EBERHARD KARLS
UNIVERSITÄT
TÜBINGEN

Faculty of Science

Department of Mathematics

PD Dr. Martin Kell

Limits of Spaces

Summer semester 2019 **11.05.2019**

Exercise sheet 3

Exercise 9

(One-dimensional non-branching spaces - 3 points) A non-branching geodesic space (M,d) is said to be *not one-dimensional* if for any unit speed geodesic $\gamma:[0,a]\to X$ and $\epsilon>0$ there is a $y\in B_a(\gamma_0)$ with $d(\gamma_a,y)<\epsilon$ but $y\notin\gamma_{[0,a]}$.

Classify all one-dimensional non-branching spaces.

Exercise 10

(One-dimensional uniquely geodesic spaces - 3 points) A geodesic space is *uniquely geodesic* if between any two points there is exactly one geodesic. We say a uniquely geodesic space (M,d) is one-dimensional if for all $\epsilon>0$ geodesic γ connecting distinct points x,y with $d(x,y)>3\epsilon$ then $\gamma([0,1])\cap\eta([0,1])\neq\varnothing$ for any geodesic η with $\eta_0\in B_\epsilon(x)$ and $\eta_1\in B_\epsilon(y)$.

Show that any one-dimensional uniquely geodesic space is a *metric tree*, i.e. a uniquely geodesic space such that for any three distinct points x,y,z not lying on a common geodesic there is a unique point m which lies on the three geodesics connecting those points.

Exercise 11

(Hyperbolic plane - 4 points) Set $M=B_1(0)\subset\mathbb{R}^2$. On M we define a metric d as follows: Let $p,q\in M$ be distinct points and connect them via a straigh line (in $\mathbb{R}^2$). The line intersects the unit sphere in points a and b. Rename the points to have the points in the order a,p,q,b, i.e. $\|a-q\|>\|a-p\|$ and $\|b-p\|>\|b-q\|$. Define the hyperbolic distanct by

$$d(p,q)=\frac{1}{2}\log\frac{\|a-q\|\|b-p\|}{\|a-p\|\|b-q\|}.$$

(3 extra points) Show d is a metric.

Assume d is a metric. Use law of the logarithm to show that d is a geodesic metric.

Show that the pointed ultralimit of $(M,\frac{1}{n}\cdot d,0)_{n\in\mathbb{N}}$ contains an uncountable metric tree. You may use the fact that the ultralimit is a $CAT(0)$-space and hence uniquely geodesic as well as the fact that (M,d) is homogeous, i.e. for any two points p,q there is an isometry mapping p to q. (Hint: Show first that any compact $K\subset\subset M$ will have bounded distance from 0 depending only on K. Then look at triangles whose vertices lie on three (Euclidean) rays passing through 0.)

(1 extra point) Show that the ultralimit is a metric tree.

Exercise 12

(additional 5-points)

Let (M,d) be a compact metric space. Assume $(\delta_n)_{n\in\mathbb{N}}$ is a sequence of ω-uniformly continuous pseudo-metrics converging uniformly to a pseudo-metric δ. Find an elementary proof to show the following: If the induced spaces $(N_{\delta_n},d_{\delta_n})$ are isometric to a fixed metric space (N,d_0) then also the induced space (N_δ,d_δ) is isometric to (N,d_0).

Deadline 17.06.2019 in mailbox C-buidling 3rd floor or office C5P08.
You reach the website of the lecture under `http://tinyurl.com/Limits-of-Spaces`.