices to show

$$d(x, \gamma_t^{(y)})^2 = d(x,y)^2 + t^2.$$

Set $y_t = \gamma_t^{(y)}$ and observe that

$$f_x : t \mapsto t^2 - d(x, y_t)^2$$

is a convex function. If f_x is constant then $d(x,y_t)^2 = t^2 + d(x,y)^2$ and we are done. In the other case there is an $a \neq 0$ and $b \in \mathbb{R}$ such that

$$f_x(t) \geq at + b$$

However,

$$\begin{aligned} 0 = b_{\gamma^{(y)}}(x) &= \lim_{t\to\pm\infty} \frac{t^2 - d(x,y_t)^2}{|t|} \\ &\geq \lim_{t\to\pm\infty} \frac{at}{|t|} = \pm a \end{aligned}$$

which is not possible if $a \neq 0$.

Pick a geodesic η_t connecting y and x with $b_{\gamma^{(y)}}(x) = 0$. Since b^+ is affine and $b^+_{\gamma^{(y)}}(x) = b^+_{\gamma^{(x)}}(y)$ it holds $b^+_{\gamma^{(\eta_s)}}(\eta_{s'}) = 0$ for all $s, s' \in [0,1]$. Therefore,

$$\cup_{s\in[0,1], t\in\mathbb{R}} \{\gamma_t^{(\eta_s)}\}$$

will span a two-dimensional Euclidean strip.

Let $x, y \in (b_\gamma^+)^{-1}(0) = \tilde{M}$. Then for all $t \in \mathbb{R}$ it holds

$$d(y, \gamma_t^{(y)}) \leq d(x, \gamma_t^{(y)}).$$

Since x, y and $\gamma_t^{(y)}$ and the geodesic connecting x and y lie in a Eucldean strip it holds

$$d(x,y) \le d(x,\gamma_t^{(y)}).$$

This shows that

$$t \mapsto t^2 - d(x,\gamma_t^{(y)})^2$$

has a minimum at $t=0$. Thus $b_{\gamma^{(x)}}(y) = 0$. Using the fact that $b^+_{\gamma^{(x)}}(\gamma_t^{(y)}) = b^+_{\gamma^{(x)}}(y) + t$ we see $b^\pm_\gamma \equiv b^\pm_{\gamma^{(x)}}$ for all $x \in M$. But then

$$d(\gamma_t^{(x)},\gamma_s^{(y)})^2 = d(x,y)^2 + |t-s|^2$$

proving that $(x,t) \mapsto \gamma_t^{(x)}$ is an isometry of $\tilde{M} \times_2 \mathbb{R}$ onto M. $\square$

4 One dimensional spaces

4.1 Non-branching spaces

Definition 4.1. A non-branching geodesic space (M,d) is not 1-dimensional if for unit speed geodesics $\gamma : [0,a] \to M$ and $\epsilon > 0$ there is a $y \in B_a(\gamma_a) \cap B_\epsilon(\gamma_0)$ with $y \notin \gamma([0,a])$.

Lemma 4.2. *If (M,d) is a geodesic space such that for each point $x \in M$ there is a $\delta > 0$ such that $B_\delta(x)$ is isometric to $(-\delta,\delta)$ or $[0,\delta)$ then M is isometric to one of the following:*

1. $\mathbb{R}$
2. $[0,\infty)$
3. $\frac{\operatorname{diam} M}{\pi} \cdot \mathbb{S}^1$
4. $[0, \operatorname{diam} M]$.

Proof. The assumption imply that M is locally uniquely geodesic and non-branching. Indeed, if $y \in B_\delta(x)$ then $y = \gamma_{d(x,y)}$ or $\gamma_{-d(x,y)}$ where γ is the geodesic representing either $(-\delta,\delta)$ or $[0,\delta)$.

Now let $\gamma : I \to M$ be a maximal unit speed local geodesic with $\gamma_0 = x$, i.e. for all local geodesics $\eta : I' \to M$ with $I \subset I'$ and $\eta\big|_I \equiv \gamma$ it holds $I = I'$ and $\eta \equiv \gamma$.

Let $z \in M$ and η be a unit speed geodesic connecting γ_0 and z. But then $B_\delta(x) \cap \eta([0,\delta)) \subset \gamma(I)$ implying that z is on γ as M is non-branching. In particular, γ is onto.

There are now three cases for I: $I = [a,b]$, $I = [0,\infty)$ and $I = \mathbb{R}$. If γ was not injective then for some $t_1 \neq t_2 \in I$ it holds $\gamma_{t_1} = \gamma_{t_0}$. But then γ can be extended to a geodesic on $\mathbb{R}$. Thus the first and second case imply that γ is injective and M isometric to $[0, \operatorname{diam} M]$ or resp. $[0,\infty)$.

Assume $I = \mathbb{R}$ and γ is not-injective such that $\gamma_0 = \gamma_{t_0}$ for $t_0 > 0$. One may easily verify that $\varphi : \frac{t_0}{\pi}\mathbb{S}^1 \to M$ defined by

$$\varphi(\alpha) = \gamma_\alpha$$

defines an isometry between $\frac{t_0}{\pi}\mathbb{S}^1$ and M.

Finally, if $I = \mathbb{R}$ and γ is injective then γ defines an isometry covering the last case. $\square$

Proposition 4.3. *A 1-dimensional non-branching space (M,d) is isometric to the following:*

1. $\mathbb{R}$

2. $[0, \infty)$
3. $\frac{\operatorname{diam} M}{\pi} \cdot \mathbb{S}^1$
4. $[0, \operatorname{diam} M]$

Proof. Let γ and ϵ be given by the converse of the definition of being not 1-dimensional. Then

$$B_a(\gamma_a) \cap B_\epsilon(\gamma_0) = \gamma((0,\epsilon)).$$

Since (M,d) is geodesic this also shows

$$\bar{B}_a(\gamma_a) \cap \bar{B}_\epsilon(\gamma_0) = \gamma([0,\epsilon]).$$

Let $x, y \in M \backslash \gamma([0,\epsilon])$ with $d(x,y) < \frac{\epsilon}{2}$ and by reversing γ and/or exchanging x and y we may assume $d(y,\gamma_0) \leq d(x,\gamma_0) \leq d(x,\gamma_\epsilon)$. Choose unit geodesics η and ξ connecting $\gamma_{\frac{\epsilon}{2}}$ and x and resp. $\gamma_{\frac{\epsilon}{2}}$ and x. Then $\eta([0,\frac{\epsilon}{2}]) = \xi([0,\frac{\epsilon}{2}]) = \gamma([0,\frac{\epsilon}{2}])$. By non-branching we see that

$$\eta\big|_{[0,d(x,\gamma_{\frac{\epsilon}{2}})} \equiv \xi\big|_{[0,d(y,\gamma_{\frac{\epsilon}{2}})]}.$$

implying that

$$\bar{B}_{\frac{\epsilon}{2}}(x) \cap \eta([0, d(x,\gamma_{\frac{\epsilon}{2}})) = \eta([d(x,\gamma_{\frac{\epsilon}{2}}) - \frac{\epsilon}{2}, d(x,\gamma_{\frac{\epsilon}{2}})]).$$

Thus if a point z lies in the interior of a geodesic then $B_\delta(z)$ is isometric the interval $(-\delta,\delta)$ if δ is sufficiently small. For all other cases we have a δ such that $B_\delta(z)$ is isometric to $[0,\delta)$. By the previous lemma we obtain the claim. □

4.2 Uniquely geodesic spaces

Definition 4.4. A uniquely geodesic space (M,d) is 1-dimensional if for all $x, y \in M$ and $\epsilon \in (0, \frac{1}{2}d(x,y))$ any geodesic connecting $x' \in B_\epsilon(x)$ and $y' \in B_\epsilon(y)$ intersects the geodesic connecting x and y.

Definition 4.5. A uniquely geodesic space (M,d) is a metric tree if for points $x, y, z \in M$ there is a point m such that

$$\begin{aligned} d(x,y) &= d(x,m) + d(m,y) \\ d(x,z) &= d(x,m) + d(m,z) \\ d(y,z) &= d(y,m) + d(m,z). \end{aligned}$$

Lemma 4.6. *Assume (M,d) is a uniquely 1-dimensional geodesic space. Then for all $x \in M$ and all geodesics η with $x \notin \eta([0,1])$ there is $\delta > 0$ and an $y \in M$ such that $y \in \gamma^{(s)}([\delta,1])$ where $\gamma^{(s)}$ is the geodesic connecting x and η_s.*

Proof. Let $3\epsilon > \inf d(x, \eta_s)$ and set $\zeta^{(s)}_{td_s} = \gamma^{(s)}_t$ where $d_s = d(x, \eta_s)$, i.e. let $\zeta^{(s)}$ be the unit speed geodesics connecting x and η_s. Choose $\epsilon \in (0, \inf d_s)$ and observe that $\zeta^{(s)}$ and $\zeta^{(s')}$ intersect whenever $d(\eta_s, \eta_{s'}) < \epsilon$. However, (M, d) is uniquely geodesics so that there must be an $r \in (0, \inf\{d_s, d_{s'}\}]$ such that $\zeta^{(s)}\big|_{[0,r]} \equiv \zeta^{(s)}\big|_{[0,r]}$. Since $\{\eta_s\}_{s\in[0,1]}$ is compact we may choose $r \in (0, \inf d_s]$ such that $\zeta^{(s)}\big|_{[0,r]} \equiv \zeta^{(s)}\big|_{[0,r]}$ for all $s, s' \in [0, 1]$. Choose $y = \zeta^{(s)}_r$ and $\delta = \frac{r}{\inf d_s}$ we obtain the claim. □

Proposition 4.7. *A uniquely* 1*-dimensional geodesic space is a metric tree.*

Proof. Choose a triple $x, y, z \in M$ and let η be the geodesic connecting x and y. Set $\gamma^{(s)}$ to be the geodesics connecting z and η_s.

The previous either $z = \eta_{s_0}$ for some $s_0 \in [0, 1]$ so that choosing $m = z$ will satisfy the definition of metric tree for x, y, z.

Otherwise $\inf_{s\in[0,1]} d(z, \eta_s) > 0$ so that the previous lemma shows all geodesics $\gamma^{(s)}$ intersect in a common point $\tilde{z} \neq z$. Choose $\tilde{z}$ to be as far away as possible from z. Then $x, y, \tilde{z}$ will be a triple and we get geodesics $\tilde{\eta}$ and $\tilde{\gamma}^{(s)}$ as above. Note $\eta = \tilde{\eta}$ and $\tilde{\gamma}$ agrees with the end part of γ. The choice of $\tilde{z}$ shows that the geodesics $\tilde{\gamma}^{(s)}$ cannot intersect in a point unequal from $\tilde{z}$. However, this only possible if $\tilde{z}$ will lie on the geodesic η. Now choosing $m = \tilde{z}$ will suffices to shows that the triple x, y, z satisfies the requirement for (M, d) to be a tree. □

5 Hausdorff and Gromov–Hausdorff distance

5.1 Hausdorff distance of subsets

Definition 5.1 (Hausdorff distance)**.** Let $A, B \subset M$ be two subset be two subsets of a (pseudo)metric space (M, d). We define the *Hausdorff distance* $d_H = d_H^{(M,d)}$ of A and B as follows:

$$d_H(A, B) = \inf\{\epsilon > 0 \,|\, A \subset \bigcup_{y \in B} B_\epsilon(y), B \subset \bigcup_{x \in A} B_\epsilon(x)\}.$$

We leave the following two lemmas as an exercise for the reader.

Lemma 5.2. *For all $A, B \subset M$ of a (pseudo)metric space (M, d) it holds*

$$d_H^{(M,d)}(A, B) = d_H^{(A \cup B, d\big|_{(A\cup B)\times(A\cup B)})}(A, B).$$

Lemma 5.3. *Let (M, d) be a pseudometric space and $(\tilde{M}, \tilde{d})$ the induced metric space, i.e. $\tilde{M} = \{[x]_d \,|\, x \in M\}$ and $\tilde{d}([x]_d, [y]_d) = d(x, y)$ where $[x]_d = \{y \in M \,|\, d(x, y) = 0\}$. Then*

$$d_H^{(M,d)}(A, B) = d_H^{(\tilde{M},\tilde{d})}(\tilde{A}, \tilde{B})$$

where $\tilde{A} = \{[x]_d \,|\, x \in A\}$ and $\tilde{B} = \{[y]_d \,|\, y \in B\}$.

Proposition 5.4. *The following holds:*

- *for all $A, B, C \subset M$*

$$d_H(A, C) \le d_H(A, B) + d_H(B, C).$$

- *for all $A, B \subset M$*

$$d(A, B) = 0 \iff \mathrm{cl}(A) = \mathrm{cl}(B).$$

Proof. Observe for all $x, y \in M$ and $\epsilon, \delta > 0$ whenever $y \in B_\epsilon(x)$ then $B_\delta(y) \subset B_{\epsilon+\delta}(x)$. Hence for $d_H(A, B) < \epsilon$ and $d_H(B, C) < \delta$ we get

$$A \subset \bigcup_{y \in B} B_\epsilon(y) \subset \bigcup_{z \in C} B_{\epsilon+\delta}(z)$$

and similarly

$$C \subset \bigcup_{y \in B} B_\delta(y) \subset \bigcup_{x \in A} B_{\epsilon+\delta}(x).$$

In particular, $d_H(A,C) \le \epsilon + \delta$. Taking the infimum over all $\epsilon, \delta > 0$ with $d_H(A,B) < \epsilon$ and $d_H(B,C) < \delta$ we obtain the triangle inequality for d_H.

Assuming $d_H(A,B) = 0$ this yields

$$\bigcup_{x\in A} B_\epsilon(x) \subset \bigcup_{y\in B} B_{\delta+\epsilon}(y) \subset \bigcup_{x\in A} B_{2\epsilon+\delta}(x)$$

for all $\delta, \epsilon > 0$. Hence

$$\bigcap_{\epsilon>0}\left(\bigcup_{x\in A} B_\epsilon(x)\right) = \bigcap_{\epsilon>0}\left(\bigcup_{y\in B} B_\epsilon(y)\right)$$

We finish the proof by observing that the left-hand side equals $\mathrm{cl}(A)$ and the right-hand side equals $\mathrm{cl}(B)$. □

Corollary 5.5. *If (M,d) is a metric space then the Hausdorff distance is a metric on the set of bounded closed subsets of (M,d).*

5.2 Gromov–Hausdorff distance

We define the disjoint union $M_1 \amalg M_2$ of those two sets M_1 and M_2 as follows

$$M_1 \amalg M_2 = \{(x_i, i) \,|\, i \in \{1,2\}, x_i \in M_i\}.$$

By identifiying M_i with the set $\{(x,i) \,|\, x \in M_i\}$ we frequently regard M_i as a subset of $M_1 \amalg M_2$.

Definition 5.6 (Gromov–Hausdorff distance)**.** For two metric spaces (M_i, d_i), $i = 1,2$, we define the Gromov–Hausdorff distance d_{GH} of (M_1,d_1) and (M_2,d_2) as follows:

$$d_{GH}((M_1,d_1),(M_2,d_2)) = \inf \left\{ \begin{array}{r} d_H^{(M,d)}(\varphi_1(M_1), \varphi_2(M_2)) \,| \\ \varphi_i : (M_i,d_i) \to (M,d) \text{ is an isometric embedding} \end{array} \right\}.$$

We also define a distance via pseudometrics on the disjoint union of M_1 and M_2:

$$\hat{d}_{GH}((M_1,d_1),(M_2,d_2)) = \inf \left\{ \begin{array}{c} d_H^{(M_1 \amalg M_2, \tilde{d})}(M_1, M_2) \,|\, \tilde{d} \text{ is a pseudometric} \\ \text{on } M_1 \amalg M_2 \text{ with } \tilde{d}\big|_{M_i \times M_i} = d_i \end{array} \right\}.$$

Lemma 5.7. *Given isometric embeddings $\varphi_i : (M_i,d_i) \to (M,d)$ we obtain a pseudometric $\tilde{d}$ as follows*

$$\tilde{d}((x_i,i),(x_j,j)) = d(\varphi_i(x_i), \varphi_j(x_j)).$$

In particular,

$$d_H^{(M_1 \amalg M_2, \tilde{d})}(M_1, M_2) = d_H^{(M,d)}(\varphi_1(M_1), \varphi_2(M_2)).$$

Proof. Symmetry follows directly from the definition. Also note that

$$\tilde{d}((x_i, i), (y_i, i)) = d(\varphi_i(x_i), \varphi_i(y_i)) = d_i(x_i, y_i)$$

and

$$\begin{aligned}\tilde{d}((x_i, i), (z_k, k)) &= d(\varphi_i(x_i), \varphi_k(z_k)) \\ &\leq d(\varphi_i(x_i), \varphi_j(y_j)) + d(\varphi_j(y_j), \varphi_k(z_k)) \\ &= \tilde{d}((x_i, i), (y_j, j)) + \tilde{d}((y_j, j), (z_k, k))\end{aligned}$$

which proves the triangle inequality.

To see that the last inequality holds just observe that the induced metric space $(\hat{M}, \hat{d})$ of $(M_1 \amalg M_2, \tilde{d})$ is isometric to $(\varphi_1(M_1) \cup \varphi_2(M_2), \check{d})$ where $\check{d}$ is the restriction of d to $\varphi_1(M_1) \cup \varphi_2(M_2)$. ☐

Lemma 5.8. *If $\tilde{d}$ is a metric on $M_1 \amalg M_2$ then there are naturally defined isometric embeddings φ_i of M_i into the induced metric space $(\hat{M}, \hat{d})$ of $(M_1 \amalg M_2, \tilde{d})$ such that*

$$d_H^{(M_1 \amalg M_2, \tilde{d})}(M_1, M_2) = d_H^{(\hat{M}, \hat{d})}(\varphi_1(M_1), \varphi_2(M_2)).$$

Proof. Denote the projection of $(M_1 \amalg M_2, \tilde{d})$ onto $(\hat{M}, \hat{d})$ by q and note that $\varphi_j = q \circ i_j$ where i_j is the inclusion of M_j into $M_1 \amalg M_2$ satisfies the requirements of the lemma: Indeed, for $x, y \in M_j$ we have

$$\begin{aligned}d_i(x, y) &= \tilde{d}(i_j(x), i_j(y)) \\ &= \hat{d}(q(i_j(x)), q(i_j(y))) \\ &= \hat{d}(\varphi_i(x), \varphi_i(y)).\end{aligned}$$

The last claim is now readily verified. ☐

Corollary 5.9. *It holds $d_{GH} = \hat{d}_{GH}$.*

Proposition 5.10. *Given two pseudometrics d_{12} and d_{23} on $M_1 \amalg M_2$ and resp. $M_2 \amalg M_3$ there is a pseudometric d_{123} on $M_1 \amalg M_2 \amalg M_3$ with $d_{123}\big|_{(M_i \amalg M_j) \times (M_i \amalg M_j)} = d_{ij}$ for $i < j$.*

Proof. We use the notation $x, \tilde{x}, \ldots \in M_1$, $y, \tilde{y}, \ldots \in M_2$ and $z, \tilde{z}, \ldots \in M_3$. Define

$$\begin{aligned}d_{123}(x, \tilde{x}) &:= d_1(x, \tilde{x}) \\ d_{123}(y, \tilde{y}) &:= d_2(y, \tilde{y}) \\ d_{123}(z, \tilde{z}) &:= d_3(z, \tilde{z})\end{aligned}$$

and

$$d_{123}(x, z) := d_{123}(z, x) := \inf_{y \in M_2} d_{12}(x, y) + d_{23}(y, z).$$

One may readily verify that d_{123} is a pseudometric on $M_1 \amalg M_2 \amalg M_3$. ☐

Corollary 5.11. *The Gromov–Hausdorff metric satisfies the triangle inequality, i.e. for (M_i, d_i), $i = 1, 2, 3$, it holds*

$$d_{GH}((M_1, d_1), (M_3, d_3)) \le d_{GH}((M_1, d_1), (M_2, d_2)) + d_{GH}((M_2, d_2), (M_3, d_3)).$$

Lemma 5.12. *Let (M_i, d_i), $i = 1, 2$, be two complete metric spaces. If $\varphi_i : (M_i, d_i) \to (\tilde{M}, \tilde{d})$ are isometric embedding into a common metric space $(\tilde{M}, \tilde{d})$ such that $\varphi_1(M_1) \cup \varphi_2(M_2) = \tilde{M}$ then $\varphi_i(M_i)$ are closed subsets of $\tilde{M}$. In particular, $(\tilde{M}, \tilde{d})$ is complete.*

Proof. Just note if $(\varphi_i(x_n^i))_{n\in\mathbb{N}}$ is convergent to $\tilde{x} \in M$ then it is Cauchy. Since φ_i is an isometric embedding also $(x_n^i)_{n\in\mathbb{N}}$ Cauchy and by completeness it converges to some $x^i \in M_i$. Using again φ_i we see that $\tilde{x} = \varphi_i(x_i)$ which proves that $\varphi_i(M_i)$ is closed. We leave the proof of completeness to the interested reader. □

Theorem 5.13. *The Gromov–Hausdorff distance induces a metric on the isometry class of complete metric spaces.*

Proof. Let (M_i, d_i) be two complete metric spaces. It suffices to show that

$$d_{GH}((M_1, d_1), (M_2, d_2)) = 0$$

if and only if they are isoemtric. The only-if part is easy as any isometry $\varphi : (M_1, d_1) \to (M_2, d_2)$ we get $d_H^{(M_2,d_2)}(\varphi(M_1), M_2) = 0$ which implies $d_{GH}((M_1, d_1), (M_2, d_2)) = 0$.

So assume $d_{GH}((M_1, d_1), (M_2, d_2)) = 0$. Then there is a $(\delta_n)_{n\in\mathbb{N}}$ be a sequence of pseudometrics with $\delta_n\big|_{M_i\times M_i} = d_i$ and

$$d_H^{(M_1 \amalg M_2, \tilde{d}_n)}(M_1, M_2) \to 0.$$

Setting $\delta = \lim_\omega \delta_n$ for some non-principle ultrafilter ω on $\mathbb{N}$ we obtain a pseudometric δ with $\delta\big|_{M_i\times M_i} = d_i$ and

$$d_H^{(M_1 \amalg M_2, \delta)}(M_1, M_2) = 0.$$

Let $(\tilde{M}, \tilde{d})$ be the induced metric space of $(M_1 \amalg M_2, \delta)$. Then there are isometric embeddings $\varphi_i : (M_i, d_i) \to (\tilde{M}, \tilde{d})$ such that $\varphi_i(M_i)$ are closed subsets of $\tilde{M}$ and

$$d_H^{(\tilde{M},\tilde{d})}(\varphi_1(M_1), \varphi_2(M_2)) = 0.$$

But $\varphi_i(M_i)$ are closed subsets so that $\varphi_1(M_1) = \varphi_2(M_2)$. This shows that φ_2^{-1} is well-defined on $\varphi_1(M_1)$. In particular, $\varphi_2^{-1} \circ \varphi_1 : (M_1, d_1) \to (M_2, d_2)$ defines an isometry between the two spaces.

□

Proposition 5.14. *Let be the set $\mathcal{X}_c$ of isometry class of compact metric spaces is closed w.r.t. d_{GH}. In particular, for all complete non-compact bounded metric spaces (M, d) there is an $\epsilon > 0$*

$$d_{GH}((M, d), (N, \delta)) \ge \epsilon$$

for all $[N, \delta] \in \mathcal{X}_c$.

Lemma 5.15. *If $d_{GH}((M,d),(N,\delta)) < \epsilon$ and (N,δ) admits a finite ϵ-net then (M,d) admits a finite 4ϵ-net.*

Proof. Let $\tilde{d}$ be a metric realizing $d_{GH}((M,d),(N,\delta)) < \epsilon$ and $A \subset N$ be a finite ϵ-net. Observe that

$$M \subset B_\epsilon^{\tilde{d}}(N) \subset B_{2\epsilon}^{\tilde{d}}(A)$$

Now for each $a \in A$ choose $x(a) \in M$ with $d(a,x) < 2\epsilon$. Let $B = \{x(a)\}_{a \in A}$. Then we have

$$A \subset B_{2\epsilon}(B)$$

so that

$$M \subset B_{4\epsilon}(B).$$

□

Proof of the proposition. Assume for a given complete metric spaces and all each $\epsilon > 0$ there is a compact metric space (N,d) with $d_{GH}((M,d),(N,\delta)) < \epsilon$. Since (N,d) is compact it admits a finite ϵ-net. The lemma implies (M,d) also admits a finite 4ϵ-net. However, this implies (M,d) is totally bounded. Then completeness implies it must be compact. This proves the claim. □

5.3 ϵ-Approximations

Definition 5.16 (ϵ-approximation)**.** A map $\Phi : (M,d) \to (N,\delta)$ is called an ϵ-*approximation* if $\Psi(M)$ is an ϵ-net of N and for all $x, y \in M$

$$|d(x,y) - \delta(\Phi(x),\Phi(y))| \le \epsilon.$$

We say (M_1,d_1) and (M_2,d_2) are ϵ-approximations of each other if there are ϵ-approximations $\Phi_{ij} : (M_i,d_i) \to (M_j,d_j)$, $i,j = 1,2$. This will give us a notion of distance

$$\hat{d}_{GH}((M_1,d_1),(M_2,d_2)) = \inf\{\epsilon > 0 \,|\, (M_1,d_1) \text{ and } (M_2,d_2) \text{ are } \epsilon\text{-approximations of each other}\}.$$

Lemma 5.17. *For every ϵ-net A of a metric space (M,d) it holds $\hat{d}_{GH}((M,d),(A,d_{A\times A})) \le 2\epsilon$.*

Proof. Define $\Psi : A \to M$ as the identity and $\Phi(x)$, $x \in M$, as any point $a \in A$ with $d(x,a) \le \epsilon$ with $\Phi(a) = a$ for all $a \in A$. The map Ψ satisfies the required properties. Since Φ is the identity on A we also see that $\Phi(M)$ is a δ-net of A for all δ. Now let $x, y \in M$ and observe

$$\begin{aligned} d(\Phi(x),\Phi(y)) &\le d(\Phi(x),x) + d(x,y) + d(y,\Phi(y)) \\ &\le 2\epsilon + d(x,y) \end{aligned}$$

and

$$\begin{aligned} d(x,y) &\le d(x,\Phi(x)) + d(\Phi(x),\Phi(y)) + d(\Phi(y),y) \\ &\le 2\epsilon + d(\Phi(x),\Phi(y)). \end{aligned}$$

□

Lemma 5.18. *If thre is an ϵ-approximation $\Phi : (M_1, d_1) \to (M_2, d_2)$ then there is a pseudometric $\tilde{d}$ on $M_1 \amalg M_2$ with $\tilde{d}\big|_{M_i \times M_i} = d_i$ and*

$$d_H^{(M_1 \amalg M_2, \tilde{d})}(M_1, M_2)) < 2\epsilon.$$

In particular, $d_{GH}((M_1, d_1), (M_2, d_2)) < 2\epsilon$

Proof. We use the following notation: $x, \tilde{x}, x', \ldots \in M_1$ and $y, \tilde{y}, y', \ldots \in M_2$. On M_i define $\tilde{d}$ by d_i and for $x \in M_1$ and $y \in M_2$ set

$$\tilde{d}(y, x) = \tilde{d}(x, y) = \inf_{\tilde{x} \in M_1} d_1(x, \tilde{x}) + \epsilon + d_2(\Phi(x), y).$$

Then

$$\begin{aligned}
\tilde{d}(x, y) &= \inf_{\tilde{x} \in M_1} d_1(x, \tilde{x}) + \epsilon + d_2(\Phi(x), y) \\
&\leq \inf_{\tilde{x} \in M_1} d_1(x, x') + d_1(x', \tilde{x}) + \epsilon + d_2(\Phi(x), y) \\
&= \tilde{d}(x, x') + \tilde{d}(x', \tilde{y})
\end{aligned}$$

and

$$\begin{aligned}
\tilde{d}(x, y) &= \inf_{\tilde{x} \in M_1} d_1(x, \tilde{x}) + \epsilon + d_2(\Phi(x), y) \\
&\leq \inf_{\tilde{x} \in M_1} d_1(x, x') + \epsilon + d_2(\Phi(x), y') + d(y', y) \\
&= \tilde{d}(x, y') + \tilde{d}(y', \tilde{y})
\end{aligned}$$

as well as

$$\begin{aligned}
\tilde{d}(x, x') &\leq \inf_{\tilde{x}, \hat{x} \in M_1} d_1(x, \tilde{x}) + d_1(\tilde{x}, \hat{x}) + d_1(\hat{x}, x') \\
&\leq \inf_{\tilde{x}, \hat{x} \in M_1} d_1(x, \tilde{x}) + 2\epsilon + d_1(\Phi(\tilde{x}), \Phi(\hat{x})) + d_1(\hat{x}, x') \\
&\leq \inf_{\tilde{x}, \hat{x} \in M_1} d_1(x, \tilde{x}) + 2\epsilon + d_1(\Phi(\tilde{x}), y) + d(y, \Phi(\hat{x})) + d_1(\hat{x}, x') \\
&= \tilde{d}(x, y) + \tilde{d}(y, x)
\end{aligned}$$

and

$$\begin{aligned}
\tilde{d}(y, y') &\leq \inf_{\tilde{x}, \hat{x} \in M_1} d_2(y, \Phi(\tilde{x})) + d_2(\Phi(\tilde{x}), \Phi(x)) + d_2(\Phi(x), \Phi(\hat{x})) + d_2(\Phi(\hat{x}), y') \\
&\leq \inf_{\tilde{x}, \hat{x} \in M_1} d_2(y, \Phi(\tilde{x})) + 2\epsilon + d_1(\tilde{x}, x) + d_1(x, \hat{x}) + d_2(\Phi(\hat{x}), x') \\
&= \tilde{d}(y, x) + \tilde{d}(x, y').
\end{aligned}$$

Because $d(x, \Phi(x)) = \epsilon$ and $\Phi(M_1)$ is an ϵ-net in M_2 we have $B_{2\epsilon}(M_1) \supset M_2$ and $B_\epsilon(M_2) \supset M_1$ proving the claim. $\square$

Lemma 5.19. *If $d_{GH}((M_1,d_1),(M_2,d_2)) < \epsilon$ then there is an ϵ-approximation $\Phi : (M_1,d_1) \to (M_2,d_2)$. In particular, $\hat{d}_{GH}((M_1,d_1),(M_2,d_2)) < 2\epsilon$.*

Proof. Let $\tilde{d}$ be a pseudometric realizing $d_{GH}((M_1,d_1),(M_2,d_2)) < \delta < \epsilon$. Define for $x \in M_1$ the sets

$$A_x = \{y \in M_2 \,|\, \tilde{d}(x,y) \le \delta\}.$$

Using the triangle inequality we see $d_1(y,y') \le 2\delta$ for all $y, y' \in A_x$. Similarly, for $y' \in A_{x'}$ and $y \in A_x$ we have

$$|\tilde{d}(x,x') - \tilde{d}(y,y')| \le \tilde{d}(x,y) + d(x',y') \le 2\delta$$

Now choose $\Phi : M_1 \to M_2$ with $\Phi(x) \in A_x$. Since $B_\delta^{\tilde{d}}(M_1) \supset M_2$ we see that

$$\bigcup_{x \in M_1} A_x = M_2.$$

Thus $\Phi(M_1)$ is an δ-net of M_2 proving it is a 2δ-approximation. □

Corollary 5.20. *It holds*

$$\hat{d}_{GH} \le 2d_{GH} \le 4\hat{d}_{GH}.$$

Proposition 5.21. *If d_1 and d_2 are two metrics on M then the identity induces an a-approximation for all $a \ge \|d_1 - d_2\|_\infty$. In particular, uniform convergence of metrics is stronger than the Gromov–Hausdorff convergence.*

5.4 Finite metric spaces and Gromov–Hausdorff convergence

5.5 Gromov Precompactness Theorem

Definition 5.22. A set $\mathcal{A}$ of (isometry classes of) compact metric spaces is called uniformly totally bounded if there is a function $M : (0,\epsilon) \to \mathbb{N}$ such that for all $(M,d) \in \mathcal{A}$ it holds $M^{(M,d)} \le M$.

Theorem 5.23. *A set $\mathcal{A}$ (of isometry classes of) compact metric spaces is precompact w.r.t. d_{GH} if and only if $\mathcal{A}$ is uniformly totally bounded.*

We first prove the theorem in a series for lemmas:

Lemma 5.24. *The set $\mathcal{X}_{N,D} = \{[M,d] \,|\, |M| \le N, \operatorname{diam}(M,d) \le D\}$ is compact w.r.t. d_{GH}.*

Proof. It suffices to prove that a sequence of finite metric spaces with exactly $k \le N$ points has a convergent subsequence. Since we talk about isometry class we may parametrize all those spaces by $M = \{x_1, \ldots, x_k\}$ and assume d_n are metrics on M. Since the diameter is bounded we also see that $\{(d_n(x_i,x_j))_{i,j}^k\}_{n\in\mathbb{N}}$ is precompact in $\mathbb{R}^{k\times k}$ equipped with the maximum norm. Note that this also shows that $\{d_n : M \times M \to \mathbb{R}\}$ is precompact in the uniform topology on $C_0(M \times M)$. Thus we may replace the sequence and assume $\|d_n - \delta\| \le \frac{1}{n}$ where δ is a pseudonorm on M. Let $q : (M, \delta) \to (\tilde{M}, d)$

We claim $\Phi_n : (M, d_n) \to (\tilde{M}, d)$ defined by $x_i \mapsto q(x_i)$ are $\frac{1}{n}$-approximations: By definition $\Phi_n(M) = \tilde{M}$ and

$$|d(\Phi_n(x_i), \Phi_n(x_j)) - d_n(x_i, x_j)| = |d(x_i, x_j) - d_n(x_i, x_j)| \le \frac{1}{n}$$

proving the claim.

We conclude using Lemma 5.18. □

Lemma 5.25. *Let $A_n \subset A_{n+1}$ be a non-decreasing sequence of sets and d a metric on $A = \bigcup_{n\in\mathbb{N}} A_n$ such that A_n is an ϵ_n-net of (A, d) with $\epsilon_n \to 0$. Then $(A_n, d\big|_{A_n \times A_n})$ GH-converges to the completion of (A, d).*

Proof of the theorem. Let $[M_n, d_n] \in \mathcal{A}$ be a sequence of isometry class of compact metric spaces in $\mathcal{A}$. For each $k \in \mathbb{N}$ let $\tilde{M}_{n,k}$ be a maximal $\frac{1}{k}$-separated set of M. Define

$$M_{n,k} = \bigcup_{k'=1}^{k} \tilde{M}_{n,k}.$$

Then $|M_{n,k}| \le \sum_{k'=1}^{k} M(\frac{1}{k'}) = M(k) < \infty$.

Now construct inductively strictly increasing functions $\varphi : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that

$$\varphi(0, n) = n$$
$$\varphi(k+1, \mathbb{N}) \subset \varphi(k, \mathbb{N})$$

and for $k \ge 1$ the sequence $([M_{\varphi(k,n),k}, d_{\varphi(k,n),k}])_{n\in\mathbb{N}}$ converges to a finite metric space $(M^{(k)}, d^{(k)})$.

Since $M_{n,k} \subset M_{n,k+1}$ we may assume $M^{(k)} \subset M^{(k+1)}$ and $d^{(k)} = d^{(k+1)}\big|_{M_k \times M_k}$. Thus there is a metric d_∞ on $\bigcup_{k\in\mathbb{N}} M^{(k)}$.

Also note that for $l > k$ it holds

$$\sup_{x\in M_{n,l}} \inf_{y\in M_{n,k}} d_n(x, y) \le \frac{1}{k}.$$

Thus we have

$$\sup_{x\in M^{(l)}} \inf_{y\in M^{(k)}} d_n(x, y) \le \frac{1}{k}.$$

But then M_k will be an $(\frac{1}{k} + \delta)$-net for all $\delta > 0$ implying the completion (M, d) of $(\bigcup_{k\in\mathbb{N}} M^{(k)}, d_\infty)$ is a compact metric space. To conclude we define $\psi(k) \in \varphi(k, \mathbb{N})$ such that $d_{GH}((M_{\psi(k),k}, d_{\psi(k),k}), (M^{(k)}, d^{(k)})) < \frac{1}{k}$ and $\psi(k) < \psi(k+1)$ and observe

$$\begin{aligned} d_{GH}((M, d), (M_{\psi(k)}, d_{\psi(k)})) &\le d_{GH}((M, d), (M^{(k)}, d^{(k)})) + d_{GH}((M^{(k)}, d^{(k)}), (M_{\psi(k),k}, d_{\psi(k),k})) \\ &\quad + d_{GH}((M_{\psi(k),k}, d_{\psi(k),k}), (M_{\psi(k)}, d_{\psi(k)})) \\ &\le \frac{3}{k}. \end{aligned}$$

where we used the fact that the distance of an ϵ-net to the full space is less that ϵ. □

5.6 Gromov–Hausdorff and ultralimits

Proposition 5.26. *Suppose $(M_n, d_n)_{n\in\mathbb{N}}$ is a sequence of compact metric spaces GH-converging to (M, d). Then any ultralimit of (M_n, d_n) is isometric to (M, d).*

Lemma 5.27. *If $(M_n, d_n) \in \mathcal{X}_{N,D}$ GH-converges to (M, d) then for any non-principle ultrafilter ω on $\mathbb{N}$ there is an index set $I \subset \mathbb{N}$, a pseudometric space $(\hat{M}, \hat{d})$ and parametrization of $M_n = \hat{M}$ for $n \in I$ such that $\lim_\omega \|d_n - \hat{d}\|_\infty = 0$ and the metric space (M, d) is isometric to the one obtained from $(\hat{M}, \hat{d})$. In particular the statement of the proposition is true for finite metric space of order at most L.*

Proof. Let $L_n = \#M_n$ and $L_\omega = \lim_\omega L_n$. Then $I = \{n \in \mathbb{N} \,|\, L_n = L_\omega\} \in \omega$ and we may assume $\hat{M} := \{x_1, \ldots, x_{L_\omega}\} = M_n$. Since $\{(d_n(x_i, x_j))_{n\in\mathbb{N}} \,|\, i, j \in \{1, \ldots, L_\omega\}\}$ is a finite set of bounded sequences we see that

$$\lim_\omega \|d_n - \hat{d}\|_\infty = \sup_{i,j\in\{1,\ldots,L_\omega\}} \lim_\omega |d_n(x_i, x_j) - \hat{d}(x_i, x_j)| = 0.$$

This implies (M, d) is isometric to the metric spaces obtained from $(\hat{M}, \hat{d})$.

To prove the last statement we claim for each $(x_n)_{n\in\mathbb{N}} \in M_\omega$ there is $k \in \{1, \ldots, L_\infty\}$ with $d_\omega((x_n)_{n\in\mathbb{N}}, (x_k)_{n\in\mathbb{N}}) = \lim_\omega d_n(x_n, x_k)$ where $M_\omega = \{(x_n) \,|\, x_n \in M_n\}$. Indeed, if this was true then the metric spaces obtained from $(\hat{M}_\omega, d_\omega)$ and $(\hat{M}, \hat{d})$ are isometric.

To prove the claim just observe that $\{I_k = \{x_n = k \,|\, n \in \mathbb{N}\}\}_{k=1}^{L_\omega}$ forms a partition of $\mathbb{N}$ so that for exactly one $I_l \in \omega$. But then

$$\lim_\omega d_n(x_n, x_k) = \lim_\omega d_n(x_l, x_l)$$

proving the claim. $\square$

Remark. If $\inf_{n,i,j} d_n(x_i, x_j) > 0$ then $\hat{d}$ will be a true metric on $\hat{M}$.

Proof of the proposition. Let $M_{n,k}$ be maximal $\frac{1}{k}$-separated sets of (M_n, d_n). Then for all we may assume $(M_{n,k}, d_n^k)$, $d_n^k = d_n\big|_{M_{n,k}\times M_{n,k}}$ will converge to a metric space (M_k, d^k) which can be seen as a $\frac{1}{k}$-separated set of (M, d) such that $d^k = d\big|_{M_k\times M_k}$. Note that any of those sets is also a $\frac{1}{k}$-net.

By Gromov Precompactness we know that $\{(M_{n,k}, d_n^k), (M_k, d^k)\}$ are finite metric spaces of bounded order. Thus any ultralimit of $(M_{n,k}, d_n)$ will agree with (M_k, d). By the previous remark we may assume $\lim_\omega \|d_n - d\| = 0$, i.e. for all $x_n, y_n \in M_{n,k}$ we have $\lim_\omega d_n(x_n, y_n) = d(((x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}})$.

Now let $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ be sequences with $x_n, y_n \in M_n$. Then for all n there are $x_{n,n}, y_{n,n} \in M_{n,n}$ and $d_n(x_n, x_{n,n}), d_n(y_n, y_{n,n}) \leq \frac{1}{n}$. Therefore,

$$|d_n(x_n, y_n) - d_n(x_{n,n}, y_{n,n}))| \leq \lim_\omega d_n(x_n, x_{n,n}) + d_n(y_n, y_{n,n})$$

implying that the metric space obtained from $\hat{M}_\infty = \{(x_n)_{n\in\mathbb{N}} \,|\, x_n \in M_{n,n}\}$ equipped with the pseudometric $d_\omega((x_n), (y_n)) = \lim_\omega d_n(x_n, y_n)$ is isometric to the ultralimit of

$(M_n, d_n)_{n\in\mathbb{N}}$. Now observe that the ultralimits (M_k, d_k) of $(M_{n,k}, d_n^k)_{n\in\mathbb{N}}$ are isometric to $\frac{1}{k}$-nets of the ultralimit. This shows that the ultralimit is compact. But then it agrees with the completion of the dense subset $\bigcup_{k\in\mathbb{N}} M_k$ where we treated M_k as subsets of the ultralimit. This proves the proposition. $\square$

Corollary 5.28. *If $(M_n, d_n)_{n\in\mathbb{N}}$ is a sequence of metric spaces such that for some non-principle ultrafilter ω its ultralimit is a compact metric space (M, d) then there is a subsequence of $(M_n, d_n)_{n\in\mathbb{N}}$ GH-converging to (M, d).*

Proof. Let M_k be a maximal $\frac{1}{k}$-separated set of (M, d). Then M_k is also a $\frac{1}{k}$-net of M.

We claim that there are index sets $I_k \in \omega$ such that M_n admits a $\frac{2}{k}$-net of cardinatility at most M_k: Set $M_{n,k} = \{x_{n,i} \,|\, i = 1, \ldots, |M_k|\}$ where $M_k = \{[x_{n,1}]_{n\in\mathbb{N}}, \ldots, [x_{n,|M_k|}]_{n\in\mathbb{N}}\}$. We claim

$$\omega \ni I_k := \bigcup_{i=1}^{|M_k|} I_{k,i}$$

where

$$I_{k,i} = \{n \in \mathbb{N} \,|\, d_n(x_{n,i}, x) \le \frac{2}{k}\}.$$

Indeed, for each $[x_n]_{n\in\mathbb{N}} \in M$ there is a i with

$$d_\omega([x_n], [x_{n,i}]) < \frac{1}{k}$$

implying $I_{k,i} \in \omega$. But then $I_k \in \omega$ because ω is a filter.

We conclude by observing because $|M_k| < \infty$ we have

$$\lim_\omega |d_n(x_{n,i}, x_{n,j}) - d([x_{n,i}], [x_{n,j}])| = |\lim_\omega d_n(x_{n,i}, x_{n,j}) - d([x_{n,i}], [x_{n,j}])| = 0$$

Hence

$$\omega \ni J_k := I_k \cap \{n \in \mathbb{N} \,|\, d_{GH}((M_{n,k}, d_n), (M_k, d)) \le \frac{1}{k}\}.$$

We conclude by observing for $n \in J_k$

$$\begin{aligned} d_{GH}((M_n, d_n), (M, d)) &\le d_{GH}((M_n, d_n), (M_{n,k}, d_n)) + d_{GH}((M_{n,k}, d_n), (M_k, d)) \\ &\quad + d_{GH}((M_k, d_k), (M, d)) \\ &\le \frac{3}{k}. \end{aligned}$$

Picking an increasing sequence $n_k \in J_k$ we see

$$d_{GH}((M_{n_k}, d_{n_k}), (M, d)) \to 0$$

as $k \to \infty$. $\square$

Remark. Without compactness on the limiting space the claim is in general wrong: Take any complete, separable, non-compact and bounded metric space (M, d). Then the ultralimit of the constant sequence $(M, d)_{n\in\mathbb{N}}$ is not separable by Proposition 2.12. Thus the ultralimit not isometric to (M, d). But the constant sequence GH-converges to itself showing that it cannot converge to the ultralimit.