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Limits of Spaces

Summer semester 2019 **06.05.2019**

Exercise sheet 2

Exercise 5

(Hopf–Rinow - 3 points) Let (M, d) be a metric space. Show that the following statements are equivalent:

1. (M, d) is a proper length space
2. (M, d) is locally compact, geodesic and whenever $\gamma : [0, 1) \to M$ is a geodesic then γ can be completed to a geodesic on all of $[0, 1]$.

Exercise 6

(2 points) Let ω be an ultrafilter on $\mathbb{N}$ and assume $(a_n), (b_n)$ are non-negative sequences. Show the following that the ulterlimit of non-negative sequences is additive. Give an example for which the ultralimit is non-additive if non-negativity is dropped.

Exercise 7

(3 points) Let (X, d) be a metric space and $x_0 \in X$. Let (X_∞, d_∞) be the pointed ultralimit of the constant sequence $((X, d))_{n\in\mathbb{N}}$. Show that if (X_∞, d_∞) is proper then it is isometric to (X, d).

Exercise 8

(2 points) Let (X, d) be a metric space. An ϵ-spanning is a set $A \subset X$ such that for all $y \in X$ there is $x \in A$ with $d(x, y) \le \epsilon$. An ϵ-separated set is a set $B \subset X$ such that for all $x, y \in B$ it holds $d(x, y) \ge \epsilon$.

Denote by $N(\epsilon)$ the minimal cardinality of an ϵ-spanning set in (X, d) and by $M(\epsilon)$ the maximal cardinality of an ϵ-separated set. Show that $N(\epsilon)$ and $M(\epsilon)$ are comparible for all $\epsilon > 0$ independent of (X, d), i.e. there is a $C > 1$ such that

$$C^{-1}N(\epsilon) \le M(\epsilon) \le CN(\epsilon).$$

Deadline 20.05.2019 in mailbox C-buidling 3rd floor or office C5P08.

You reach the website of the lecture under `http://tinyurl.com/Limits-of-Spaces`.