EBERHARD KARLS UNIVERSITÄT TÜBINGEN

Faculty of Science

Department of Mathematics

PD Dr. Martin Kell

Limits of Spaces

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Exercise sheet 4

Exercise 13

(Synthetic definition of a manifold - 5 points) A function $f : M \to \mathbb{R}$ on a geodesic space (M, d) is called *affine* if for all geodesics γ it holds

$$f(\gamma_t) = (1-t)f(\gamma_0) + tf(\gamma_1).$$

We say that a metric space has *finite dimensional* if all its tangent space are proper metric spaces.

Fact. *A finite dimensional geodesically complete space (M, d) is isometric to a finite dimensional normed space if for all distinct $x, y \in M$ there is an affine function $f_{x,y}$ such that $f_{x,y}(x) \neq f_{x,y}(y)$.*

Assume (M, d) is geodesically complete, has finite dimension and for all $x_0 \in M$ there is a function $C = C_{x_0} : (0, \infty) \to (0, \infty]$ with $\limsup_{\epsilon \to 0} C(\epsilon) < \infty$ such that for all $x, y, z \in B_\epsilon(x_0)$ and all midpoints m of y and z and all geodesics γ and η connecting x and y and resp. x and z it holds

$$d^2(x, m) \leq \frac{1}{2}d^2(x, y) + \frac{1}{2}d^2(x, z) - C^{-1}d^2(x, y)$$

and

$$d(\gamma_t, \eta_t) \geq (1 - C\epsilon^2)td(\gamma_1, \eta_1).$$

Show that (M, d) is a finite dimensional manifold.

Exercise 14

(Metric trees - 3 points) Show that a metric tree is a CAT(0)-space.

Exercise 15

(Submetries - 4 points) A map φ between metric spaces (M, d) and $(N, \tilde{d})$ is called a *submetry* if for all $x \in M$ and $r > 0$ it holds $\varphi(B_r(x)) = B_r(\varphi(x))$.

Let (M, d) be a proper geodesic space and $(N, \tilde{d})$ a metric space. Assume φ is a submetry. Show that $(N, \tilde{d})$ is a proper geodesic space and if (M, d) is either non-branching or uniquely geodesic then so is $(N, \tilde{d})$.

(2 extra point) Give an example that shows geodesic completeness is not preserved. Is it possible that $(N, \tilde{d})$ is uniquely geodesic but not (M, d)?

Exercise 16

(Group actions and quotient spaces - 3 points) We say G a group acty by isometries on metric space (M, d) if there is a map $i : g \mapsto \varphi_g$ with $\varphi_g \in C^0(M, M)$ such that for all $g, h \in G$ it holds $d(\varphi_g(x), \varphi_g(y)) = d(x, y)$ and $\varphi_{g \circ h} = \varphi_g \circ \varphi_h$. We may equip G with the unique topology making $g \mapsto \varphi_g$ into a topological embedding.

Let G be a compact group acting by isometries on the proper metric space (M, d). Construct a metric $\hat{d}$ on the space of orbits $\hat{X}$ making the natural embedding $x \mapsto [x]$ into a submetry. Here the orbit of x is defined by $[x] = y \in M \mid \exists g \in G : \varphi_g(x) = y$

Deadline 01.07.2019 in mailbox C-buidling 3rd floor or office C5P08.
You reach the website of the lecture under `http://tinyurl.com/Limits-of-Spaces`.